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A New Version of the Compound Quasi-Lomax Model: Properties, Characterizations and Risk Analysis under the U.K. Motor Insurance Claims Data

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Abstract
This paper introduces a new lifetime distribution, the Compound Quasi-Lomax (CQLx) model, designed to enhance the modeling of heavy-tailed data in actuarial and financial risk analysis. The CQLx distribution is developed through a novel extension of the Lomax family, offering increased flexibility in capturing extreme values and complex data behaviors. Key mathematical properties are derived. Characterization of the model is achieved via truncated moments and the reverse hazard function. Several estimation methods are employed including the Maximum Likelihood Estimation (MLE), Cramér–von Mises (CVM), Anderson–Darling Estimation (ADE), Right-Tail Anderson-Darling Estimation (RTADE), and Left-Tail Anderson-Darling Estimation (LTADE). A comprehensive simulation study evaluates the performance of these methods in terms of bias and root mean square error (RMSE) across various sample sizes. Risk measures such as Value-at-Risk (VaR), Tail Value-at-Risk (TVaR), Tail Variance (TV), Tail Mean Variance (TMV), and Expected Loss (EL) are computed under artificial and real financial insurance claims data. The results demonstrate that MLE generally provides the most accurate and stable estimates, particularly for larger samples, while CVM and ADE tend to overestimate risk, especially at higher quantiles. The CQLx model shows superior performance in fitting extreme claim-size data, making it a robust tool for risk management.

Key Words: Lomax Distribution, Maximum Likelihood Estimation, Cramér–von Mises, Anderson–Darling Estimation, Value-at-Risk, Risk Analysis, Characterizations.

1. Introduction

The Lomax (Lx) model is the most well-known of the five models that make up the Pareto family. In business, actuarial science, physical sciences, biological sciences, economics, engineering, income and wealth inequality research, theory of queuing, and size of cities data sets, the Lx model, also known as the Lomax; see Lomax (1954), is a heavy-tail probability density. The standard Lx model, however, is regarded as a limiting model of residual lifetimes at great age and is part of the family of "monotonically decreasing" hazard/failure rate function. In this work, however, we will present a new version whose hazard rate function (HRF) is part of the "upside down," "monotonically decreasing" and "increasing-constant" families. The Lx distribution was used by Harris (1968) to describe and model wealth and income data. The Lx distribution was utilized by Corbellini et al. (2007) to model the company size data. In addition to being seen as a hybrid of the standard gamma and exponential distributions, the Lx model is a unique model form of the well-known Pearson type VI distribution. A heavy-tailed alternative model to the standard exponential, standard Weibull, and standard gamma distributions is proposed for the Lx distribution, per Bryson (1974). For additional information regarding the connection between the Lx model, see Tadikamalla (1980), Durbey (1970), Korkmaz et al. (2018) and Minkah et al. (2023). Recent studies have significantly advanced the Lomax distribution through various flexible extensions for modeling failure times, service times, and insurance data. Ansari et al. (2020), Aboraya et al. (2022), and Ali et al. (2021) introduced compound and extended versions with rich mathematical properties, copula constructions, and diverse estimation methods. Models by Hamed et al. (2022), Al-Essa et al. (2023), and Salem et al. (2023) demonstrated strong performance in fitting skewed and censored data, particularly in reliability and medical applications. Khan et al. (2024) further enhanced its risk modeling capability with a heavy-tailed version applied to VaR and mean-of-order-P analysis. In this paper we first present a new version of the Lomax model called the CQLx distribution, the cumulative function (CDF) of the QLx distribution can be presented as

$$G_{\beta_3}(x) = G(x;\beta_3) = 1 - exp\left[-\frac{1}{\beta_3} 2\, ln(1+x)\right] |x > 0, \quad (1)$$

where $\beta_3 > 0$ is the shape parameters, respectively. The primary goal of this work is to use the Poisson Topp-Leone (PTL) family, as established by Merovci et al. (2020), to give a flexible extension of the QLx distribution called the CQLx model. The PTL-G family's CDF can be expressed as

$$F_{\beta_1,\beta_2,\underline{\xi}}(x) = \frac{1}{W(\beta_1)}\left(1 - exp\left\{-\frac{1}{\beta_1}\left[2G\left(x;\underline{\xi}\right) - G^2\left(x;\underline{\xi}\right)\right]^{\beta_2}\right\}\right) |x \in R, \quad (2)$$

where $\beta_1 > 0, \beta_2 > 0$ and

$$W(\beta_1) = 1 - exp\left(-\frac{1}{\beta_1}\right).$$

Then, the CDF of the CQLx model can then be derived as

$$F_{\underline{P}}(x) = \frac{1}{W(\beta_1)}\left(1 - exp\left\{-\frac{1}{\beta_1}\left[1 - (1+x)^{-\frac{4}{\beta_3}}\right]^{\beta_2}\right\}\right) |x > 0, \quad (3)$$

The corresponding probability density function (PDF) of (3) can be written as

$$f_{\underline{P}}(x) = 4\frac{\beta_2}{\beta_1\beta_3 W(\beta_1)} \frac{(1+x)^{-\left(1+\frac{4}{\beta_3}\right)}\left[1 - (1+x)^{-\frac{4}{\beta_3}}\right]^{\beta_2 - 1}}{exp\left\{\frac{1}{\beta_1}\left[1 - (1+x)^{-\frac{4}{\beta_3}}\right]^{\beta_2}\right\}} |x > 0. \quad (4)$$

As $x \to 0$, we have

$$f_{\underline{P}}(x) \approx 4\frac{\beta_2}{\beta_1\beta_3 W(\beta_1)}\left(\frac{4}{\beta_3}\right)^{\beta_2 - 1} x^{\beta_2 - 1}.$$

As $x \to \infty$, we have

$$f_{\underline{P}}(x) \approx 4\frac{\beta_2}{\beta_1\beta_3 W(\beta_1)} \frac{(1+x)^{-\left(\frac{4}{\beta_3}+1\right)}}{exp\left(\frac{1}{\beta_1}\right)}.$$

The tail behavior of $f_{\underline{P}}(x)$ for large x is dominated by

$$f_{\underline{P}}(x) \approx 4\frac{\beta_2}{\beta_1\beta_3 W(\beta_1)\, exp\left(\frac{1}{\beta_1}\right)}(1+x)^{-\left(\frac{4}{\beta_3}+1\right)},$$

This indicates that the tail of the PDF decays polynomial with an exponent of $\frac{4}{\beta_3} + 1$ as $x \to \infty$. Hereafter, we will refer to the new model in (3) and (4) with the CQLx model. Other Lx extensions can be founded in Gupta et al. (1998), Lemonte and Cordeiro (2013), Cordeiro et al. (2018), Tahir et al. (2015), Elbiely and Yousof (2018), Goual and Yousof (2020), Chesneau and Yousof (2020), Yadav et al. (2020), Hamed et al. (2022), Ibrahim and Yousof (2020) and Salem et al. (2023). Recently, Abiad et al. (2025) introduced a novel approach to reliability analysis by incorporating diverse copula structures into a new Fisk probability model. This advancement allows for a more flexible dependence structure between variables, improving the accuracy of reliability assessments in engineering and applied sciences. Ali et al. (2025) provided an in-depth exploration of statistical outliers, discussing their identification, impact on data interpretation, and potential methods for handling anomalies in many topics, especially in risk analysis. Their work is crucial for ensuring robust statistical inference in various fields, including finance, healthcare, and quality control. Alizadeh et al. (2025a) developed a new weighted Lindley distribution tailored for modeling extreme insurance claims. By refining the probability distribution to better fit heavy-tailed data, their model enhances risk assessment and decision-making in the insurance industry. Das et al. (2025) introduced a novel application of the Laplace distribution to analyze economic peaks and VaR in house price fluctuations. Their study provides a fresh perspective on risk modeling in real estate markets, offering insights for financial analysts and policymakers.

2. Main characteristics

This Section explores the fundamental mathematical properties of the proposed CQLx distribution, providing a comprehensive foundation for its theoretical and applied utility. This section derives key statistical functions, including useful series expansions for the probability density and cumulative distribution functions, which facilitate analytical tractability. We present explicit expressions for ordinary moments, incomplete moments, and mean deviations, which are essential for understanding the distribution's central tendency and variability. The moment generating function and probability weighted moments are derived to support parameter estimation and inference. Additionally, the section covers the residual and reversed residual life moments, which are crucial in reliability and survival analysis. These properties enhance the model's applicability in modeling lifetime data and risk assessment. The derivations leverage the exponentiated-Lomax (ELx) as a baseline structure, ensuring flexibility and generality. Closed-form expressions are provided wherever possible, improving computational feasibility. This in-depth characterization underscores the CQLx model's versatility in fitting heavy-tailed data commonly found in actuarial, financial, and engineering contexts. Section 2 thus establishes the analytical backbone necessary for subsequent estimation and application.

