Bootstrap Confidence Intervals of the Difference between Two Process Capability Indices for Half Logistic Distribution

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Abstract

The process capability indices are important numerical measures in statistical quality control. Well-known process capability indices are constructed under the process distribution is normal. Unfortunately, this situation is rather not realistic. This paper focuses on the half logistic distribution. The bootstrap confidence intervals for the difference between two process capability indices for the mentioned distribution are proposed. The bootstrap confidence intervals considered in this paper consist of the standard bootstrap confidence interval, the percentile bootstrap confidence interval and the bias-corrected percentile bootstrap confidence interval. A Monte Carlo simulation has been used to investigate the estimated coverage probabilities and average widths of the bootstrap confidence intervals. Simulation results showed that the estimated coverage probabilities of the percentile bootstrap confidence interval and the bias-corrected percentile bootstrap confidence interval get closer to the nominal confidence level than those of the standard bootstrap confidence interval.

Keywords: Process capability index, Bootstrap confidence interval, Half logistic distribution.

1. Introduction

The half logistic distribution, which is the distribution of the absolute logistic random variable, was introduced by Balakrishnan (1985). The main references about the half logistic distribution include Balakrishnan and Chan (1992), Balakrishnan and Wong (1994) and Balakrishnan and Aggarwala (1996). If Y is a logistic random variable, then $X = |Y|$ has a half logistic distribution. The probability density function $(f(x))$ and the cumulative distribution function $(F(x))$ are

$$f(x) = \frac{2\exp\{-(x-\mu)/\sigma\}}{\sigma\left[1+\exp\{-(x-\mu)/\sigma\}\right]^2}, \tag{1.1}$$

and

$$F(x) = \frac{1-\exp\{-(x-\mu)/\sigma\}}{\left[1+\exp\{-(x-\mu)/\sigma\}\right]}, \quad x \ge \mu \ge 0, \quad \sigma > 0, \tag{1.2}$$

where μ and σ are the location and the scale parameters, respectively. Characterizations of the half logistic distribution were described in Olapade and Ojo (2002).The graph of the probability density function for half logistic distribution is shown in Fig. 1.1 The mean and the variance of X are defined as

$$E(X) = \mu + \sigma \ln(4) \text{ and } Var(X) = \sigma^2 \left[\frac{\pi^2}{3} - (\ln(4))^2 \right].$$

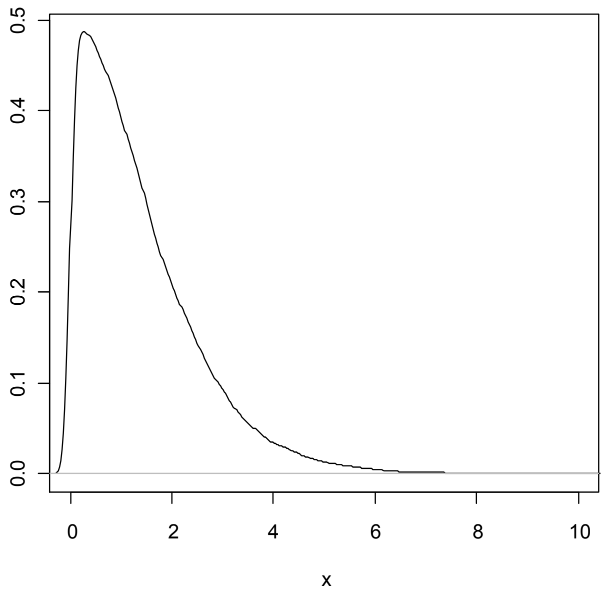


Figure 1.1 The probability density function for half logistic distribution with $\mu = 0$ and $\sigma = 1$.

Several studies have applied the half logistic distribution. For instance, Balakrishnan (1985) has suggested the usage of this distribution as a possible life-time model with an increasing hazard rate. In addition, Balakrishnan and Chan (1992) have shown that the failure times of air conditioning equipment in a Boeing 720 airplane fits the half logistic distribution quite well. This distribution was also applied to environmental and sports records data (Mbah and Tsokos, 2008). In recent papers, many authors have applied the half logistic distribution under progressive Type-II censoring (see Kang et al., 2008, Balakrishnan and Saleh, 2011, Jang et al. 2011). As mentioned above, it is known that the half logistic distribution is an increasing failure rates model with reasonable importance in statistical quality control and reliability studies (see Kantam and Rosaiah 1998,Kantam et al. 2000, Srinivasa, 2004, Satyaprasad, 2007, Rosaiah et al., 2009, Kantam et al., 2010).

