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A Bimodal Extension of the *Tanh* Skew Normal Distribution: Properties and Applications

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Abstract

This article introduces a novel family of skew distributions namely bimodal *Tanh* skew normal (BTSN) distributions, which incorporates a new skew function with the help of hyperbolic tangent function. This new distribution is designed to accommodate data sets with two modes. Besides, the article presents various essential mathematical properties, such as moments, moment generating function, characteristic function, mean deviation, characterizations and the method for maximum likelihood estimation of this distribution. A simulation study is also conducted using Metropolis–Hastings algorithm to examine the behavior of the obtained parameters. Furthermore, the practical utility of this new distribution is demonstrated through a real life application involving a specific data set. To assess the suitability of the BTSN distribution, the article employs Akaike information criterion (AIC) and Bayesian information criterion (BIC). Finally, a likelihood ratio test is conducted to distinguish between the new model and the existing competing models.

Key Words: Skewness, Skew Normal Distribution, Simulation, Bi-modality, AIC

Mathematical Subject Classification: 60E05, 62E15.

1 Introduction

Though the Gaussian distribution is an important and widely used model of probability distribution, in many practical scenarios the normal distribution may not accurately represent the data when the data appears to be asymmetric or with heavy tails. In order to tackle these type of issues, Azzalini (1985) proposed the skew normal (SN) distribution. The pdf of skew normal distribution is

$$f(x;\lambda) = 2\phi(x)\Phi(\lambda x), \qquad x \in R, \lambda \in R, \tag{1}$$

where, $\phi(\cdot)$, $\Phi(\cdot)$ are the probability density function (pdf) and cumulative distribution function (cdf) of standard normal distribution respectively. The additional parameter λ is also known as asymmetry parameter regulates the asymmetry in the distribution. Undoubtedly the said distribution was a path-breaking introduction towards the families of probability distribution with asymmetric behavior. It is elaborately used in the areas like financial risk management, epidemiology, climate science, environmental studies, biomedical research, quality Control and manufacturing, insurance companies etc. Later on, numerous families of skew distributions were developed by adopting the idea of this ground-breaking distribution. Some perfect examples of those distributions are skew logistic distribution (Nadarajah, 2009), skew laplace distribution (Aryal and Nadarajah, 2005), skew uniform distribution (Nadarajah and Aryal, 2004), skew-t distribution (Theodossiou, 1998) etc. Following the skew normal distribution (Azzalini, 1985), continuous research studies have been carried out by many researchers to develop some new directions towards the domain of skew distribution. For example Chakraborty et al. (2012) introduced a new skew logistic distribution using a new skew function which isn't a cdf. Similarly, using hyperbolic tangent function Mahmoud et al. (2020) discussed a new class of skew normal distribution namely $Tanh$ skew normal distribution. The pdf of $Tanh$ skew normal distribution is given by

$$f(x;\lambda) = 2\phi(x)G(\lambda x); \qquad x \in R, \lambda \in R, \tag{2}$$

where, $G(\lambda x) = \frac{1}{2}\Big(1 + Tanh\Big[\frac{\lambda x}{2}\Big]\Big)$ and $G(x) + G(-x) = 1$.

Nevertheless, the primary goal of many of the previous works were to improve unimodal skew distribution models for diverse applications. However, there are number of practical applications exists in the literature which provides the sufficient evident of data having more than one mode. For example Lim et al. (2002) and Rushforth et al. (1971) showed the presence of bimodality in Blood Glucose distribution. Ochab-Marcinek and Tabaka (2010) found that when the concentration of inducers varies, the gene expression may alternate between being unimodal and bimodal. On the other hand Duarte et al. (2018) discussed alternative methods for crop insurance pricing based on parametric distributions reflecting the skewness as well as bimodality of the data. Lima et al. (2002) showed the nanostructured coatings having a bimodal distribution in their Weibull plots. Besides, McGee et al. (2011) found the presence of bimodality in twitter handling while Li et al. (2017) showed an bimodal distributional pattern in the posting rates of reviewers. Additionally, beyond the works mentioned, numerous real-life scenarios illustrate both the applications and necessity of data exhibiting bimodality.

In the present context, researchers appear to be increasingly focused on exploring probability distribution models that exhibit both unimodality and bimodality, as well as trimodality and multimodality. There are several authors recently suggesting families of distributions for modelling data that exhibit bi-modality. Elal-Olivero et al. (2020) proposed several bimodal symmetric as well as asymmetric distribution adding some new parameters to the Azzalini's SN (Azzalini, 1985) model. During the study, Elal-Olivero et al. (2020) introduced a novel family of skew distribution for supporting bimodal data namely two parameter bimodal skew normal distribution with the pdf

$$f(x;\alpha,\lambda) = 2\frac{1+\alpha x^2}{1+\alpha}\phi(x)\Phi(\lambda x); \qquad x \in R, \lambda \in R, \alpha \geq 0, \tag{3}$$

Besides alpha skew normal (Elal-Olivero, 2010), alpha skew logistic (Hazarika and Chakraborty, 2014), alpha skew laplace (Harandi and Alamatsaz, 2013), generalised alpha skew normal (Sharafi et al., 2017) etc. are some noticeable model of probability distributions which allow to fit data with uni-bimodality character. Additionally, using Balakrishnan (Arnold et al., 2002) mechanism some new model of skew probability distribution were proposed which includes Balakrishnan alpha skew normal distribution (Hazarika et al., 2020), Balakrishnan alpha skew logistic distribution (Shah et al., 2020a), Balakrishnnan alpha skew laplace distribution Shah et al. (2020c) etc.

Nonetheless, there are also some examples of research studies that took into account the bimodal behaviour of the data sets, such as the Balakrishnan log alpha skew normal distribution (Shah et al., 2020b), Bimodal skew-symmetric normal distribution (Harandi and Alamatsaz, 2013), skew bimodal distribution proposed by Braga et al. (2018) and many more.

Furthermore alpha beta skew normal distribution was another new family of skew distribution introduced by Shafiei et al. (2016) which was able to fit data up to three modes. Alpha beta skew logistic distribution (Esmaeili et al., 2020) and alpha beta skew generalized t distribution (Lak et al., 2019) were another two models under the same umbrella for fitting the data up to three modes. Additionally, some new classes of flexible skewed models were proposed by Martínez-Flórez et al. (2022) allowing to fit data up to trimodality. Following this idea there are some other research works were carried out resulting the models like trimodal skew logistic distribution (Pathak et al., 2023) , flexible alpha skew normal distribution (Das et al., 2023) etc.

In this article a novel family of skew distribution is introduced using the hyperbolic tangent ($Tanh$) function instead of cdf of normal distribution. This new distribution is an extension of $Tanh$ skew normal distribution (Mahmoud et al., 2020) which allows to fit data with bimodality. Additionally, the adaptability as well as usefulness of the new model is checked through real life data set.

The rest of the article is organized as follows: In Section 2 the new family of skew normal distribution is introduced which is an extension of $Tanh$ skew normal (Mahmoud et al., 2020) distribution. Some Pictorial visualizations of the new distribution along with its special cases are also included in this section. Furthermore, Some important mathematical properties of the distribution are discussed in Section 3. Section 4 is responsible for the characterizations of the new distribution via truncated moments while the parameter estimation is included in Section 5. Section 6 is devoted to the simulation results and Section 7 provides the real life application to check the flexibility as well as adaptability of the new distribution. On the other hand results of the hypothesis testing are summarized in the Section 8. Finally, Section 9 includes the article.

2 Bimodal $Tanh$ Skew Normal Distribution

A novel extension of the Tanh skew normal distribution is introduced in this section along with the discussion of some significant mathematical properties.

Definition

The random variable X is said to follow the bimodal $Tanh$ skew normal (BTSN) distribution if its probability density function (pdf) is given as

$$f(x;\lambda,\alpha) = \frac{1+\alpha x^2}{C(\alpha)}\phi(x)G(\lambda x), \qquad x \in R, \lambda \in R, \alpha \geq 0 \tag{4}$$

where, $C(\alpha)$ is the normalizing constant and it is calculated as $C(\alpha) = (1+\alpha)$. It is denoted as $X \in BTSN(\alpha,\lambda)$. The skew function $G(\lambda x)$ is defined using the $Tanh$ and obtained as

$$G(\lambda x) = \frac{1}{2}\Big(1 + Tanh\Big[\frac{\lambda x}{2}\Big]\Big), \tag{5}$$

where, λ is the asymmetry parameter. Mahmoud et al. (2020) showed that using the Taylor series expansion, the skew function $G(\lambda x)$ can be rewritten as

$$G(\lambda x) = \begin{cases} \sum_{k=0}^{\infty}(-1)^k \exp(-k\lambda x), & x \geq 0, \\ \sum_{k=0}^{\infty}(-1)^k \exp((k+1)\lambda x), & x < 0, \end{cases} \tag{6}$$

for $\lambda > 0$. While for $\lambda < 0$, it was written as

$$G(-\lambda x) = \begin{cases} \sum_{k=0}^{\infty}(-1)^k \exp((k+1)\lambda x), & x \geq 0, \\ \sum_{k=0}^{\infty}(-1)^k \exp(-k\lambda x), & x < 0. \end{cases} \tag{7}$$

Hence, the density function of the BTSN distribution can be obtained as,

$$f(x;\alpha,\lambda)=\begin{cases}\dfrac{1+\alpha x^2}{C(\alpha)\sqrt{2\pi}}\exp\Big(-\dfrac{x^2}{2}\Big)\sum_{k=0}^{\infty}(-1)^k\exp(-k\lambda x), & x\geq 0,\\ \\ \dfrac{1+\alpha x^2}{C(\alpha)\sqrt{2\pi}}\exp\Big(-\dfrac{x^2}{2}\Big)\sum_{k=0}^{\infty}(-1)^k\exp((k+1)\lambda x), & x<0,\end{cases} \tag{8}$$

for $\lambda > 0$. And for $\lambda < 0$,

$$f(x;\alpha,\lambda)=\begin{cases}\dfrac{1+\alpha x^2}{C(\alpha)\sqrt{2\pi}}\exp\Big(-\dfrac{x^2}{2}\Big)\sum_{k=0}^{\infty}(-1)^k\exp((k+1)\lambda x), & x\geq 0,\\ \\ \dfrac{1+\alpha x^2}{C(\alpha)\sqrt{2\pi}}\exp\Big(-\dfrac{x^2}{2}\Big)\sum_{k=0}^{\infty}(-1)^k\exp(-k\lambda x), & x<0,\end{cases} \tag{9}$$

2.1 Properties of BTSN distribution

(i) If $\lambda = 0$, then $X \sim$ two parameter bimodal normal distribution (Elal-Olivero et al., 2020).

