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Approximation Methods for the Bivariate Compound Zero-Truncated Poisson-Gamma Distribution

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Abstract

In certain situations, probability computations can become complex, particularly when dealing with compound distributions. This computational complexity can be simplified by using approximation techniques such as the "saddle-point" approximation. In this paper, the authors have proposed the bivariate compound zero-truncated Poisson-Gamma distribution. This distribution is obtained by compounding the zero-truncated Poisson distribution with independent Gamma variates. To demonstrate the effectiveness of the proposed approach, the authors have provided an illustrative example to showcase the approximate computation of the bivariate compound zero-truncated Poisson−Gamma distribution. Furthermore, an extensive simulation study has been conducted to evaluate the performance of the proposed saddle-point approximation. The results indicate that the proposed saddle-point approximation is an excellent approximation of the distribution function of the bivariate compound zero-truncated Poisson-Gamma distribution which validates the effectiveness of the proposed approach. The high accuracy of the saddle-point approximation method is demonstrated by comparisons among saddle-point approximations, normal approximations, and exact calculations.

Keywords: Saddle-point, Poisson Distribution, Gamma Distribution, Compound Distribution, Approximation Method.

Mathematical Subject Classification:

1. Introduction

Modelling complex phenomena often requires the use of compound probability distributions. One area where compound distributions have a crucial role is in actuarial problems, particularly in the modelling of the total amount of claims or losses. Unfortunately, there are very few compound distributions of the claim amount for which a closed-form approximation is available. Additionally, the distribution of the collective risk model does not have a closed form, posing a challenge for researchers and practitioners.

The compounding of probability distributions has been an area of interest for many authors. Several new distributions have been obtained by compounding a discrete distribution such as the Poisson distribution with a univariate distribution such as the Gamma distribution. The Poisson−Gamma compound distribution is one such example that has been widely used in various contexts. The distribution has been used, among others, by Christensen et al. (2003) to obtain data on efforts; by Christopher et al. (2011) to obtain multiple lesions per patient, recruitment for multicentre studies, insurance, pump failure, etc.; and by Aitken and Elton (1986) to study the effect of gamete concentration on sperm−oocyte fusion and for two-stage cluster sampling.

Furthermore, the Poisson−Gamma model has been used to model a variety of other phenomena, such as human capital distribution, mortality data, mall visit frequency, distribution of microorganisms in a food matrix, mine equipment injury rates, and recruitment in multicentre trials. In summary, the modelling of complex phenomena using compound probability distributions, particularly the Poisson−Gamma model, has been a subject of significant interest and has found numerous applications across various fields.

Numerous researchers have delved into the Poisson−Gamma model, with notable contributions from Buishand (1977), who focused on maximum likelihood estimates in exponential rainfall and Ozturk (1981), who explored related aspects. Revfeim (1982) introduced a moment estimate and a seasonality allowance within this model. Furthermore, the Poisson−Gamma model has been extended and generalized by various scholars such as Nahmias and Demmy (1982), Fukasawa and Basawa (2002), Christensen et al. (2003), Henderson and Shimakura (2003), Galue (2007), and Choo and Walker (2008). A recent study by Nascimento et al. (2023) investigated the compound truncated Poisson−Gamma distribution to understand multimodal SAR intensities. Abdelghani et al. (2021) proposed applications involving bivariate compound distributions based on Poisson maxima of Gamma variates. Ausaina et al. (2023) examined features of the Gamma zero-truncated Poisson distribution and studied the asymptotic properties of maximum likelihood estimators.

Often, a bivariate compound distribution is required when joint modelling of two phenomena is necessary. For example, in risk theory, a bivariate compound distribution might be required when comparing the total claims for two different portfolios of independent auto insurance policies at a given time. This type of problem is typically solved by assuming that the arrival of claims follows a Poisson distribution, and the amount of individual claims follows a Gamma distribution, resulting in an overall bivariate compound Poisson−Gamma distribution for the total claim amount. In insurance premium prediction problems, the total claim amount for a covered risk typically has a continuous distribution of positive values, with the possible exception of being exactly zero when the claim does not occur. The resulting distribution in this case is referred to as a bivariate compound zero-truncated Poisson−Gamma (BCZTPG) distribution.

This study focuses on modelling aggregate claim amounts commonly found in car insurance policies. The objective is to estimate the cumulative probability at a specific claim value. To achieve this, we have employed analytical approximation techniques, more specifically, the saddle-point approximation method. This method is renowned for its computational efficiency and ability to provide highly accurate tail probabilities. Saddle-point approximations are crucial in obtaining precise expressions for distribution functions that lack closed-form solutions. They are considered superior to other methods due to their robustness and efficiency. Furthermore, the simplicity of implementation and minimal computational requirements make this approach highly practical. One of the key advantages of saddle-point approximations is their ability to generate precise probabilities, particularly in the tails of distributions, even with limited data points or a single observation, distinguishing them from other asymptotic approximations. This method is particularly valuable for understanding complex and unknown distributional behaviours such as bivariate compound random variables. The versatility and effectiveness of saddle-point approximation methods have been demonstrated across a wide range of applications. Saddle-point approximations can also be a powerful tool for approximating the cumulative distribution function (CDF) of various statistical quantities. The key idea is to leverage the cumulant generating function (CGF) of the statistic of interest to obtain an accurate approximation of the CDF.

This paper explores several applications of the saddle-point approximation method: In Section 2, the authors present a modified version of Wang's saddle-point approximation that is tailored for zero-truncated Poisson sums. Section 3 focuses on the BCZTPG distribution. The authors derive the saddle-point approximation for the CDF of this distribution. Section 4 includes a simulation study to assess the accuracy of the saddle-point approximations developed in the paper. This simulation can serve as a useful reference for practitioners when working with similar data. The paper concludes in Section 5 with a summary of the key findings and potential applications of the saddle-point approximation techniques presented.

2. Bivariate Saddle-Point for the Sum of Zero-Truncated Poisson Distribution

In this section, we have given a saddle-point approximation method for the sum of zero-truncated Poisson random variables. Let Z_i be a sequence of independent and identically distributed (*iid*) bivariate random vectors, wherein $\mathbf{z}_i = \left[X_i \quad Y_i\right]'$ and X_i and Y_i can either be dependent or independent. The *bivariate compound distribution* is defined as distribution of the sum as follows:

$$\mathbf{S}_N = \mathbf{z}_1 + \mathbf{z}_2 + \cdots + \mathbf{z}_n = \left(S_1 \quad S_2\right)' = \left[\sum_{i=1}^{N} X_i \quad \sum_{i=1}^{N} Y_i\right]', \tag{1}$$

where, N is independent of the Z_i's. Let the CGF of S_N be defined as:

$$K_{S_N}\left(t,s\right) = \ln\left[E\left\{\exp\left(t\sum_{i=1}^{N} X_i + s\sum_{i=1}^{N} Y_i\right)\right\}\right] = K_N\left[K_Z\left(t,s\right)\right], \tag{2}$$

where, $K_Z\left(t,s\right) = \ln M_Z\left(t,s\right)$ is the CGF of $\mathbf{z}$.

