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The Statistical Distributional Validation under a Novel Generalized Gamma Distribution with Value-at-Risk Analysis for the Historical Claims, Censored and Uncensored Real-life Applications

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Abstract

This study introduces and examines a new probability distribution, presenting various characterizations. Key financial risk measures, including the value-at-risk (VaR), tail-value-at-risk (TVaR), also referred to as conditional tail expectation or conditional value-at-risk (CVaR), tail variance (TV), tail mean-variance (TMV), and mean excess loss (MExL) function are evaluated using maximum likelihood estimation, ordinary least squares, weighted least squares, and the Anderson-Darling estimation methods. These methods are applied for actuarial analysis in both a simulation study and an insurance claims data application. For validation of the distribution using complete data, the widely recognized Nikulin-Rao-Robson statistic is utilized and assessed through simulations and three real data sets. Two uncensored real-life data sets for failure times and remission times are used in uncensored validation. Additionally, for censored data validation, a modified version of the Nikulin-Rao-Robson statistic is proposed and evaluated through extensive simulations and three censored real data sets.

Key Words: Characterizations; Distributional Validation; Nikulin-Rao-Robson; Risk Assessment; Value-at-risk; Statistical Modeling.

1. Introduction

This paper introduces a novel probability distribution called Burr X generalized gamma (BXGG) distribution characterized by only two parameters, offering a straightforward and practical alternative to complex algebraic derivations and purely theoretical frameworks. The simplicity of the proposed distribution enhances its applicability across various fields, particularly where ease of implementation and interpretability are crucial. The study emphasizes the significance of the new distribution by focusing on two key areas of application. The first area is distributional verification, which encompasses both complete datasets and datasets subject to uncensored observations. This aspect evaluates the adequacy of the proposed distribution in accurately representing real-life data, ensuring it provides a reliable fit across different data scenarios. Such verification is critical in statistical modeling as it underpins the practical utility and robustness of the distribution in diverse empirical contexts. The second area addresses actuarial risk analysis, specifically utilizing historical insurance data. In this domain, the new distribution is applied to assess and model risks associated with insurance claims, premiums, and other actuarial variables. By incorporating the proposed distribution into risk analysis frameworks, this study demonstrates its capacity to capture the inherent variability and uncertainty in insurance datasets, facilitating informed decision-making in risk management, pricing, and financial forecasting.

Modeling right-censored data using probability distributions involves the process of fitting a statistical distribution to datasets that contain observations which are only partially observed or censored. Specifically, right-censored data refers to instances where the exact value of some observations is unknown but is known to exceed a certain threshold. This type of data is common in various fields, such as survival analysis, reliability engineering, and insurance, where events or measurements are only partially recorded due to time or measurement constraints. In this study, we introduce and explore a new continuous probability distribution using innovative approaches that diverge from traditional methodologies employed by most researchers. While theoretical findings and algebraic derivations are valuable, our focus shifts towards practical aspects to allow for a deeper exploration of risk analysis, distributive validation, and their applications. By emphasizing these practical dimensions, we aim to address challenges in both complete and censored datasets, shedding light on actionable insights and methods for handling right-censored data effectively. This approach underscores the importance of bridging theoretical understanding with practical utility in fields such as actuarial science, insurance modeling, and statistical analysis.

By analyzing a set of widely used financial indicators, including the VaR, TVaR, TV, TMV, and MExL function, one can effectively assess and evaluate the risks faced by insurance companies. These indicators provide critical insights into the behavior of losses, especially in extreme scenarios, aiding in comprehensive risk management. Four prominent estimation methods are explored for these key risk indicators (KRIs): maximum likelihood estimation (MLE), ordinary least squares estimation (OLSE), weighted least squares estimation (WLSE), and Anderson-Darling estimation (ADE). These methodologies are applied in two distinct frameworks of financial and actuarial assessment. The first involves simulation studies conducted under three different confidence levels (CLs) and across varying sample sizes. The second focuses on practical applications to insurance claims data, ensuring the robustness of the methods in real-life scenarios. To meet the demands of actuarial risk analysis, a simulation study is conducted to compare the performance of VaR estimators based on insurance data, allowing for an in-depth evaluation of their effectiveness.

For distributional validation and statistical hypothesis testing of complete data, the well-established Rao-Robson-Nikulin (RRNI) statistic, denoted as Y_Q^2, is employed. This statistic is based on uncensored maximum likelihood estimators (UMLEs) derived from initial non-grouped data and is applied under a probability model known as the BXGG distribution. The RRNI statistic builds on the foundational work of Rao and Robson (1974) and Nikulin (1973a, b, c). A simulation study is performed to evaluate the Y_Q^2 statistic using three real-life datasets, providing insights into its reliability and applicability. For censored data, a modified version of the RRNI statistic, denoted as $M_Q^2(r-1)$, is introduced. This variant relies on censored maximum likelihood estimators (CMLEs) derived from initial non-grouped data and is specifically designed for distributional validation and hypothesis testing under the BXGG model. A comprehensive simulation study evaluates the $M_Q^2(r-1)$ statistic using three real datasets with censored observations. This thorough investigation of both complete and censored data underscores the effectiveness of the BXGG model and the proposed methodologies in addressing real-life actuarial and statistical challenges. Following Yousof et al. (2017a), the cumulative distribution function (CDF) of the BXGG model is given as

$$F_{\underline{\mho}}(z) = \left(1 - exp\left\{-\left[\frac{1}{\varsigma_z^{\theta}} - 1\right]^{-2}\right\}\right)^{a} |_{z\geq 0}, \quad (1)$$

where $\underline{\mho} = (a, \theta)$, $a, \theta > 0$,

$$\varsigma_z = \varsigma(z) = 1 - (1 + z)\, exp(-z), \quad (2)$$

and ς_z^{θ} refers to the CDF of the generalized gamma model (GG) proposed by Gupta et al. (1998). The exponentiated gamma model is flexible enough to accommodate both monotonic as well as nonmonotonic failure rates. The probability density function (PDF) corresponding to (1) can be expressed as

$$f_{\underline{\mho}}(z) = 2a\theta x \frac{exp(-z)\, \varsigma_z^{2\theta-1}\left(1 - \varsigma_z^{\theta}\right)^{-3}}{exp\left[\left(\frac{1}{\varsigma_z^{\theta}} - 1\right)^{-2}\right]} \left\{1 - exp\left[\left(\frac{1}{\varsigma_z^{\theta}} - 1\right)^{-2}\right]\right\}^{a-1} |_{z>0}. \quad (3)$$

The BXGG distribution can accommodate various failure rate behaviors. The general form of the distribution allows it to model both monotonic and non-monotonic failure rates. This property is especially useful in reliability analysis, where the risk of failure may increase or decrease over time or under different conditions. The distribution's flexibility comes from its structure, combining components of the generalized gamma distribution and exponential functions. By adjusting the parameters aa and θ, the BXGG distribution can model various data patterns and provide insights into the behavior of different systems under stress or over time. In general, the validity of a statistical model can be evaluated using a variety of criteria, particularly those aimed at assessing the model's goodness of fit to the data. For unfiltered, complete data, widely used methods include tests based on empirical functions, such as the likelihood ratio test, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and chi-square tests. Additionally, statistical tests such as the Anderson-Darling test, Kolmogorov-Smirnov test, and similar approaches are frequently employed to measure the agreement between observed data and the theoretical model. Among these goodness-of-fit measures, the Rao-Robson-Nikulin (RRNI) statistic holds particular significance. It is based on maximum likelihood estimators (MLEs) calculated from initial, non-grouped data and provides a robust framework for validating statistical distributions. However, the presence of censorship in data, where certain observations are only partially known, renders traditional goodness-of-fit tests, including the RRNI statistic in its original form, invalid. This introduces several practical challenges, as conventional methods fail to account for the information loss caused by censorship. To address these challenges, researchers have developed modified versions of existing goodness-of-fit tests specifically tailored for censored data. Bagdonavicius and Nikulin (2011) proposed a revised version of the RRNI statistic to handle statistical distributions with unknown parameters in the presence of right censoring. This modified RRNI statistic is designed to recover information lost during data censoring or regrouping, making it particularly suitable for fields like survival analysis, reliability engineering, and other domains where censoring is common. In this study, we extend these advancements by introducing modified RRNI chi-square goodness-of-fit test statistics for both complete and right-censored data. These adaptations are tailored to fit the proposed probability model, following the foundational work of Nikulin (1973a, b, c) and Rao and Robson (1974). The modified statistics are particularly useful for validating the fit of statistical distributions in cases where traditional tests are inadequate due to censoring, offering a more comprehensive tool for data analysis in fields where such challenges frequently arise.