(ii) If $\alpha = 0$, then $X \sim Tanh$ skew normal Distribution (Mahmoud et al., 2020).

(iii) If $\alpha = 0, \lambda = 0$, then $X \sim$ N(0,1).

(iv) If $\alpha \to \infty$, then $X \sim$ bimodal $Tanh$ skew normal Distribution with the pdf

$$f(x;\lambda) = x^2\phi(x)\Big[1 + Tanh\Big(\frac{\lambda x}{2}\Big)\Big]$$

(v) If $\lambda \to \infty$, then $X \sim$ two parameter bimodal normal distribution (Elal-Olivero et al., 2020).

(vi) If $X \sim BTSN(\alpha, \lambda)$, then $-X \sim BTSN(-\alpha, -\lambda)$.

2.2 Plots of density function

The plot of density function of $BTSN(\alpha, \lambda)$ distribution for different values of α and λ are illustrated in Figure 1. From the figure it may be visualized that with different choices of the parameters, the said distribution may possess at most two modes. It is also observed that when $\lambda = 0$, then the density function yield symmetric bimodal behavior (fig: 1(c)). On the other hand, the density function becomes unimodal when $\alpha = 0, \lambda = 0$ and asymmetric normal when $\alpha = 0$ (fig: 1(d)). Besides, from the fig: 1(a), it may be observed that the density function exhibits high peak to the left tail with increasing value of α with fixed value of $\lambda(< 0)$. Similarly, fig: 1(b) depicts that the density function exhibits high peak to the right tail with increasing value of α with fixed $\lambda(> 0)$.

3 Mathematical Properties

This section includes some important mathematical properties of the $BTSN(\alpha, \lambda)$ distribution as well as related results. These properties includes moment generating function, r^{th} order moment, mean deviation, characterizations etc. Besides cumulative distribution function (cdf) of the novel distribution is calculated.

Lemma 3.1. Let, $X \sim BTSN(\alpha, \lambda)$, then the cdf of X is obtained as

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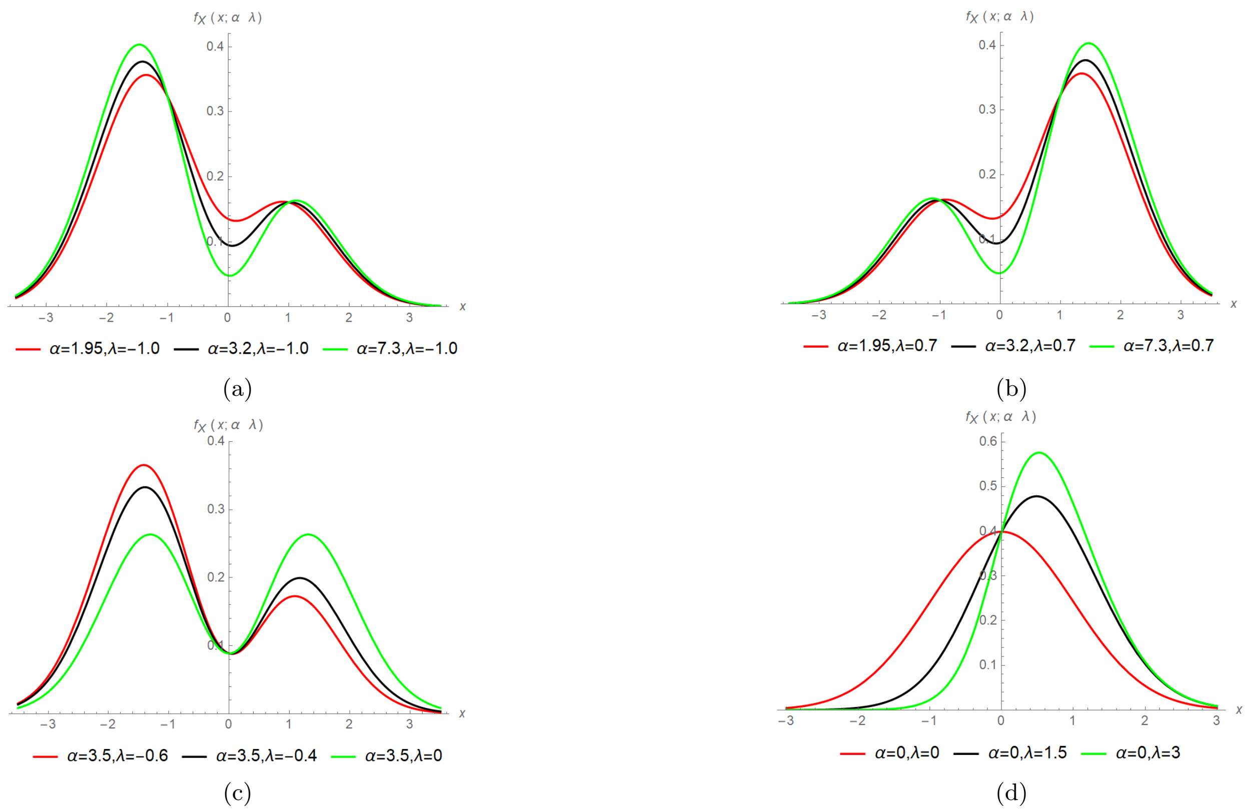

Figure 1: Plots of the probability density function of $BTSN(\alpha, \lambda)$ for different choices of α and λ

Case I: When $\lambda > 0$,

$$F(X) = \begin{cases} \frac{1}{C(\alpha)\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^k \Bigg[\Bigg(2\exp\left(\frac{k^2\lambda^2}{2}\right) \sqrt{\frac{\pi}{2}} \Big(\Big(1 - \Phi(k\lambda)\Big) \\ \quad - \Big(1 - \Phi(x + k\lambda)\Big)\Big)\Bigg) + \alpha \Bigg(\frac{1}{2} \Bigg(-2k\lambda + \exp\left(\frac{k^2\lambda^2}{2}\right) \\ \quad \sqrt{2\pi}\left(1 + k^2\lambda^2\right) \Phi(k\lambda) + \exp\left(\frac{-x(x+2k\lambda)}{2}\right) \\ \quad \Bigg(2k\lambda - 2x - \exp\left(\frac{(x+k\lambda)^2}{2}\right) \sqrt{2\pi}\left(1 + k^2\lambda^2\right) \\ \quad \Big(2(1 - \Phi(x + k\lambda) - 1)\Big)\Bigg)\Bigg)\Bigg)\Bigg], & x \geq 0, \\ \frac{1}{C(\alpha)\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^k \Bigg[\Bigg(\exp\left(\frac{(k+1))^2\lambda^2}{2}\right) \sqrt{\frac{\pi}{2}} \Big(2 \Big(1 - \\ \quad \Phi\left(x - (1+k)\lambda\right)\Big)\Big)\Bigg) + \frac{\alpha}{2} \Bigg(\exp\left(\frac{(k+1))^2\lambda^2}{2}\right) \\ \Bigg(-2\exp\left(\frac{(x-(1+k)\lambda)^2}{2}\right)(x + \lambda + k\lambda) + \sqrt{2\pi}\left(1 + (k+1)^2\lambda^2\right) \\ \quad \left(2\left(1 - \Phi\left(x - (1+k)\lambda\right)\right)\right)\Bigg)\Bigg)\Bigg], & x < 0, \end{cases} \tag{10}$$

Proof: Using the expression of density function mentioned on (8), the results for the cdf of the distribution (for $x \geq 0$) is given as,

$$
\begin{aligned}
F(x) &= \int_0^x \frac{1+\alpha x^2}{C(\alpha)\sqrt{2\pi}} \exp\Big(-\frac{x^2}{2}\Big) \sum_{k=0}^{\infty} (-1)^k \exp(-k\lambda x) dx \\
&= \frac{1}{C(\alpha)\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^k \int_0^x \left(1+\alpha x^2\right) \exp\left(-\frac{x^2}{2} - k\lambda x\right) dx \\
&= \frac{1}{C(\alpha)\sqrt{2\pi}} \sum_{k=0}^{\infty} \left[\int_0^x \exp\left(-\frac{x^2}{2} - k\lambda x\right) dx + \alpha \int_0^x x^2 \exp\left(-\frac{x^2}{2} - k\lambda x\right) dx\right] \\
&= \frac{1}{C(\alpha)\sqrt{2\pi}} \sum_{k=0}^{\infty} [I_1 + \alpha I_2]
\end{aligned}
$$

Now, using the method of integration by parts I_1 and I_2 are calculated and hence the final results of the cdf is obtained as

$$
\begin{aligned}
F(x) = \frac{\sum_{k=0}^{\infty}(-1)^k}{C(\alpha)\sqrt{2\pi}} &\Bigg[\Bigg(2\exp\left(\frac{k^2\lambda^2}{2}\right)\sqrt{\frac{\pi}{2}}\Big(\Big(1-\Phi(k\lambda)\Big) - \Big(1-\Phi(x+k\lambda)\Big)\Big)\Bigg) \\
&+ \alpha\Bigg(\frac{1}{2}\Bigg(-2k\lambda + \exp\left(\frac{k^2\lambda^2}{2}\right)\sqrt{2\pi}\left(1+k^2\lambda^2\right)\Phi(k\lambda) + \exp\left(\frac{-x(x+2k\lambda)}{2}\right) \\
&\Big(2k\lambda - 2x - \exp\left(\frac{(x+k\lambda)^2}{2}\right)\sqrt{2\pi}\left(1+k^2\lambda^2\right)\Big(2(1-\Phi(x+k\lambda)-1)\Big)\Big)\Bigg)\Bigg)\Bigg].
\end{aligned}
$$