Now, suppose that N in equation (1) has a truncated Poisson distribution, truncated at zero; then the distribution of $S_{\boldsymbol{N}}$ in equation (1) is not continuous due to a point probability at zero, which is $P(N=0) = e^{-\lambda}$. In addition, when N has a zero-truncated Poisson distribution, then the CGF in (2) becomes

$$K_{S_N}\left(t,s\right) = \ln\left[\exp\left\{\lambda M_{X,Y}\left(t,s\right) - 1\right\} - \mathrm{e}^{-\lambda} \Big/ \left(1 - \mathrm{e}^{-\lambda}\right)\right].$$

Now, assume as in Wang (1990) a saddle-point approximation for a bivariate CDF. The convergent domain U of the CGF of $K_{\boldsymbol{N}}(t_0, s_0)$ contains an open neighbourhood of the origin. Moreover, for a fixed (x, y), suppose that there exists a unique $(t_0, s_0) \in U$ such that

$$\left.\begin{aligned} K_N'\left[K_Z\left(\hat{t}_0,\hat{s}_0\right)\right]K_Z^{(t)}\left(\hat{t}_0,\hat{s}_0\right) = x \\ K_N'\left[K_Z\left(\hat{t}_0,\hat{s}_0\right)\right]K_Z^{(s)}\left(\hat{t}_0,\hat{s}_0\right) = y \end{aligned}\right\}, \tag{3}$$

where, $t = \hat{t}_0$ and $s = \hat{s}_0$ are the solution of

$$K_N'\left[K_Z\left(\hat{t}_0,0\right)\right]K_Z^{(t)}\left(\hat{t}_0,0\right) = x, \text{ and } K_N'\left[K_Z\left(0,\hat{s}_0\right)\right]K_Z^{(s)}\left(0,\hat{s}_0\right) = y, \tag{4}$$

with $K_Z^{(t)} = \left(\partial/\partial t\right)K_Z\left(t,s\right)$ and $K_Z^{(s)} = \left(\partial/\partial s\right)K_Z\left(t,s\right)$.

Under the above general conditions, the joint distribution function $F_n\left(x,y\right)$ of $\left(X,Y\right)$ can be approximated by the bivariate saddle-point formula as follows:

$$F_n\left(x,y\right) = I_{11} + I_{12} + I_{21} + I_{22}, \tag{5}$$

where,

$$I_{11} \square\ \Phi\left(\tau_1, v_1, \rho_1\right), \tag{6}$$

$$I_{12} \square\ \Phi\left(w_{s_0}\right)\phi\left(v_0\right)\left\{v_0^{-1} - \left(s_0 G\right)^{-1}\right\}, \tag{7}$$

$$I_{21} = \Phi\left(v_0\right)\phi\left(\tau_1\right)\left[w_{s_0}^{-1} - t_0^{-1}\left\{K_{tt}\left(t_0,s_0\right)\right\}^{-1/2}\right], \tag{8}$$

$$I_{22} = \exp\left[K\left(t_0,s_0\right) - t_0 x - s_0 y\right]\left[w_{s_0}^{-1} - t_0^{-1}\left\{K_{tt}\left(t_0,s_0\right)\right\}^{-1/2}\right]\left[v_0^{-1} - \left(s_0 G\right)^{-1}\right]\Big/ 2\pi. \tag{9}$$

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In addition, in the above expressions, ϕ and Φ are, respectively, the standard normal density and distribution functions; moreover, $\sim$ indicates the error of approximation is of $O(n^{-1})$ relative to the main term as $n \to \infty$. The other quantities appearing in equations (6)–(9) are defined as follows:

$$\tau_1 = \operatorname{sgn}(t_0)\left[-2\ln\left\{\exp\left[\lambda M_X(\hat{t}_0)-1\right]-e^{-\lambda}\Big/\left(1-e^{-\lambda}\right)\right\}-\hat{t}_0 x\right]^{1/2},\ b=\frac{w_{s_0}-\tau_1}{v_0},$$

$$v_1=\frac{v_0-bx_1}{\left(1+b^2\right)^{1/2}},\ G=\sqrt{\left(K_{ss}-K_{ts}^2/K_{tt}\right)\Big|_{(t_0,s_0)}},\ \rho_1=-\frac{b}{\left(1+b^2\right)^{1/2}},$$

$$w_{s_0}=\operatorname{sgn}\left[-2\left\{\ln\left[\exp\left\{\lambda\left(M_{X,Y}(t_0,s_0)-1\right)\right\}-e^{-\lambda}\Big/\exp\left\{\lambda\left(M_Y(s_0)-1\right)\right\}-e^{-\lambda}\right]-t_0 x\right\}\right]^{1/2},$$

$$v_0=\operatorname{sgn}(s_0)\left[-2\left\{\ln\left[\exp\left\{\lambda\left(M_{X,Y}(t_0,s_0)-1\right)\right\}-e^{-\lambda}\Big/\exp\left\{\lambda\left(M_X(\hat{t}_0)-1\right)\right\}-e^{-\lambda}\right]-\left(t_0-\hat{t}_0\right)x-s_0 y\right\}\right]^{1/2}.$$

If $t_0 = 0$, then the factor $w_{s_0}^{-1} - t_0^{-1}\{K_{tt}(t_0, s_0)\}^{-1/2}$ should be replaced by $K_{tt}(0, s_0)/\left[6\{K_{tt}(0, s_0\}^{3/2}\right]$. Moreover, if $s_0 = 0$, the same argument applies to the quantity $v_0^{-1} - (s_0 G)^{-1}$. The limits can be well approximated by the corresponding values evaluated at $t = t_0$, which are very close to zero (see Wang, 1990).

For $d > 2$, Kolassa (2003) has provided another approximation. In the case where X and Y are independent, then $K(t, s) = K(t)K(s)$, and, in this case, the first-order saddle-point approximations by Lugannani and Rice (1980) are applicable for both continuous and discrete cases.