The RRNI statistic test is a well-established variant of the traditional chi-squared tests applied in the context of complete data. It relies on the differences between two estimators of the probability of falling within specific grouping intervals. One estimator is derived from the empirical distribution function, while the other is based on the maximum likelihood estimates (MLEs) of the unobserved parameters of the tested model, calculated using ungrouped initial data. For further details, refer to the foundational work of Nikulin (1973a, b, c) and Rao and Robson (1974), as well as applications under uncensored schemes discussed by Goual and Yousof (2020a), Goual et al. (2019), Goual and Yousof (2020b), and Yousof et al. (2021a,b,c). In general, statistical methods for hypothesis testing and evaluating the validity of parametric distributions under censoring are continuously evolving. However, handling censored data remains a significant challenge. The statistical literature contains numerous notable contributions in this area, particularly for verification tests involving censored data. For instance, based on the well-known Kaplan-Meier estimators, Habib and Thomas (1986) and Hollander and Peña (1992) proposed a modified chi-squared test for randomly censored data and demonstrated its usefulness with real-life applications. Similarly, Galanova (2012) explored various nonparametric modifications to goodness-of-fit tests, such as the Anderson-Darling, Kolmogorov-Smirnov, and Cramér-von Mises statistics, in the context of accelerated failure time models. These advancements highlight the ongoing efforts to adapt statistical methodologies to the challenges posed by censorship in data analysis.

Studies focusing on the RRNI test are relatively limited in the statistical literature, but several notable contributions have been made in recent years. Goual et al. (2019) examined the odd Lindley exponentiated exponential distribution using a modified RRNI goodness-of-fit test with applications to both censored and uncensored data, while Abouelmagd et al. (2019a) and Abouelmagd et al. (2019b) addressed the distributional validity of the zero-truncated Poisson-Burr-X G family and the zero Topp-Leone Poisson G family of distributions, respectively. Ibrahim et al. (2019) introduced a modified validation test for an extended Lindley distribution, incorporating characterizations and various estimation methods, and Goual and Yousof (2020) validated the Burr type XII inverse Rayleigh model using a modified RRNI chi-squared test. Yadav et al. (2020) evaluated the Topp-Leone-Lomax distribution using the RRNI test under classical estimation techniques, Goual et al. (2020) validated the Lomax inverse Weibull model, and Ibrahim et al. (2020a) developed a modified RRNI test for censored data with a new Burr type XII model, applying classical estimation methods and censored regression modeling. Yousof et al. (2021a) proposed a new inverted Rayleigh model with copulas, properties, and a modified RRNI right-censored test for distributional validation, while Yadav et al. (2022) explored the xgamma exponential model's validity via the RRNI test for censored and uncensored samples under different estimation methods. Recently, Emam et al. (2023) and Yousof et al. (2023a) introduced further refinements and applications of the Nikulin and Rao-Robson statistic tests. Emam et al. (2023) used the BB algorithm to compare Bayesian approaches with the censored maximum likelihood method, providing a comprehensive construction of the RRNI statistic for a new model under uncensored conditions and deriving the Bagdonavicius and Nikulin statistic for censored cases. Yousof et al. (2023b) extended this by demonstrating four applications of a new probability model under censored conditions, deriving an updated Nikulin statistic test, and conducting simulation experiments using real-life, fabricated, and censored datasets to evaluate the original and updated tests. These advancements underscore the evolving role of RRNI tests in addressing challenges in censored and uncensored data analysis. In this study, the BXGG distribution is derived and used, the complete and right censored scenarios are used to validate a modified chi-squared goodness-of-fit test statistic based on the RRNI test (Y_Q^2) and the modified RRNI test ($M_Q^2(r-1)$) respectively.

First, the Y_Q^2 statistic test is used for testing the null hypothesis H_0 according to which a certain complete sample belongs to a BXGG model. The RRNI statistic test is evaluated using a simulation study via the Barzilai-Borwein (BB) algorithm in the case of complete data and a simulation study in the case of censored data. In the simulation studies, we have relied on the standard mean square error (MSEs) in the evaluation process, taking into account different sample sizes to help us evaluate the behavior of the test with an increase in the sample size. The Barzilai and Borwein gradient methodology has received a lot of interest from a variety of optimization areas. This is due to its practical usefulness, computer affordability, and simplicity. Using spectral analysis techniques, this paper proves root-linear global convergence for the Barzilai and Borwein method for strictly convex quadratic problems presented in infinite-dimensional Hilbert spaces. The application of these discoveries to two optimization problems controlled by partial differential equations is demonstrated.

2. Risk indicators

Below, we provide a literature review synthesizes advancements in statistical modeling and risk analysis across various domains, including insurance, finance, reliability, and medicine. Korkmaz et al. (2018) emphasized VaR estimation with their Burr X Pareto distribution. Shrahili et al. (2021) presented an asymmetric density model for claim-size variability. Ahmed, Ali, and Yousof (2022) introduced a novel G family for single acceptance sampling plans, enhancing quality and risk decisions, while Hamed et al. (2022) developed the Compound Lomax Model for negatively skewed insurance claims. Rasekhi et al. (2022) contributed the Odd Log-Logistic Weibull-G Family for financial risk modeling. Alizadeh et al. (2023) proposed the XGamma extension, offering insights into actuarial risk analysis in reinsurance, and Hamedani et al. (2023) introduced a right-skewed distribution for actuarial risk analysis. Salem et al. (2023) developed a Lomax extension for risk analysis in censored medical and insurance data, while Ibrahim et al. (2023) proposed the Compound Reciprocal Rayleigh Extension for left-skewed insurance data. Hashempour et al. (2023) applied a Lindley extension to bimodal precipitation risk assessment, and Teghri et al. (2024) expanded on this with a Lindley-frailty model for censored and uncensored reliability datasets. Alizadeh et al. (2024) introduced the Extended Gompertz Model for extreme stress risk analysis. Elbatal et al. (2024) incorporated entropy-based methods for VaR modeling in their new probability model, while Aljadani et al. (2024) presented a model tailored for financial peaks over random threshold VaR analysis. Loubna et al. (2024) tackled survival analysis using the quasi-xgamma frailty model for emergency care data. Yousof et al. (2024) developed a Pareto model integrating MOOP and PORT-VaR methods for financial and reliability applications, complementing their earlier work on the Reciprocal Weibull Extension for heavy-tailed data. Finally, Yousof et al. (2023c) introduced the Bimodal Heavy-Tailed Burr XII Model for extreme insurance risks, collectively advancing the field of risk analysis with innovative statistical distributions and methodologies tailored to specific challenges, enabling precise decision-making in high-stakes environments.

Recently, Alizadeh et al. (2024) proposed the extended Gompertz model, focusing on statistical threshold risk analysis for extreme stress data. This model provides a robust framework for reliability and risk analysis, incorporating the Mean of Order P (MOOP) assessment to evaluate its statistical properties and applications. Similarly, Yousof et al. (2024) introduced a new Pareto model designed for risk applications and reliability analysis, particularly emphasizing the MOOP methodology and Peaks Over Random Threshold (PORT) analysis. These studies demonstrate the efficacy of these models in addressing extreme value scenarios and quantifying associated risks. Aljadani et al. (2024) presented a novel model tailored for financial and reliability applications. Their work integrates theoretical underpinnings with practical implementations, highlighting the analysis of financial peaks over random threshold VaR. This approach enables precise risk assessment in volatile financial environments. Building on this, Shehata et al. (2024a,b) proposed the Reciprocal-Weibull Model, which incorporates statistical properties and reliability applications. Their study extends the use of MOOP and PORT-VaR analyses to reliability data, providing a comprehensive tool for extreme value assessment in finance and reliability. Yousof et al. (2024) introduced a discrete generator with broad applicability, including reliability, medicine, agriculture, and biology. This study underscores the importance of count statistical modeling and inference in analyzing complex datasets from diverse disciplines. Khan et al. (2024) proposed a heavy-tailed Lomax model, emphasizing its characterizations, applications, and risk analysis capabilities through PORT-VaR and MOOP methods. This model is particularly effective in addressing extreme value problems and capturing heavy-tailed phenomena in datasets. Finally, see Ali et al. (2022) and Ali et al. (2025) for more related works.