One of the statistical quality control tools widely used is the process capability index (PCI). This index uses both process variability and process specification to determine whether the process is capable (Peng, 2010). Even though there are many process capability indices, the two most commonly used indices are C_p and C_{pk} (Kane, 1986, Zhang, 2010). In this paper, we focus only on the popular process capability index C_{pk} defined as follows

$$C_{pk} = \min\left\{ \frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma} \right\}, \tag{1.3}$$

where USL and LSL denote respectively the upper and lower specification limits of the process, μ is the process mean and σ is the process standard deviation. As the process standard deviation and the process mean are unknown, they must be estimated from the sample data $\{X_1,...,X_n\}$. Under the normality assumption, the sample mean $\bar{X}$; $\bar{X}=n^{-1}\sum_{i=1}^{n}X_i$ and the sample standard deviation S; $S=\sqrt{(n-1)^{-1}\sum_{i=1}^{n}(X_i-\bar{X})^2}$ are used to estimate the unknown parameters μ and σ in Eq. (1.3), respectively. Therefore, the natural estimator of the process capability index C_{pk} can be obtained as

$$\tilde{C}_{pk}=\min\left\{\frac{USL-\bar{X}}{3S},\frac{\bar{X}-LSL}{3S}\right\}.$$

However, the underlying process distribution is non-normal in some situations. Hence, it may be a skewed distribution. To deal with these phenomena, Clements (1989) proposed a new method for computing the estimator of the process capability index C_{pk} when the process distribution is non-normal. This estimator is defined as

$$\breve{C}_{pk}=\min\left\{\frac{USL-M}{U_p-M},\frac{M-LSL}{M-L_p}\right\}, \tag{1.4}$$

where U_p,L_p and M denote the 99.865th percentile, the 0.135th percentile and the 50th percentile of the distribution, respectively. The advantage of $\breve{C}_{pk}$ shown in Eq.(1.4) is that it can be applied to any distribution. Kantam et al. (2010) discussed the relationship between $\breve{C}_{pk}$ and the probability that a product falling outside the specification limits when X has a half logistic distribution. This probability is given by

$$P_t=P(X\le LSL)+P(X\ge USL)=1+F(LSL)-F(USL),$$

where $F(\cdot)$ is the cumulative distribution function of a half logistic distribution shown in Eq.(1.2). Using the open source statistical package R (Ihaka and Gentleman 1996), some values of L_p,U_p and M for the half logistic distribution are shown in Table 1.

Table 1: The values of L_p,U_p and M for the half logistic distribution

μ	σ	L_p	U_p	M
0	1	0.002700002	7.300123	1.098612
0	1.5	0.00405	10.95018	1.647918
0.5	1	0.5027	7.800123	1.598612
0.5	1.5	0.50405	11.45018	2.147918
1	1	1.0027	8.300123	2.098612
1	1.5	1.00405	11.95018	2.647918
1.5	1	1.5027	8.800123	2.611111
1.5	1.5	1.50405	12.45018	3.147918
2	1	2.0027	9.300123	3.098612
2	1.5	2.00405	12.95018	3.647918

The estimator $\breve{C}_{pk}$ in Eq. (1.4) can be applied when the scale parameter σ of the half logistic distribution is equal to 1. Therefore, if a scale parameter σ is introduced and known, i.e., $\sigma \neq 1$ in Eq.(1.1), the optimal estimator of C_{pk} is given by

$$\breve{C}'_{pk} = \min\left\{ \frac{USL - \sigma M}{\sigma(U_p - M)}, \frac{\sigma M - LSL}{\sigma(M - L_p)} \right\}.$$

In practice, the scale parameter σ is unknown. Therefore, we must estimate the unknown σ by its estimator. In this paper, we use the moment method for calculating this estimator. The estimator of C_{pk} for a half logistic distribution is

$$\hat{C}_{pk} = \min\left\{ \frac{USL - \hat{\sigma} M}{\hat{\sigma}(U_p - M)}, \frac{\hat{\sigma} M - LSL}{\hat{\sigma}(M - L_p)} \right\}, \tag{1.5}$$

where $\hat{\sigma}$ is the estimator of σ. Here, we use the simple estimator which is computed by the moment method given by $\hat{\sigma} = \frac{1}{\ln(4)}\left[\bar{X} - \hat{\mu} \right]$ and the maximum likelihood estimator for μ is $\hat{\mu} = X_{(1)}$, the smallest sample order statistics. The moments of the half logistic distribution were shown in Giles (2012).

In this paper, our focus is on the difference between two process capability indices, $C_{pk1} - C_{pk2}$, which we will denote by δ which

$$\delta = \min\left\{ \frac{USL - \mu_1}{3\sigma_1}, \frac{\mu_1 - LSL}{3\sigma_1} \right\} - \min\left\{ \frac{USL - \mu_2}{3\sigma_2}, \frac{\mu_2 - LSL}{3\sigma_2} \right\},$$

where μ_1, μ_2 and σ_1, σ_2 are the process mean and standard deviation of the first and the second population, respectively. Similar to Eq.(1.5), we can get the estimator of δ which is