3. Saddle-point Approximation for the CDF of BCZTPG Distribution

The BCZTPG distribution, derived from the sum of random vectors with a Gamma distribution under a truncated Poisson sample size, is a versatile distribution with diverse applications. In climatology, it can model rainfall and snow patterns, for instance, in analysing rainy days, where N represents the number of rainy days following a Poisson distribution and the rainfall amounts (x, y) at different stations or seasons follow a Gamma distribution. Similarly, in snowfall data analysis, it can represent total snowfall over a period. In manufacturing, it can model total production from two renewal systems after a set number of renewals. Additionally, in risk theory, it can describe the total claim amount from car insurance by distinct groups. The utility of this distribution spans various fields, making it a valuable tool for statistical modelling and analysis. The distribution is introduced below:

Let the random sum $S_N = \left(S_1 \quad S_2\right)^{/}$ be the aggregate claims generated in a fixed period of time by an independent group. Suppose that the number of claims, N, follows a zero-truncated Poisson distribution with parameter λ (see Chattamvelli & Shanmugam, 2020) and is written as $ZTP(\lambda)$. The moment generating function (MGF) of N (see Appendix A) is as follows:

$$M_N(t)=\left\{\exp\left(\lambda e^t-1\right)-e^{-\lambda}\right\}\Big/\left(1-e^{-\lambda}\right).$$

In addition, if (x_i's, y_i's) are *iid* random variables having Gamma distributions – $G(\alpha_1, \beta_1)$ and $G(\alpha_2, \beta_2)$ – then the joint MGF of (x_i's, y_i's) is as follows:

$$M_{X,Y}(t,s)=\left(1-\beta_1 t\right)^{-\alpha_1}\left(1-\beta_2 t\right)^{-\alpha_2}. \tag{10}$$

The sum S_N, in this case, is said to have a BCZTPG distribution. It is easy to show that the CGF of S_N is

$$K_N\left[K_Z(t,s)\right]=\ln\left[\left\{\exp\left[\lambda\left(M_{X,Y}(t,s)-1\right)\right]-e^{-\lambda}\right\}\Big/\left(1-e^{-\lambda}\right)\right]. \tag{11}$$

Using the joint MGF of X and Y from equation (10), the CGF of S_N becomes

$$K_N\left[K_Z\left(t,s\right)\right]=\ln\left[\exp\left(\lambda\left\{\left(1-\beta_1 t\right)^{-\alpha_1}\left(1-\beta_2 s\right)^{-\alpha_2}\right\}-1\right)-\mathrm{e}^{-\lambda}\right]-\ln\left[\left(1-\mathrm{e}^{-\lambda}\right)\right].$$

The saddle-point equations, from equation (3) in this instance, are given as follows:

$$\frac{\lambda\alpha_1\beta_1\exp\left\{\lambda\left(1-\beta_1 t\right)^{-\alpha_1}\left(1-\beta_2 s\right)^{-\alpha_2}-\lambda\right\}}{\left(1-\beta_1 t\right)^{\alpha_1}\left(1-\beta_2 s\right)^{\alpha_2}\left(1-\beta_1 t\right)\left[\exp\left\{\lambda\left(1-\beta_1 t\right)^{-\alpha_1}\left(1-\beta_2 s\right)^{-\alpha_2}-\lambda\right\}-\mathrm{e}^{-\lambda}\right]}-x=0, \tag{12}$$

and

$$\frac{\lambda\alpha_2\beta_2\exp\left\{\lambda\left(1-\beta_1 t\right)^{-\alpha_1}\left(1-\beta_2 s\right)^{-\alpha_2}-\lambda\right\}}{\left(1-\beta_1 t\right)^{\alpha_1}\left(1-\beta_2 s\right)^{\alpha_2}\left(1-\beta_2 s\right)\left[\exp\left\{\lambda\left(1-\beta_1 t\right)^{-\alpha_1}\left(1-\beta_2 s\right)^{-\alpha_2}-\lambda\right\}-\mathrm{e}^{-\lambda}\right]}-y=0. \tag{13}$$

We solve these equations to obtain the values of t_0 and s_0 (Appendix B). The values of t_0 and s_0 are as follows:

$$t_0=\frac{1}{\beta_1}\left[1-\left\{\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 y}{\alpha_2\beta_2 x}\right)^{\alpha_2}\right\}^{1/(\alpha_1+\alpha_2)}\right],\quad s_0=\frac{1}{\beta_2}\left[1-\left\{\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 y}{\alpha_2\beta_2 x}\right)^{\alpha_1}\right\}^{1/(\alpha_1+\alpha_2)}\right].$$

Now, to find $\hat{t}_0$, we solve $\frac{\partial}{\partial t}K_1\left(t\right)=0,$ (14)

and

$$\frac{\lambda\alpha_1\beta_1\exp\left\{\lambda\left(1-\beta_1 t\right)^{-\alpha_1}-\lambda\right\}}{\left(1-\beta_1 t\right)^{\alpha_1}\left(1-\beta_1 t\right)\left[\exp\left\{\lambda\left(1-\beta_1 t\right)^{-\alpha_1}-\lambda\right\}-\mathrm{e}^{-\lambda}\right]}-x=0. \tag{15}$$

From Appendix C, we obtain $\hat{t}_0=\frac{1}{\beta_1}\left[1\pm\left(\frac{\lambda}{z}\right)^{1/\alpha_1}\right].$

Moreover,

$$M_X\left(\hat{t}_0\right)=\ln\left[\exp\left\{\lambda^2\left(1-\beta_1\hat{t}_0\right)^{-\alpha_1}-\lambda\right\}-\mathrm{e}^{-\lambda}\right]-\ln\left(1-\mathrm{e}^{-\lambda}\right),$$

$$K\left(t_0,s_0\right)=\ln\left[\exp\left(\lambda\left\{\left(1-\beta_1\hat{t}_0\right)^{-\alpha_1}\left(1-\beta_2\hat{s}_0\right)^{-\alpha_2}\right\}-\lambda\right)-\mathrm{e}^{-\lambda}\right]-\ln\left[\left(1-\mathrm{e}^{-\lambda}\right)\right],$$

$$K\left(0,s_0\right)=\ln\left[\exp\left(\lambda\left\{\left(1-\beta_2\hat{s}_0\right)^{-\alpha_2}\right\}-\lambda\right)-\mathrm{e}^{-\lambda}\right]-\ln\left[\left(1-\mathrm{e}^{-\lambda}\right)\right].$$

The saddle-point approximation given in equation (5) becomes

$$\hat{F}_{S_N}\left(x,y\right)=\Phi\left(\tau_1,v_1,\rho_1\right)+\Phi\left(w_{s_0}\right)\phi\left(v_0\right)\left[v_0^{-1}-\left(s_0 G\right)^{-1}\right]+\Phi\left(v_0\right)\phi\left(\tau_1\right)\left[w_{s_0}^{-1}-t_0^{-1}K_{tt}^{-1/2}\left(t_0,s_0\right)\right]$$
$$+\exp\left[K\left(t_0,s_0\right)-t_0 x-s_0 y\right]\left[w_{s_0}^{-1}-t_0^{-1}K_{tt}^{-1/2}\left(t_0,s_0\right)\right]\left[v_0^{-1}-\left(s_0 G\right)^{-1}\right]\Big/2\pi,$$

where, $x,y\in\left[1,\infty\right)$.