Similarly, for $x < 0$, the results for the cdf is obtained as

$$
\begin{aligned}
F(x) = \frac{\sum_{k=0}^{\infty}(-1)^k}{C(\alpha)\sqrt{2\pi}} &\Bigg[\Bigg(\exp\left(\frac{(k+1))^2\lambda^2}{2}\right)\sqrt{\frac{\pi}{2}}\Big(2\Big(1-\Phi\left(x-(1+k)\lambda\right)\Big)\Big)\Bigg) \\
&+ \frac{\alpha}{2}\Bigg(\exp\left(\frac{(k+1))^2\lambda^2}{2}\right)\Bigg(-2\exp\left(\frac{(x-(1+k)\lambda)^2}{2}\right)(x+\lambda+k\lambda) + \\
&\sqrt{2\pi}\left(1+(k+1)^2\lambda^2\right)\left(2\left(1-\Phi\left(x-(1+k)\lambda\right)\right)\right)\Bigg)\Bigg)\Bigg].
\end{aligned}
$$

Case II: When $\lambda < 0$,

$$
F(X) = \begin{cases}
\frac{1}{C(\alpha)\sqrt{2\pi}} \sum_{k=0}^{\infty}(-1)^k \Bigg[\Bigg(2\exp\left(\frac{((1+k)\lambda)^2}{2}\right)\sqrt{\frac{\pi}{2}}\Big(\Phi\left(1+k\right)\lambda + \Phi\left(x-(1+k)\lambda\right)\Big)\Bigg) \\
+\alpha\Bigg(x+k\lambda - \exp\left(\frac{-x^2}{2} + (1+k)\lambda x\right)(x+\lambda+k\lambda) + \exp\left(\frac{((1+k)\lambda)^2}{2}\right) \\
\sqrt{\frac{\pi}{2}}\left(1+\left((1+k)\lambda\right)^2\right)\Big(\Phi\left((1+k)\lambda\right) + \Phi\left(x-(1+k)\lambda\right)\Big)\Bigg)\Bigg], & x \geq 0, \\
\frac{1}{C(\alpha)\sqrt{2\pi}} \sum_{k=0}^{\infty}(-1)^k \Bigg[\Bigg(2\exp\left(\frac{((k\lambda)^2}{2}\right)\sqrt{\frac{\pi}{2}}(1-\Phi(x-k\lambda))\Bigg)\alpha\Bigg(\frac{1}{2}\exp\left(\frac{-x\left(x+2k\lambda\right)}{2}\right) \\
\Big(-2x+2k\lambda+\exp\left(\frac{(x+k\lambda)^2}{2}\right)\sqrt{2\pi}\left(1+k^2\lambda^2\right)\left(1-\Phi\left(x+k\lambda\right)\right)\Big)\Bigg)\Bigg], & x < 0.
\end{cases} \tag{11}
$$

In this context, the density function expression for the BTSN distribution mentioned in equation (9) will be utilized, and the subsequent steps in the proof will follow a similar approach as in Case I.

Remark 1:

Throughout this paper, unless otherwise specified, the statistical results for the assumption $\lambda > 0$ will be considered because for the corresponding results of $\lambda < 0$, $-X$ has the pdf $f(x) = 2\phi(x)G(-\lambda x)$.

Lemma 3.2. If $X \sim BTSN(\alpha, \lambda)$ then the r^{th} order moment of the distribution is obtained as

$$E(X^r) = 2^{\frac{r}{2}}\left[\frac{1}{\sqrt{2}}\Gamma\left(\frac{1+r}{2}\right)\Big(H_F(1) + (-1)^r H_F(2) + (1+r)\alpha\Big(H_F(3) + (-1)^r H_F(4)\Big)\Big) + \lambda\Gamma\left(1+\frac{r}{2}\right)\Big(-kH_F(5) - \alpha k(2+r)H_F(6) + (-1)^{1+r}(1+k)\Big(H_F(7) + (2+r)H_F(8)\Big)\Big)\right], \tag{12}$$

where, $H_F(\cdot)$ is the Kummer confluent hypergeometric functions (Gasaneo et al., 2001) which is defined as ${}_1F_1(b,c,x) = \sum_{m=0}^{\infty}\frac{(b)_m x^m}{(c)_m m!}$. More specifically it is referred here as

$$H_F(1) = {}_1F_1\left[\frac{1+r}{2}, \frac{1}{2}, \frac{(k\lambda)^2}{2}\right], \quad H_F(2) = {}_1F_1\left[\frac{1+r}{2}, \frac{1}{2}, \frac{((k+1)\lambda)^2}{2}\right],$$

$$H_F(3) = {}_1F_1\left[\frac{3+r}{2}, \frac{1}{2}, \frac{(k\lambda)^2}{2}\right], \quad H_F(4) = {}_1F_1\left[\frac{3+r}{2}, \frac{1}{2}, \frac{((k+1)\lambda)^2}{2}\right],$$

$$H_F(5) = {}_1F_1\left[1+\frac{r}{2}, \frac{3}{2}, \frac{(k\lambda)^2}{2}\right], \quad H_F(6) = {}_1F_1\left[2+\frac{r}{2}, \frac{3}{2}, \frac{(k\lambda)^2}{2}\right],$$

$$H_F(7) = {}_1F_1\left[\frac{2+r}{2}, \frac{3}{2}, \frac{((1+k)\lambda)^2}{2}\right], \quad H_F(8) = {}_1F_1\left[\frac{1+r}{2}, \frac{3}{2}, \frac{((1+k)\lambda)^2}{2}\right].$$

Proof:

$$\begin{aligned} E(X^r) &= \int_{-\infty}^{\infty} x^r f(x)dx \\ &= \int_{-\infty}^{0} x^r f(x)dx + \int_{0}^{\infty} x^r f(x)dx \\ &= I_3 + I_4 \end{aligned} \tag{13}$$

Now,

$$\begin{aligned} I_3 = \int_{-\infty}^{0} x^r f(x)dx &= \int_{-\infty}^{0} x^r \frac{1+\alpha x^2}{C(\alpha)\sqrt{2\pi}} \exp\Big(-\frac{x^2}{2}\Big)\sum_{k=0}^{\infty}(-1)^k \exp((k+1)\lambda x)dx \\ &= \frac{\sum_{k=0}^{\infty}(-1)^k}{C(\alpha)\sqrt{2\pi}}\left[\int_{-\infty}^{0} x^r \exp\left(\frac{-x^2}{2}\lambda(k+1)x\right)dx \right. \\ &\qquad \left. + \alpha\int_{-\infty}^{0} x^{r+2}\exp\left(\frac{-x^2}{2}\lambda(k+1)x\right)dx\right] \end{aligned}$$

Using the expressions mentioned in the Section 2.3 of (Prudnikov et al., 1986), the results for the integration are evaluated and I_3 is obtained as

$$I_3 = \frac{\sum_{k=0}^{\infty}(-1)^{r+k}}{C(\alpha)\sqrt{2\pi}}\left[\frac{1}{\sqrt{2}}\Gamma\left(\frac{1+r}{2}\right)\Big(H_F(2) + (1+r)\alpha H_F(4)\Big) - (1+k)\lambda\Gamma\left(1+\frac{r}{2}\right)\Big(H_F(7) + (2+r)\alpha H_F(8)\Big)\right].$$

Again,

$$I_4 = \int_{0}^{\infty} x^r f(x)dx = \int_{0}^{\infty} x^r \frac{1+\alpha x^2}{C(\alpha)\sqrt{2\pi}} \exp\Big(-\frac{x^2}{2}\Big)\sum_{k=0}^{\infty}(-1)^k \exp(-k\lambda x)dx.$$

Similarly the calculation of I_3, Using the expressions mentioned in the Section 2.3 of (Prudnikov et al., 1986), the results for the I_4 is obtained as

$$I_4 = \frac{\sum_{k=0}^{\infty}(-1)^k}{C(\alpha)\sqrt{2\pi}}\left[\frac{1}{\sqrt{2}}\Gamma\left(\frac{1+r}{2}\right)\Big(H_F(1)+(1+r)\alpha H_F(3)\Big)-\sqrt{2}k\lambda\Gamma\left(1+\frac{r}{2}\right)\Big(H_F(5)+(2+r)\alpha H_F(9)\Big)\right],$$

where, $H_F(9) = {}_1F_1\left[\frac{4+r}{2}, \frac{3}{2}, \frac{(k\lambda)^2}{2}\right]$.