The probability-based distributions might offer an appropriate overview of risk exposure. To represent the degree of risk exposure, one value, or at the very least a small set of values, is typically employed. These risk exposure data, which are usually referred to as KRIs, are plainly functions of a particular model. These KRIs give actuaries and risk managers knowledge of how much a company is exposed to various kinds of risk. There are many KRIs that can be considered and studied, including the VAR, TVAR (also known as CVAR), TV indication, TMV, and MExL function, among others. A quantile of the distribution of total losses in particular is the VaR. Actuaries and risk managers usually concentrate on calculating the chance of a bad outcome, which can be expressed using the VaR indicator at a specific probability/confidence level. This indicator is typically used to determine how much capital will be required to handle such likely adverse circumstances. The VAR of the BXGG distribution at the $100Q\%$ level, say $\text{VaR}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}})$ or $\pi(Q)$, is the $100Q\%$ quantile (or percentile). Then, we can simply write

$$\text{VAR}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}}) = \Pr(\text{Ȥ} > Q(U)), \tag{4}$$

where $Q(U)$ is from (3), for a one-year time when $Q = 0.99$, the interpretation is that there is only a very small chance (1%) that the insurance company will be bankrupted by an adverse outcome over the next year. If the distribution of gains (or losses) is limited to the normal distribution, it is acknowledged that the number $\text{VAR}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}})$ meets all coherence requirements. The data sets for insurance such as the insurance claims and reinsurance revenues are typically skewed whether to the right or to the left, though. Using normal distribution to describe the revenues from reinsurance and insurance claims is not suitable. The $\text{TVAR}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}})$ of Ȥ at the $100Q\%$ confidence level is the expected loss given that the loss exceeds the $100Q\%$ of the distribution of Ȥ, then the $\text{TVAR}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}})$ of Ȥ can be expressed as

$$\text{TVAR}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}}) = \mathbf{E}(\text{Ȥ}|\text{Ȥ} > \pi(Q)). \tag{5}$$

The quantity $\text{TVAR}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}})$, which gives further details about the tail of the BXGG distribution, is therefore the average of all the VaR values mentioned above at the confidence level q. Moreover, the TVAR $(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}})$ can also be expressed as $\text{TVAR}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}})$= $e(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}})$ $\text{VaR}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}})$ where $e(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}})$ is the mean excess loss ($\text{MExL}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}})$) function evaluated at the $100Q\%^{th}$ quantile (see Acerbi and Tasche 2002; Tasche, 2002; Wirch, 1990). When the $e(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}})$ value vanishes, then $\text{TVAR}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}}) = \text{VaR}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}})$ and for the very small values of $e(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}})$, the value of $\text{TVAR}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}})$ will be very close to $\text{VaR}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}})$ The $\text{TV}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}})$ risk indicator, which Furman and Landsman (2006) developed, calculates the loss's deviation from the average along a tail. Explicit expressions for the TV risk indicator under the multivariate normal distribution were also developed by Furman and Landsman (2006). The $\text{TV}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}})$ risk indicator ($\text{TV}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}})$) can then be expressed as

$$\text{TV}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}}) = \mathbf{E}(\text{Ȥ}^2|\text{Ȥ} > \pi(Q)) - [\text{TVAR}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}})]^{\mathbf{2}}, \tag{6}$$

As a statistic for the best portfolio choice, Landsman (2010) developed the TMV risk indicator, which is based on the TV risk indicator. Consequently, the TMV risk indicator may be written as

$$\text{TMV}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}}) = \text{TVAR}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}}) + c\text{TV}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}}). \tag{7}$$

Then, for any RV, TMV $(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}}) > \text{TV}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}})$ and, for $c = 1$, $\text{TMV}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}}) = \text{TVAR}(\text{Ȥ}|_{[Q]};\underline{\widehat{\text{CƆ}}})$.

3. Risk analysis using different estimation methods with validation

3.1 Risk assessment under artificial data

In this section, we consider the above-mentioned estimation methods for calculating the KRIs using $N = 1{,}000$ with different sample sizes ($n = 50{,}150{,}300{,}500$) and three CLs ($Q = (0.70{,}0.90{,}0.99)$). Tables 1, 2, 3 and 4 present a comprehensive analysis of KRIs under artificial data for different sample sizes and four estimation methods: MLE, OLSE, WLSE, and ADE. A clear observation across all tables is that as the sample size increases, the estimates for VaR, TVaR, TV, TMV, and MExL stabilize, suggesting that larger datasets provide more reliable and consistent risk estimates. While the risk measures show minor fluctuations across the different methods, the general trend reveals that these variations are relatively small, demonstrating the robustness and consistency of the estimation techniques used. These results indicate that for practical purposes, the choice of estimation method has minimal impact on the risk indicators, particularly as the sample size grows. This consistency suggests that even with varying data sizes, the methods employed (MLE, OLSE, WLSE, ADE) are effective in estimating the key risk measures, providing a reliable foundation for risk analysis in insurance and financial sectors. However, further statistical testing could be useful to determine whether these small differences in estimates are significant enough to influence real-life decision-making.

The numerical results in the Table 1 show very small differences across the estimation methods (MLE, OLSE, WLSE, and ADE) for the KRIs at different confidence levels (Q=0.70, 0.90, 0.99), suggesting that the choice of method has minimal impact on the results for the given artificial dataset with $n = 50$. This could indicate that the data is relatively simple or homogeneous, leading to similar estimates across methods, or it may reflect the limitations of the small sample size, which may not provide enough power to reveal significant differences. The precision of the estimators might also be a factor, with all methods converging on similar values due to the nature of the dataset. To better understand the performance of these methods, further analysis with larger or more complex datasets, including those with outliers or more varied characteristics, would be necessary, and statistical tests could be performed to assess whether any differences are statistically significant.

The numerical results in Table 2 show minimal differences across the estimation methods (MLE, OLSE, WLSE, and ADE) for the KRIs at various confidence levels (Q=0.70, 0.90, 0.99) with $n = 150$. The values for VaR, TVaR, TV, TMV, and MExL are very similar across methods, suggesting that, in this case, the choice of estimation method does not significantly affect the results. This could indicate that the underlying data structure is such that all methods are yielding comparable results, or it may reflect the higher sample size, which provides more stable estimates across methods. However, these small differences are still observable, especially at higher confidence levels, and further examination with more diverse datasets or using more complex models could help identify any substantial distinctions between the methods. Additionally, statistical tests could be applied to determine whether the observed differences are statistically significant.

The numerical results in Table 3 show that, similar to the previous cases, the KRIs under different estimation methods (MLE, OLSE, WLSE, and ADE) are highly consistent at various confidence levels (0.70, 0.90, 0.99) with $n = 300$. Although there are slight variations in the values of VaR, TVaR, TV, TMV, and MExL across methods, the overall differences are minimal, indicating that the chosen estimation method does not substantially affect the results. The increased sample size of 300 seems to lead to even more stable estimates across all methods, further supporting the observation that, for this data, the results are robust to the method employed. These small differences might be due to the slight variability in estimation, which could become more pronounced in more complex datasets or with higher levels of censoring. However, additional statistical tests could be useful to assess the significance of these differences and provide further insights into the performance of the estimation methods under various scenarios.

Table 4 presents the KRIs under artificial data for $n = 500$, displaying results across different estimation methods: MLE, OLSE, WLSE, and ADE. As with smaller sample sizes, the estimates of VaR, TVaR, TV, TMV, and MExL remain consistent across methods at varying confidence levels (0.70, 0.90, 0.99). The values are stable with slight variations across the methods, particularly in the risk measures like VaR, TVaR, and TMV, indicating that larger sample sizes, such as $n = 500$, tend to produce more reliable and precise estimates. However, the differences remain relatively small, which suggests that the method used has minimal impact on the final risk indicators, highlighting the robustness of the estimation methods under this dataset. These results could be further examined with statistical tests to evaluate whether these slight variations have any practical significance, especially in more complex or heterogeneous datasets.

Table 1: KRIs generated using artificial data with n=50.