$$\hat{\delta} = \min\left\{ \frac{USL - \hat{\sigma}_1 M_1}{\hat{\sigma}_1(U_{p1} - M_1)}, \frac{\hat{\sigma}_1 M_1 - LSL}{\hat{\sigma}_1(M_1 - L_{p1})} \right\} - \min\left\{ \frac{USL - \hat{\sigma}_2 M_2}{\hat{\sigma}_2(U_{p2} - M_2)}, \frac{\hat{\sigma}_2 M_2 - LSL}{\hat{\sigma}_2(M_2 - L_{p2})} \right\}, \tag{1.6}$$

where $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are the moment estimator of σ_1 and σ_2 given by $\hat{\sigma}_1 = \frac{1}{\ln(4)}\left[\bar{X}_1 - \hat{\mu}_1 \right]$ and $\hat{\sigma}_2 = \frac{1}{\ln(4)}\left[\bar{X}_2 - \hat{\mu}_2 \right]$, respectively, $\hat{\mu}_1 = X_{1(1)}$ and $\hat{\mu}_2 = X_{2(1)}$.

The remainder of the paper is organized as follows. Section 2 describes the bootstrap confidence intervals. Some simulation evidence on the performance of bootstrap confidence intervals is provided in Section 3. In Section 4, all bootstrap confidence intervals are illustrated and compared through numerical example. A discussion of the results and conclusions are presented in the final section. The conclusions are offered in the final section.

2. Bootstrap Confidence Intervals

The bootstrap is a computer-based and resampling method for assigning measures of accuracy to statistical estimates (Efron and Tibshirani, 1993). Many types of bootstrap methods for constructing confidence intervals have been introduced; for example, the standard bootstrap method (SB), the percentile bootstrap method (PB) and the bias-corrected percentile bootstrap method (BCPB).

For a sequence of independent and identically distributed (i.i.d.) random variables, the bootstrap procedure can be defined as follows (Tosasukul et al., 2009). Let the random variables $\{X^*_{k,j}, 1 \le j \le m\}$ be the result from sampling m times from the k^{th} population with replacement from the n observations $X_{k,1},...,X_{k,n}$. The random variables $\{X^*_{k,j}, 1 \le j \le m\}$ are called the bootstrap samples from original data $X_{k,1},...,X_{k,n}$. In what follows, we describe the constructions for the confidence interval of the difference between two process capability indices δ using bootstrap techniques.

2.1 Standard Bootstrap (SB) Confidence Interval

Let $X^*_{k,b}$, where $1 \le b \le B$, and $k = 1,2$ be the b^{th} bootstrap samples and let $X^*_{k,1},...,X^*_{k,B}$ be the B bootstrap samples. The b^{th} bootstrap estimator of δ is computed by

$$\hat{\delta}^{*(b)} = \min\left\{\frac{USL-\hat{\sigma}_1^{*(b)}M_1}{\hat{\sigma}_1^{*}(U_{p1}-M_1)}, \frac{\hat{\sigma}_1^{*(b)}M_1-LSL}{\hat{\sigma}_1^{*(b)}(M_1-L_{p1})}\right\} - \min\left\{\frac{USL-\hat{\sigma}_2^{*(b)}M_2}{\hat{\sigma}_2^{*(b)}(U_{p2}-M_2)}, \frac{\hat{\sigma}_2^{*(b)}M_2-LSL}{\hat{\sigma}_2^{*(b)}(M_2-L_{p2})}\right\}, \qquad (2.1)$$

where $\hat{\sigma}_1^{*(b)} = \bar{X}_1^{*(b)}/\ln(4)$, $\bar{X}_1^{*(b)} = n_1^{-1}\sum_{j=1}^{n_1} X^*_{1,j}$, $\hat{\sigma}_2^{*(b)} = \bar{X}_2^{*(b)}/\ln(4)$, $\bar{X}_2^{*(b)} = n_2^{-1}\sum_{j=1}^{n_2} X^*_{2,j}$, Thus, the standard bootstrap $(1-\alpha)100\%$ confidence interval is

$$CI_{SB} = \left(\bar{\hat{\delta}}^* - Z_{1-\frac{\alpha}{2}} S^*_{\delta}, \bar{\hat{\delta}}^* + Z_{1-\frac{\alpha}{2}} S^*_{\delta}\right), \qquad (2.2)$$

where $Z_{1-\alpha/2}$ is a $(1-\alpha/2)^{th}$ quantile of the standard normal distribution, $\bar{\hat{\delta}}^* = B^{-1}\sum_{i=1}^{B}\hat{\delta}^{*(i)}$ and

$$S^*_{\delta} = \sqrt{\frac{1}{B-1}\sum_{i=1}^{B}\left(\hat{\delta}^{*(i)} - \bar{\hat{\delta}}^{*(i)}\right)^2}.$$

2.2 Percentile Bootstrap (PB) Confidence Interval

The percentile bootstrap $(1-\alpha)100\%$ confidence interval is given by

$$CI_{PB} = \left(\hat{\delta}^*_{\left(\frac{\alpha}{2}B\right)}, \hat{\delta}^*_{\left(\left[1-\frac{\alpha}{2}\right]B\right)} \right), \tag{2.3}$$

where $\hat{\delta}^*_{(r)}$ is the r^{th} ordered value on the list of the B bootstrap estimator of δ.