4. Simulation Study

In this section, we present two illustrative examples demonstrating the saddle-point approximation technique for the CDF of the BCZTPG distribution. Let the number of claims, N, follow a zero-truncated Poisson distribution with parameter λ, written as $ZTP(\lambda)$, and let the amounts of claims X_i's and Y_i's $(i = 1, 2, 3, \dots, N)$ be *iid* Gamma variates, denoted as $G(\alpha_1, \beta_1) \vee G(\alpha_2, \beta_2)$. It is assumed that N is independent of X_i's and Y_i's. The random variable S_N, as defined in equation (1), has a BCZTPG distribution in this scenario. In insurance contexts, S_N can be interpreted as the total amount of the claim, N as the number of reported claims, X_i's and Y_i's as the insurance payment for the *i*th claim. When $N = 0$, then S_N, has a probability mass at zero for non-users, where $P(S_N = 0) = e^{-\lambda}$. For $N > 0$, S_N is the sum of N *iid* bivariate Gamma random variables. An illustrative example for saddle-point approximation of the CDF of the BCZTPG distribution is presented next. In the first example, Figure 1 illustrates the bivariate scatter plots of the simulated BCZTPG data, accompanied by marginal histogram distributions that depict the range and scatter of the BCZTPG distribution, as well as the skewness of the marginal distributions. Let $x = 7.5, y = 13.5, \beta_1 = 2, \beta_2 = 4, \alpha_1 = 1, \alpha_2 = 1,$ and $\lambda = 4$. Moreover, in this case, $\alpha_1 + \alpha_2 = 2$ is even, and we will use the double-sign versions of the solution that are given below:

$$\left[\pm(\alpha_1\beta_1)^{\alpha_2/(\alpha_1+\alpha_2)}\left(\alpha_2\beta_2 x / \lambda^{\alpha_2^{-1}} y\right)^{\alpha_2/(\alpha_1+\alpha_2)} z^{(\alpha_1+\alpha_2+1)/(\alpha_1+\alpha_2)} - x\right]e^z + x = 0, \tag{16}$$

$$t = \frac{1}{\beta_1}\left[1 \pm \left\{(\lambda/z)(\alpha_1\beta_1 y/\alpha_2\beta_2 x)^{\alpha_2}\right\}^{1/(\alpha_1+\alpha_2)}\right], \tag{17}$$

$$s = \frac{1}{\beta_2}\left[1 \pm \left\{(\lambda/z)(\alpha_2\beta_2 x/\alpha_1\beta_1 y)^{\alpha_1}\right\}^{1/(\alpha_1+\alpha_2)}\right]. \tag{18}$$

Using the specified values of the parameters in (16), we have

$$\left[\pm 2^{1/2}\left(\frac{5}{9}\right)^{1/2} z^{3/2} - 7.5\right]e^z + 7.5 = 0 \text{ or } \left(\pm\sqrt{10}z^{3/2} - 22.5\right)e^z + 22.5 = 0.$$

It is to be noted that the case $\left(-\sqrt{10}z^{3/2} - 22.5\right)e^z + 22.5 = 0,$ has the unique exact solution $z = 0$. As z appears in the denominator in equations (17) and (18), this solution will be discarded.

In contrast, the case $\left(\sqrt{10}z^{3/2} - 22.5\right)e^z + 22.5 = 0,$ has an exact solution of $z = 0$ and an approximate solution of $z \approx 3.63388$. Moreover, as z appears in the denominator of equations (17) and (18), the exact solution $z = 0$ will be discarded. The possible values of t and s are as follows:

$$t = \frac{1}{2}\left[1 \pm \left\{\frac{4}{z}\left(\frac{9}{10}\right)\right\}^{1/2}\right] = \frac{1}{2}\left[1 \pm \left(\frac{18}{5z}\right)^{1/2}\right],$$

$$s = \frac{1}{4}\left[1 \pm \left\{\frac{4}{z}\left(\frac{10}{9}\right)\right\}^{1/2}\right] = \frac{1}{4}\left[1 \pm \left(\frac{40}{9z}\right)^{1/2}\right].$$

Furthermore, as $\frac{\alpha_2\beta_2 x}{\alpha_1\beta_1 y} = \frac{10}{9} > 0,$ it follows that s and v have the same sign. The pairs

$$\begin{aligned}(t_1, s_1) &= \left(\frac{1}{2}\left[1-\left(\frac{18}{5z}\right)^{1/2}\right], \frac{1}{4}\left[1-\left(\frac{40}{9z}\right)^{1/2}\right]\right),\\ &\approx \left(\frac{1}{2}\left[1-\left(\frac{18}{5\times 3.63388}\right)^{1/2}\right], \frac{1}{4}\left[1-\left(\frac{40}{9\times 3.63388}\right)^{1/2}\right]\right) = (0.00234, -0.02647).\end{aligned} \tag{19}$$

and

$$\begin{aligned}(t_2, s_2) &= \left(\frac{1}{2}\left[1+\left(\frac{18}{5z}\right)^{1/2}\right], \frac{1}{4}\left[1+\left(\frac{40}{9z}\right)^{1/2}\right]\right),\\ &\approx \left(\frac{1}{2}\left[1+\left(\frac{18}{5\times 3.63388}\right)^{1/2}\right], \frac{1}{4}\left[1+\left(\frac{40}{9\times 3.63388}\right)^{1/2}\right]\right) = (0.99766, 0.52647),\end{aligned} \tag{20}$$

are the possible solutions for the system. Introducing these values of (t_1, s_1) and (t_2, s_2) in the original equations, we have found that (t_1, s_1) is a solution but (t_2, s_2) is not a solution. Hence, the solution of the system is as follows:

$$(t_0, s_0) \approx (0.00234, -0.02648). \tag{21}$$

In addition, the value of $\hat{t}_0$ is $\hat{t}_0 = \frac{1}{\beta_1}\left[1 \pm \left(\frac{\lambda}{z}\right)^{1/\alpha_1}\right] = \frac{1}{2}\left[1 \pm \left(\frac{4}{z}\right)\right]$, where, z satisfies

$$\left[\pm\alpha_1\beta_1 z^{(\alpha_1+1)/\alpha_1} - \lambda^{1/\alpha_1}x\right]e^z + \lambda^{1/\alpha_1}x = 0 \ or \ \left(\pm 2z^2 - 4\times 7.5\right)e^z + 4\times 7.5 = 0,$$

or $z \approx 3.83074$, and then $\hat{t}_0 = \frac{1}{2}\left(1 - \frac{4}{3.83074}\right) = -0.02209$, and hence

$$\begin{aligned}&M_X(\hat{t}_0) = -0.17270,\ \tau_1 = -0.37439,\ K(0, s_0) = -0.39184,\ v_0 = -0.23424;\ b = -0.55860,\\ &\rho_1 = 0.48789,\ v_1 = -0.26225,\ K_{tt}(t_0, s_0) = 28.65512,\ K_{ts}(t_0, s_0) = 24.45243,\\ &K_{ss}(t_0, s_0) = 92.84258,\ G = \sqrt{\left(K_{ss} - K_{ts}^2/K_{tt}\right)\Big|_{(t_0, s_0)}} = 8.48389.\end{aligned}$$

Using these values, we have $I_{11} = 0.25779, I_{12} = 0.03570, I_{21} = 0.03068,$ and $I_{22} = 0.00532,$ hence, an approximate value of the CDF, by using the saddle-point approximation, is $F(x, y) = I_{11} + I_{12} + I_{21} + I_{22} = 0.32950.$