Method	$\hat{a}$	$\hat{\theta}$	$\mathrm{VaR}(Z\|Q,\underline{\widehat{\mho}})$	$\mathrm{TVaR}(Z\|Q,\underline{\widehat{\mho}})$	$\mathrm{TV}(Z\|Q,\underline{\widehat{\mho}})$	$\mathrm{TMV}(Z\|Q,\underline{\widehat{\mho}})$	$\mathrm{MExL}(Z\|Q,\underline{\widehat{\mho}})$
MLE	5.1204	5.9836					
0.70			4.225582	4.371918	0.013539	4.378688	0.146336
0.90			4.401435	4.507392	0.008163	4.511474	0.105958
0.99			4.634055	4.704714	0.006137	4.707783	0.070659
OLSE	5.1048	5.9880					
0.70			4.225939	4.372384	0.013557	4.379162	0.146445
0.90			4.401928	4.507947	0.008175	4.512034	0.106018
0.99			4.634673	4.705631	0.004868	4.708065	0.070958
WLSE	5.0813	5.992					

0.70			4.225946	4.372557	0.013583	4.379349	0.146611
0.90			4.402144	4.508256	0.00819	4.512351	0.106112
0.99			4.635086	4.706078	0.004918	4.708537	0.070991
ADE	5.093	5.990					
0.70			4.226005	4.372530	0.013570	4.379315	0.146525
0.90			4.402095	4.508158	0.008183	4.512250	0.106063
0.99			4.634935	4.705908	0.004899	4.708357	0.070973

Table 2: KRIs generated using artificial data with n=150.

Method	$\hat{\alpha}$	$\hat{\theta}$	$VaR(\mathcal{Z}\|Q,\widehat{\omega})$	$TVaR(\mathcal{Z}\|Q,\widehat{\omega})$	$TV(\mathcal{Z}\|Q,\widehat{\omega})$	$TMV(\mathcal{Z}\|Q,\widehat{\omega})$	MExL$(\mathcal{Z}\|Q,\widehat{\omega})$
MLE	5.0372	5.9921					
0.70			4.224497	4.371426	0.013634	4.378243	0.14693
0.90			4.401094	4.507389	0.008205	4.511491	0.106294
0.99			4.634419	4.705216	0.006231	4.708331	0.070797
OLSE	5.02823	5.9956					
0.70			4.224876	4.37187	0.013645	4.378692	0.146994
0.90			4.401555	4.507884	0.008213	4.511991	0.106330
0.99			4.634951	4.706046	0.004907	4.708500	0.071095
WLSE	5.0247	5.9932					
0.70			4.224293	4.371313	0.013649	4.378137	0.14702
0.90			4.401004	4.50735	0.008211	4.511455	0.106346
0.99			4.634436	4.705257	0.006234	4.708373	0.07082
ADE	5.0240	5.9960					
0.70			4.224824	4.371848	0.013650	4.378673	0.147025
0.90			4.401541	4.507888	0.008216	4.511996	0.106347
0.99			4.634973	4.706076	0.004910	4.708531	0.071103

Table 3: KRIs generated using artificial data with n=300.

Method	$\hat{a}$	$\hat{\theta}$	$VaR(\mathcal{Z}\|Q,\widehat{\omega})$	$TVaR(\mathcal{Z}\|Q,\widehat{\omega})$	$TV(\mathcal{Z}\|Q,\widehat{\omega})$	$TMV(\mathcal{Z}\|Q,\widehat{\omega})$	$MExL(\mathcal{Z}\|Q,\widehat{\omega})$
MLE	5.0164	5.9971					
0.70			4.224788	4.371867	0.013658	4.378697	0.147079
0.90			4.401574	4.507952	0.00822	4.512062	0.106378
0.99			4.635071	4.706186	0.004922	4.708647	0.071115
OLSE	5.0072	5.9981					
0.70			4.224677	4.371823	0.013669	4.378657	0.147146
0.90			4.401547	4.507962	0.008225	4.512075	0.106416
0.99			4.635123	4.706254	0.004928	4.708718	0.071131
WLSE	5.024	5.995					
0.70			4.224745	4.37177	0.013649	4.378594	0.147024
0.90			4.401462	4.507809	0.008215	4.511916	0.106347
0.99			4.634894	4.705999	0.004900	4.708449	0.071105
ADE	5.007	5.998					
0.70			4.22474	4.371882	0.013668	4.378717	0.147142
0.90			4.401605	4.508018	0.008225	4.512131	0.106413
0.99			4.635176	4.706305	0.004935	4.708772	0.071129

Table 4: KRIs generated using artificial data with n=500.

Method	$\hat{\alpha}$	$\hat{\theta}$	$VaR(\mathcal{Z}\|Q,\widehat{\omega})$	$TVaR(\mathcal{Z}\|Q,\widehat{\omega})$	$TV(\mathcal{Z}\|Q,\widehat{\omega})$	$TMV(\mathcal{Z}\|Q,\widehat{\omega})$	$MExL(\mathcal{Z}\|Q,\widehat{\omega})$
MLE	5.0262	5.9973					
0.70			4.22516	4.372168	0.013647	4.378991	0.147008
0.90			4.401856	4.508193	0.008217	4.512302	0.106337
0.99			4.635269	4.706359	0.004945	4.708832	0.071091
OLSE	5.0223	5.998					

0.70			4.225284	4.372319	0.013651	4.379145	0.147036
0.90			4.402015	4.508367	0.008221	4.512477	0.106352
0.99			4.63546	4.706553	0.004969	4.709037	0.071093
WLSE	5.0310	5.995					
0.70			4.224938	4.371912	0.013641	4.378733	0.146974
0.90			4.401592	4.507910	0.008212	4.512016	0.106318
0.99			4.634964	4.706054	0.004908	4.708508	0.071090
ADE	5.021	5.998					
0.70			4.225271	4.372312	0.013652	4.379138	0.147041
0.90			4.402009	4.508364	0.008221	4.512475	0.106355
0.99			4.635461	4.706555	0.004969	4.70904	0.071094

3.2 Risk assessment and validation under insurance claims data

The historical progression of insurance claims over time for specific exposure (or origin) periods is often depicted in insurance data using a triangular presentation. The development period of a given origin period is referred to as the "claim age" or "claim lag." In practice, data from different insurance policies are often aggregated to represent consistent company lines, division levels, or risk categories. As a practical example, this article examines an insurance claims payment triangle from a U.K. Motor Non-Comprehensive account. The selected origin period spans from 2007 to 2013. The data is organized in a structure resembling how a database typically stores it, including columns for the origin year (ranging from 2007 to 2013), the development year, and the incremental payments. Initially, this insurance claims data was analyzed using a probability-based distribution. The ability of the insurance company to effectively manage such claims is of significant interest to actuaries, regulators, investors, and rating agencies. This work introduces specific KRI quantities for left-skewed insurance claims data under the BXGG distribution framework. These include VAR, TVAR, TV, and TMV, building upon the foundation laid by Artzner (1999). For analyzing heavy-tailed distributions, one of the most effective approaches is the t-Hill method, an upper order statistic adjustment of the t-estimator. This technique has been discussed extensively in the works of Stehlík et al. (2010) and Figueiredo et al. (2017). These methodologies enable more robust risk assessment for claims with complex statistical characteristics, ensuring greater accuracy and reliability in evaluating the performance and risk profile of insurance portfolios.

Tables 5 displays the KRIs for the BXGG model under different estimation methods: MLE, OLSE, WLSE, and AE, respectively, based on insurance claims data. A prominent trend across all tables is the consistent ordering of risk metrics by quantile values, with VaR $(Ꝣ|Q = 0.30)$ being the smallest and VaR $(Ꝣ|Q = 0.01)$ the largest, for all methods. Similarly, TVaR $(Ꝣ|Q = 0.30)$ is consistently less than TVaR $(Ꝣ|Q = 0.01)$, while TV $(Ꝣ|Q = 0.30)$ and TMV $(Ꝣ|Q = 0.30)$ exhibit a reverse pattern, increasing as the quantile decreases. Furthermore, the MExL $(Ꝣ|Q)$ measure shows a similar trend, greater for smaller quantiles. For each method, both VaR $(Ꝣ|Q)$ and TVaR $(Ꝣ|Q)$ exhibit a monotonically increasing behavior across quantiles, with the smallest starting values at higher quantiles and the largest values at lower quantiles. Conversely, TV $(Ꝣ|Q)$, TMV $(Ꝣ|Q)$, and MExL $(Ꝣ|Q)$ exhibit monotonically decreasing trends as quantiles decrease. Among the methods, the OLSE method shows the most favorable results, providing a more acceptable risk exposure analysis than MLE, followed by AE and WLSE. This consistency in the risk measures across methods and quantiles highlights the reliability of these risk assessment techniques for evaluating insurance claims data. Consequently, OLSE stands out as the recommended method for risk exposure analysis.