2.3 Bias-Corrected Percentile Bootstrap (BCPB) Confidence Interval

The obtained bootstrap distributions using only a sample of the complete bootstrap distribution may be shifted higher or lower than would be expected. Therefore, this approach has been introduced in order to correct for the potential bias. Firstly, using the ordered distribution of $\hat{\delta}^*$, compute the probability $P_0 = P(\hat{\delta}^* \leq \hat{\delta})$. Then, $Z_0 = \Phi^{-1}(P_0)$. Therefore, the percentile of the ordered distribution $G^*(\hat{\delta}^*)$, $P_L = \Phi\left(2Z_0 - Z_{1-\alpha/2}\right)$ and $P_U = \Phi\left(2Z_0 + Z_{1-\alpha/2}\right)$ are obtained, where $\Phi(\cdot)$ is the standard normal cumulative function. Finally, the bias-corrected percentile bootstrap $(1-\alpha)100\%$ confidence interval is defined as follows

$$CI_{BCPB} = \left(\hat{\delta}^*_{(P_L B)}, \hat{\delta}^*_{(P_U B)} \right), \tag{2.4}$$

where $\hat{\delta}^*_{(r)}$ is the r^{th} ordered value on the list of the B bootstrap estimator of δ.

To study the different confidence intervals, we consider their estimated coverage probabilities and average widths. For each of the methods considered, the probability that the true value of C_{pk} is covered by the $(1-\alpha)100\%$ bootstrap confidence interval, which is called the "coverage probability", can be obtained. In addition, the average width of the bootstrap confidence interval is calculated based on the $M = 5,000$ different trials .The estimated coverage probability and the average width are given by

$$\widehat{1-\alpha} = \frac{\#(L \leq \delta \leq U)}{M},$$

and

$$\widehat{Width} = \frac{\sum_{i=1}^{M} (U_i - L_i)}{M},$$

where L and U denote the lower and upper bound of the bootstrap confidence interval.

In the following section, a simulation study is presented in order to evaluate the performance of the confidence intervals CI_{SB}, CI_{PB}, and CI_{BCPB} based on their estimated coverage probabilities and average widths.

3. Simulation Study

A simulation study on the behavior of three bootstrap confidence intervals of the difference between two process capability indices for half logistic distribution is described. The statistical package R (Ihaka and Gentleman, 1996) is used to carry out the simulation study in this section. In addition, the sample sizes and parameter values of half logistic distribution that we used in this simulation are listed in Table 2. Similar to previous experiments of Kantam et al. (2010), we set the lower and upper specification limits are 1 and 29, respectively. The process capability indices and the difference between the two process capability indices of twelve designs are shown in Table 2. For each design, $B = 1,000$ bootstrap samples with each of size n are drawn from the original sample. Additionally, the simulation is replicated 5,000 times. The 90% and 95% bootstrap confidence intervals are constructed by each of the three methods, i.e., SB, PB, and BCPB confidence intervals.

The simulation results are summarized in Tables 3 and 4. These two tables present the results on the estimated coverage probabilities and average widths of 90% and 95% bootstrap confidence intervals, respectively. We begin with the results for Designs 1a-f, 2a-f, 3a-f and 4a-f ($\mu_1 < \sigma_1$ and $\mu_2 < \sigma_2$), the estimated coverage probabilities of CI_{SB} are larger than the nominal confidence level. In addition, the estimated coverage probabilities of CI_{BCPB} get reasonably closer to the nominal confidence level than those of CI_{SB} and CI_{PB} for all sample sizes. When Designs 6a-f and 7a-f ($\mu_1 = \sigma_1$ and $\mu_2 < \sigma_2$) are considered, the estimated coverage probabilities of CI_{SB} are not less than the nominal confidence level. The CI_{PB} provides the estimated coverage probabilities closer to the nominal confidence level than those of CI_{SB} and CI_{BCPB}.

As one can see, under Designs 5a-f, 8a-f, 9a-f, 10a-f, 11a-f and 12a-f ($\mu_1 > \sigma_1$ and $\mu_2 \le \sigma_2$), the CI_{SB}, CI_{PB}, and CI_{BCPB} give poor coverage probabilities than the nominal confidence level for large sample sizes. On the other hand, the estimated coverage probabilities of CI_{SB} are significantly above the nominal confidence level for small sample size $(n_1 = n_2 = 10)$.

The CI_{BCPB} provides the shortest average width for all situations. Additionally, the average widths of all the bootstrap confidence intervals get shorter when n_1 and (or) n_2 increases.

Table 2: Sample sizes and parameter values of half logistic distribution used in the simulation study.