It is seen that the bivariate saddle-point formulas are easily computed once the CGF is available. We will now present a simulation study to see the performance of the saddle-point approximation for the CDF of the BCZTPG distribution. The simulation study has been conducted by generating one million exact and approximate values of the CDF. The approximate value is computed as follows:

$$F\left(S_N \le \begin{pmatrix} x \\ y \end{pmatrix}\right) = \frac{1}{10^6}\sum_{1}^{10^6} I\left(\begin{bmatrix}\sum_{i=1}^{N} x_i \\ \sum_{i=1}^{N} x_i\end{bmatrix} \le \begin{pmatrix} x \\ y \end{pmatrix}\right), \tag{22}$$

where, we have generated N from $ZTP(4)$, X_i's are generated from $G(1,2)$ and Y_i's are generated from $G(1,4)$. Based on the generated value of N, the independent Gamma variates are generated. For example, if N = 4, then we have generated four independent values from the Gamma distributions, and so on. The simulation code has been written using the R language. The results of the simulation study are given in Table 1 below. For each pair of values, (x, y), the first value in each cell of Table 1 is the *exact* value of the CDF, the *second* value is the saddle-point approximation, and the *third* one is the normal approximation. From Table 1, we can see that the saddle-point approximation provides values which are close to the exact value of the CDF of the BCZTPG distribution.

To have a better insight of the approximation, we have used a much wider range of the values of x and y and have computed the exact value of the CDF of the BCZTPG distribution alongside the approximate values of the CDF by using the saddle-point and the normal approximations. The values have been computed by compounding a zero-truncated $ZTP(4)$ variate with $G(1,2)$ and $G(1,4)$ variates $- ZTP(4) \vee \{G(1,2), G(1,4)\}$. The plot of the values is given in Figure 2, where panel (a) contains the plot for exact values method, panel (b) is the plot of CDF using saddle-point approximation method, and panel (c) contains the plot of CDF by using the normal approximation method. The plots clearly show that the saddle-point approximation for the CDF of the BCZTPG distribution is an excellent approximation for the true CDF of the BCZTPG distribution.

Figure 1: Bivariate Scatter Plots of BCZTPG Distribution: $\boldsymbol{ZTP(4)} \vee \{\boldsymbol{G(1,2)}, \boldsymbol{G(1,4)}\}$

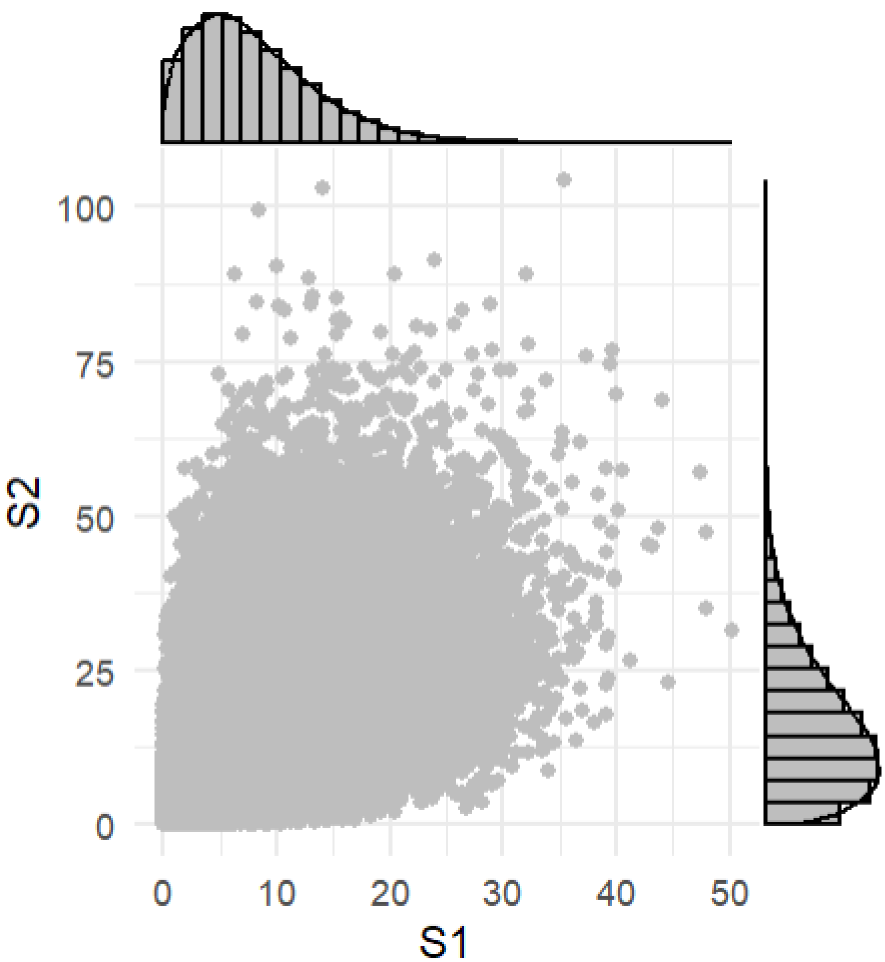

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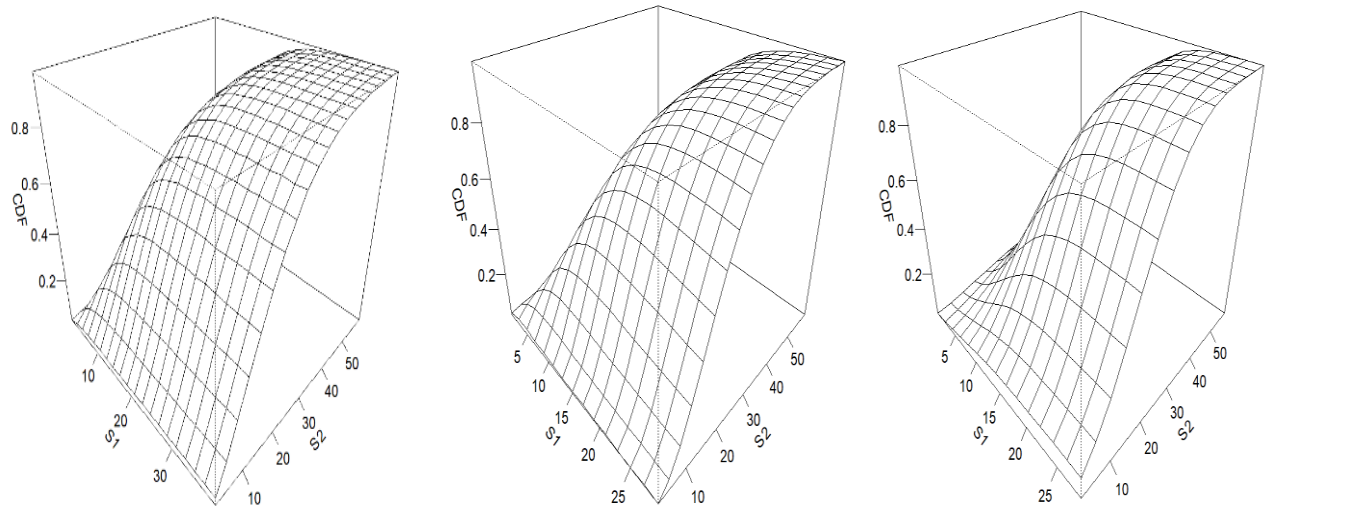
Figure 2: Plots of the CDF of the BCZTPG Distribution: $ZTP(4) \vee \{G(1,2), G(1,4)\}$