Using the results of I_3 and I_4 in (13), the r^{th} order moment is derived as

$$E(X^r) = \frac{\sum_{k=0}^{\infty}(-1)^k 2^{\frac{r}{2}}}{C(\alpha)\sqrt{2\pi}}\left[\frac{1}{\sqrt{2}}\Gamma\left(\frac{1+r}{2}\right)\Big(H_F(1)+(-1)^r H_F(2)+(1+r)\alpha\Big(H_F(3)+(-1)^r H_F(4)\Big)\Big)\right.$$
$$\left.+\lambda\Gamma\left(1+\frac{r}{2}\right)\Big(-kH_F(5)-\alpha k(2+r)H_F(6)+(-1)^{1+r}(1+k)\Big(H_F(7)+(2+r)H_F(8)\Big)\Big)\right].$$

Remark 2:

From the equation (12), the first four moments of X can be obtained as

$$E[X] = \frac{\sum_{k=0}^{\infty}(-1)^k}{C(\alpha)\sqrt{2\pi}}\frac{\lambda}{2}\left[-\sqrt{2\pi}kE(1)\left(A(1)\right)-2\alpha(2k+1)\lambda+\sqrt{2\pi}\Big(kE(1)\left(A(1)\right)\right.$$
$$\left.\Phi_1(\lambda)+E(2)(1+k)\left(\alpha\left((k+1)^2\lambda^2+3\right)+1\right)\Phi_2(\lambda)\Big)\right],$$

$$E[X^2] = \frac{\sum_{k=0}^{\infty}(-1)^k}{2C(\alpha)\sqrt{2\pi}}\left[E(1)\sqrt{2\pi}A(2)-2(2k+1)\lambda\left(1+\alpha\left(\left(k^2+k+1\right)\lambda^2+5\right)\right)\right.$$
$$\left.+\sqrt{2\pi}\Big(-E(1)A(1)\Phi_1(\lambda)+E(2)\Big(1+3\alpha+(k+1)^2\lambda^2\alpha+(k+1)^4\lambda^4\Big)\Phi_2(\lambda)\Big)\right],$$

$$E[X^3] = \frac{\sum_{k=0}^{\infty}(-1)^k}{C(\alpha)\sqrt{2\pi}}\frac{\lambda}{2}\left[-E(1)\sqrt{2\pi}kA(3)-2(2k+1)\lambda\Big(1+\alpha\Big((2k(k+1)+1)\lambda^2+9\Big)\Big)\right.$$
$$\left.+\sqrt{2\pi}\Big(E(1)kA(3)\Phi_1(\lambda)+E(2)(k+1)\Big(15\alpha+\alpha(k+1)^4\lambda^4+(10\alpha+1)(k+1)^2\lambda^2+3\Big)\Phi_2(\lambda)\Big)\right],$$

$$E[X^4] = \frac{\sum_{k=0}^{\infty}(-1)^k}{C(\alpha)\sqrt{2\pi}}\frac{\lambda}{2}\left[(2k+1)\lambda\Big(-33\alpha+\alpha\left(-\left(k(k+1)\left(k^2+k+3\right)+1\right)\right)\lambda^4-(14\alpha+1)\left(k^2+k+1\right)\lambda^2-5\Big)\right.$$
$$+E(1)\sqrt{\frac{\pi}{2}}A(4)+\sqrt{\frac{\pi}{2}}\Big(-E(1)A(4)\Phi_1(\lambda)+E(2)\Big(15\alpha+(15\alpha+1)(k+1)^4\lambda^4+3(15\alpha+2)\lambda^2$$
$$\left.(k+1)^2+3\alpha+(k+1)^6\lambda^6\Big)\Phi_2(\lambda)\Big)\right].$$

Where, $E(1) = \exp\left(\frac{k^2\lambda^2}{2}\right)$, $E(2) = \exp\left(\frac{(k+1)^2\lambda^2}{2}\right)$, $\Phi_1(\lambda) = 2(\Phi(k\lambda)-1)$, $\Phi_2(\lambda) = 1-\Phi((1+k)\lambda)$, $A(1) = \left[\alpha\left(k^2\lambda^2+3\right)+1\right]$, $A(2) = \left[3\alpha+\alpha k^4\lambda^4+(6\alpha+1)k^2\lambda^2+1\right]$, $A(3) = \left[15\alpha+\alpha k^4\lambda^4+(10\alpha+1)k^2\lambda^2+3\right]$ and $A(4) = \left[15\alpha+\alpha k^6\lambda^6+(15\alpha+1)k^4\lambda^4+3(15\alpha+2)k^2\lambda^2+3\right]$.

Additionally, the variance of $BTSN(\alpha,\lambda)$ also can be calculated using the above special results which involve very complicated mathematical form. So for some particular values of the parameter, mean and variance of the said distribution are calculated and listed in Table 1.

Table 1: Mean and Variance of $BTSN(\alpha, \lambda)$ distribution for difrerent values of parameter

$\lambda \rightarrow$	-1		-2		1		3		5	
$\alpha \downarrow$	$E(X)$	$Var(X)$	$E(X)$	$Var(X)$	$E(X)$	$Var(X)$	$E(X)$	$Var(X)$	$E(X)$	$Var(X)$
0	-0.4132	0.8293	-0.6057	0.6331	0.4132	0.8293	0.6890	0.5253	0.7514	0.4354
1	-0.7641	1.4161	-1.0296	0.9399	0.7641	1.4162	1.1171	0.7521	1.1685	0.6346
2	-0.8810	1.5538	-1.1709	0.9590	0.8810	1.5538	1.2597	0.7432	1.3075	0.6204
3	-0.9395	1.6174	-1.2416	0.9584	0.9395	1.6173	1.3311	0.7282	1.3770	0.6039
4	-0.9745	1.6503	-1.2839	0.9516	0.9746	1.6502	1.3739	0.7124	1.4187	0.5873

Lemma 3.3. Let, $X \sim BTSN(\alpha, \lambda)$, then the Moment Generating Function (mgf) of X is given as

$$M_X(t) = \frac{\sum_{k=0}^{\infty}(-1)^k}{C(\alpha)\sqrt{2\pi}}\Bigg[\alpha(t-k\lambda) - \alpha(k\lambda+\lambda+t) + \exp\left(\frac{1}{2}(t-k\lambda)^2\right)\sqrt{\frac{\pi}{2}}\Big(\alpha+\alpha(t-k\lambda)^2+1\Big)2\Phi(t-k\lambda)$$
$$+\exp\left(\frac{1}{2}(t+\lambda+k\lambda)^2\right)\sqrt{\frac{\pi}{2}}\Big(\alpha+\alpha(k\lambda+\lambda+t)^2+1\Big)\Big(1-\Phi(t+\lambda+k\lambda)\Big)\Bigg]. \tag{14}$$

Proof:

$$\begin{aligned} M_X(t) = E(e^{xt}) &= \int_{-\infty}^{\infty} e^{xt} f(x)dx \\ &= \int_{-\infty}^{0} e^{xt} f(x)dx + \int_{0}^{\infty} e^{xt} f(x)dx \\ &= I_5 + I_6 \end{aligned}$$

Now,

$$\begin{aligned} I_5 &= \int_{-\infty}^{0} e^{xt} f(x)dx \\ &= \int_{-\infty}^{0} e^{xt}\frac{1+\alpha x^2}{C(\alpha)\sqrt{2\pi}}\exp\Big(-\frac{x^2}{2}\Big)\sum_{k=0}^{\infty}(-1)^k\exp((k+1)\lambda x)dx \\ &= \frac{\sum_{k=0}^{\infty}(-1)^k}{C(\alpha)\sqrt{2\pi}}\int_{-\infty}^{0}(1+\alpha x^2)\exp(-\frac{x^2}{2}+xt+\lambda x(k+1))dx \\ &= \frac{\sum_{k=0}^{\infty}(-1)^k}{C(\alpha)\sqrt{2\pi}}\Bigg[\int_{-\infty}^{0}\exp((-\frac{x^2}{2}+xt+\lambda x(k+1))dx + \alpha\int_{-\infty}^{0}x^2\exp((-\frac{x^2}{2}+xt+\lambda x(k+1))dx\Bigg] \end{aligned}$$

So, I_5 can be written as,

$$I_5 = \frac{\sum_{k=0}^{\infty}(-1)^k}{C(\alpha)\sqrt{2\pi}}\Big[I_5^1 + I_5^2\Big].$$

Now, I_5^1 is calculated with the help of the results involved in the moment generating function Tanh skew normal distribution (Mahmoud et al., 2020). For the remaining part of the integration, the method of integration by parts is employed. Consequently, the results for I_5 is determined as

$$I_5 = \frac{\sum_{k=0}^{\infty}(-1)^k}{C(\alpha)\sqrt{2\pi}}\Bigg[-\alpha(k\lambda+\lambda+t) + \exp\left(\frac{1}{2}(k\lambda+\lambda+t)^2\right)\sqrt{\frac{\pi}{2}}\Big(\alpha+\alpha(k\lambda+\lambda+t)^2+1\Big)\Big(1-\Phi(t+\lambda+k\lambda)\Big)\Bigg].$$

Again,

$$\begin{aligned} I_6 &= \int_{0}^{\infty} e^{xt} f(x)dx \\ &= \frac{\sum_{k=0}^{\infty}(-1)^k}{C(\alpha)\sqrt{2\pi}}\int_{0}^{\infty}(1+\alpha x^2)\exp(-\frac{x^2}{2}+xt+\lambda x(k+1))dx \end{aligned}$$

In a similar manner to the procedure for computing I_5, the integration for I_6 is also performed, resulting in the determination of the outcomes for I_6 as

$$I_6 = \frac{\sum_{k=0}^{\infty}(-1)^k}{C(\alpha)\sqrt{2\pi}}\left[\alpha(t-k\lambda)+\exp\left(\frac{1}{2}(t-k\lambda)^2\right)\sqrt{\frac{\pi}{2}}\Big(\alpha+\alpha(t-k\lambda)^2+1\Big)2\Phi(t-k\lambda)\right].$$

By combining the outcomes of both I_5 and I_6, the ultimate results for the moment generating function (MGF) are derived as

$$M_X(t) = \frac{\sum_{k=0}^{\infty}(-1)^k}{C(\alpha)\sqrt{2\pi}}\Bigg[\alpha(t-k\lambda)-\alpha(k\lambda+\lambda+t)+\exp\left(\frac{1}{2}(t-k\lambda)^2\right)\sqrt{\frac{\pi}{2}}\Big(\alpha+\alpha(t-k\lambda)^2+1\Big)2\Phi(t-k\lambda)$$
$$+\exp\left(\frac{1}{2}(t+\lambda+k\lambda)^2\right)\sqrt{\frac{\pi}{2}}\Big(\alpha+\alpha(k\lambda+\lambda+t)^2+1\Big)\Big(1-\Phi(t+\lambda+k\lambda)\Big)\Bigg].$$

Corollary 3.3.1. By substituting (it) instead of t in equation (14), the characteristic function of the $BTSN(\lambda,\alpha)$ distribution can be computed. Consequently, the characteristic function is determined as

$$\phi_X(t) = \frac{\sum_{k=0}^{\infty}(-1)^k}{C(\alpha)\sqrt{2\pi}}\Bigg[\alpha(it-k\lambda)-\alpha(k\lambda+\lambda+it)+\exp\left(\frac{1}{2}(it-k\lambda)^2\right)\sqrt{\frac{\pi}{2}}\Big(\alpha+\alpha(it-k\lambda)^2+1\Big)2\Phi(it-k\lambda)$$
$$+\exp\left(\frac{1}{2}(it+\lambda+k\lambda)^2\right)\sqrt{\frac{\pi}{2}}\Big(\alpha+\alpha(k\lambda+\lambda+it)^2+1\Big)\Big(1-\Phi(it+\lambda+k\lambda)\Big)\Bigg]. \quad (15)$$