Design	n_1	n_2	μ_1	σ_1	μ_2	σ_2	C_{pk1}	C_{pk2}	δ
1a	10	10	0.5	1	0	1	0.5462	0.0900	0.4562
1b	30	30							
1c	50	50							
1d	10	30							
1e	10	50							
1f	30	50							
2a	10	10	0.5	1.5	0	1.5	0.9011	0.5969	0.3042
2b	30	30							
2c	50	50							
2d	10	30							
2e	10	50							
2f	30	50							
3a	10	10	0.5	1.5	0	1	0.9011	0.0900	0.8111
3b	30	30							
3c	50	50							
3d	10	30							
3e	10	50							
3f	30	50							
4a	10	10	0.5	1	0	1.5	0.5462	0.5969	-0.0507
4b	30	30							
4c	50	50							
4d	10	30							
4e	10	50							
4f	30	50							
5a	10	10	1.5	1	0.5	1	1.4535	0.5462	0.9073
5b	30	30							
5c	50	50							
5d	10	30							
5e	10	50							
5f	30	50							
6a	10	10	1.5	1.5	0.5	1.5	1.5094	0.9011	0.6083
6b	30	30							
6c	50	50							
6d	10	30							
6e	10	50							
6f	30	50							

Table 2: (Continued)

Design	n_1	n_2	μ_1	σ_1	μ_2	σ_2	C_{pk1}	C_{pk2}	δ
7a	10	10	1.5	1.5	0.5	1	1.5094	0.5462	0.9632
7b	30	30							
7c	50	50							
7d	10	30							
7e	10	50							
7f	30	50							
8a	10	10	1.5	1	0.5	1.5	1.4535	0.9011	0.5525
8b	30	30							
8c	50	50							
8d	10	30							
8e	10	50							
8f	30	50							
9a	10	10	2	1	1	1	1.9149	1.0025	0.9125
9b	30	30							
9c	50	50							
9d	10	30							
9e	10	50							
9f	30	50							
10a	10	10	2	1.5	1	1.5	1.6862	1.2052	0.4810
10b	30	30							
10c	50	50							
10d	10	30							
10e	10	50							
10f	30	50							
11a	10	10	2	1.5	1	1	1.6862	1.0025	0.6837
11b	30	30							
11c	50	50							
11d	10	30							
11e	10	50							
11f	30	50							
12a	10	10	2	1	1	1.5	0.9149	1.2052	0.7097
12b	30	30							
12c	50	50							
12d	10	30							
12e	10	50							
12f	30	50							

Table 3: The estimated coverage probabilities and average widths of a 90% bootstrap confidence intervals of the difference between two process capability indices.

Design	Coverage probabilities			Average widths		
	SB	PB	BCPB	SB	PB	BCPB
1a	0.9860	0.9142	0.9434	2.7235	2.3111	2.3092
1b	0.9178	0.8908	0.8986	0.7999	0.7885	0.7864
1c	0.9052	0.8876	0.8926	0.5622	0.5579	0.5570
1d	0.9698	0.8504	0.8864	1.9125	1.6208	1.3842
1e	0.9676	0.8088	0.8644	1.8768	1.5627	1.2446
1f	0.9202	0.8992	0.9000	0.6867	0.6773	0.6691
2a	0.9886	0.9176	0.9356	1.2442	1.0618	1.0666
2b	0.9252	0.9018	0.9092	0.3601	0.3547	0.3543
2c	0.9088	0.8942	0.8988	0.2535	0.2515	0.2512
2d	0.9732	0.8386	0.8678	0.8648	0.7358	0.6210
2e	0.9708	0.8018	0.8516	0.8467	0.7057	0.5488
2f	0.9186	0.9002	0.9008	0.3089	0.3047	0.3004
3a	0.9784	0.8714	0.9126	2.1340	1.7884	1.6881
3b	0.9138	0.8750	0.8982	0.6330	0.6204	0.6030
3c	0.9016	0.8784	0.8936	0.4437	0.4386	0.4310
3d	0.9624	0.8990	0.9158	1.0235	0.9222	0.8829
3e	0.9622	0.8632	0.8920	0.9557	0.8143	0.7239
3f	0.9138	0.8946	0.9002	0.4776	0.4724	0.4713
4a	0.9816	0.8906	0.9242	2.0281	1.7091	1.6198
4b	0.9224	0.8996	0.9042	0.6065	0.5949	0.5792
4c	0.9044	0.8952	0.8894	0.4254	0.4208	0.4141
4d	0.9676	0.7926	0.8490	1.8307	1.5141	1.1599
4e	0.9698	0.7712	0.8378	1.8084	1.4869	1.0934
4f	0.9172	0.8898	0.8932	0.5737	0.5609	0.5311
5a	0.9762	0.8756	0.8960	2.3311	1.9775	1.9818
5b	0.7140	0.6696	0.6810	0.6680	0.6570	0.6535
5c	0.5408	0.5024	0.5144	0.4660	0.4619	0.4603
5d	0.9732	0.9346	0.8624	1.4293	1.2390	1.0725
5e	0.9796	0.9296	0.8416	1.3667	1.1621	0.9149
5f	0.7028	0.6944	0.6408	0.5410	0.5347	0.5318
6a	0.9950	0.9674	0.8834	1.1822	1.0331	1.0809
6b	0.9668	0.9480	0.8308	0.4766	0.4697	0.4637
6c	0.9522	0.9302	0.8152	0.3898	0.3826	0.3727
6d	0.9800	0.8584	0.8008	0.8248	0.7233	0.6169
6e	0.9780	0.8014	0.7718	0.7997	0.6879	0.5377
6f	0.9580	0.9224	0.8080	0.4366	0.4264	0.4004
7a	0.9844	0.9242	0.8854	2.0087	1.7083	1.6297
7b	0.9392	0.9266	0.8388	0.6882	0.6804	0.6778
7c	0.9034	0.8900	0.7992	0.5258	0.5224	0.5210
7d	0.9654	0.8962	0.8520	0.9642	0.8884	0.8558
7e	0.9526	0.8218	0.8088	0.8807	0.7930	0.7105
7f	0.9160	0.8940	0.8084	0.5602	0.5560	0.5532