(a) Exact Method (b) Saddle-Point Approximation Method (c) Normal Approximation Method

Table 1: The Exact, Saddle-Point, and Normal Approximation CDFs for the BCZTPG Distribution: $ZTP(4) \vee \{G(1,2), G(1,4)\}$

X/Y	0.5	5.5	9.5	17.5	21.5	45.5
0.5	0.0021	0.0137	0.0178	0.0198	0.0205	0.0205
	0.0056	0.0137	0.0167	0.0189	0.0191	0.0184
	0.0078	0.0163	0.0261	0.0360	0.0634	0.0920
2.5	0.0067	0.0645	0.0959	0.1262	0.1321	0.1388
	0.0104	0.0681	0.0931	0.1206	0.1299	0.1311
	0.0141	0.0292	0.0467	0.0914	0.1135	0.1647
6.5	0.0094	0.1264	0.2213	0.3534	0.3911	0.4451
	0.0057	0.1186	0.2182	0.3517	0.3884	0.4419
	0.0337	0.0698	0.1118	0.2185	0.2714	0.3936
10.5	0.0098	0.1479	0.2809	0.4981	0.5692	0.6958
	0.0033	0.1158	0.2562	0.4982	0.5751	0.7019
	0.0572	0.1185	0.1896	0.3707	0.4605	0.6676
22.5	0.0099	0.1565	0.3105	0.5996	0.7068	0.9451
	0.0014	0.0801	0.2163	0.5459	0.6849	0.9723
	0.0848	0.1757	0.2813	0.5498	0.6830	0.99027
26.5	0.0099	0.1572	0.3116	0.6011	0.7115	0.9879
	0.0091	0.1520	0.2907	0.5978	0.6720	0.9876
	0.0836	0.1765	0.2826	0.5524	0.6861	0.9949

To verify the simulation results, we will now examine another example of a BCZTPG distribution with different parameters. Let $\beta_1 = 3$, $\beta_2 = 4$, $\alpha_1 = 2$, $\alpha_2 = 1$, and $\lambda = 3$. The bivariate scatter plots of the simulated BCZTPG data are shown in Figure 3, along with marginal histogram distributions that show the skewness of the marginal distributions and the range and scatter of the BCZTPG distribution. Table 2 below lists the outcomes of the simulation study. For every pair of values (x, y), the exact value of the CDF is the first value in each cell of Table 2, the saddle-point approximation is the second value, and the normal approximation is the third value that approximates the value of the CDF. Table 2 shows that the values obtained from the saddle-point approximation are close to the exact value of the CDF of the BCZTPG distribution. The values are plotted in Figure 4, with panel (a) showing the plot for exact values method, panel (b) showing the CDF plot using the saddle-point approximation method, and panel (c) showing the CDF plot using the normal approximation method. Plotting the data makes it evident that the saddle-point approximation of the CDF of the BCZTPG distribution is an excellent approximation of the true CDF of the distribution.

We checked with the previous two examples by optimizing the simulation algorithm and increasing the sample size and found that it is achievable for the simulated values to closely match the theoretical distribution and the values of the distribution parameters, such as the mean and variance. This optimization guarantees that the generated random values accurately reflect the desired distribution, highlighting the superiority of the saddle-point approximation method over the normal approximation method.

Figure 3: Bivariate Scatter Plots of BCZTPG Distribution: $\boldsymbol{ZTP}(\mathbf{3}) \vee \{\boldsymbol{G}(\mathbf{2},\mathbf{3}), \boldsymbol{G}(\mathbf{1},\mathbf{4})\}$

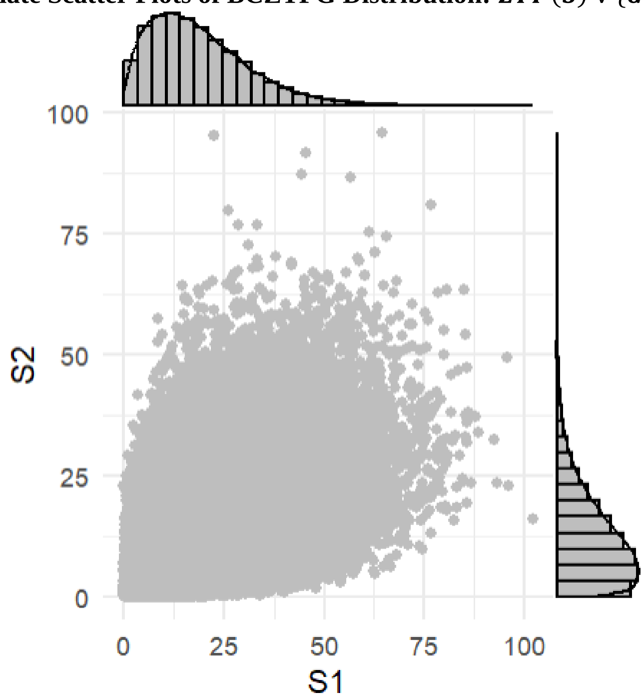
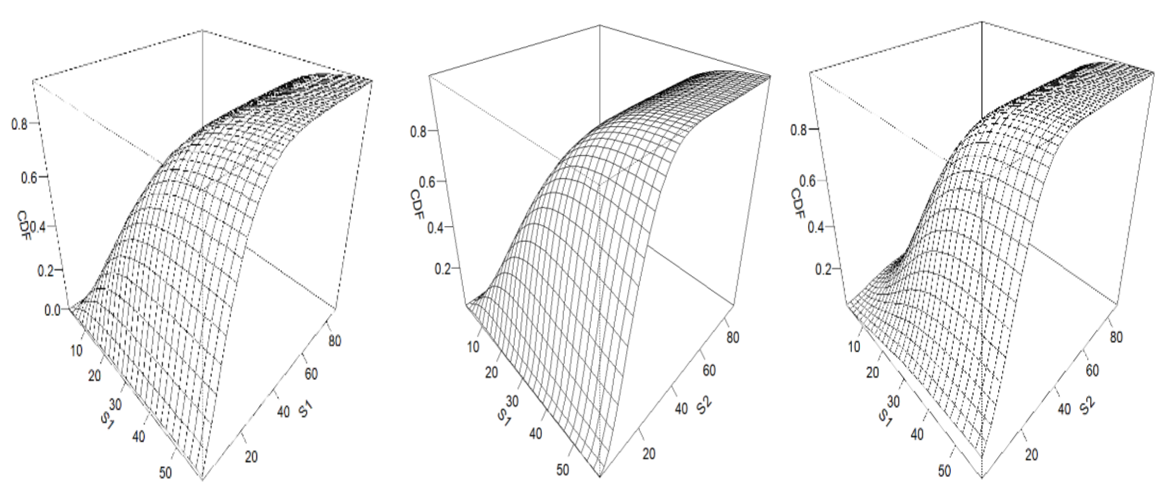