Table 5: Risk assessment results under insurance claims data for all methods.

Method	$\hat{\alpha}$	$\hat{\theta}$	$VaR(Ꝣ\|Q,\underline{\widehat{\mho}})$	$TVaR(Ꝣ\|Q,\underline{\widehat{\mho}})$	$TV(Ꝣ\|Q,\underline{\widehat{\mho}})$	$TMV(Ꝣ\|Q,\underline{\widehat{\mho}})$	$MExL(Ꝣ\|Q,\underline{\widehat{\mho}})$
MLE	2.566	3.665					
0.70			3.510764	3.687710	0.018864	3.697143	0.176947
75%			3.554621	3.718768	0.016818	3.727177	0.164147
80%			3.602649	3.75391	0.014799	3.76131	0.151261
85%			3.657577	3.795423	0.012756	3.801801	0.137846
0.90			3.725107	3.848188	0.010592	3.853484	0.12308
95%			3.822001	3.926759	0.008068	3.930793	0.104758
0.99			3.993511	4.072405	0.004908	4.074859	0.078893
OLSE	1.959	3.996					
0.70			3.554376	3.746031	0.021647	3.756855	0.191655
75%			3.602501	3.779605	0.019175	3.789192	0.177103

80%			3.654937	3.81744	0.016754	3.825817	0.162503
85%			3.714562	3.861928	0.014323	3.86909	0.147366
0.90			3.78737	3.918167	0.011777	3.924055	0.130797
95%			3.890906	4.001314	0.008849	4.005738	0.110408
0.99			4.071709	4.153772	0.005267	4.156406	0.082063
WLSE	2.390	3.675					
0.70			3.498943	3.679562	0.019558	3.689341	0.180619
75%			3.543858	3.711251	0.017404	3.719953	0.167393
80%			3.592983	3.747064	0.015308	3.754718	0.154081
85%			3.649084	3.789316	0.013214	3.795923	0.140232
0.90			3.717938	3.842966	0.010933	3.848433	0.125028
95%			3.816513	3.922645	0.008552	3.92692	0.106132
0.99			3.990403	4.070041	0.00533	4.072706	0.079638
ADE	2.2478	3.837					
0.70			3.537095	3.721000	0.020166	3.731083	0.183905
75%			3.582965	3.753248	0.017929	3.762212	0.170283
80%			3.633075	3.78966	0.015752	3.797535	0.156585
85%			3.690226	3.832584	0.013535	3.839351	0.142358
0.90			3.760258	3.886989	0.011241	3.89261	0.126732
95%			3.86031	3.967736	0.008558	3.972015	0.107426
0.99			4.036264	4.116461	0.005973	4.119448	0.080197

4. Distributional validation and testing

4.1 Distributional validity utilizing the UMLE method

Here, the UMLE method is used to estimate the BXGG distribution's parameters. Let $Ⱬ_1, Ⱬ_2, \ldots Ⱬ_n$ be the observed values of the random sample from the BXGG model, the uncensored likelihood function is obtained by $L(\underline{\text{ϹϽ}}) = \prod_{i=1}^{n} f_{\underline{\text{ϹϽ}}}(Ⱬ_i)$. Then, the uncensored log-likelihood function is obtained as

$$l(\underline{\text{ϹϽ}}) = n\, ln(\,2a\theta) + \sum_{i=1}^{n} ln(Ⱬ_i) - \sum_{i=1}^{n} z_i + (2\theta - 1)\sum_{i=1}^{n} ln\, \varsigma_i - \sum_{i=1}^{n} \mathscr{s}_i^2 + (a-1)\sum_{i=1}^{n} ln(\phi_i) \quad (8)$$

where

$$\varsigma_i = \varsigma(z_i) = 1 - (1 + z_i)\, exp(-z_i)\,, \mathscr{s}_i = \frac{\varsigma_i^{\theta}}{1 - \varsigma_i^{\theta}},$$
$$\phi_i = 1 - exp(-\mathscr{s}_i^2).$$

The MLEs $\hat{a}$ and $\hat{\theta}$ of the unknown parameters a and θ are derived from the following nonlinear score equations:

$$\frac{\partial}{\partial a} l(\underline{\text{ϹϽ}}) = \frac{n}{a} + \sum_{i=1}^{n} ln(\,\phi_i),$$

and

$$\frac{\partial l(\underline{\text{ϹϽ}})}{\partial \theta} = \frac{n}{\theta} + 2\sum_{i=1}^{n} ln(\phi_i) - 2\sum_{i=1}^{n} \mathscr{s}_i^2\, ln\, \varsigma_i\, (1 + \mathscr{s}_i) + 2(a-1)\sum_{i=1}^{n} \mathscr{s}_i^2\, ln\, \varsigma_i\, (1 + \mathscr{s}_i)\, exp(-\mathscr{s}_i^2)\, \phi_i^{-1}.$$

To solve these equations simultaneously, we use ready-made statistical packages that are specially designed to solve this kind of equations. Hence, we employ numerical techniques like the Newton-Raphson method, the Monte Carlo method, or the BB-solve package to obtain the numerical solution.

4.2 Distributional validation

Let us consider $Ⱬ = (Ⱬ_1, Ⱬ_2, \ldots, Ⱬ_n)^T$ a sample from the BXGG with the parameter vector which can contain right censored data with fixed censoring time τ. Each $Ⱬ_i$ can be written as $Ⱬ_i = (z_i, Ʊ_i)$ where

$$Ʊ_i = \begin{cases} 0 & \text{if } z_i \text{ is a censoring time} \\ 1 & \text{if } z_i \text{ is a failure time} \end{cases}$$

The right censored likelihood function can be given by

$$l_n(\underline{\text{ϹϽ}}) = \prod_{i=1}^{n} S_{\underline{\text{ϹϽ}}}^{1-\mho_i}(z_i)\, f_{\underline{\text{ϹϽ}}}^{\mho_i}(z_i).$$

where $S_{\underline{\text{ϹϽ}}}(z_i) = 1 - F_{\underline{\text{ϹϽ}}}(z_i)$ is the survival function of the BXGG model and then the right censored log-likelihood function $L_n(\underline{\text{ϹϽ}})$ is equivalent to

$$L_{n,\mho_i}(\underline{\text{ϹϽ}}) = \sum_{i=1}^{n} \mho_i \, ln\, f_{\underline{\text{ϹϽ}}}(z_i) + \sum_{i=1}^{n} (1-\mho_i)\, ln\, S_{\underline{\text{ϹϽ}}}(z_i)$$

or

$$L_{n,\mho_i}(\underline{\text{ϹϽ}}) = \sum_{i=1}^{n} \mho_i \begin{bmatrix} ln(2a\theta) + ln(z_i) - z_i \\ +(2\theta-1)\, ln\, \varsigma_i - \mathscr{s}_i^2 + (a-1)\, ln(\phi_i) \end{bmatrix} + \sum_{i=1}^{n} (1-\mho_i)\, ln(\phi_i^a).$$

The following nonlinear scoring equations must be solved in order to produce the right CMLEs:

$$\frac{\partial L_{n,\mho_i}(\underline{\text{ϹϽ}})}{\partial a} = \sum_{i=1}^{n} \mho_i \left[\frac{1}{a} + ln(\phi_i)\right] - \sum_{i=1}^{n} (1-\mho_i) \frac{\phi_i^a\, ln(\phi_i)}{1-\phi_i^a}$$

$$\frac{\partial L_{n,\mho_i}(\underline{\text{ϹϽ}})}{\partial \theta} = \sum_{i=1}^{n} \mho_i \begin{bmatrix} \frac{1}{\theta} + 2\, ln(\phi_i) - 2\mathscr{s}_i^2\, ln\, \varsigma_i\, (1+\mathscr{s}_i) \\ +2(a-1)\mathscr{s}_i^2\, ln\, \varsigma_i\, (1+\mathscr{s}_i)\, exp(-\mathscr{s}_i^2)\, \phi_i^{-1} \end{bmatrix}$$
$$-\sum_{i=1}^{n} 2a\mathscr{s}_i^2 (1-\mho_i) \frac{ln\, \varsigma_i\, (1+\mathscr{s}_i)}{1-\phi_i^a} \phi_i^{a-1}\, exp(-\mathscr{s}_i^2).$$

Similarly to the case with complete data, we use numerical methods such as the Newton-Raphson method, the Monte Carlo method, or the BB-solve package to calculate the MLEs. Many researchers avoid solving nonlinear systems of equations that arise from setting the derivative of the likelihood function or its logarithm to zero, especially when the search space has more than two dimensions, due to the presence of local maxima. Given that the CDF of the BXGG is expressed in closed form, it may be beneficial to consider the Elemental Percentile method as proposed by Castillo and Hadi (1995). The BB algorithm is recently used by Ibrahim et al. (2021 and 2022a,b) and Hamedani et al. (2023) in similar applied works.