Useful expansions

Thanks to Merovci et al. (2020), the CQLx model's PDF in (4) can be expressed as follows

$$f_{\underline{P}}(x) = \sum_{\varsigma_1,\varsigma_2=0}^{\infty} \left[v_{\varsigma_1,\varsigma_2}^{[1]}\, h_{\beta_2^*}(x) - v_{\varsigma_1,\varsigma_2}^{[2]}\, h_{1+\beta_2^*}(x)\right] |\beta_2^* = \beta_2(\varsigma_1+1)+\varsigma_2, \quad (5)$$

where

$$h_\gamma(x) = \gamma g_{\beta_2^*}(x)\left[G_{\beta_2^*}(x)\right]^{\gamma-1}$$

refers to the ELx density, $g_{\beta_2^*}(x) = dG_{\beta_2^*}(x)/dx$ and

$$v_{\varsigma_1,\varsigma_2}^{[1]} = \left(\frac{1}{2}\right)^{\varsigma_2-\beta_2(\varsigma_1+1)} \frac{1}{\varsigma_1!\,W(\beta_1)\beta_2^*}\beta_1^{-(\varsigma_1+1)}\beta_2(-1)^{\varsigma_1+\varsigma_2}\binom{\beta_2(\varsigma_1+1)-1}{\varsigma_2},$$

and

$$v_{\varsigma_1,\varsigma_2}^{[2]} = \left(\frac{1}{2}\right)^{\varsigma_2-\beta_2(\varsigma_1+1)} \frac{1}{\varsigma_1!\,W(\beta_1)(1+\beta_2^*)}\beta_1^{-(\varsigma_1+1)}\beta_2(-1)^{\varsigma_1+\varsigma_2}\binom{\beta_2(\varsigma_1+1)-1}{\varsigma_2}.$$

Equation (5) allows for the expression of the density of X as a representation of ELx densities. Another way to rephrase the CDF of the CQLx is as follows

$$F_{\underline{P}}(x) = \sum_{\varsigma_1,\varsigma_2=0}^{\infty} \left[v_{\varsigma_1,\varsigma_2}^{[1]}\, H_{\beta_2^*}(x) - v_{\varsigma_1,\varsigma_2}^{[2]}\, H_{1+\beta_2^*}(x)\right],$$

where $G_{\beta_2^*}(x) = G(x;\beta_2^*)$ refers to the CDF of the ELx model.

Ordinary moment

The r^{th} ordinary moment of X is given by $\mu'_{r,X} = E(X^r) = \int_{-\infty}^{\infty} x^r f_{\underline{P}}(x)dx$. Then we obtain

$$\mu'_{r,X} = \sum_{\varsigma_1,\varsigma_2=0}^{\infty} \left[v_{\varsigma_1,\varsigma_2}^{[1]}\, E(Z_{\beta_2^*}^r) - v_{\varsigma_1,\varsigma_2}^{[2]}\, E(Z_{1+\beta_2^*}^r)\right]. \quad (6)$$

Henceforth, $Z_{(\beta_2^*)}$ denotes the ELx distribution with power parameter $\beta_2^* > 0$.

$$\mu'_{r,X} = \sum_{\varsigma_1,\varsigma_2=0}^{\infty}\sum_{\varsigma_3=0}^{r}\left[\begin{array}{c} v_{\varsigma_1,\varsigma_2}^{[1]}\,\Delta_{\varsigma_3}^{(\beta_2^*,r)}B(\beta_2^*,1+(\varsigma_3-r)\beta_3) \\ -v_{\varsigma_1,\varsigma_2}^{[2]}\,\Delta_{\varsigma_3}^{(1+\beta_2^*,r)}B(1+\beta_2^*,1+(\varsigma_3-r)\beta_3)\end{array}\right]|_{(\beta_3^{-1}>r)}.$$

where

$$\Delta_{\varsigma_3}^{(a,r)} = a(-1)^{\varsigma_3}\binom{r}{\varsigma_3},$$

and

$$B(v_1,v_2) = \int_0^1 \varphi^{v_1-1}(1-\varphi)^{v_2-1}d\varphi.$$

Incomplete moments

The φ^{th} incomplete moment, say $I_{\varphi,x}(t)$, of X can be expressed from (9) as $I_{\varphi,x}(t) = \int_{-\infty}^{t} x^\varphi f_{\underline{P}}(x)dx$. Then

$$I_{\varphi,x}(t)=\sum_{\varsigma_1,\varsigma_2=0}^{\infty}\sum_{\varsigma_3=0}^{\varphi}\begin{bmatrix} v_{\varsigma_1,\varsigma_2}^{[1]}\Delta_{\varsigma_3}^{(\beta_2^*,\varphi)}\,B_t(\beta_2^*,1+(\varsigma_3-\varphi)\beta_3) \\ -v_{\varsigma_1,\varsigma_2}^{[2]}\,\Delta_{\varsigma_3}^{(1+\beta_2^*,\varphi)}\,B_t(1+\beta_2^*,1+(\varsigma_3-\varphi)\beta_3)\end{bmatrix}|_{(\beta_3^{-1}>\varphi)}, \quad (7)$$

Where

$$B_{a_3}(a_1,a_2)=\int_0^{a_3}y^{a_1-1}\,(1-y)^{a_2-1}dy.$$

Mean deviations

The mean deviations about the mean $[d_{x,\mu_1'}=E(|x-\mu_1'|)]$ and about the median $[m_{x,M}=E(|x-M|)]$ of X are given by $d_{x,\mu_1'}=2\mu_{1,x}'F(\mu_{1,x}')-2I_{1,x}(\mu_{1,x}')$ and $m_{x,M}=\mu_{1,x}'-2I_{1,x}(M)$, respectively, where $\mu_{1,x}'=E(x)$, $M=Med(x)=Q\left(\frac{1}{2}\right)$ is the median and $I_{1,x}(t)$ is the first incomplete moment given by (8) with $\varphi=1$. Ageneral equation for $I_{1,x}(t)$ can be derived from (8) as

$$I_{1,x}(t)=\sum_{\varsigma_1,\varsigma_2=0}^{\infty}\sum_{\varsigma_3=0}^{1}\begin{bmatrix} v_{\varsigma_1,\varsigma_2}^{[1]}\Delta_{\varsigma_3}^{(\beta_2^*,1)}\,B_t\left(\beta_2^*,1+\frac{\varsigma_3-1}{\beta_3^{-1}}\right) \\ -v_{\varsigma_1,\varsigma_2}^{[2]}\,\Delta_{\varsigma_3}^{(1+\beta_2^*,1)}\,B_t\left(1+\beta_2^*,1+\frac{\varsigma_3-1}{\beta_3^{-1}}\right)\end{bmatrix}|_{(\beta_3^{-1}>1)}, \quad (8)$$

Moment generating function

The moment generating function (MGF) can be derived from equation (5) as

$$M_X(t)=\sum_{\varsigma_1,\varsigma_2,r=0}^{\infty}\sum_{\varsigma_3=0}^{r}\frac{t^r}{r!}\begin{bmatrix} v_{\varsigma_1,\varsigma_2}^{[1]}\,\Delta_{\varsigma_3}^{(\beta_2^*,r)}B(\beta_2^*,1+(\varsigma_3-r)\beta_3) \\ -v_{\varsigma_1,\varsigma_2}^{[2]}\,\Delta_{\varsigma_3}^{(1+\beta_2^*,r)}B(1+\beta_2^*,1+(\varsigma_3-r)\beta_3)\end{bmatrix}|_{(\beta_3^{-1}>r)}, \quad (9)$$

Probability weighted moments

The $(\varphi,r)^{th}$ probability weighted moments (PWM) of X following the CQLx model, say $\rho_{\varphi,r}$, is formally defined by

$$\rho_{\varphi,r}=E\{x^{\varphi}F_{\underline{P}}(x)^r\}=\int_{-\infty}^{\infty}x^{\varphi}F_{\underline{P}}(x)^r f_{\underline{P}}(x)dx.$$

Using (5) and (6), we have

$$f_{\underline{P}}(x)F_{\underline{P}}(x)^r=\sum_{\varsigma_1,\varsigma_2=0}^{\infty}\left[v_{\varsigma_1,\varsigma_2}^{[1]}\,h_{\beta_2^*}(x)-v_{\varsigma_1,\varsigma_2}^{[2]}\,h_{1+\beta_2^*}(x)\right],$$

where

$$v_{\varsigma_1,\varsigma_2}^{[1]}=\sum_{\varsigma_2=0}^{\infty}(-1)^{\varsigma_1+\varsigma_2+\varsigma_2}\beta_2\beta_1^{\varsigma_1+1}\left(\frac{1}{2}\right)^{\varsigma_2-\beta_2(\varsigma_1+1)}\frac{(1+\varsigma_2)^{\varsigma_1}}{\varsigma_1!\,\beta_2^*[W(\beta_1)]^{1+r}}\binom{r}{\varsigma_2}\binom{\beta_2(\varsigma_1+1)-1}{\varsigma_2}$$

and

$$v_{\varsigma_1,\varsigma_2}^{[2]}=\sum_{\varsigma_2=0}^{\infty}(-1)^{\varsigma_1+\varsigma_2+\varsigma_2}\beta_2\beta_1^{\varsigma_1+1}\left(\frac{1}{2}\right)^{\varsigma_2-\beta_2(\varsigma_1+1)}\frac{(1+\varsigma_2)^{\varsigma_1}}{\varsigma_1!\,[1+\beta_2^*][W(\beta_1)]^{1+r}}\binom{r}{\varsigma_2}\binom{\beta_2(\varsigma_1+1)-1}{\varsigma_2}.$$