Table 3: (Continued)

Design	Coverage probabilities			Average widths		
	SB	PB	BCPB	SB	PB	BCPB
8a	0.9758	0.9174	0.8748	1.6024	1.3555	1.3264
8b	0.5388	0.5572	0.4896	0.4535	0.4457	0.4379
8c	0.2726	0.2882	0.2382	0.3155	0.3125	0.3094
8d	0.9808	0.9274	0.8100	1.3519	1.1249	0.8379
8e	0.9854	0.9358	0.7972	1.3362	1.1008	0.7626
8f	0.5542	0.5912	0.4688	0.4171	0.4090	0.3888
9a	0.9888	0.9222	0.8988	2.0488	1.7439	1.7968
9b	0.8388	0.8056	0.6588	0.6149	0.6072	0.6120
9c	0.6604	0.6472	0.4630	0.4449	0.4410	0.4390
9d	0.9960	0.9616	0.8372	1.2700	1.1016	0.9289
9e	0.9942	0.9442	0.8006	1.2314	1.0381	0.7750
9f	0.8540	0.8584	0.5928	0.5128	0.5064	0.4934
10a	0.9898	0.9904	0.7504	1.2419	1.1133	1.1157
10b	0.7200	0.7948	0.4616	0.7284	0.7174	0.6746
10c	0.4158	0.4882	0.2512	0.6060	0.6027	0.5677
10d	0.9524	0.9718	0.6680	0.9600	0.8664	0.7998
10e	0.9450	0.9634	0.6406	0.9339	0.8387	0.7491
10f	0.6578	0.7382	0.4466	0.7040	0.6928	0.6577
11a	0.9804	0.9554	0.7346	1.8674	1.6222	1.4736
11b	0.6454	0.6536	0.3460	0.8340	0.8251	0.7719
11c	0.3336	0.3432	0.1622	0.6690	0.6652	0.6296
11d	0.8978	0.8184	0.5940	1.0482	0.9763	0.9339
11e	0.8648	0.7000	0.5480	0.9843	0.9032	0.8342
11f	0.5406	0.5200	0.2708	0.7608	0.7525	0.7173
12a	0.9876	0.9534	0.7706	1.4432	1.2372	1.2690
12b	0.3914	0.4640	0.3372	0.4655	0.4575	0.4430
12c	0.1342	0.1946	0.1726	0.3400	0.3334	0.3137
12d	0.9974	0.9804	0.7068	1.1992	0.9949	0.7418
12e	0.9978	0.9736	0.6778	1.1826	0.9680	0.6520
12f	0.3990	0.5064	0.2948	0.4276	0.4158	0.3755

Table 4: The estimated coverage probabilities and average widths of a 95% bootstrap confidence intervals of the difference between two process capability indices.