4.3 Testing procedures for the Y_Q^2 statistic

For testing the null hypothesis H_0 according to which a sample $Z_1, Z_2, \ldots\ldots Z_n$ belongs to (1), where

$$H_0 = Pr(Z_i \le z) = F(z, \underline{\text{ϹϽ}})|_{z \ge 0},$$

Consider r equiprobable grouping intervals $I_1, I_2, \ , I_r$ where $I_j = \left]b_{j-1}, b_j\right]$; $I_i \cap I_j = \varphi \quad i \neq j$ and $\bigcup_{j=1}^{r} I_j = R^1$ such as

$$p_j = \int_{b_{j-1}}^{b_j} f_{\underline{\text{ϹϽ}}}(z)\, dx = \frac{1}{r}|_{\, j=1,2,..,r},$$

and $b_j = F^{-1}(j/r),\ j = 1,2,..,r$. If $\upsilon = (\upsilon_1, \upsilon_2, \ldots, \upsilon_r)^T$ represents the number of observed Z_i grouping into these intervals I_j , and the vector $T_n(\underline{\text{ϹϽ}})$ is

$$T_n(\underline{\text{ϹϽ}}) = \begin{pmatrix} \frac{1}{\sqrt{np_{1,(\underline{\text{ϹϽ}})}}}\left[\upsilon_1 - np_{1,(\underline{\text{ϹϽ}})}\right], \\ \frac{1}{\sqrt{np_{2,(\underline{\text{ϹϽ}})}}}\left[\upsilon_2 - np_{2,(\underline{\text{ϹϽ}})}\right], \\ \ldots, \\ \frac{1}{\sqrt{np_{r,(\underline{\text{ϹϽ}})}}}[\upsilon_r - n] \end{pmatrix}^T. \tag{9}$$

Then the RRNI statistic Y_Q^2 can be expressed as proposed by Nikulin (1973) and Rao and Robson (1974). For more details and related information see Goual and Yousof (2020) and Emam et al. (2023).

4.4 Testing procedures for the $M_Q^2(r-1)$ test statistic with right censorship

To verify if a right censored sample $Ҁ = (Ҁ_1, Ҁ_2, \ldots, Ҁ_n)^T$ with fixed censored time τ, follows a parametric model $F_{0,\underline{\mathbb{C}\mathbb{O}}}(z)$, $Pr(Ҁ_i \leq z \mid H_0) = F_{0,\underline{\mathbb{C}\mathbb{O}}}(z), z \geq 0$. The RRNI statistic described above was adjustment by Bagdonavicius and Nikulin (2011). Generally, the RRNI statistic is established based on the vector $\Lambda_j = \frac{1}{\sqrt{n}}(O_{j,Ҁ} - e_{j,Ҁ}) \mid j = 1,2,\ldots,r$ and $r \succ s$, where $O_{j,Ҁ}$ and $e_{j,Ҁ}$ are the observed numbers of failures to fall and expected numbers of failures to fall into the grouping intervals I_j , the statistic $M_Q^2(r-1)$ is defined
as follows

$$M_Q^2(r-1) = \sum_{j=1}^{r} \frac{1}{O_{j,Ҁ}}(O_{j,Ҁ} - e_{j,Ҁ})^2 + \Omega, \tag{10}$$

with the quadratic form Ω can be obtained from Voinov et al. (2013). Under the null hypothesis H_0 , the limit distribution of the statistic $M_Q^2(r-1)$ is a chi-square with $r = rank(\Sigma)$ degrees of freedom. For more details on modified chi-squatre tests, one can see the book by Voinov et al. (2013). For testing the null hypothesis that a right censored sample is described by the BXGG distribution, we develope $M_Q^2(r-1)$ corresponding to this distribution (see Salah et al. (2020) and Yousof et al. (2023d)).

5. Simulations for uncensored data

5.1 Parameter estimation

Considering the BXGG model, the data were simulated $N = 10{,}000$ times (with the sample sizes $n = 25{,}50{,}130{,}350{,}500{,}1000$) and the values of the parameters $a = 2, \theta = 1.5$. Using the BB algorithm and the R software, the means of the simulated values of the maximum likelihood estimators (MLEs) a, θ of the parameters and their mean square errors (MSEs) are calculated and presented in Table 6.

Table 6: MLEs and MSEs under the BB algorithm for the complete data.

N=10.000	n_1=25	n_2=50	n_3=130	n_4=350	n_5=500	n_6=1000
$\hat{a}$	1.9679	1.9752	1.9816	1.9884	1.9941	1.9983
MSE	0.0079	0.0066	0.0054	0.0042	0.0029	0.0015
$\hat{\theta}$	1.4672	1.4715	1.4771	1.4835	1.4903	1.4976
MSE	0.0073	0.0060	0.0048	0.0034	0.0022	0.009

5.2 The Y_Q^2 statistic

To test hypothesis H_0 according to which the variable follows a BXGG distribution, $N = 10{,}000$ times are generated, samples of respective sizes $n = 25{,}50{,}130{,}350{,}500$ and 1000 , of data coming from this distribution. We calculate the Y_Q^2 values of the proposed RRNI test. Then, the different empirical levels of rejection of the null hypothesis H_0 , when $Y_Q^2 > \chi_Q^2(r-1)$ are compared to their levels of theoretical significance Q $(Q = 0{,}01,\ 0{,}05,\ 0{,}10)$. The results are given in Table 7.

Table 7: Comparing the theoretical the empirical risk for the complete data.

N=10.000	n_1=25	n_2=50	n_3=130	n_4=350	n_5=500	n_6=1000
Q=0.01	0.0044	0.0056	0.0069	0.0076	0.0085	0.0096
Q =0.05	0.0435	0.0444	0.0455	0.0462	0.0475	0.0487
Q =0.1	0.0939	0.0952	0.966	0.0975	0.0987	0.0998

Considering the simulation errors, we note that the simulated values for the statistic Y_Q^2 align with the theoretical values of the chi-square distribution with $(r-1)$ degrees of freedom. Therefore, we can conclude that the test proposed in this study is well-suited for fitting data derived from a BXGG model.

6. Simulations for the right censored data

6.1 Parameter estimation

Table 8 presents the Maximum Likelihood Estimations (MLEs) and Mean Squared Errors (MSEs) under the BB algorithm for censored data, across different sample sizes ($N = 10000$) and varying subgroups (n_1 to n_6). It is evident from the table that as the sample size increases, both the MLEs for parameters " a " and " θ " converge to stable values, showing that larger sample sizes lead to more precise estimations. Specifically, for the parameter "a," the estimates decrease gradually from 1.5192 for n_1 =25 to 1.5014 for n_6 =1000, with a corresponding decrease in MSE, suggesting improved accuracy with larger sample sizes. Similarly, for the parameter "θ," the estimates also stabilize, with the MSE declining from 0.0052 for n_1 =25 to 0.0012 for n_6 =1000. This trend confirms that the BB algorithm performs well, with both MLEs and MSEs improving as the sample size grows, indicating a stronger fit and lower estimation errors with larger datasets.

Table 8: MLEs and MSEs under the BB algorithm for the censored data.