Lemma 3.4. Let, $X \sim BTSN(\alpha,\lambda)$, then the Mean Deviation of X about mean (μ) is given as

$$\mu(x) = \frac{\sum_{k=0}^{\infty}(-1)^k}{C(\alpha)\sqrt{2\pi}}\Bigg[\exp\left((k+1)\lambda\mu-\frac{\mu^2}{2}\right)\Big(\alpha\left((k+1)^2\lambda^2+2\right)+1\Big)-\sqrt{2\pi}\exp\left(\frac{1}{2}(k+1)^2\lambda^2\right)$$
$$\Big(1-\Phi(\lambda+k\lambda-\mu)\Big)\Big((k+1)\lambda\left(\alpha\left((k+1)^2\lambda^2+3\right)+1\right)-\mu\left(\alpha+\alpha(k+1)^2\lambda^2+1\right)\Big)$$
$$+\frac{1}{2}\exp\left(-\frac{1}{2}\mu(2k\lambda+\mu)\right)\Bigg(-2\alpha\left(k^2\lambda^2+2\right)-2+2\sqrt{2\pi}\exp\left(\frac{1}{2}(k\lambda+\mu)^2\right)\Big(\alpha\mu+$$
$$\alpha k^3\lambda^3+\alpha k^2\lambda^2\mu+3\alpha k\lambda+k\lambda+\mu\Big)(1-\Phi(k\lambda+\mu))\Bigg)\Bigg] \quad (16)$$

Proof:

$$\begin{aligned}\mu(x) &= \int_{-\infty}^{\mu}(\mu-x)f(x)dx+\int_{\mu}^{\infty}(\mu-x)f(x)dx\\ &= I_7+I_8.\end{aligned}$$

Now,

$$\begin{aligned}I_7 &= \int_{-\infty}^{\mu}(\mu-x)f(x)dx\\ &= \int_{-\infty}^{\mu}(\mu-x)\frac{1+\alpha x^2}{C(\alpha)\sqrt{2\pi}}\exp\Big(-\frac{x^2}{2}\Big)\sum_{k=0}^{\infty}(-1)^k\exp((k+1)\lambda x)dx\\ &= \frac{\sum_{k=0}^{\infty}(-1)^k}{C(\alpha)\sqrt{2\pi}}\left[\int_{-\infty}^{\mu}\mu\left(1+\alpha x^2\right)\exp\Big((k+1)\lambda x-\frac{x^2}{2}\Big)dx-\int_{-\infty}^{\mu}x\left(1+\alpha x^2\right)\exp\Big((k+1)\lambda x-\frac{x^2}{2}\Big)dx\right]\\ &= \frac{\sum_{k=0}^{\infty}(-1)^k}{C(\alpha)\sqrt{2\pi}}\left[I_7^1-I_7^2\right]\end{aligned}$$

For calculating the integration I_7^1 and I_7^2 involved in the above calculation, method of integration by parts is employed. Thus the integration are calculated and the putting the results I_7 can be obtained as follows

$$I_7 = \frac{\sum_{k=0}^{\infty}(-1)^k}{C(\alpha)\sqrt{2\pi}}\left[\left(-\alpha\mu(k\lambda+\lambda+\mu)\exp\left((k+1)\lambda\mu-\frac{\mu^2}{2}\right)+(1-\Phi(\lambda+k\lambda-\mu))\sqrt{2\pi}\mu\left(\alpha+\alpha(k+1)^2\lambda^2+1\right)\right)\right.$$
$$-\left(-\exp\left((k+1)\lambda\mu-\frac{\mu^2}{2}\right)\left(\alpha\Big((k+1)^2\lambda^2+(k+1)\lambda\mu+\mu^2+2\Big)+1\right)+(1+k)\sqrt{2\pi}\lambda\Big(\alpha\left((k+1)^2\lambda^2+3\right)+1\Big)\right.$$
$$\left.\left.\Big(1-\Phi(\lambda+k\lambda-\mu)\Big)\exp\left(\frac{1}{2}(k+1)^2\lambda^2\right)\right)\right]$$

Therefore,

$$I_7 = \frac{\sum_{k=0}^{\infty}(-1)^k}{C(\alpha)\sqrt{2\pi}}\left[\Big(\alpha\left((k+1)^2\lambda^2+2\right)+1\Big)\exp\left((k+1)\lambda\mu-\frac{\mu^2}{2}\right)-\sqrt{2\pi}\Big((k+1)\lambda\Big(\alpha\left((k+1)^2\lambda^2+3\right)+1\Big)\right.$$
$$\left.-\Big(\mu\left(\alpha+\alpha(k+1)^2\lambda^2+1\right)\Big)\Big)\Big(1-\Phi(\lambda+k\lambda-\mu)\Big)\right].$$

Similarly,

$$I_8 = \int_{\mu}^{\infty}(\mu-x)f(x)dx$$
$$= \frac{\sum_{k=0}^{\infty}(-1)^k}{C(\alpha)\sqrt{2\pi}}\left[\frac{1}{2}\exp\left(-\frac{1}{2}\mu(2k\lambda+\mu)\right)\left(-2\alpha\left(k^2\lambda^2+2\right)+\exp\left(\frac{1}{2}(k\lambda+\mu)^2\right)2\sqrt{2\pi}\Big(\alpha\mu+\alpha k^3\lambda^3+\alpha k^2\lambda^2\mu+\right.\right.$$
$$\left.\left.3\alpha k\lambda+k\lambda+\mu\Big)\Big(1-\Phi(k\lambda+\mu)\Big)-2\right)\right].$$

Now, combining the I_7 and I_8, final results of the mean deviation about mean is obtained as

$$\mu(x) = \frac{\sum_{k=0}^{\infty}(-1)^k}{C(\alpha)\sqrt{2\pi}}\left[\exp\left((k+1)\lambda\mu-\frac{\mu^2}{2}\right)\Big(\alpha\left((k+1)^2\lambda^2+2\right)+1\Big)-\sqrt{2\pi}\exp\left(\frac{1}{2}(k+1)^2\lambda^2\right)\right.$$
$$\Big(1-\Phi(\lambda+k\lambda-\mu)\Big)\Big((k+1)\lambda\left(\alpha\left((k+1)^2\lambda^2+3\right)+1\right)-\mu\left(\alpha+\alpha(k+1)^2\lambda^2+1\right)\Big)$$
$$+\frac{1}{2}\exp\left(-\frac{1}{2}\mu(2k\lambda+\mu)\right)\left(-2\alpha\left(k^2\lambda^2+2\right)-2+2\sqrt{2\pi}\exp\left(\frac{1}{2}(k\lambda+\mu)^2\right)\Big(\alpha\mu+\right.$$
$$\left.\left.\alpha k^3\lambda^3+\alpha k^2\lambda^2\mu+3\alpha k\lambda+k\lambda+\mu\Big)(1-\Phi(k\lambda+\mu))\right)\right]$$

Corollary 3.4.1. Replacing μ by M in equation (17), the expression for the mean deviation about median (M) can be obtained.

4 Characterizations Results

In this section the characterizations of the BTSN distribution via two truncated moments are discussed. For these characterizations, the cdf need not to have a closed form.

4.1 Characterizations based on two truncated moments

This sub-section addresses the characterization of BTSN distribution, relying on the connection between two truncated moments. The characterization leverage Lemma 4.1.1 from Glänzel (1987), as presented below. Notably, the outcome remains valid even if H is not a closed interval. This characterization exhibits stability in terms of weak convergence, as detailed in reference (Glanzel, 1990).

Lemma 4.1.1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ ($d = -\infty,\ e = \infty$ might as well be allowed). Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let k and h be two real functions defined on H such that

$$\mathbf{E}\left[k\left(X\right)\ \mid X\geq x\right]=\mathbf{E}\left[h\left(X\right)\ \mid X\geq x\right]\eta\left(x\right),\quad x\in H,$$

is defined with some real function η. Assume that $k,\, h\in C^{1}\left(H\right)$, $\eta\in C^{2}\left(H\right)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta h=k$ has no real solution in the interior of H. Then F is uniquely determined by the functions k, h and η , particularly

$$F\left(x\right)=\int_{a}^{x}C\left|\frac{\eta^{\prime}\left(u\right)}{\eta\left(u\right)h\left(u\right)-k\left(u\right)}\right|\exp\left(-s\left(u\right)\right)\ du\ ,$$

where the function s is a solution of the differential equation $s^{\prime}=\dfrac{\eta^{\prime}\ h}{\eta\ h\ -\ k}$ and C is the normalization constant, such that $\int_{H}dF=1$.

Proposition 4.1.1. Let the random variable $X:\Omega\rightarrow\mathbb{R}$ be continuous, and let $h\left(x\right)=\dfrac{1}{\left[(1+\alpha x^{2})G(\lambda x)\right]}$ and $k\left(x\right)=h\left(x\right)\Phi(x)$ for $x\in\mathbb{R}$. Then, the density of X is given in (4) if and only if the function η defined in Lemma 4.1.1 is

$$\eta\left(x\right)=\frac{1}{2}\left\{1+\Phi(x)\right\},\qquad x\in\mathbb{R}.$$

Proof: If X has pdf (4), then

$$\left(1-F\left(x\right)\right)E\left[h\left(X\right)\ \mid X\geq x\right]=\frac{1}{C(\alpha)}\left\{1-\Phi(x)\right\},\qquad x\in\mathbb{R},$$

and

$$\left(1-F\left(x\right)\right)E\left[k\left(X\right)\ \mid X\geq x\right]=\frac{1}{2C(\alpha)}\left\{1-\Phi^{2}(x)\right\},\qquad x\in\mathbb{R},$$

and hence,

$$\eta(x)=\frac{\dfrac{1}{2C(\alpha)}\left\{1-\Phi^{2}(x)\right\}}{\dfrac{1}{C(\alpha)}\left\{1-\Phi(x)\right\}}=\frac{1}{2}\left\{1+\Phi(x)\right\}.$$

So, finally,

$$\eta\left(x\right)h\left(x\right)-k\left(x\right)=\frac{1}{2}h\left(x\right)\left\{1-\Phi(x)\right\}>0\quad for\ \ x\in\mathbb{R}.$$

Conversely, if η has the above form, then

$$s^{\prime}\left(x\right)=\frac{\eta^{\prime}\left(x\right)h\left(x\right)}{\eta\left(x\right)h\left(x\right)-k\left(x\right)}=\frac{\phi(x)}{1-\Phi(x)},$$

So,

$$s\left(x\right)=-\log\left\{1-\Phi(x)\right\},\quad x\in\mathbb{R}.$$

In view of Lemma 4.1.1, X has pdf (4).