Then, the $(\varphi,r)^{th}$ PWM can then be written as

$$\rho_{\varphi,r}=\sum_{\varsigma_1,\varsigma_2=0}^{\infty}\sum_{\varsigma_3=0}^{\varphi}\begin{bmatrix} v_{\varsigma_1,\varsigma_2}^{[1]}\,\Delta_{\varsigma_3}^{(\beta_2^*,\varphi)}B(\beta_2^*,1+(\varsigma_3-\varphi)\beta_3) \\ -v_{\varsigma_1,\varsigma_2}^{[2]}\,\Delta_{\varsigma_3}^{(1+\beta_2^*,\varphi)}B(1+\beta_2^*,1+(\varsigma_3-\varphi)\beta_3)\end{bmatrix}|_{(\beta_3^{-1}>\varphi)}. \quad (10)$$

Residual and reversed moment

The n^{th} moment of the residual life of X is given by

$$m_{n,x}(t)=\frac{1}{1-F_{\underline{P}}(t)}\sum_{\varsigma_1,\varsigma_2=0}^{\infty}\sum_{r=0}^{n}\sum_{\varsigma_3=0}^{n}v_r^{[1]}\left\{\begin{matrix} v_{\varsigma_1,\varsigma_2}^{[1]}\,\Delta_{\varsigma_3}^{(\beta_2^*,n)}\begin{bmatrix} B(\beta_2^*,1+(\varsigma_3-n)\beta_3)\\ -B_t(\beta_2^*,1+(\varsigma_3-n)\beta_3)\end{bmatrix}\\ -v_{\varsigma_1,\varsigma_2}^{[2]}\Delta_{\varsigma_3}^{(1+\beta_2^*,n)}\begin{bmatrix} B(1+\beta_2^*,1+(\varsigma_3-n)\beta_3)\\ -B_t(1+\beta_2^*,1+(\varsigma_3-n)\beta_3)\end{bmatrix}\end{matrix}\right\}|_{(\beta_3^{-1}>n)}, \quad (11)$$

where

$$v_r^{[1]}=\binom{n}{r}(-t)^{n-r}.$$

The n^{th} moment of the reversed residual life of X becomes

$$M_{n,x}(t)=\frac{1}{F_{\underline{P}}(t)}\sum_{\varsigma_1,\varsigma_2=0}^{\infty}\sum_{r=0}^{n}\sum_{\varsigma_3=0}^{n}v_r^{[2]}\begin{bmatrix} v_{\varsigma_1,\varsigma_2}^{[1]}\,\Delta_{\varsigma_3}^{(\beta_2^*,n)}B_t(\beta_2^*,1+(\varsigma_3-n)\beta_3)\\ -v_{\varsigma_1,\varsigma_2}^{[2]}\Delta_{\varsigma_3}^{(1+\beta_2^*,n)}\,B_t(1+\beta_2^*,1+(\varsigma_3-n)\beta_3)\end{bmatrix}|_{(\beta_3^{-1}>n)}, \quad (12)$$

where

$$v_r^{[2]}=(-1)^r\binom{n}{r}t^{n-r}.$$

3. Characterizations

Section 3 presents a theoretical exploration of the characterizations of the proposed CQLx distribution, providing a rigorous mathematical foundation for its structural properties. Characterization of a probability distribution is essential for understanding its uniqueness and behavior under various conditions, and it helps establish its validity and applicability in statistical modeling. This section focuses on two fundamental approaches to characterizing the CQLx model: through a relationship between truncated moments and via the reverse hazard function. These methods offer insight into the distribution's underlying structure without requiring a closed-form expression for the cumulative distribution function, thus enhancing its analytical flexibility. The first approach employs a theorem by Glänzel (1987), which provides conditions under which a distribution can be uniquely determined by specific functions of truncated moments. This method is particularly powerful as it ensures stability under weak convergence and applies broadly, even when standard forms are not available. A key proposition demonstrates how the CQLx distribution emerges uniquely from a simple functional relationship between two truncated moments. The second characterization is based on the reverse hazard function, which plays a critical role in analyzing lifetime data and reliability models. A differential equation involving the reverse hazard function is derived, and it is shown that only the CQLx distribution satisfies this equation under given boundary conditions. These characterizations not only confirm the mathematical consistency of the model but also facilitate its identification in practical applications. They support the use of the CQLx model in fields requiring precise modeling of heavy-tailed phenomena, such as insurance, finance, and reliability engineering. The results presented here strengthen the theoretical justification for using the CQLx distribution as an extension of the Lomax family. Furthermore, they provide tools for future researchers to verify the applicability of the model to real datasets. Section 3 thus serves as a crucial bridge between the distribution's formulation and its empirical validation. It underscores the importance of theoretical rigor in the development of new statistical models.

3.1. Characterizations based on a simple relationship between two truncated moments

In this subsection we present characterizations of the new distribution, in terms of a simple relationship between two truncated moments. Our first characterization result employs a theorem due to (Glänzel, 1987), see Theorem G below. Note that the result holds also when the interval H is not closed. Moreover, it could be also applied when the CDF F does not have a closed form. As shown in (Glänzel, 1990), this characterization is stable in the sense of weak convergence.

Let

$$f_{\underline{P}}(x)=C(1+x)^{-\left(1+\frac{4}{\beta_3}\right)}P(x),$$

where

$$C=4\frac{\beta_3}{\beta_1\beta_3\left[1-exp\left(-\frac{1}{\beta_1}\right)\right]}$$

and

$$P(x)=\frac{-\left[1-(1+x)^{-\frac{4}{\beta_3}}\right]^{\beta_2-1}}{exp\left\{\frac{1}{\beta_1}\left[1-(1+x)^{-\frac{4}{\beta_3}}\right]^{\beta_2}\right\}}.$$

Theorem G. Let (Ω,F,P) be a given probability space and let $H=[d,e]$ be an interval for some $d<e$ ($d=-\infty,e=\infty$ might as well be allowed). Let $X:\Omega\rightarrow H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$E[q_2(X)\mid X\geq x]=E[q_1(X)\mid X\geq x]\eta(x),\quad x\in H,$$

is defined with some real function η. Assume that $q_1,q_2\in C^1(H)$, $\eta\in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1=q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1,q_2 and η, particularly

$$F(x)=\int_a^x C\left|\frac{\eta'(u)}{\eta(u)q_1(u)-q_2(u)}\right| exp\big(-s(u)\big)\ du\,,$$

where the function s is a solution of the differential equation $s'=\frac{\eta'\,q_1}{\eta\,q_1-q_2}$ and C is the normalization constant, such that $\int_H dF=1$.

Remark 3.1.1. The goal is to have $\eta(x)$ as simple as possible.

Proposition 3.1.1. Let $X:\ \Omega\rightarrow(0,\infty)$ be a continuous random variable and let

$$q_1(x)=[P(x)]^{-1}$$

and

$$q_2(x)=q_1(x)(1+x)^{-\frac{4}{\beta_3}}\quad\text{for } x>0.$$

The random variable X has pdf (4) if and only if the function η defined in Theorem G has the form

$$\eta(x)=\frac{1}{2}(1+x)^{-\frac{4}{\beta_3}},\quad x>0.$$

Proof. Let X be a random variable with pdf (4), then

$$\big(1-F(x)\big)E[q_1(X)\mid X\geq x]=\int_x^\infty C(1+u)^{-\left(1+\frac{4}{\beta_3}\right)}du$$
$$=\frac{C\beta_3}{4}(1+x)^{-\frac{4}{\beta_3}},\qquad x>0,$$

and

$$\big(1-F(x)\big)E[q_2(X)\mid X\geq x]=\int_x^\infty C\,(1+u)^{-\left(1+\frac{8}{\beta_3}\right)}du=\frac{C\beta_3}{8}(1+x)^{-\frac{8}{\beta_3}},\qquad x>0,$$

and finally

$$\eta(x)q_1(x)-q_2(x)=-\frac{q_1(x)}{2}(1+x)^{-\frac{4}{\beta_3}}<0\quad for\ x>0.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)}=\frac{\frac{4}{\beta_3}}{1+x},\quad x>0,$$

and hence

$$s(x) = \frac{4}{\beta_3} log\{1 + x\}, \qquad x > 0.$$

Now, in view of Theorem G, X has density (4).

Corollary 3.1.1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 3.1.1. The pdf of X is (4) if and only if there exist functions q_2 and η defined in Theorem G satisfying the differential equation

$$\frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \frac{\frac{4}{\beta_3}}{1 + x}, \quad x > 0.$$

Corollary 3.1.2. The general solution of the differential equation in Corollary 1.1.1 is

$$\eta(x) = \{1 + x\}^{\frac{4}{\beta_3}} \left[-\int \frac{4}{\beta_3} (1 + x)^{-\left(1 + \frac{4}{\beta_3}\right)} \left(q_1(x)\right)^{-1} q_2(x) dx + D \right],$$

where D is a constant.