Design	Coverage probabilities			Average widths		
	SB	PB	BCPB	SB	PB	BCPB
1a	0.9954	0.9570	0.9728	3.3069	3.1170	3.1227
1b	0.9654	0.9462	0.9528	0.9512	0.9606	0.9571
1c	0.9528	0.9400	0.9494	0.6699	0.6728	0.6715
1d	0.9908	0.9022	0.9394	2.2775	2.1122	1.7941
1e	0.9908	0.8662	0.9146	2.1888	2.0067	1.5792
1f	0.9654	0.9520	0.9558	0.8164	0.8209	0.8103
2a	0.9976	0.9626	0.9732	1.4716	1.3986	1.4073
2b	0.9678	0.9514	0.9552	0.4299	0.4344	0.4330
2c	0.9648	0.9568	0.9576	0.3019	0.3033	0.3029
2d	0.9932	0.8930	0.9280	1.0231	0.9455	0.7918
2e	0.9916	0.8532	0.8974	1.0166	0.9146	0.7025
2f	0.9630	0.9496	0.9484	0.3695	0.3714	0.3661
3a	0.9942	0.9206	0.9510	2.5140	2.3620	2.2337
3b	0.9640	0.9278	0.9426	0.7512	0.7508	0.7304
3c	0.9554	0.9330	0.9450	0.5298	0.5287	0.5191
3d	0.9886	0.9492	0.9632	1.2211	1.1845	1.1150
3e	0.9898	0.9178	0.9454	1.1204	1.0527	0.9234
3f	0.9650	0.9496	0.9514	0.5718	0.5740	0.5720
4a	0.9956	0.9346	0.9636	2.4300	2.2786	2.1623
4b	0.9684	0.9474	0.9564	0.7191	0.7197	0.7015
4c	0.9494	0.9444	0.9458	0.5041	0.5036	0.4954
4d	0.9936	0.8400	0.8986	2.2065	1.9790	1.5015
4e	0.9910	0.8202	0.8826	2.1534	1.9355	1.4052
4f	0.9644	0.9458	0.9468	0.6838	0.6804	0.6448
5a	0.9934	0.9454	0.9552	2.7516	2.6256	2.6406
5b	0.8296	0.7678	0.7804	0.7938	0.7998	0.7941
5c	0.6688	0.6198	0.6314	0.5564	0.5580	0.5559
5d	0.9908	0.9634	0.9294	1.7202	1.6203	1.3767
5e	0.9922	0.9562	0.9172	1.6604	1.5180	1.1776
5f	0.8142	0.8048	0.7668	0.6440	0.6486	0.6442
6a	0.9992	0.9894	0.9458	1.3956	1.3285	1.4170
6b	0.9890	0.9826	0.9100	0.5677	0.5631	0.5496
6c	0.9770	0.9678	0.8984	0.4672	0.4571	0.4406
6d	0.9962	0.9226	0.8902	0.9935	0.9049	0.7555
6e	0.9946	0.8868	0.8668	0.9711	0.8665	0.6609
6f	0.9822	0.9634	0.8942	0.5201	0.5062	0.4711
7a	0.9966	0.9572	0.9382	2.4163	2.2504	2.1523
7b	0.9710	0.9646	0.9074	0.8234	0.8279	0.8173
7c	0.9642	0.9554	0.8904	0.6273	0.6280	0.6234
7d	0.9882	0.9520	0.9104	1.1507	1.1127	1.0578
7e	0.9866	0.8954	0.8880	1.0548	0.9853	0.8695
7f	0.9682	0.9558	0.8952	0.6677	0.6683	0.6609

Table 4L: (Continued)

Design	Coverage probabilities			Average widths		
	SB	PB	BCPB	SB	PB	BCPB
8a	0.9930	0.9688	0.9396	1.9218	1.8175	1.7892
8b	0.6534	0.6918	0.6328	0.5400	0.5434	0.5338
8c	0.3640	0.4066	0.3496	0.3752	0.3762	0.3722
8d	0.9942	0.9638	0.8946	1.6007	1.4503	1.0628
8e	0.9972	0.9598	0.8894	1.5901	1.4303	0.9883
8f	0.6582	0.7190	0.6034	0.4964	0.4962	0.4710
9a	0.9968	0.9644	0.9570	2.4256	2.3099	2.3948
9b	0.9206	0.8984	0.7922	0.7335	0.7439	0.7455
9c	0.8010	0.7932	0.6028	0.5315	0.5371	0.5302
9d	0.9992	0.9790	0.9142	1.5307	1.4002	1.1714
9e	0.9996	0.9684	0.8758	1.4467	1.3200	0.9618
9f	0.9328	0.9392	0.7234	0.6127	0.6184	0.5933
10a	0.9972	0.9974	0.8516	1.4609	1.3769	1.3909
10b	0.8066	0.8806	0.5968	0.8681	0.8325	0.7888
10c	0.5388	0.6260	0.3794	0.7216	0.7039	0.6671
10d	0.9828	0.9928	0.8020	1.2153	1.0196	0.9819
10e	0.9790	0.9868	0.7968	1.1313	0.9767	0.8953
10f	0.7652	0.8430	0.5828	0.8449	0.8013	0.7703
11a	0.9950	0.9788	0.8222	2.2604	2.0928	1.8903
11b	0.7542	0.7744	0.4498	0.9973	0.9760	0.9097
11c	0.4502	0.4714	0.2392	0.7959	0.7826	0.7412
11d	0.9652	0.9098	0.6998	1.2288	1.1565	1.1258
11e	0.9468	0.8250	0.6356	1.1739	1.0682	1.0338
11f	0.6846	0.6726	0.3768	0.9061	0.8765	0.8383
12a	0.9982	0.9888	0.8848	1.7307	1.6358	1.6927
12b	0.5478	0.6848	0.4284	0.5510	0.5582	0.5325
12c	0.2080	0.3270	0.2272	0.4057	0.4078	0.3776
12d	0.9996	0.9892	0.8044	1.4398	1.2943	0.9522
12e	0.9994	0.9818	0.7588	1.4080	1.2479	0.8424
12f	0.5546	0.7336	0.3822	0.5026	0.4998	0.4411