Corollary 4.1.1 If $X:\Omega\rightarrow\mathbb{R}$ is a continuous random variable and $h\left(x\right)$ is as in Proposition 4.1.1 Then, X has pdf (4) if and only if there exist functions k and η defined in Lemma 4.1.1 satisfying the following first order differential equation

$$\frac{\eta^{\prime}\left(x\right)h\left(x\right)}{\eta\left(x\right)h\left(x\right)-k\left(x\right)}=\frac{\phi(x)}{1-\Phi(x)}.$$

Corollary 4.1.2 The general solution of the above differential equation is

$$\eta\left(x\right)=\left\{1-\Phi(x)\right\}^{-1}\left[-\int\phi(x)\left(h\left(x\right)\right)^{-1}k\left(x\right)+D\right],$$

where D is a constant. A set of functions satisfying this differential equation is presented in Proposition 4.1.1 with

$D = \frac{1}{2}$. Clearly, there are other triplets (h, k, ξ) satisfying the conditions of Lemma 4.1.1.

5 Parameter Estimation

5.1 Location and Scale Extension

If $X \sim BTSN(\alpha, \lambda)$, then the location and scale extension of X is defined as $Y = \mu + X\beta$. As a result, a location-scale generalised bimodal tanh skew normal distribution gets generated, and its pdf is represented as $Y \sim BTSN(\mu, \beta, \alpha, \lambda)$.

$$f(y; \alpha, \lambda, \mu, \beta) = \left(\frac{1 + \alpha \left(\frac{y-\mu}{\beta} \right)^2}{1 + \alpha} \right) \frac{\exp\left(-\frac{(y-\mu)^2}{2\beta^2} \right)}{\beta\sqrt{2\pi}} \left(1 + Tanh\left[\frac{\lambda \left(\frac{y-\mu}{\beta} \right)}{2} \right] \right) \tag{17}$$

where, $y \in R$, $\mu \in R$, $\lambda \in R$, $\alpha \geq 0$ and $\beta > 0$.

5.2 Maximum Likelihood Estimation

Assume that $y_1, y_2, ...y_n$ is a random sample of size n that chosen from the $BTSN(\mu, \beta, \alpha, \lambda)$ distribution. Then, for the set of parameters $\theta = (\mu, \beta, \alpha, \lambda)$, the log-likelihood function is given by

$$\begin{aligned} l(\theta) = & -n \log(1+\alpha) - \frac{n}{2} \log(2\pi) - n \log(\beta) + \sum_{i=1}^{n} \left[1 + \alpha \left(\frac{y_i - \mu}{\beta} \right)^2 \right] - \sum_{i=1}^{n} \left(\frac{y_i - \mu}{\beta} \right)^2 \\ & - \sum_{i=1}^{n} \left[1 + \exp\left(-\lambda \left(\frac{y_i - \mu}{\beta} \right) \right) \right]. \end{aligned} \tag{18}$$

Now, differentiating the equation (19) with respect to the set of parameters,

$$\begin{aligned} \frac{\partial l(\theta)}{\partial \mu} &= \frac{\lambda}{\beta} \sum_{i=1}^{n} \frac{A(y_i; \mu, \beta, \lambda)}{\left[1 + A(y_i; \mu, \beta, \lambda) \right]} + \sum_{i=1}^{n} \left(\frac{y_i - \mu}{\beta} \right) - \frac{2\alpha}{\beta^2} \sum_{i=1}^{n} \frac{y_i - \mu}{C(y_i; \mu, \beta, \alpha)}, \\ \frac{\partial l(\theta)}{\partial \beta} &= -\frac{n}{\beta} - \sum_{i=1}^{n} \frac{(y_i - \mu)^2}{\beta^3} - \frac{\lambda}{\beta^2} \sum_{i=1}^{n} \frac{y_i - \mu}{\left[1 + B(y_i; \mu, \beta, \lambda) \right]}, \\ \frac{\partial l(\theta)}{\partial \lambda} &= \frac{1}{\beta} \sum_{i=1}^{n} \frac{y_i - \mu}{\left[1 + B(y_i; \mu, \beta, \lambda) \right]}, \\ \frac{\partial l(\theta)}{\partial \alpha} &= -\frac{n}{1+\alpha} + \frac{1}{\beta^2} \sum_{i=1}^{n} \frac{(y_i - \mu)^2}{C(y_i; \mu, \beta, \alpha)}. \end{aligned}$$

Where, $A(y_i; \mu, \beta, \lambda) = \exp\left(-\frac{\lambda(y_i - \beta)}{\beta} \right)$, $B(y_i; \mu, \beta, \lambda) = \exp\left(\frac{\lambda(y_i - \beta)}{\beta} \right)$ and
$C(y_i; \mu, \beta, \alpha) = \left(1 + \alpha \left(\frac{y_i - \mu}{\beta} \right)^2 \right)$.
The calculation of the above equation isn't mathematically sound. Therefore, using the numerical maximisation method of equation (19) with respect to the set of parameters, $\theta = (\mu, \beta, \alpha, \lambda)$, the maximum likelihood estimator for the parameters is derived. The *GenSA* packages in R-software is used for the process.

6 Simulation Study

A simulation study is conducted to assess the effectiveness of maximum likelihood estimates for the parameters of the $BTSN(\mu, \beta, \alpha, \lambda)$ model. The Metropolis-Hastings (M-H) algorithm is employed to generate a set of random numbers. During the study, process is replicated 10,000 times, incorporating three distinct sample sizes ($n = 100$, 300, and 500). The algorithm(s) for generating the random samples are given in the Appendix. Subsequently, the maximum likelihood estimates were computed for each generated sample using the *GenSA* package in the R software.

Finally, the estimated statistics are presented in terms of biases and mean square errors (MSEs) of the estimates and the formula are given by

$$Bias(\hat{\theta}) = E(\hat{\theta}) - \theta \quad and \quad MSE(\hat{\theta}) = V(\hat{\theta}) + Bias(\hat{\theta})^2 \quad Where, \ \hat{\theta} = (\hat{\mu}, \hat{\beta}, \hat{\alpha}, \hat{\lambda})$$

From the results in $Table\ 2 - 7$, it is observed that the maximum likelihood estimators (MLEs) effectively estimate the model parameters. Besides the results also showed that with an increase in sample size, the bias and mean-square error of the MLEs decrease which indicates the asymptotic consistency of the MLEs' of $BTSN(\mu, \beta, \alpha, \lambda)$ distribution.

Table 2: Results of Simulation

			$\mu = 0,\quad \beta = 1$							
			μ		β		λ		α	
α	λ	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
	-2	100	-0.0653	0.0500	-0.0706	0.0450	-0.0555	0.0489	-0.0433	0.0398
		300	-0.0421	0.0410	-0.0408	0.0411	0.0299	0.0401	0.0421	0.0233
		500	0.0337	0.0329	0.0159	0.0207	-0.0167	0.0178	-0.0092	0.0162
	-1	100	-0.0480	0.0325	-0.0898	0.0580	-0.0757	0.0427	-0.0853	0.0642
		300	-0.0465	0.0249	-0.0657	0.0610	-0.0461	0.0321	0.0598	0.0489
		500	-0.0176	0.0198	0.0340	0.0431	0.0329	0.0184	-0.0360	0.0309
0.5	0	100	0.0456	0.0399	-0.0875	0.0855	-0.0707	0.0498	-0.0695	0.0450
		300	-0.0258	0.0253	0.0654	0.0754	-0.0462	0.0356	0.0472	0.0370
		500	0.0123	0.0169	0.0310	0.0451	0.0320	0.0123	-0.0340	0.0229
	1	100	0.0732	0.0369	-0.0849	0.0459	-0.0647	0.0532	-0.0870	0.0755
		300	0.0449	0.0478	0.0560	0.0307	0.0434	0.0454	-0.0610	0.0720
		500	-0.0189	0.0167	-0.0267	0.0188	-0.0355	0.0169	0.0309	0.0465
	2	100	-0.0547	0.0483	0.0750	0.0637	-0.0855	0.0730	0.0459	0.0301
		300	0.0289	0.0460	-0.0522	0.0480	0.0578	0.0496	-0.0470	0.0239
		500	0.0190	0.0172	-0.0397	0.0323	-0.0336	0.0339	-0.0153	0.0176