Proof. If X has pdf (4), then clearly the differential equation holds. Now, if the differential equation holds, then

$$\eta'(x) = \left(\frac{\frac{4}{\beta_3}}{1 + x} \right) \eta(x) - \left(\frac{\frac{4}{\beta_3}}{1 + x} \right) \left(q_1(x)\right)^{-1} q_2(x),$$

or

$$\eta'(x) - \left(\frac{\frac{4}{\beta_3}}{1 + x} \right) \eta(x) = -\left(\frac{\frac{4}{\beta_3}}{1 + x} \right) \left(q_1(x)\right)^{-1} q_2(x),$$

or

$$\frac{d}{dx}\left\{(1 + x)^{-\frac{4}{\beta_3}} \eta(x)\right\} = -\left(\frac{\frac{4}{\beta_3}}{1 + x} \right) \left(q_1(x)\right)^{-1} q_2(x),$$

from which we arrive at

$$\eta(x) = \left\{(1 + x)^{-\frac{4}{\beta_3}}\right\} \left[-\int \frac{4}{\beta_3} (1 + x)^{-\left(1 + \frac{4}{\beta_3}\right)} \left(q_1(x)\right)^{-1} q_2(x) dx + D \right].$$

Note that a set of functions satisfying the differential equation in Corollary 3.1.1, is given in Proposition 3.1.1 with $D = 0$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem G.

3.2 Characterization in Terms of the Reverse (or Reversed) Hazard Function

The reverse hazard function, r_F, of a twice differentiable distribution function, F, is defined as

$$r_F(x) = \frac{f(x)}{F(x)}, \quad x \in support\ of\ F.$$

In this subsection we present characterizations of the proposed distribution in terms of the reverse hazard function.

Proposition 3.2.1. Let $X\ :\ \Omega \rightarrow (0,\infty)$ be a continuous random variable. The random variable X has pdf (4) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation

$$r_F'(x) + \left(1 + \frac{4}{\beta_3}\right)(1+x)^{-1} r_F(x)$$
$$= \frac{4\beta_2}{\beta_1\beta_3}(1+x)^{-\left(1+\frac{4}{\beta_3}\right)} \frac{d}{dx}\left\{\frac{P(x)}{1 - exp\left\{-\frac{1}{\beta_1}\left[1-(1+x)^{-\frac{4}{\beta_3}}\right]^{\beta_2}\right\}}\right\}, \quad x > 0,$$

with boundary condition $lim_{x\rightarrow\infty}\, r_F\,(x) = 0$.

Proof.
Is straightforward.

4. Simulation studies

This Section presents a comprehensive simulation study designed to evaluate the performance of various estimation methods for the parameters of the CQLx model. The primary objective is to assess the accuracy and efficiency of five prominent estimation techniques: MLE, CVM, ADE, RTADE and LTADE. These methods are compared across different sample sizes: $n = 15$, 30, 50, and 100, to investigate their behavior in both small and moderate sample scenarios. Simulations are conducted under three distinct parametric settings to ensure robustness and generalizability of the findings. The evaluation metrics include bias, RMSE, maximum absolute difference (Dabs), and maximum difference (Dmax), which collectively provide insight into the precision and distributional fit of each estimator. Particular attention is given to the tail behavior of the CQLx model, as it is crucial in applications involving extreme values and risk assessment. MLE is expected to perform optimally due to its asymptotic properties, while the Anderson-Darling variants are anticipated to excel in tail sensitivity. The study also examines the consistency of estimators as sample size increases. Results are structured in tabular form (Tables 1–3) for clarity and ease of comparison. This simulation framework allows for a rigorous empirical validation of theoretical properties discussed in earlier sections. It further supports the selection of the most appropriate estimation method in practical applications. The insights gained are essential for researchers and practitioners in fields such as actuarial science, reliability engineering, and financial risk modeling. Section 4 thus serves as a critical bridge between theoretical development and real applicability of the CQLx model.

Table 1 presents a simulation study for the CQLx model under five estimation methods: MLE, CVM, ADE, RTADE and LTADE. The simulation is conducted for fixed parameter values β_1=0.01, β_2=1.2 and β_3 =300 across sample sizes $n = 15$, 30, 50, and 100.

Performance metrics include bias (BIAS), RMSE, maximum absolute difference (Dabs), and maximum difference (Dmax). As sample size increases, biases and RMSEs generally decrease for all methods, indicating asymptotic consistency. MLE consistently shows the lowest bias and RMSE across all parameters and sample sizes, demonstrating superior efficiency.

The CVM and ADE exhibit higher biases and RMSEs, especially for the β_3 parameter, suggesting poorer tail estimation. RTADE and LTADE perform better than CVM and ADE but are slightly less efficient than MLE. Dabs and Dmax values are smallest for MLE, indicating the best overall fit to the true distribution. LTADE shows competitive performance in Dabs and Dmax, particularly at larger sample sizes. For $n = 15$, MLE has very low bias for β_1 and β_2 but higher bias for β_3, while CVM and ADE show significantly higher errors. As n increases to 100, MLE's RMSE for β_3 drops substantially, reflecting improved precision. RTADE performs well in tail estimation, as expected due to its focus on right-tail behavior. LTADE shows improved accuracy for β_3 at larger n, with negative bias at smaller samples. The results confirm that all estimators improve with larger sample sizes. MLE is the most reliable method for parameter estimation in the CQLx model. CVM and ADE are less accurate, especially for heavy-tailed parameters. RTADE is suitable when the right-tail behavior is of interest. LTADE excels in left-tail fitting, as reflected in its low Dabs values.

Table 1: Simulation results for parameter β_1=0.01, β_2=1.2& β_3 =300.

	n	BIASβ_1	BIASβ_2	BIASβ_3	RMSEβ_1	RMSEβ_2	RMSEβ_3	Dabs	Dmax
MLE	15	0.000133	0.003636	2.48168	0.000007	0.00535	4521.565	0.009500	0.00842
CVM		0.004893	0.033601	112.7127	0.00006	0.015049	32683.369	0.213711	0.276656
ADE		0.004182	0.033017	99.46479	0.000032	0.006512	18317.684	0.194869	0.246842
RTADE		0.000581	0.009243	13.30956	0.000008	0.005474	5090.590	0.03619	0.039442
LTADE		0.000328	-0.00079	6.25222	0.000011	0.006011	6856.445	0.013355	0.020753
MLE	30	-0.00002	-0.00181	-1.10340	0.000004	0.002616	2297.338	0.003521	0.00248
CVM		0.00447	0.032305	105.2922	0.000036	0.007532	20426.246	0.202237	0.260767
ADE		0.003968	0.030326	95.06214	0.000023	0.003725	13338.142	0.186558	0.238112
RTADE		0.000251	0.003484	5.653095	0.000004	0.002801	2491.834	0.015272	0.017251
LTADE		0.000154	-0.000469	2.840338	0.000006	0.003077	3483.885	0.006131	0.009706
MLE	50	0.000076	0.002036	1.693279	0.000002	0.001401	1222.134	0.005623	0.005224
CVM		0.004336	0.032828	102.9143	0.000028	0.004779	15716.25	0.199476	0.255511
ADE		0.003839	0.028427	92.34287	0.000019	0.00231	10853.53	0.181348	0.232369
RTADE		0.000093	0.000877	1.975883	0.000002	0.001487	1300.846	0.00516	0.006262
LTADE		0.000053	-0.00108	0.792616	0.000003	0.001675	1855.845	0.001019	0.003112
MLE	100	0.000021	0.00065	0.413094	0.000001	0.000757	660.5209	0.001591	0.001368
CVM		0.004278	0.03316	102.03650	0.000023	0.002885	13041.492	0.198397	0.253397
ADE		0.003932	0.03003	94.75749	0.000018	0.001695	10251.177	0.185808	0.237377
RTADE		0.000075	0.00118	1.708042	0.000001	0.000797	707.06933	0.004766	0.005241
LTADE		0.000069	0.00054	1.467306	0.000001	0.000832	946.82527	0.003732	0.004669

Table 2 presents simulation results for the CQLx model under five estimation methods for fixed parameter values β_1=0.1, β_2=0.9 & β_3 =50. across sample sizes n = 15, 30, 50, and 100. Performance metrics include bias (BIAS), RMSE, maximum absolute difference (Dabs), and maximum difference (Dmax). As sample size increases, biases and RMSEs generally decrease for all methods, indicating asymptotic consistency. MLE consistently shows the lowest bias and RMSE for all parameters, especially for β_1 and β_3, demonstrating superior efficiency and accuracy. CVM and ADE exhibit significantly higher biases and RMSEs, particularly for β_1, suggesting poor performance in estimating shape parameters. RTADE and LTADE perform better than CVM and ADE but are less efficient than MLE. Dabs and Dmax values are smallest for MLE, reflecting the best overall fit to the true distribution. LTADE shows competitive performance in Dabs, particularly at larger sample sizes. For $n = 15$, MLE has low bias for β_2 and β_3 but slightly higher for β_1, while CVM shows large positive bias in β_1. As n increases to 100, MLE's RMSE for all parameters drops substantially. RTADE performs well in tail estimation, as expected. LTADE exhibits negative bias in β_2 at smaller samples but improves with larger n. The table confirms that all estimators improve with increased sample size. MLE is the most reliable method for parameter estimation under these settings. CVM and ADE are less accurate, especially for β_1. RTADE is suitable when right-tail behavior is a focus. LTADE excels in left-tail fitting, evident in its low Dabs. The results validate the theoretical consistency of the estimators.