Figure 4: Plots of the CDF of the BCZTPG Distribution: $\boldsymbol{ZTP}(\mathbf{3}) \vee \{\boldsymbol{G}(\mathbf{2},\mathbf{3}), \boldsymbol{G}(\mathbf{1},\mathbf{4})\}$



(a) Exact Method (b) Saddle-Point Approximation Method (c) Normal Approximation Method

Table 2: The Exact, Saddle-Point, and Normal Approximation CDFs for the BCZTPG Distribution: $\boldsymbol{ZTP}(\mathbf{3}) \vee \{\boldsymbol{G}(\mathbf{2},\mathbf{3}), \boldsymbol{G}(\mathbf{1},\mathbf{4})\}$

X/Y	2.5	12.5	20.5	30.5	44.5	52.5
2.5	0.0129	0.0312	0.0321	0.0335	0.0327	0.0327
	0.0187	0.0320	0.0334	0.0331	0.0325	0.0321
	0.0138	0.0460	0.0596	0.0609	0.0930	0.0930
8.5	0.064	0.1789	0.1963	0.2038	0.2009	0.2016
	0.0726	0.1764	0.1973	0.2016	0.2003	0.1993
	0.0599	0.0992	0.1585	0.1938	0.2004	0.2005
12.5	0.0816	0.2798	0.3222	0.3344	0.3374	0.3370
	0.0813	0.2805	0.3262	0.3399	0.3405	0.3395
	0.0451	0.1496	0.2390	0.2922	0.3020	0.3022
16.5	0.0923	0.3612	0.4376	0.4616	0.4694	0.4706
	0.0773	0.3705	0.4489	0.4755	0.4796	0.4790
	0.0630	0.2090	0.3338	0.4081	0.4219	0.4222
32.5	0.1060	0.5242	0.7049	0.7933	0.8161	0.8181
	0.0969	0.5075	0.7410	0.8353	0.8564	0.8577
	0.1288	0.4268	0.6818	0.8336	0.8618	0.8622
50.5	0.1051	0.5492	0.7692	0.8850	0.9647	0.9892
	0.1389	0.4912	0.7779	0.9443	0.9817	0.9815
	0.1485	0.4923	0.7863	0.9614	0.9939	0.9944

5. Conclusion

The saddle-point approximation, as discussed in Daniels (1954), holds significant influence in the field of statistics. In this study, we have introduced a BCZTPG distribution and approximated its CDF using the saddle-point method. We have seen that the resulting distribution can be applied in insurance scenarios, particularly in modelling aggregate claim amounts or total losses in car insurance policies. The saddle-point approximation effectively estimates the CDF of the BCZTPG distribution, demonstrating its utility in providing an excellent approximation. The saddle-point approximation is a valuable tool for approximating functions of distributions lacking closed-form representations. Numerical examples show that saddle-point approximations work remarkably better than other approximation methods. This method is straightforward to implement and demands minimal computational effort. Numerous unexplored benefits of this technique exist within statistics, such as parameter estimation in time series models, approximating joint distributions of definite or bivariate quadratic forms, and extending the technique to include stable distributions without moments. Additionally, the potential applications of this technique beyond statistics remain untapped, with possibilities in domains such as wireless networks, visual communications, and image processing. It is challenging to comprehensively list all the uses of saddle-point techniques due to their diverse applications and potential.

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Appendix A

The zero-truncated Poisson (ZTP) is a truncated Poisson (TP) distribution when $P(X=0)$ is discarded. The probability mass function of a ZTP distribution is as follows:

$$P\left(X=x \mid X>0\right)=\frac{P\left(X=x\right)}{1-P\left(X=0\right)}=\frac{\lambda^{x}\mathrm{e}^{-\lambda}}{x!\left(1-\mathrm{e}^{-\lambda}\right)}\ ;\ x=1,2,3,\ldots,\infty.$$

The MGF of the ZTP distribution is as follows:

$$M_X(t) = E\left(e^{tX}\right) = \sum_{x=1}^{\infty} e^{tx} \frac{\lambda^x e^{-\lambda}}{x!\left(1-e^{-\lambda}\right)} = \frac{e^{-\lambda}}{1-e^{-\lambda}} \sum_{x=1}^{\infty} \frac{1}{x!}\left(\lambda e^t\right)^x,$$

$$= \frac{e^{-\lambda}}{1-e^{-\lambda}}\left[\sum_{x=0}^{\infty} \frac{1}{x!}\left(\lambda e^t\right)^x - 1\right] = \frac{e^{-\lambda}}{1-e^{-\lambda}}\left[\exp\left(\lambda e^t\right)-1\right],$$

$$= \frac{\exp\left(\lambda e^t - 1\right) - e^{-\lambda}}{1-e^{-\lambda}}, \; t > 0.$$

Appendix B

The solution of the saddle-point equations is given in equations (12) and (13) as follows:

$$\frac{\lambda\alpha_1\beta_1 \exp\left\{\lambda(1-\beta_1 t)^{-\alpha_1}(1-\beta_2 s)^{-\alpha_2} - \lambda\right\}}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}(1-\beta_1 t)\left[\exp\left\{\lambda(1-\beta_1 t)^{-\alpha_1}(1-\beta_2 s)^{-\alpha_2} - \lambda\right\} - e^{-\lambda}\right]} - x = 0,$$

and

$$\frac{\lambda\alpha_2\beta_2 \exp\left\{\lambda(1-\beta_1 t)^{-\alpha_1}(1-\beta_2 s)^{-\alpha_2} - \lambda\right\}}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}(1-\beta_2 s)\left[\exp\left\{\lambda(1-\beta_1 t)^{-\alpha_1}(1-\beta_2 s)^{-\alpha_2} - \lambda\right\} - e^{-\lambda}\right]} - y = 0.$$

The above equations can be written as

$$\frac{\lambda\alpha_1\beta_1 \exp\left\{\lambda(1-\beta_1 t)^{-\alpha_1}(1-\beta_2 s)^{-\alpha_2} - \lambda\right\}}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}(1-\beta_1 t)\left[\exp\left\{\lambda(1-\beta_1 t)^{-\alpha_1}(1-\beta_2 s)^{-\alpha_2} - \lambda\right\} - e^{-\lambda}\right]} = x, \tag{B.1}$$

and

$$\frac{\lambda\alpha_2\beta_2 \exp\left\{\lambda(1-\beta_1 t)^{-\alpha_1}(1-\beta_2 s)^{-\alpha_2} - \lambda\right\}}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}(1-\beta_2 s)\left[\exp\left\{\lambda(1-\beta_1 t)^{-\alpha_1}(1-\beta_2 s)^{-\alpha_2} - \lambda\right\} - e^{-\lambda}\right]} = y. \tag{B.2}$$