Table 3: Results of Simulation

			$\mu = 0,\quad \beta = 1$							
			μ		β		λ		α	
α	λ	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
	-2	100	0.0802	0.0650	-0.0477	0.0618	-0.0766	0.0419	-0.0670	0.0539
		300	0.0549	0.0466	0.0501	0.0203	0.0498	0.0440	-0.0439	0.0407
		500	-0.0368	0.0348	-0.0357	0.0199	-0.0150	0.0247	0.0359	0.0300
	-1	100	-0.0455	0.0300	0.0763	0.0420	0.0754	0.0482	-0.0716	0.0451
		300	0.0469	0.0230	-0.0548	0.0299	-0.0460	0.0355	-0.0418	0.0424
		500	-0.0144	0.0179	-0.0240	0.0233	- 0.0333	0.0163	0.0149	0.0201
1.5	0	100	-0.0916	0.0450	-0.0557	0.0462	0.0632	0.0393	0.0875	0.0654
		300	0.0410	0.0465	-0.0460	0.0421	0.0499	0.0388	-0.0547	0.0456
		500	-0.0140	0.0098	0.0326	0.0170	-0.0198	0.0100	0.0335	0.0357
	1	100	-0.0490	0.0355	-0.0871	0.0685	0.0691	0.0532	-0.0900	0.0435
		300	-0.0245	0.0264	-0.0698	0.0640	-0.0461	0.0330	0.0468	0.0365
		500	0.0143	0.0109	0.0348	0.0400	0.0332	0.0109	-0.0190	0.0198
	2	100	0.0523	0.0407	-0.0490	0.0378	-0.0491	0.0357	-0.0666	0.0458
		300	- 0.0290	0.0311	-0.0467	0.0219	0.0254	0.0266	0.0448	0.0444
		500	-0.0197	0.0198	0.0192	0.0157	-0.0166	0.0101	-0.0100	0.0204

Table 4: Results of Simulation

			$\mu = 0,\quad \beta = 1$							
			μ		β		λ		α	
α	λ	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
	-2	100	0.0540	0.0215	-0.0644	0.0545	-0.0466	0.0321	-0.0588	0.0498
		300	-0.0471	0.0244	0.0467	0.0360	-0.0424	0.0231	0.0430	0.0233
		500	-0.0155	0.0165	-0.0319	0.0120	-0.0093	0.0165	-0.0266	0.0189
	-1	100	0.0490	0.0447	-0.0800	0.0497	0.0645	0.0444	0.0754	0.0630
		300	-0.0407	0.0294	0.0460	0.0397	0.0348	0.0423	-0.0529	0.0484
		500	0.0163	0.0155	-0.0290	0.0187	-0.0159	0.0314	-0.0367	0.0318
2.0	0	100	-0.0489	0.0308	-0.0616	0.0449	-0.0809	0.0434	0.1021	0.0500
		300	-0.0455	0.0274	-0.0432	0.0490	0.0561	0.0317	-0.0629	0.0497
		500	0.0112	0.0189	0.0330	0.0357	-0.0200	0.0180	0.0358	0.0357
	1	100	0.1097	0.0602	0.0729	0.0468	0.0801	0.0480	-0.0687	0.0505
		300	0.0654	0.0497	-0.0427	0.0234	0.0487	0.0333	0.0450	0.0315
		500	-0.0207	0.0344	-0.0268	0.0187	-0.0255	0.0131	-0.0300	0.0197
	2	100	-0.1008	0.0587	-0.0465	0.0351	0.0580	0.0490	0.0658	0.0456
		300	-0.0459	0.0478	-0.0320	0.0250	0.0438	0.0230	-0.0391	0.0378
		500	0.0298	0.0257	0.0097	0.0131	-0.0261	0.0181	-0.0209	0.0217

7 Real Life Application

This section examines the applicability of the novel probability distribution using one real life data set. A comparative analysis between the newly introduced model along with several alternative model of distributions is conducted. Those alternative models include normal distribution $N(\mu, \beta^2)$, skew normal distribution $SN(\mu, \beta, \lambda)$ proposed by (Azzalini, 1985), Tanh skew normal distribution $TSN(\mu, \beta, \lambda)$ introduced by (Mahmoud et al., 2020) and alpha skew normal distribution $ASN(\mu, \beta, \alpha)$ given by (Elal-Olivero, 2010) ,two parameter bimodal skew normal distribution $BSN(\mu, \beta, \lambda, \alpha)$ proposed by Elal-Olivero et al. (2020) and alpha beta skew normal distribution $ABSN(\mu, \beta, \alpha, b)$ introduced by Shafiei et al. (2016). The value of the fitted models has been calculated using maximum likelihood techniques using the *GenSA* package in R software. Additionally, in order to compare the models, the Akaike information criterion (AIC) and the Bayesian information criterion (BIC) are taken into consideration.

Table 5: Results of Simulation

			$\mu = 1, \quad \beta = 1.5$							
			μ		β		λ		α	
α	λ	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
	-2	100	0.0456	0.0287	- 0.0650	0.0409	0.0655	0.0529	-0.0491	0.0328
		300	-0.0317	0.0240	-0.0391	0.0355	-0.0487	0.0399	0.0420	0.0250
		500	0.0197	0.0154	0.0287	0.0207	-0.0349	0.0387	-0.0212	0.0163
	-1	100	-0.0659	0.0507	0.0699	0.0453	0.0704	0.0470	0.1201	0.0802
		300	0.0424	0.0367	-0.0484	0.0311	0.0537	0.0230	0.0600	0.0457
		500	-0.0333	0.0108	-0.0281	0.0203	-0.0298	0.0257	-0.0215	0.0398
0.5	0	100	-0.0621	0.0487	-0.0490	0.0323	0.0759	0.0638	-0.0652	0.0597
		300	-0.0301	0.0374	0.0427	0.0259	-0.0520	0.0480	-0.0409	0.0357
		500	0.0237	0.0311	-0.0230	0.0178	0.0361	0.0398	-0.0329	0.0233
	1	100	-0.0680	0.0537	-0.1000	0.0611	-0.0536	0.0244	-0.0623	0.0455
		300	0.0451	0.0385	-0.0620	0.0468	-0.0397	0.0237	-0.0491	0.0331
		500	0.0344	0.0190	-0.0397	0.0420	0.0190	0.0107	0.0280	0.0297
	2	100	0.0608	0.0409	-0.0459	0.0377	0.0619	0.0420	0.0658	0.0444
		300	-0.0337	0.0323	0.0408	0.0262	-0.0400	0.0364	-0.0457	0.0293
		500	0.0259	0.0307	0.0190	0.0203	-0.0283	0.0190	0.0250	0.0398

Table 6: Results of Simulation

			$\mu = 1, \quad \beta = 1.5$							
			μ		β		λ		α	
α	λ	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
	-2	100	0.0421	0.0268	0.0541	0.0412	-0.0488	0.0398	0.1501	0.0930
		300	-0.0400	0.0259	-0.0432	0.0308	-0.0435	0.0259	-0.0498	0.0390
		500	0.0164	0.0176	-0.0109	0.0200	-0.0216	0.0185	-0.0347	0.0245
	-1	100	0.0721	0.0622	0.0687	0.0431	0.0645	0.0511	0.1109	0.0530
		300	-0.0567	0.0423	-0.0391	0.0395	-0.0481	0.0364	-0.0640	0.0568
		500	0.0360	0.0211	0.0297	0.0200	-0.0320	0.0258	-0.0340	0.0222
1.5	0	100	-0.0658	0.0334	0.0618	0.0432	-0.0400	0.0371	-0.0499	0.0389
		300	0.0437	0.0288	-0.0307	0.0320	-0.0418	0.0260	-0.0452	0.0256
		500	-0.0206	0.0139	0.0211	0.0317	0.0100	0.0254	0.0161	0.0285
	1	100	0.0523	0.0413	-0.0701	0.0600	-0.0568	0.0410	-0.0398	0.0256
		300	-0.0471	0.0345	0.0567	0.0413	-0.0430	0.0360	0.0410	0.0297
		500	-0.0209	0.0233	-0.0397	0.0355	0.0245	0.0254	-0.0169	0.0161
	2	100	-0.0560	0.0437	0.0659	0.0298	-0.1021	0.0732	0.0566	0.0464
		300	-0.0403	0.0398	-0.0430	0.0271	0.0666	0.0459	-0.0453	0.0363
		500	-0.0147	0.0267	-0.0256	0.0298	-0.0394	0.0357	-0.0149	0.0236

Illustration 1.

For this illustration, a data set of failure times of 84 Aircraft Windshield (El-Bassiouny et al., 2015) is considered. here, Table 8 reflects the maximum likelihood estimate of the fitted models along with their corresponding log-likelihood, AIC and BIC. On the other hand Figure 2 depicts the performance as well as behaviour of the fitted models. According to the Table 8, value of AIC and BIC of $BTSN(\mu, \beta, \alpha, \lambda)$ distribution is less than that of the other rival distributions. Additionally, Figure 2 reflects the good fit of the $BTSN(\mu, \beta, \alpha, \lambda)$ distribution. Thus, it may be concluded that the Bimodal Tanh Skew Normal distribution provides better fits for the data set under consideration.

8 Hypothesis Testing

In this section, to discriminate between some nested models like $N(\mu, \beta)$, $SN(\mu, \beta, \lambda)$, $TSN(\mu, \beta, \lambda)$ $ASN(\mu, \beta, \alpha)$ and $BTSN(\mu, \beta, \alpha, \lambda)$ Likelihood Ratio (LR) test is deployed. The test statistics as well as the null hypothesis are as follows:

(i) To discriminate $N(\mu, \beta)$ and $BTSN(\mu, \beta, \alpha, \lambda)$ distribution, the null hypothesis $H_0 : \alpha = 0, \lambda = 0$ have to test

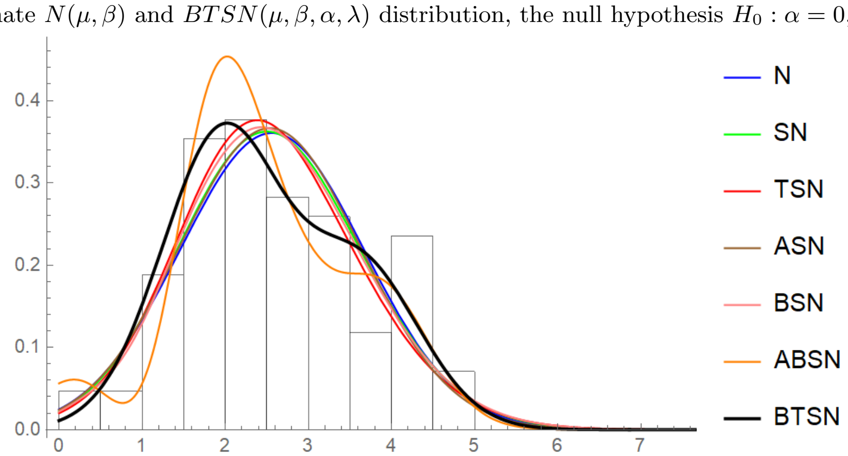

Figure 2: Plots of observed and expected densities of some distributions for 84 Aircraft Windshield.