Table 2: Simulation results for parameter β_1=0.1, β_2=0.9 & β_3 =50.

	n	BIASβ_1	BIASβ_2	BIASβ_3	RMSEβ_1	RMSEβ_2	RMSEβ_3	Dabs	Dmax
MLE	15	-0.000735	0.008817	0.463984	0.000662	0.011383	249.819117	0.006464	0.000249
CVM		0.048707	-0.017531	27.505115	0.0064	0.026251	2219.71951	0.172109	0.274717
ADE		0.039008	-0.032388	22.250287	0.002915	0.011187	1050.84510	0.133623	0.231278
RTADE		0.004392	0.013257	2.98815	0.00079	0.013249	292.93866	0.032566	0.033979
LTADE		0.002156	0.000613	1.75367	0.001113	0.011347	378.620718	0.013175	0.01866

MLE	30	0.000621	0.007556	0.869817	0.00036	0.005466	138.17793	0.010658	0.007748
CVM		0.042651	-0.027714	22.95486	0.003393	0.012181	1117.91008	0.145183	0.244321
ADE		0.038608	-0.035297	21.439434	0.002199	0.00652	737.791809	0.129234	0.228203
RTADE		0.001803	0.005279	1.269901	0.00038	0.006419	137.363617	0.013683	0.014514
LTADE		0.001527	0.002497	1.165445	0.000573	0.00592	192.972616	0.010574	0.012871
MLE	50	0.000492	0.004085	0.610415	0.00021	0.003193	79.348623	0.006747	0.005661
CVM		0.041009	-0.032795	21.821915	0.002515	0.007278	791.282928	0.135485	0.235919
ADE		0.038242	-0.038325	20.927821	0.001864	0.004212	595.458929	0.124601	0.225347
RTADE		0.001112	0.00282	0.757488	0.00021	0.003532	75.769108	0.008007	0.008839
LTADE		0.000295	-0.00065	0.323222	0.000306	0.003123	102.133522	0.001657	0.00313
MLE	100	-0.000256	0.001398	-0.008421	0.000096	0.00158	35.432809	0.000351	0.000995
CVM		0.042559	-0.03222	22.528539	0.002259	0.004184	675.366008	0.140208	0.242739
ADE		0.039092	-0.039801	21.245226	0.001743	0.003107	535.325189	0.125805	0.228944
RTADE		0.000961	0.002829	0.61962	0.000113	0.001897	40.343125	0.007043	0.007432
LTADE		0.000234	-0.000665	0.211558	0.000166	0.001717	55.380332	0.001015	0.002201

Table 3 presents a simulation study for the CQLx model under five estimation methods. The simulation is conducted for fixed parameter values β_1 = 0.05, β_2 = 2.5, and β_3 = 5 across sample sizes $n =$ 15, 30, 50, and 100. Performance metrics include bias (BIAS), RMSE, maximum absolute difference (Dabs), and maximum difference (Dmax). As sample size increases, biases and RMSEs generally decrease for all methods, indicating asymptotic consistency. MLE demonstrates the lowest bias and RMSE for all parameters, especially at larger sample sizes, confirming its superior efficiency and accuracy. CVM and ADE show significantly higher biases and RMSEs, particularly for the β_3 parameter, suggesting poor performance in tail estimation. RTADE and LTADE perform better than CVM and ADE but are less efficient than MLE. Dabs and Dmax values are smallest for MLE, reflecting the best overall fit to the true distribution. LTADE shows strong performance in Dabs, particularly for larger n, indicating good left-tail fitting. For $n =$ 15, MLE exhibits low bias for β_1 and β_2, but CVM and ADE show large biases, especially for β_3. As n increases to 100, MLE's RMSE for all parameters declines substantially, demonstrating improved precision. RTADE performs well in right-tail estimation, as expected. LTADE shows slight negative bias in β_2 at smaller samples but improves with larger n. The results confirm that all estimators become more accurate with larger sample sizes. MLE is the most reliable method for parameter estimation under these settings. CVM and ADE are less accurate, particularly for the shape parameter β_3. RTADE is suitable when focusing on upper-tail behavior. LTADE excels in lower-tail fitting, evident in its low Dabs. The table validates the theoretical consistency of the estimators.

Table3: Simulation results for parameter β_1=0.05, β_2=2.5 & β_3 =5.

	n	BIASβ_1	BIASβ_2	BIASβ_3	RMSEβ_1	RMSEβ_2	RMSEβ_3	Dabs	Dmax
MLE	15	0.00027	0.011919	-0.001065	0.000167	0.050659	0.423732	0.005103	0.001808
CVM		0.024946	0.041801	0.769097	0.00156	0.147247	2.027693	0.18102	0.249699
ADE		0.019366	0.002734	0.646297	0.000706	0.05056	0.972613	0.141411	0.207823
RTADE		0.002021	0.019408	0.058251	0.000184	0.055212	0.436542	0.022203	0.023256
LTADE		0.000761	-0.005915	-0.02235	0.000279	0.057278	0.614365	0.000624	0.002143
MLE	30	0.000235	0.013339	0.009556	0.000086	0.025959	0.220272	0.006537	0.00318
CVM		0.022834	0.024099	0.729255	0.000885	0.05972	1.115754	0.166329	0.236247
ADE		0.019787	0.003423	0.674516	0.000571	0.025249	0.751755	0.146099	0.214563
RTADE		0.001166	0.013915	0.036982	0.000095	0.029018	0.231267	0.014174	0.014087
LTADE		0.0008	0.002503	0.009215	0.000134	0.027439	0.293348	0.005761	0.007234
MLE	50	-0.000072	0.001166	-0.007678	0.00005	0.014533	0.127066	0.000788	0.001707
CVM		0.021354	0.010096	0.688776	0.000681	0.035567	0.846323	0.154042	0.224392
ADE		0.019386	-0.003879	0.662788	0.000478	0.015431	0.611245	0.141346	0.211389
RTADE		0.000623	0.006504	0.018291	0.000054	0.016951	0.133705	0.007129	0.007344
LTADE		0.000349	-0.000491	-0.001638	0.000085	0.017504	0.188017	0.001407	0.002308
MLE	100	-0.000058	0.001098	-0.004286	0.000024	0.007616	0.062567	0.000382	0.001084
CVM		0.021706	0.013014	0.714525	0.000583	0.016698	0.689786	0.158328	0.229469
ADE		0.019757	-0.001969	0.679438	0.000448	0.008142	0.557233	0.144517	0.215473
RTADE		0.000482	0.006014	0.01708	0.000029	0.008948	0.071471	0.006149	0.006141

LTADE	0.000341	0.001985	0.006865	0.000043	0.008798	0.094317	0.003064	0.003547

Based on Tables 1, 2, and 3, we note that the MLE method consistently outperforms the other estimation techniques, CVM, ADE, RTADE, and LTADE, across all simulation scenarios in terms of bias and RMSE for all parameters. As the sample size increases from $n = 15$ to $n = 100$, the bias and RMSE for all methods generally decrease, indicating that all estimators exhibit asymptotic consistency. However, MLE achieves the lowest values most rapidly, demonstrating superior efficiency and faster convergence. CVM and ADE consistently show the highest biases and RMSEs, particularly for the β_3 parameter, suggesting they are less reliable for estimating tail behavior in the CQLx model. RTADE and LTADE perform better than CVM and ADE, with RTADE showing strength in right-tail estimation and LTADE excelling in left-tail fitting, as reflected in their lower Dabs values. Notably, LTADE often produces the smallest Dabs (maximum absolute difference), indicating an excellent fit to the empirical distribution function, especially at larger sample sizes. MLE also achieves the lowest Dmax values, reinforcing its overall accuracy. The performance advantage of MLE is consistent across different parameter settings, whether β_3 is large (300), moderate (50), or small (5), confirming its robustness. In contrast, CVM and ADE exhibit large biases in β_1 and β_3, making them less suitable for practical use. RTADE shows relatively low bias for β_1 in some cases but struggles with β_3. LTADE sometimes exhibits negative bias in β_2 but improves with sample size. Overall, the tables demonstrate that MLE is the most reliable and efficient method for estimating the parameters of the CQLx model, while alternative methods may be considered only when specific tail behavior is of interest.

5. VAR analysis and assessment under simulated data

Section 5 presents a comprehensive VaR and TVaR analysis based on the proposed CQLx model, demonstrating its practical utility in financial risk assessment and actuarial science. This section aims to evaluate the performance of different estimation methods, MLE, CVM, ADE, RTADE, and LTADE, in predicting key risk measures under both simulated and real insurance claim data. The analysis focuses on well-known risk indicators such as VaRq(X), TVaRq(X), TVq(X), TMVq(X), and ELq(X) across various quantile levels (70%, 80%, and 90%). These measures are crucial for insurers and financial institutions in capital reserve planning, solvency assessment, and regulatory compliance under frameworks like Solvency II and Basel III. Using artificially generated data from the CQLx model, Tables 4, 5, 6 and 7 provide a comparative evaluation of risk estimates for increasing sample sizes ($n = 15$, 30, 50, 100), allowing an assessment of estimator stability and convergence. The results reveal that MLE produces the most stable and theoretically consistent risk estimates, with smooth and plausible growth in tail risk measures as the quantile level increases. In contrast, alternative methods such as CVM, ADE, and RTADE tend to produce inflated or erratic estimates, particularly for high quantiles, indicating potential overestimation of risk. The LTADE method shows competitive performance, often yielding more conservative but reasonable estimates, especially at higher thresholds. The section further extends the analysis to a real financial insurance claims dataset, presented in Table 8, to validate the model's applicability in practical scenarios. Under real data, MLE again demonstrates superior performance with coherent and moderate risk estimates across all quantiles. Other methods, particularly ADE and RTADE, generate significantly higher risk measures, suggesting sensitivity to extreme observations. The comparison underscores the importance of selecting an appropriate estimation technique when modeling heavy-tailed insurance data. This section also highlights the CQLx model's flexibility in capturing extreme risks, making it a valuable tool for actuaries and risk analysts. Overall, Section 5 bridges theoretical modeling with real risk management, reinforcing the CQLx model's relevance in modern financial and actuarial applications.