4. Illustrative example

In this section, a simulated example is presented to illustrate the bootstrap confidence intervals of the difference between two process capability indices developed in Section 2. The first and second random samples of sizes $n_1 = n_2 = 20$ are generated from the half logistic distribution with $\mu_1 = 0.5, \sigma_1 = 1.5$ and $\mu_2 = 0, \sigma_2 = 1,$ respectively. In this case, we set $LSL = 1$ and $USL = 29$, the true difference between two process capability indices, δ, is 0.8111. The first random sample (x_1) generated is

0.02	0.05	0.32	0.44	0.64	0.64	0.67	0.87	1.05	1.05
1.32	1.33	1.40	2.52	2.80	3.21	3.66	4.11	4.12	5.04.

The second random sample (x_2) generated is

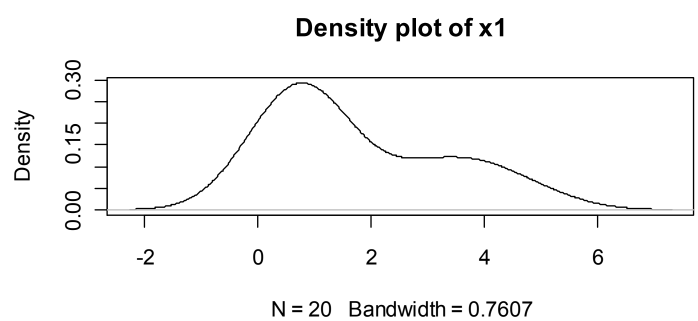
0.04	0.14	0.19	0.20	0.23	0.44	0.75	0.81	0.88	1.07
1.07	1.09	1.29	1.50	1.62	1.83	1.91	3.56	5.04	5.15.

In addition, the density plot of generated samples is displayed in Fig. 4.1. Assuming the half logistic distribution for corresponding random variables X_1 and X_2, three bootstrap confidence intervals of the difference between two process capability indices with confidence level 95% are constructed, and they are shown in the following table.

Table 5: Bootstrap confidence intervals and widths of the difference between two process capability indices

Methods	Confidence intervals	Widths
SB	(0.2441 , 1.2891)	1.0450
PB	(0.3025 , 1.2891)	0.9866
BCPB	(0.2654 , 1.3003)	1.0349

As presented in Table 5, the true difference between C_{pk1} and C_{pk2} $(\delta = 0.8111)$ lies in the proposed bootstrap confidence intervals. Additionally, the width of CI_{PB} is shorter than any other confidence intervals by about 5%.



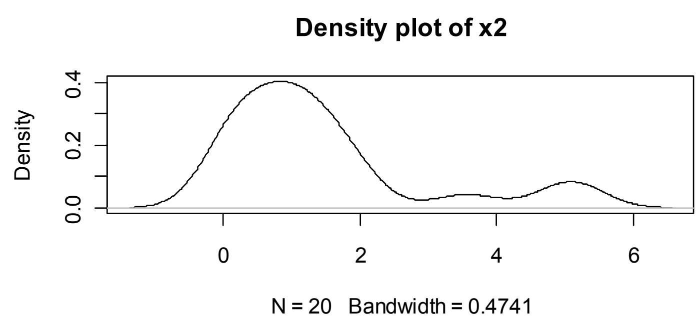


Figure 4.1 The density plot of generated random samples x_1 and x_2.

5. Conclusions

In this paper, we have proposed bootstrap confidence intervals of the difference between two process capability indices for half logistic distribution. Three bootstrap confidence intervals are considered: the standard bootstrap confidence interval (CI_{SB}), the percentile bootstrap confidence interval (CI_{PB}) and the bias-corrected percentile bootstrap confidence interval (CI_{BCPB}).The performances of bootstrap confidence intervals are compared by considering their coverage probabilities and average widths using Monte Carlo experiments. Based on simulation studies, the CI_{BCPB} achieves better coverage probability than the other bootstrap confidence intervals when $\mu_1 < \sigma_1$ and $\mu_2 < \sigma_2$. In addition, the CI_{PB} performs well with respect to the coverage criterion when $\mu_1 = \sigma_1$ and $\mu_2 < \sigma_2$. On the other hand, proposed bootstrap confidence intervals are not suitable in terms of coverage probability for other situations ($\mu_1 > \sigma_1$ and $\mu_2 \le \sigma_2$).

It would be of interest to propose confidence intervals of the difference between two process capability indices for half logistic distribution when $\mu_1 > \sigma_1$ and $\mu_2 \le \sigma_2$, and this is left as a topic for future work.

Acknowledgements

The authors acknowledge the excellent suggestions provided by Professor Dr. R.R.L. Kantam and Professor Dr. David E. Giles. The authors are grateful to Mark Zentz for his careful editing of the manuscript, and to two referees for their constructive comments on an earlier version of this paper.

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