Table 7: Results of Simulation

			$\mu = 1, \quad \beta = 1.5$							
			μ		β		λ		α	
α	λ	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
	-2	100	0.0520	0.0410	-0.0771	0.0661	0.0598	0.0400	0.0497	0.0354
		300	0.0421	0.0331	-0.0501	0.0410	-0.0434	0.0339	-0.0424	0.0254
		500	-0.0239	0.0255	0.0337	0.0300	-0.0278	0.0257	-0.0236	0.0168
	-1	100	-0.0477	0.0209	0.1089	0.0630	0.0699	0.0501	-0.0401	0.0311
		300	0.0417	0.0245	-0.0570	0.0408	-0.0491	0.0345	0.0324	0.0251
		500	-0.0161	0.0100	-0.0348	0.0332	0.0327	0.0288	0.0098	0.0100
2.0	0	100	-0.0609	0.0400	0.0898	0.0547	-0.0541	0.0353	-0.0493	0.0349
		300	-0.0409	0.0330	-0.0574	0.0390	0.0293	0.0278	-0.0250	0.0257
		500	0.0281	0.0197	-0.0270	0.0215	-0.0187	0.0155	0.0173	0.0189
	1	100	-0.0407	0.0350	0.0631	0.0359	- 0.0760	0.0494	0.0501	0.0630
		300	0.0424	0.0255	0.0299	0.0308	-0.0540	0.0329	-0.0490	0.0343
		500	-0.0090	0.0102	-0.0180	0.0167	-0.0233	0.0201	-0.0390	0.0240
	2	100	-0.0621	0.0533	0.0702	0.0423	0.0568	0.0498	0.0609	0.0370
		300	0.0497	0.0412	-0.0499	0.0369	-0.0457	0.0367	- 0.0422	0.0362
		500	-0.0364	0.0345	-0.0245	0.0201	0.0119	0.0249	0.0090	0.0209

Table 8: MLE's, log-likelihood, AIC and BIC for failure times of 84 Aircraft Windshield.

Distributions	μ	β	λ	α	b	*logL*	*AIC*	*BIC*
$N(\mu, \beta)$	2.5626	1.1066	–	–	–	-129.22	262.44	267.30
$SN(\mu, \beta, \lambda)$	1.8669	1.3071	0.8933	–	–	-129.19	264.38	271.67
$TSN(\mu, \beta, \lambda)$	1.6001	1.4666	2.4493	–	–	-129.27	264.54	271.83
$ASN(\mu, \beta, \alpha)$	2.7018	1.1065	–	0.1268	–	-129.21	264.42	271.71
$BSN(\mu, \beta, \lambda, \alpha)$	0.5617	1.4314	2.9474	3.4591	–	-128.77	265.54	275.26
$ABSN(\mu, \beta, \alpha, b)$	2.2303	0.7229	–	0.3515	-0.2561	-126.71	261.42	271.15
$BTSN(\mu, \beta, \alpha, \lambda)$	2.9751	0.8479	-0.6080	0.8442	–	-124.56	257.12	266.84

against the alternative hypothesis $H_1 : \alpha \neq 0, \lambda \neq 0$ and the test statistic is

$$-2\log(LR) = -2[\log L(\hat{\mu}_1, \hat{\beta}_1, \alpha = 0, \lambda = 0|x) - \log L(\hat{\mu}_2, \hat{\beta}_2, \hat{\alpha}_2, \hat{\lambda}_2)] \sim \chi^2_2,$$

where, $\hat{\mu}_1$, $\hat{\beta}_1$ and $\hat{\mu}_2$, $\hat{\beta}_2$, $\hat{\lambda}_2$, $\hat{\alpha}_2$ are the MLEs' of $N(\mu, \beta)$ and $BTSN(\mu, \beta, \alpha, \lambda)$ distribution and $r = 2$ (difference between the numbers of parameters).

(ii) To discriminate $SN(\mu, \beta, \lambda)$ and $BTSN(\mu, \beta, \alpha, \lambda)$ distribution, the null hypothesis $H_0 : \alpha = 0$ have to test against the alternative hypothesis $H_1 : \alpha \neq 0$ and the test statistic is

$$-2\log(LR) = -2[\log L(\hat{\mu}_1, \hat{\beta}_1, \hat{\lambda}_1, \alpha = 0|x) - \log L(\hat{\mu}_2, \hat{\beta}_2, \hat{\alpha}_2, \hat{\lambda}_2)] \sim \chi^2_1,$$

where, $\hat{\mu}_1$, $\hat{\beta}_1$, $\hat{\lambda}_1$ and $\hat{\mu}_2$, $\hat{\beta}_2$, $\hat{\lambda}_2$, $\hat{\alpha}_2$ are the MLEs' of $SN(\mu, \beta, \lambda)$ and $BTSN(\mu, \beta, \alpha, \lambda)$ distribution and $r = 1$ (difference between the numbers of parameters).

(iii) To discriminate $TSN(\mu, \beta, \lambda)$ and $BTSN(\mu, \beta, \alpha, \lambda)$ distribution, the null hypothesis $H_0 : \alpha = 0$ have to test against the alternative hypothesis $H_1 : \alpha \neq 0$ and the test statistic is

$$-2\log(LR) = -2[\log L(\hat{\mu}_1, \hat{\beta}_1, \hat{\lambda}_1, \alpha = 0|x) - \log L(\hat{\mu}_2, \hat{\beta}_2, \hat{\alpha}_2, \hat{\lambda}_2)] \sim \chi^2_1,$$

where, $\hat{\mu}_1$, $\hat{\beta}_1$, $\hat{\lambda}_1$ and $\hat{\mu}_2$, $\hat{\beta}_2$, $\hat{\lambda}_2$, $\hat{\alpha}_2$ are the MLEs' of $TSN(\mu, \beta, \lambda)$ and $BTSN(\mu, \beta, \alpha, \lambda)$ distribution and $r = 1$ (difference between the numbers of parameters).

(iv) To discriminate $ASN(\mu, \beta, \alpha)$ and $BTSN(\mu, \beta, \alpha, \lambda)$ distribution, the null hypothesis $H_0 : \lambda = 0$ have to test against the alternative hypothesis $H_1 : \lambda \neq 0$ and the test statistic is

$$-2\log(LR) = -2[\log L(\hat{\mu}_1, \hat{\beta}_1, \hat{\alpha}_1, \lambda = 0|x) - \log L(\hat{\mu}_2, \hat{\beta}_2, \hat{\alpha}_2, \hat{\lambda}_2)] \sim \chi^2_1,$$

where, $\hat{\mu}_1$, $\hat{\beta}_1$, $\hat{\alpha}_1$ and $\hat{\mu}_2$, $\hat{\beta}_2$, $\hat{\lambda}_2$, $\hat{\alpha}_2$ are the MLEs' of $TSN(\mu, \beta, \lambda)$ and $BTSN(\mu, \beta, \alpha, \lambda)$ distribution and $r = 1$ (difference between the numbers of parameters).

From the table 9 it can be noted that the value of the LR test statistics is higher than the tabulated critical value at the 5% level of significance for the entire null hypothesis. Consequently, it is established that, given the proposed models, the BTSN distribution was the best fit according to the criteria used.

9 Conclusion

This article introduces a groundbreaking family of continuous probability distributions known as the Bimodal *Tanh* Skew Normal (BTSN). Unlike traditional distributions, the BTSN model is specifically designed to adeptly cap-

Table 9: The value of the likelihood ratio test for the respective hypotheses of of 84 Aircraft Windshield.

Hypothesis	LR value	df	Critical values at 5 %
$H_0 : \alpha = 0, \lambda = 0$ Vs $H_1 : \alpha \neq 0, \lambda \neq 0$	9.32	2	5.99
$H_0 : \alpha = 0$ Vs $H_1 : \alpha \neq 0$	9.26	1	3.84
$H_0 : \alpha = 0$ Vs $H_1 : \alpha \neq 0$	9.42	1	3.84
$H_0 : \lambda = 0$ Vs $H_1 : \lambda \neq 0$	9.30	1	3.84

ture data characterized with two modes. The article meticulously delves into various essential statistical properties inherent to the BTSN distribution, shedding light on its mathematical properties and providing a comprehensive understanding of its behavior. The study employs the Maximum Likelihood method to rigorously examine the challenges associated with estimating the distribution's parameters, offering insights into the precision and reliability of parameter estimation techniques within the BTSN framework. Furthermore to examine the behavior of the estimated parameter, a simulation study is performed using Metropolis Hastings algorithm and it is found that the estimated parameters are asymptotically consistent with the increasing number of sample sizes. To assess the distribution's versatility and comparative performance, the article conducts extensive studies comparing BTSN with rival distributions. By utilizing a real-life dataset, the research demonstrates the superior flexibility of the BTSN distribution in comparison to other competing models, showcasing its ability to effectively accommodate diverse patterns in empirical data. Finally, a Likelihood Ratio (LR) test is executed to discern and establish distinctions between the BTSN distribution and other rival distributions, providing a robust statistical assessment of the uniqueness and suitability of the proposed model in relation to its counterparts.

Appendix

The MH algorithm for generating random samples:

- Consider BTSN as target density denoted as f.
- Consider normal distribution as proposal density denoted as g.
- Initialize y_0 (an arbitrary value).

For each iteration i from 1 to N,

- Generate x^* from g.
- Compute acceptance ratio $r = \dfrac{f(x^*)}{f(y_t)} \times \dfrac{g(y_t)}{g(x^*)}$.
- Generate a uniform random number u.
- If $u < min(1, r)$, then accept x^* ($y_{t+1} = x^*$); otherwise $y_{t+1} = y_t$.
- Store the accepted sample.
- Discard the first $N - n$ sample values (Burn-in values) and use the rest sample value as filtered sample.

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