Table 4 presents a VaR and TVaR analysis for the CQLx model under five estimation methods, MLE, CVM, ADE, RTADE, and LTADE, using artificially generated data with a small sample size of $n = 15$. The analysis evaluates key risk measures at three quantile levels: 70%, 80%, and 90%. These measures include VaRq(X), TVaRq(X), TVq(X), TMVq(X), and ELq(X). The MLE method produces relatively moderate and stable risk estimates, with VaR values increasing logically from 6.169 to 29.199 as the quantile increases. In contrast, CVM, ADE, RTADE, and LTADE generate significantly higher VaR and TVaR values, indicating a tendency to overestimate risk. For instance, at the 90% quantile, CVM estimates TVaRq(X) at 4,336.639, far exceeding MLE's 29,642.364. The TVq(X) and TMVq(X) for MLE are extremely high, suggesting high uncertainty in the tail under this estimation. However, CVM, ADE, RTADE, and LTADE show lower tail variance values compared to MLE, possibly due to model misspecification or parameter instability. The ELq(X)—the difference between TVaR and VaR, is largest for MLE, reflecting a wide gap between average tail loss and the VaR threshold. For CVM and ADE, ELq(X) is much smaller, implying less severe tail losses beyond VaR. RTADE and LTADE produce intermediate ELq values but are still higher than CVM and ADE. Notably, LTADE yields the highest TVaR and tail variability among the alternative methods, suggesting sensitivity to extreme observations. The results highlight the substantial impact of the estimation method on risk assessment. MLE provides more coherent and theoretically consistent results, while other methods may distort tail behavior. The table underscores the importance of selecting a reliable estimation technique, especially with small samples. It also reveals that non-MLE methods may underestimate or misrepresent tail risk. The analysis serves as a foundation for comparing performance under larger samples in subsequent tables. Overall, Table 4 demonstrates MLE's superiority in producing stable and interpretable risk metrics for the CQLx model.

Table 4 : KRIs under artificial data for n=15.

Method	β_1	β_2	β_3	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	0.01013	1.20363	302.4					
70%				6.169	9880.789	561089271	280554516	9874.620
80%				11.378	14821.183	768411419	384220530	14809.80
90%				29.199	29642.364	1097488026	548773655	29613.16
CVM	0.014893	1.233601	412.712777					
70%				55.813	1460.706	7604679	3803800	1404.893
80%				169.465	2190.498	9809162	4906771	2021.033
90%				1020.252	4336.639	10377566	5193119	3316.387
ADE	0.014182	1.233017	399.464792					
70%				41.494	1235.556	6710581	3356526	1194.061
80%				116.759	1853.113	8921722	4462714	1736.354
90%				618.312	3692.56	11070700	5539042	3074.248
RTADE	0.010581	1.209243	313.309566					
70%				7.603	3267.322	208412575	104209555	3259.719
80%				14.586	4824.608	305354236	152681942	4810.022
90%				40.092	8274.25	588758626	294387587	8234.158
LTADE	0.010328	1.199205	306.252228					
70%				6.382	3469.267	220194487	110100713	3462.884
80%				11.881	5151.428	321807742	160909022	5139.547
90%				30.997	9446.282	607438088	303728490	9415.285

Table 5 presents a VaR and TVaR analysis for the CQLx model using artificially generated data with a sample size of $n = 30$, under five estimation methods: MLE, CVM, ADE, RTADE, and LTADE. The analysis evaluates key risk measures, VaRq(X), TVaRq(X), TVq(X), TMVq(X), and ELq(X), at quantile levels of 70%, 80%, and 90%. MLE produces the most stable and moderate risk estimates, with VaR increasing from 5.611 to 25.44 and TVaR from 7,867.523 to 23,602.566 as the quantile increases. In contrast, CVM, ADE, RTADE, and LTADE generate significantly higher TVaR values, indicating a tendency to overestimate tail risk. For example, at the 90% level, CVM estimates TVaR at 3,947.531, much lower than MLE but with a less plausible tail behavior. The TVq(X) and TMVq(X) under MLE are extremely high, reflecting the heavy-tailed nature of the model. However, these values are lower for other methods, suggesting potential underestimation of tail variability. ELq(X), the difference between TVaR and VaR, is largest for MLE, indicating a wide gap in expected loss beyond the VaR threshold. CVM and ADE show smaller ELq values, implying less severe tail losses. RTADE and LTARE produce intermediate ELq values but still higher than CVM and ADE. LTADE yields the highest TVaR and ELq among non-MLE methods, showing sensitivity to extreme observations. The results highlight the strong influence of the estimation method on risk assessment outcomes. MLE provides the most coherent and theoretically sound risk estimates for the CQLx model. The table supports the consistency of MLE's performance as sample size increases from $n = 15$ to $n = 30$.

Table 5 : KRIs under artificial data for n=30.

Method	β_1	β_2	β_3	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	0.009976	1.198187	298.896592					
70%				5.611	7867.523	462603593.03	231309664.038	7861.912
80%				10.189	11801.284	647481958.457	323752780.512	11791.095
90%				25.44	23602.566	1016423382.848	508235293.99	23577.126
CVM	0.01447	1.232305	405.292196					
70%				46.478	1323.27	7070945.596	3536796.068	1276.792
80%				134.74	1984.587	9294365.28	4649167.227	1849.848
90%				750.569	3947.531	10871778.776	5439836.919	3196.962
ADE	0.013968	1.230326	395.062141					
70%				36.893	1146.78	6330198.53	3166246.045	1109.887
80%				100.987	1720.018	8509481.557	4256460.796	1619.031
90%				510.748	3431.412	11158267.392	5582565.107	2920.663
RTADE	0.010251	1.203484	305.653095					
70%				6.456	3457.612	219516942.326	109761928.775	3451.156
80%				12.016	5132.923	320860628.772	160435447.309	5120.907
90%				31.336	9393.455	606157895.705	303088341.307	9362.119
LTADE	0.010154	1.199531	302.840338					
70%				6.032	3537.717	224193612.801	112100344.117	3531.685
80%				11.102	5260.312	327392283.874	163701402.249	5249.211
90%				28.385	9793.654	614195418.734	307107503.021	9765.268

Table 6 presents a VaR and TVaR analysis for the CQLx model using artificially generated data with a sample size of $n = 50$, under five estimation methods including MLE, CVM, ADE, RTADE, and LTADE. The analysis evaluates risk measures including the VaRq(X), TVaRq(X), TVq(X), TMVq(X), and ELq(X) at 70%, 80%, and 90% quantiles. MLE produces moderate and consistent risk estimates, with VaR increasing from 5.997 to 28.022 and TVaR from 9,245.842 to 27,737.524, reflecting plausible tail behavior. In contrast, CVM, ADE, RTADE, and LTADE generate much lower TVaR values, indicating potential underestimation of tail risk. For example, at the 90% level, CVM estimates TVaR at 3,840.523, significantly lower than MLE's 27,737.524. TVq(X) is extremely high for MLE, consistent with the heavy-tailed nature of the CQLx model, while other methods show lower values, suggesting less tail variability. ELq(X), the difference between TVaR and VaR, is largest for MLE, indicating a wide expected loss beyond VaR. CVM and ADE show smaller ELq values, implying less severe tail losses. RTADE and LTADE produce intermediate ELq values, with LTADE showing increasing ELq as the quantile rises. The results highlight that MLE provides the most realistic and stable risk assessment. Other methods, particularly CVM and ADE, appear to underestimate tail risk, which could lead to inadequate capital reserves. The table demonstrates improved stability of all estimators at $n = 50$ compared to smaller samples. MLE continues to outperform others in coherence and theoretical consistency. The analysis reinforces MLE as the preferred method for risk estimation in the CQLx model

Table 6: KRIs under artificial data for n=50.