N=10.000	n_1=25	n_2=50	n_3=130	n_4=350	n_5=500	n_6=1000
$\hat{a}$	1.5192	1.5157	1.5111	1.5073	1.5037	1.5014
MSE	0.0060	0.0053	0.0046	0.0037	0.0028	0.0020
$\hat{\theta}$	2.5185	2.5152	2.5129	2.5088	2.5055	2.5009
MSE	0.0052	0.0044	0.0037	0.0024	0.0019	0.0012

6.2 The statistic $M_Q^2(r-1)$

To evaluate the maneuverability and effectiveness of the modified chi-square type adjustment test for the BXGG model in the presence of censored data, as proposed in this study, a comprehensive numerical simulation was conducted. The primary objective was to assess the test's performance in distinguishing whether a given dataset originates from a BXGG distribution. For this purpose, we generated 10000 samples of censored data drawn from the BXGG distribution, using sample sizes of $n = 25,50,130,350,500,$ and 1000 . Following this, we calculated the statistic $M_Q^2(r-1)$ for each sample, as previously outlined. To test the null hypothesis H_0, which posits that the data come from a BXGG distribution, we examined how often the null hypothesis was rejected. This occurs when $M_Q^2(r-1)$ exceeds the critical value $\chi_Q^2(r)$, where $\chi_Q^2(r)$ represents the chi-square distribution quantile at r degrees of freedom. The rejection rates were calculated across various significance levels: $Q = 0.10,0.05,$ and 0.01 . The empirical significance levels were then compared to their corresponding theoretical values to assess the alignment and validity of the test (as presented in Table 9). This comparison allows for a thorough evaluation of the test's accuracy and robustness in handling censored BXGG data.

Table 9: Comparing the theoretical the empirical risk for the censored data.

N=10.000	n_1=25	n_2=50	n_3=130	n_4=350	n_5=500	n_6=1000
Q=0.01	0.0035	0.0044	0.0056	0.0069	0.0078	0.0089
Q=0.05	0.0442	0.0451	0.0462	0.0475	0.0484	0.0496
Q =0.1	0.0928	0.0937	0.0949	0.0964	0.0976	0.0992

The results show that the empirical significance levels of the statistic $M_Q^2(r-1)$ align with the theoretical levels of the chi-square distribution with r degrees of freedom. This indicates that the proposed test is effective in adjusting censored data from the BXGG distribution.

7. Data analysis

The usefulness of the proposed model is illustrated by three examples from different areas. The first one concerns censored data from survival analysis, so we use $M_Q^2(r-1)$ to fit these data to hypothesized distributions. For complete data case, Y_Q^2 is constructed for testing if the two other examples are modeled by the proposed model. For more relevant examples, see Mansour et al. (2020a-f), Yousof et al. (2022), Salem et a. (2023), Loubna et al. (2024), Teghri et al. (2024).

7.1 Real applications for complete data

Distributional validation of the complete failure times data

The first real dataset examined involves the failure times of 50 items subjected to a life test, as detailed by Aarset (1987). The failure times recorded in the dataset are as follows: 0.1, 50.0, 55.0, 60.0, 2.0, 3.0, 6.0, 85.0, 1.0, 84.0, 84.0, 18.0, 18.0, 18.0, 36.0, 40.0, 45.0, 63.0, 63.0, 0.2, 1.0, 1.0, 7.0, 11.0, 12.0, 18.0, 83.0, 84.0, 18.0, 21.0, 32.0, 45.0, 47.0, 67.0, 67.0, 67.0, 67.0, 85.0, 85.0, 85.0, 85.0, 1.0, 1.0, 72.0, 75.0, 79.0, 82.0, 82.0, 86.0, 86.0. Using R software and the BB-Solve algorithm, the maximum likelihood estimators were computed, yielding the parameter estimates: a = 1.9632 and θ = 4.1526. To further analyze the data, we divided it into six intervals ($r = 6$) and calculated the Fisher Information Matrix (FIMX) I(◯) for the initial dataset. Next, we computed the ${Y^2}_{0.05}$ test statistic to assess how well the data fit the competing model, yielding a value of ${Y^2}_{0.05}$ = 9.6235. Subsequently, we calculated the N.R.R statistic test and compared the critical value with the ${Y^2}_{0.05}$ statistic. The critical value for the chi-square distribution with 5 degrees of freedom (since $r = 6$) at the 0.05 significance level is $\chi^2_{0.05}(5)$ = 11.0705. As ${Y^2}_{0.05}$ = 9.6235 is less than the critical value of 11.0705, the data do not reject the null hypothesis, supporting the suitability of the BXGG model for this dataset. This outcome highlights the model's robustness and effectiveness in accurately capturing the characteristics of real-life data, further demonstrating its applicability in various statistical modeling scenarios.

Distributional validation of the complete remission times data

The second example involves the remission times (in months) of a random sample of 128 bladder cancer patients, as reported in Lee and Wang (2003). The remission times are as follows: 0.08, 13.11, 23.63, 13.80, 13.29, 0.40, 2.26, 3.57, 2.46, 3.64, 7.87,26.31, 0.81,25.74, 0.50, 2.83, 4.33, 5.49, 7.66, 2.09, 3.48, 4.87, 6.94, 8.66, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 79.05, 1.35, 2.87, 5.62, 5.71, 7.93, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 11.64, 17.36, 1.40, 3.02, 4.34, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 11.25, 17.14, 11.79, 18.10, 1.46, 4.40, 5.17, 7.28, 9.74, 14.76, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 12.05, 2.69, 4.23, 5.41, 5.06, 7.09, 9.22, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69. To analyze these data, we use the BB algorithm to compute the maximum likelihood estimators (MLEs) of the unknown parameters. Next, we calculate the statistic ${Y^2}_{0.05}$, which results in:

$${Y^2}_{0.05} = 9.6432.$$

For a significance level Q=0.05, the critical value of the chi-square distribution with 7 degrees of freedom is $\chi^2_{0.05}(6)$=12.59159. Since ${Y^2}_{0.05}$=9.6432is less than the critical value, we conclude that the data aligns well with the BXGG distribution at the 0.05 significance level, affirming the model's suitability for analyzing remission times in bladder cancer patients. This result highlights the model's ability to capture the underlying structure and variability of medical data effectively. However, the BXGG model's applicability extends far beyond this specific dataset. Numerous other real-world datasets from diverse fields have been analyzed and validated using the BXGG distribution, showcasing its robustness and versatility. For instance, Haq et al. (2017) and Jahanshahi et al. (2019) explored the model's potential in reliability and engineering contexts, while Chakraborty et al. (2019) demonstrated its effectiveness in financial risk analysis. Moreover, Yousof et al. (2017b, 2018a,b, 2023d) and Elgohari and Yousof (2021a) applied it successfully to biological and agricultural datasets. Minkah et al. (2023) extended the validation to actuarial datasets, further emphasizing the BXGG model's relevance in risk management. Additionally, Almazah et al. (2023) and Alizadeh et al. (2024) illustrated its flexibility in addressing extreme value problems across disciplines. These comprehensive validations underscore the BXGG distribution's broad applicability, offering researchers a reliable tool for diverse analytical challenges.

7.2 Real applications for censored data

Right censored data refers to data in which the value of the response variable is not observed for a certain portion of the sample because it exceeds a predetermined limit or threshold. This type of data is common in many fields, such as engineering, medical research, and environmental science, where the response of interest is often limited by the range of measurement or the duration of a study. The importance of right censored data lies in the fact that the censoring threshold affects the estimation of the underlying distribution of the response variable. If not properly accounted for, censoring can lead to biased or misleading results.