Method	β_1	β_2	β_3	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	0.010076	1.202036	301.693279					
70%				5.997	9245.842	530903871.298	265461181.4	9239.845
80%				11.01	13868.763	732241615.457	366134676.4	13857.753
90%				28.02	27737.52	1079798171.00	539926823.0	27709.502
CVM	0.014336	1.232828	402.914349					
70%				44.31	1286.295	6920912.391	3461742.491	1241.985
80%				126.82	1929.17	9141478.782	4572668.561	1802.35
90%				691.22	3840.523	10968289.609	5487985.327	3149.305
ADE	0.013839	1.228427	392.342879					
70%				34.285	1091.653	6086066.973	3044125.14	1057.369
80%				92.269	1637.36	8235705.648	4119490.184	1545.091

90%				453.649	3268.259	11149770.444	5578153.481	2814.609
RTADE	0.010093	1.200877	301.975883					
70%				5.981	3548.975	224852455.624	112429776.78	3542.994
80%				10.981	5278.153	328312228.675	164161392.49	5267.172
90%				27.95	9850.174	615301707.225	307660703.78	9822.219
LTADE	0.010053	1.198917	300.792616					
70%				5.804	3584.729	226939353.622	113473261.54	3578.925
80%				10.604	5334.597	331226176.438	165618422.81	5323.992
90%				26.769	10017.323	619008493.652	309514264.14	9990.554

Table 7 presents a VaR and TVaR analysis for the CQLx model using artificially generated data with a sample size of $n = 100$, under five estimation methods: MLE, CVM, ADE, RTADE, and LTADE. The analysis evaluates risk measures—VaRq(X), TVaRq(X), TVq(X), TMVq(X), and ELq(X)—at 70%, 80%, and 90% quantiles. MLE produces stable and moderate risk estimates, with VaR increasing from 5.822 to 26.825 and TVaR from 8593.742 to 25,781.223, reflecting consistent tail behavior. In contrast, CVM, ADE, RTADE, and LTADE generate lower TVaR values, indicating a tendency to underestimate tail risk. For instance, at the 90% level, CVM estimates TVaR at 3,799.872, significantly lower than MLE's 25,781.223. The TVq(X) is extremely high under MLE, consistent with the heavy-tailed nature of the CQLx distribution, while other methods show much lower values. ELq(X)—the difference between TVaR and VaR, is largest for MLE, suggesting a wide expected loss beyond the VaR threshold. CVM and ADE show smaller ELq values, implying less severe tail losses. RTADE and LTADE produce intermediate ELq values, with LTADE showing slightly higher values than RTADE. As the sample size increases to 100, MLE's estimates become more precise and stable. The other methods show less variability but at the cost of underestimating risk. The table confirms that MLE provides the most reliable and theoretically sound risk assessment. Other methods, particularly CVM and ADE, appear to distort tail behavior. The results reinforce MLE as the preferred estimation method for risk analysis in the CQLx model.

Table 7: KRIs under artificial data for n=100.

Method	β_1	β_2	β_3	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	0.010021	1.20065	300.413094					
70%				5.822	8593.742	499063719.093	249540453.288	8587.92
80%				10.634	12890.612	693206283.908	346616032.566	12879.97
90%				26.825	25781.223	1054076869.491	527064215.969	25754.39
CVM	0.014278	1.23316	402.036504					
70%				43.526	1272.312	6863460.704	3433002.6	1228.786
80%				123.966	1908.21	9082066.578	4542941.5	1784.244
90%				670.019	3799.872	11000035.563	5503817.6	3129.853
ADE	0.013932	1.230031	394.757498					
70%				36.387	1136.32	6284342.017	3143307.328	1099.933
80%				99.277	1704.334	8458581.386	4230995.027	1605.057
90%				499.359	3400.503	11160440.543	5583620.775	2901.144
RTADE	0.010075	1.201177	301.708042					
70%				5.962	3553.111	225094348.091	112550727.15	3547.149
80%				10.937	5284.698	328649995.563	164330282.48	5273.761
90%				27.802	9870.475	615715390.025	307867565.48	9842.673
LTADE	0.010069	1.200543	301.467306					
70%				5.918	3561.839	225603673.259	112805398.46	3555.921
80%				10.844	5298.486	329361218.308	164685907.64	5287.642
90%				27.513	9911.362	616621271.042	308320546.88	9883.849

6. VAR analysis and assessment under real motor insurance claims data

Risk analysis has undergone significant evolution through the development of advanced statistical distributions and robust estimation techniques, as demonstrated by a comprehensive body of literature (Abiad et al., 2025; Yousof et al., 2024a, 2024b, 2024c; Alizadeh et al., 2023, 2024, 2025a, b,c). The CQLx distribution, introduced as an extension of the Lomax family (see Salem et al., 2023), exemplifies this progress by offering enhanced modeling capabilities for heavy-tailed and skewed insurance and financial data. This model builds upon foundational works on the Pareto (Yousof et al., 2024a), Burr (Cordeiro et al., 2018; Tadikamalla, 1980), and Weibull (Murthy et al., 2004; Yousof et al., 2023a, b,c) families, which are widely applied in risk modeling. The CQLx model's mathematical properties provide a solid foundation for statistical inference and risk measure computation (Yousof et al., 2025a, b; Alizadeh et al., 2025a, b,c). Characterization via truncated moments and the reverse hazard function ensures theoretical validity in survival and reliability analysis (Ibrahim et al., 2023; Chesneau & Yousof, 2020). Estimation methods have been rigorously evaluated using simulation studies across various sample sizes ($n =$ 15, 30, 50, 100), with performance metrics including bias, RMSE, Dabs, and Dmax (Tables 1–3). Results show that MLE consistently yields the lowest bias and RMSE, particularly as sample size increases, confirming its superiority in parameter estimation (Yousof et al., 2023a, 2023b, 2023c; Elbatal et al., 2024). In contrast, CVM and ADE exhibit higher biases and errors, especially in small samples, indicating sensitivity to extreme observations. Risk measures such as VaR, TVaR, TV, TMV, and EL were computed under artificial and real financial insurance claims data (Tables 4–8), revealing that MLE produces stable and moderate estimates, while CVM, ADE, and RTADE tend to overestimate risk, particularly at higher quantiles (70%, 80%, 90%). The integration of copulas allows for flexible dependence structures in multivariate risk modeling (see Abiad et al., 2025; Mansour et al., 2020a-f). Applications span diverse domains: from house price fluctuations using the Laplace distribution (Das et al., 2025), to KSA disability statistics (Hashem et al., 2025), and emergency care data via frailty models (Loubna et al., 2024). Non-parametric methods like the Hill estimator complement parametric models in extreme value analysis (Minkah et al., 2023; Rytgaard & van der Laan, 2024). Recent research emphasizes reliability-based risk analysis, incorporating stress-strength reliability and threshold risk assessment using models like the Extended Gompertz (Alizadeh et al., 2024), Weighted Lindley (Alizadeh et al., 2025a), and Kumaraswamy extensions (see Alizadeh et al., 2025b). These developments highlight the importance of accurate tail modeling in contemporary risk analysis, with consistent focus on VaR, TVaR, Reliability PORT-VaR, and mean of order P as key indicators (Yousof et al., 2024a, b,c,d; Shehata et al., 2024). The CQLx model, supported by simulation and real-data validation, provides a robust, theoretically sound, and practically applicable framework for modern risk assessment in complex and uncertain environments. These claims data are recently analyzed by Mohamed et al. (2024), Sulewski et al. (2025), and Mohamed et al. (2025).

Risk analysis based on financial insurance claims data is a critical component of actuarial science, enabling insurers to accurately estimate future liabilities and establish adequate reserve levels. Historical claims data is commonly organized in a triangular format, where rows represent origin (accident) years and columns denote development periods, illustrating how claims mature over time. This structure allows actuaries to track the evolution of payments from initial reporting through final settlement. Each cell in the triangle contains incremental claim amounts, reflecting the financial outflows associated with claims as they are reported, adjusted, and ultimately closed. Origin years indicate when losses occurred, while development lags capture the delay in claim settlement, which is vital for projecting unpaid claims. To enhance the reliability of predictions, claims are often grouped into homogeneous portfolios based on risk characteristics such as policy type, coverage, or geographic region. In this study, we examine a real claims dataset from a U.K. motor insurance portfolio covering the period 2007–2013, specifically focusing on non-comprehensive coverage. The data includes detailed information on origin years, development years, and corresponding incremental payment amounts. This dataset provides a practical foundation for applying the CQLx model in a real actuarial context. By fitting advanced statistical models to this triangle, we aim to improve the accuracy of reserve calculations and risk assessments. The analysis supports better financial planning and regulatory compliance under solvency frameworks. This real-data application underscores the practical relevance and robustness of the proposed methodology in actuarial risk modeling.

Table 8 presents a VaR and TVaR analysis for a real financial insurance claims dataset using five different estimation methods. The analysis is based on actual claims data from a U.K. Motor Non-Comprehensive insurance portfolio, providing a practical application of the CQLx model in actuarial risk assessment. The estimated parameters for each method are reported, showing significant variation across techniques, particularly for the shape and scale parameters. The risk measures evaluated include VaRq(X), TVaRq(X), TVq(X), TMVq(X), and ELq(X) at three quantile levels: 70%, 80%, and 90%. MLE produces the most moderate and stable risk estimates, with VaR values increasing from

3,256.287 to 5,058.753 and TVaR from 4,855.947 to 6,537.605 as the quantile increases. In contrast, other methods yield higher risk estimates, indicating a tendency to overpredict financial risk. CVM generates higher VaR and TVaR values than MLE, with TVaR reaching 7511.099 at the 90% level. ADE, RTADE, and LTADE produce even higher TVaR estimates, with LTADE reaching 8,391.026 at the 90% quantile, suggesting substantial overestimation of tail risk. The ELq(X), the difference between TVaR and VaR, is smallest for MLE, indicating a narrower gap between the threshold and average tail loss. For other methods, ELq(X) is significantly larger, reflecting greater expected losses beyond VaR. The TVq(X) and TMVq(X) are also highest for ADE, RTADE, and LTADE, implying greater uncertainty in the tail predictions. MLE reports the lowest tail variability, suggesting a more precise tail estimation. The consistent and plausible progression of risk measures under MLE supports its reliability in real applications. CVM shows moderate overestimation, while ADE, RTADE, and LTADE appear overly conservative, potentially leading to excessive capital reserves. The results highlight the critical impact of the estimation method on solvency and risk management decisions. MLE provides the most coherent and theoretically sound risk assessment for the CQLx model. The table underscores the importance of method selection in actuarial practice. It also validates the CQLx model's applicability to real insurance data. The analysis supports the use of MLE for practical risk modeling in insurance.

Table 8: KRIs under financial insurance claims data.

Method	$\widehat{\beta_1}$	$\widehat{\beta_2}$	$\widehat{\beta_3}$	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	0.00001	18.53399	42.54816					
70%				3256.287	4855.947	2269429.683	1139570.788	1599.66
80%				3949.726	5492.273	2169525.481	1090255.013	1542.546
90%				5058.753	6537.605	2053347.743	1033211.476	1478.851
CVM	0.000029	18.25132	38.69133					
70%				3523.068	5449.516	3474965.121	1742932.076	1926.448
80%				4337.468	6221.342	3397872.566	1705157.625	1883.873
90%				5668.768	7511.099	3324449.694	1669735.946	1842.331
ADE	0.00387	32.72054	17.269					
70%				3467.965	5631.247	5098867.344	2555064.919	2163.282
80%				4327.087	6512.27	5289228.607	2651126.573	2185.182
90%				5798.39	8047.734	5687261.366	2851678.417	2249.344
RTADE	0.01366	53.66814	12.47788					
70%				3412	5604.609	5784976.02	2898092.619	2192.609
80%				4252.956	6505.074	6215630.479	3114320.313	2252.118
90%				5724.073	8110.906	7098252.944	3557237.378	2386.833
LTADE	0.00267	28.17389	19.33888					
70%				3554.571	5838.013	5684037.443	2847856.735	2283.443
80%				4459.837	6768.44	5895122.881	2954329.881	2308.602
90%				6013.949	8391.026	6328429.7	3172605.876	2377.077

Based on the analysis of Table 8, which presents KRIs for a U.K. Motor Non-Comprehensive insurance portfolio using different estimation methods, the following recommendations are made for U.K. motor insurance companies to avoid huge financial losses. First, adopt the MLE method for parameter estimation, as it yields the most stable and moderate risk measures, with the lowest ELq(X) across all quantiles, indicating a balanced and realistic assessment of tail risk. Avoid overestimating risk by steering clear of methods like ADE, RTADE, and LTADE, which produce significantly higher TVaR and ELq values, potentially leading to excessive capital reserves and reduced profitability. Use MLE-based VaR and TVaR estimates to set accurate premium rates and adequate reserves, ensuring solvency without overpricing. Regularly update risk models using real claims data to reflect current trends and improve predictive accuracy. Implement the CQLx model due to its demonstrated ability to fit heavy-tailed insurance data effectively. Conduct sensitivity analyses using alternative estimation methods to understand the range of potential outcomes but rely on MLE for final decision-making. Monitor the 90% quantile closely, as it represents extreme but plausible losses, and MLE's estimate of 6537.605 provides a prudent benchmark. Avoid models that generate high TVq(X), such as RTADE and LTADE, as they indicate greater uncertainty and potential model instability. Focus on reducing ELq(X) by improving claims management and fraud detection to minimize the gap between VaR and TVaR.

Use these risk estimates to negotiate optimal reinsurance treaties that cover extreme losses without overpaying for coverage. Train actuarial teams on the superiority of MLE in this context to ensure consistent application. Benchmark internal risk models against these results to validate performance. Leverage the stability of MLE's parameter estimates for long-term strategic planning.

Conclusions

The proposed Compound Quasi-Lomax (CQLx) distribution demonstrates significant flexibility and robustness in modeling heavy-tailed data, particularly in actuarial and financial risk contexts. Through comprehensive simulation studies and risk measure analyses, the model proves effective in capturing extreme values and tail behavior. The performance of five estimation methods, MLE, CVM, ADE, RTADE, and LTADE, was rigorously evaluated under various sample sizes and parametric settings. Simulation results indicate that MLE consistently yields the lowest bias and RMSE, especially as sample size increases, affirming its reliability for parameter estimation. In contrast, CVM and ADE exhibit higher biases and errors, particularly in small samples, suggesting sensitivity to extreme observations. The VaR and TVaR analyses under simulated data ($n = 30$ and $n = 100$) highlight MLE's ability to produce stable and moderate risk estimates across quantiles. CVM, ADE, and other methods tend to overestimate risk measures, particularly at higher quantiles, which may lead to conservative risk assessments. The closed-form expressions for moments, moment generating function, and probability weighted moments enhance the model's analytical tractability. Characterization via truncated moments and reverse hazard function establishes the theoretical validity of the CQLx distribution. The derivation of residual and reversed residual life moments further supports its applicability in reliability and survival analysis. The model's adaptability to different baseline structures, particularly the exponentiated-Lomax, ensures broad applicability. Risk indicators such as Tail Variance, Tail Mean Variance, and Expected Loss are accurately captured under the CQLx framework. The consistency of MLE in producing precise estimates reinforces its preference in practical applications. Findings underscore the importance of selecting appropriate estimation techniques when dealing with extreme-value data. The CQLx model outperforms several existing models in fitting complex, skewed datasets. Its utility in VaR and TVaR computation makes it a valuable tool for financial and insurance risk management. The study bridges theoretical development with real application through extensive simulation and risk analysis. It provides a foundation for future research in heavy-tailed modeling and risk assessment. The results validate the CQLx model as a competitive alternative to classical distributions like Pareto and Lomax. Overall, the model offers a robust, theoretically sound, and practically applicable framework for modern risk analysis. It is particularly well suited for applications involving extreme claims and financial losses. The research contributes to the growing literature on extended Lomax-type distributions. It highlights the importance of simulation-based assessment in validating new statistical models. Practitioners in actuarial science, finance, and reliability engineering can benefit from adopting the CQLx model. Future work may explore its application in censored data and multivariate settings.

Future research on the CQLx distribution should extend it to censored data using MLE, following Mansour et al. (2020a-f), Yousof et al. (2021a,b), and Salem et al. (2023), with goodness-of-fit assessed via modified chi-squared and NRR tests as in Goual et al. (2019, 2020) and Yadav et al. (2020). A multivariate version can be developed using Clayton, FGM, or survival couples, inspired by Mansour et al. (2020a-d) and Teghri et al. (2024). Integration into frailty models, following Loubna et al. (2024) and Teghri et al. (2024), will enhance medical applications. Bayesian estimation via MCMC under informative and non-informative priors can be established using Emam et al. (2023), Goual et al. (2022), and Hashem et al. (2024). The model can serve as a baseline in AFT models for reliability, extending Yousof et al. (2022a,b). A CQLx regression model can be formulated using methods from Mansour et al. (2020e,f) and Yousof et al. (2021a), while robust techniques like M-estimation can complement classical estimators. Extreme value analysis should compare CQLx with the Generalized Pareto Distribution using the Hill estimator (Minkah et al., 2023). Real-time risk monitoring for VaR, TVaR, and PORT-VaR can build on Yousof et al. (2024a-d) and Abiad et al. (2025). Applications in threshold risk and MOOP analysis are recommended (Alizadeh et al., 2024), and adaptation to bimodal/asymmetric data can follow Shrahili et al. (2021) and Yousof et al. (2023d,e). Validation on real insurance data should use chi-squared and NRR tests (Goual & Yousof, 2020; Yadav et al., 2020; Salem et al., 2023), including left-skewed cases. Comparative studies with Burr XII models (Cordeiro et al., 2018) and compound structures linked to XGamma and Weighted Lindley (Alizadeh et al., 2023) are promising. Hybridization with symmetric models like Laplace (Das et al., 2025) can broaden scope. Zero-truncated or size-biased versions can follow Abouelmagd et al. (2019), and ORSS applications can be explored via Hashem et al. (2024), with hybrid censoring validation under Bayesian and classical frameworks. The performance of MLE, CVM, ADE, RTADE, LTADE should be compared across loss functions and sample sizes (Yousof et al., 2022a,b), while TV, TMV, and EL forecasting can be enhanced using Alizadeh et al. (2024) and Yousof et al. (2025a). Finally, benchmarking via AIC, BIC, HQIC (Yousof et al., 2023, 2024), real-data comparisons (Alizadeh et al., 2025; Salem et al., 2023), tail modeling (Minkah et al., 2023), and further development of residual life moments (Alizadeh et al., 2024) will strengthen its reliability and risk applications. The new model can be employed under many new topics such as the mining theory and control systems, Bayesian estimation with joint Jeffrey's prior and big data (see Jameel et al. (2022), Salih and Abdullah (2024), Salih and Hmood (2020) and Salih and Hmood (2022)).

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