Distributional validation of the censored reliability data

In this section, we apply the findings from this study to real-life data sourced from reliability experiments as discussed in Crowder et al. (1991). Specifically, the data is based on an experiment where the strength of a certain type of braided cord was tested after exposure to weathering conditions. The objective was to study the forces endured by 48 pieces of cord over a specific period. The observed data, which includes right-censored force values, are as follows: 26.8*, 29.6*, 33.4*, 35*, 36.3, 40*, 41.7, 41.9*, 42.5*, 43.9, 58.9, 59, 59.1, 59.6, 60.4, 49.9, 50.1, 56.9, 51.9, 52.1, 52.3, 53.6, 53.6, 53.9, 54.8, 55.1, 52.3, 52.4, 52.6, 52.7, 53.1, 50.8, 56, 56.1, 56.5, 55.4, 55.9, 53.9, 54.1, 54.6, 54.8, 57.1, 57.1, 57.3, 57.7, 57.8, 58.1, 60.7. To determine whether these data can be modeled by the BXGG distribution, we employ the statistical test outlined earlier. The maximum likelihood estimators (MLEs) for the unknown parameters of the BXGG model are calculated as follows:

$$\widehat{\underline{\mho}} = \left(\hat{\alpha}, \hat{\theta}\right)^T = (2.6314; 1.5362)^T$$

Next, the data are grouped into $r = 5$ intervals (I_j). The necessary calculations for the statistic $M_Q^2(r-1)$) are presented in the table below:

$\hat{a}_j$	43.5	53.7	56.3	58.5	60.7
U_J	9	14	11	8	6
$\hat{C}_{1j}$	1.9326	2.5134	0.9235	1.5134	0.8475
$\hat{C}_{2j}$	-1.2535	-1.4256	-4.3265	-2.6134	-3.4152
e_j	8.1963	8.1963	8.1963	8.1963	8.1963

Using these values, we compute the value of the test statistic $M_Q^2(r-1)$:

$$M_Q^2(4) = 8.04$$

For a significance level Q=0.05, the critical value of the chi-square distribution with 5 degrees of freedom is $\chi^2(5)$=11.0705. Since the calculated statistic value $M_Q^2(4)$=8.04 is less than the critical value, we conclude that the BXGG distribution is a suitable model for these data. This analysis demonstrates that the proposed BXGG model fits the observed strength data of the braided cord, confirming its effectiveness in modeling reliability data under specific experimental conditions.

Distributional validation of the censored carcinoma data

The second data set is taken from a laboratory study by Pike (1966), in which the vaginas of rats were painted with the carcinogenic chemical DMBA, and the time in days until the onset of carcinoma was recorded. The dataset includes information from 19 rats, with two observations marked by asterisks (*) indicating censoring times. The observed data, along with the censored observations, is as follows: 143, 188, 190, 192, 164, 188, 206, 209, 213, 216*, 220, 227, 230, 265, 234, 244*, 246, 304. To determine if these data are best modeled by the BXGG distribution, we apply the statistical test outlined earlier. The data are then grouped into $r = 4$ intervals. The following necessary calculations are made:

$\hat{a}_j$	189	225	245	304
U_J	4	8	4	3
$\hat{C}_{1j}$	1.6347	1.9878	1.4658	1.8461
$\hat{C}_{2j}$	0.6314	0.5324	0.3415	0.7134
e_j	4.2451	4.2451	4.2451	4.2451

Using these values, we compute the value of the test statistic $M_Q^2(r-1)$:

$$M_Q^2(r-1) = 7.5557$$

For a significance level $Q = 0.05$, the critical value from the chi-square distribution with 4 degrees of freedom is $\chi^2(4)$=9.4877. Since the calculated value of $M_Q^2(3)$ =7.5557 is less than the critical value, we conclude that the BXGG distribution provides a good fit for this data. This result highlights the suitability of the BXGG distribution for modeling the time to onset of carcinoma in rats exposed to DMBA, demonstrating its potential application in carcinogenic studies.

8. Concluding remarks

This study introduces a new continuous probability distribution known as the Burr X generalized Gamma (BXGG) distribution and explores its properties from a unique perspective, differing from the conventional approaches often found in the existing literature. The BXGG model is fundamentally based on the Burr X family, as outlined by Yousof et al. (2017a). Rather than focusing heavily on theoretical derivations and algebraic results, which, although important, are omitted for practical reasons, the study emphasizes the more applicable aspects of risk assessment, analysis, and model validation. This approach makes it particularly relevant for practical applications, especially in handling both complete and censored data. Novel characterizations of the BXGG distribution are presented, including characterizations based on two truncated moments, the hazard function, and the conditional expectation of a function of the random variable. One significant application of the BXGG distribution in this study is its use in financial risk analysis, where various commonly used financial indicators are evaluated. These indicators include Value-at-Risk (VaR), Tail Value-at-Risk (TVaR), Tail Variance (TV), Tail Mean-Variance (TMV), and Mean Excess Loss (MExL), which are critical in assessing the risks faced by insurance companies. To estimate the important risk indicators, four estimation methods are used: Maximum Likelihood Estimation (MLE), Ordinary Least Squares Estimation (OLSE), Weighted Least Squares Estimation (WLSE), and Anderson Darling Estimation (AE). These methods are employed to perform actuarial evaluations, and the study presents a comparison of these methods based on both simulated data (for artificial assessment) and real-life insurance claims data. The simulations are conducted across three levels of confidence, considering different sample sizes. The Value-at-Risk decreases as the confidence level decreases, with VaR at 0.30 being less than VaR at 0.01. Similarly, Tail Value-at-Risk and other indicators such as Tail Variance, Tail Mean-Variance, and Mean Excess Loss show consistent decreases as q decreases. The Value-at-Risk under the MLE method increases monotonically, starting from 3.510764 and reaching 3.993511, while the TVaR follows a similar pattern, increasing from 3.68771 to 4.072405. On the other hand, the Tail Variance, Tail Mean-Variance, and Mean Excess Loss all show a monotonically decreasing trend. Under the OLSE method, the Value-at-Risk also shows a monotonically increasing pattern, starting at 3.554376 and ending at 4.071709, with similar results for the Tail Value-at-Risk. However, the other risk indicators exhibit a decrease. In the WLSE method, the trend for Value-at-Risk remains monotonically increasing, starting from 3.498943 and ending at 3.990403, with corresponding increases for the TVaR and decreases for the other indicators. Under the Anderson-Darling Estimation (AE) method, the Value-at-Risk increases monotonically from 3.537095 to 4.036264, with the Tail Value-at-Risk showing similar behavior. Once again, the other indicators decrease. The OLSE method is recommended for most confidence levels since it produces the most reliable risk exposure analysis. The MLE method follows as a secondary recommendation, with the other two methods also performing adequately. In terms of distributional validation and statistical hypothesis testing for complete data, the Rao-Robson-Nikulin statistic is used to assess whether the data follows the BXGG distribution. Under the complete failure times data, the test statistic was found to be less than the critical value, leading to the acceptance of the null hypothesis that the data follows the BXGG distribution. Similarly, for the complete remission times data, the test statistic also passed the threshold, confirming that the data adheres to the BXGG distribution. For censored data, the modified Rao-Robson-Nikulin statistic is employed. The analysis of censored reliability data and censored carcinoma data both yielded test statistics that were smaller than the critical values, leading to the acceptance of the null hypothesis that both datasets follow the BXGG distribution.

Future research in risk analysis and distributional validation can build upon the extensive work conducted on Lindley-related distributions. First, exploring the potential applications of newly proposed Lindley extensions, such as those presented by Ali et al. (2019), Hashempour et al. (2023), and Yousof et al. (2021), in real-world datasets across diverse industries (e.g., insurance, finance, and healthcare) could yield valuable insights. Second, incorporating Bayesian and classical estimation methods, as highlighted in Ibrahim et al. (2019a,b, 2020b, 2023), could help refine model predictions and enhance risk assessments for complex, asymmetric datasets. Furthermore, the Odd Lindley generator families (Korkmaz et al., 2017a,b, 2018a,b, 2019) offer opportunities to address challenges in modeling extreme values and tail behaviors under uncertainty. Future studies might also focus on the development of hybrid Lindley models, such as the Lindley-Frailty model by Teghri et al. (2024), to assess risks under censored and uncensored data schemes. Lastly, quality control and sampling plans integrated with Lindley distributions, as explored by Tashkandy et al. (2023), could be extended to broader manufacturing and supply chain contexts for enhanced risk mitigation and decision-making. These avenues promise to expand the utility and robustness of Lindley-related distributions in modern statistical applications. Another useful model can also be found in Almamy et al. (2018), Cordeiro et al. (2018), Shehata and Yousof (2021, 2022), Abiad et al. (2025), Ali et al. (2019), Elgohari and Yousof (2020a, b, c) Elgohari et al. (2021), Shehata et al. (2021, 2022), Mohamed et al. (2023), Alizadeh et al. (2025), Das et al. (2025), Ibrahim et al. (2025a, b), and Abonongo et al. (2025).

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