Now, dividing (B.1) by (B.2) we have $\frac{\alpha_1\beta_1(1-\beta_2 s)}{\alpha_2\beta_2(1-\beta_1 t)} = \frac{x}{y}$. (B.3)

Substituting $(1-\beta_1 t) = w$ and $(1-\beta_2 s) = v$ in (B.3), we have $\frac{\alpha_1\beta_1 v}{\alpha_2\beta_2 w} = \frac{x}{y} \Rightarrow v = \frac{\alpha_2\beta_2 wx}{\alpha_1\beta_1 y}$. (B.4)

Using the values of *w* and *v*, equation (B.1) can be written as follows:

$$\frac{\lambda\alpha_1\beta_1 \exp\left[\lambda w^{-\alpha_1}\left(\alpha_2\beta_2 wx/\alpha_1\beta_1 y\right)^{-\alpha_2} - \lambda\right]}{w^{\alpha_1+1}\left(\alpha_2\beta_2 wx/\alpha_1\beta_1 y\right)^{\alpha_2}\left[\exp\left\{\lambda w^{-\alpha_1}\left(\alpha_2\beta_2 wx/\alpha_1\beta_1 y\right)^{-\alpha_2} - \lambda\right\} - e^{-\lambda}\right]} = x. \tag{B.5}$$

Making another substitution $z = \lambda w^{-\alpha_1}\left(\alpha_2\beta_2 wx/\alpha_1\beta_1 y\right)^{-\alpha_2}$, we have

$$z = \frac{\lambda}{w^{\alpha_1+\alpha_2}}\left(\frac{\alpha_1\beta_1 y}{\alpha_2\beta_2 x}\right)^{\alpha_2} \Rightarrow w = \left[\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 y}{\alpha_2\beta_2 x}\right)^{\alpha_2}\right]^{1/(\alpha_1+\alpha_2)}, \tag{B.6}$$

with the caution that a double sign may appear if $\alpha_1 + \alpha_2$ is even. Using (B.6) in (B.5), we have

$$\frac{z\alpha_1\beta_1 \exp(z-\lambda)}{\left[\left\{(\lambda/z)\left(\alpha_1\beta_1 y/\alpha_2\beta_2 x\right)\right\}^{1/(\alpha_1+\alpha_2)}\right]\left[\exp(z-\lambda) - e^{-\lambda}\right]} = x,$$

then

$$\frac{z\alpha_1\beta_1 \exp(z-\lambda)}{\left[\left\{(\lambda/z)(\alpha_1\beta_1 y/\alpha_2\beta_2 x)\right\}^{1/(\alpha_1+\alpha_2)}\right]} = x\left[\exp(z-\lambda)-\mathrm{e}^{-\lambda}\right],$$

or
$$\frac{z\alpha_1\beta_1 \exp(z-\lambda)}{\left[\left\{(\lambda/z)(\alpha_1\beta_1 y/\alpha_2\beta_2 x)\right\}^{1/(\alpha_1+\alpha_2)}\right]} = x\exp(z-\lambda)-x\mathrm{e}^{-\lambda},$$

or
$$\frac{z^{(\alpha_1+\alpha_2+1)/(\alpha_1+\alpha_2)}\alpha_1\beta_1 \exp(z-\lambda)}{\lambda^{1/(\alpha_1+\alpha_2)}}\left(\frac{\alpha_2\beta_2 x}{\alpha_1\beta_1 y}\right)^{\alpha_2/(\alpha_1+\alpha_2)} - x\exp(z-\lambda)+x\mathrm{e}^{-\lambda}=0,$$

or
$$\left[(\alpha_1\beta_1)^{\alpha_1/(\alpha_1+\alpha_2)}\left(\frac{\alpha_2\beta_2 x}{\lambda^{1/\alpha_2} y}\right)^{\alpha_2/(\alpha_1+\alpha_2)} z^{(\alpha_1+\alpha_2+1)/(\alpha_1+\alpha_2)} - x\right]\exp(z-\lambda)+x\mathrm{e}^{-\lambda}=0,$$

or
$$\left[(\alpha_1\beta_1)^{\alpha_1/(\alpha_1+\alpha_2)}\left(\frac{\alpha_2\beta_2 x}{\lambda^{1/\alpha_2} y}\right)^{\alpha_2/(\alpha_1+\alpha_2)} z^{(\alpha_1+\alpha_2+1)/(\alpha_1+\alpha_2)} - x\right]\exp(z)+x=0. \tag{B.7}$$

which is an equation that implicitly depends only on *t* and not on *s*. Again, the caution of a double sign before the coefficient of $z^{(\alpha_1+\alpha_2+1)/(\alpha_1+\alpha_2)}$ may appear if $\alpha_1+\alpha_2$ is even. Once a value of *z* that satisfies equation (B.7) is obtained, the corresponding value of *t* is found by substitution as follows:

Recall equation (B.6); $w=\left[\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 y}{\alpha_2\beta_2 x}\right)^{\alpha_2}\right]^{1/(\alpha_1+\alpha_2)}$.

and using the substitution $(1-\beta_1 t)=w$, we obtain

$$1-\beta_1 t=\left[\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 y}{\alpha_2\beta_2 x}\right)^{\alpha_2}\right]^{1/(\alpha_1+\alpha_2)} \Rightarrow \beta_1 t = 1-\left[\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 y}{\alpha_2\beta_2 x}\right)^{\alpha_2}\right]^{1/(\alpha_1+\alpha_2)},$$

or

$$t_0=\frac{1}{\beta_1}\left[1-\left\{\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 y}{\alpha_2\beta_2 x}\right)^{\alpha_2}\right\}^{1/(\alpha_1+\alpha_2)}\right]. \tag{B.8}$$

with the caution of a double sign before the power if $\alpha_1+\alpha_2$ is even.

To obtain the value of *s*, observe that *z* can also be written in terms of only *v* as follows. We note that equation (B.4) can be manipulated to obtain the value of *v* as follows:

$$w=\frac{\alpha_1\beta_1 vy}{\alpha_2\beta_2 x} \Rightarrow z=\frac{\lambda}{v^{\alpha_2}}\left(\frac{\alpha_1\beta_1 vy}{\alpha_2\beta_2 x}\right)^{-\alpha_1} \Rightarrow v=\left[\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 x}{\alpha_2\beta_2 y}\right)^{\alpha_1}\right]^{1/(\alpha_1+\alpha_2)}.$$

with the caution of a double sign if $\alpha_1+\alpha_2$ is even. This implies that equation (B.7) can also be seen as an equation that implicitly depends only on *s* and not on *t*. Once a value of *z* is found, the corresponding value of *s* can be found by substitution as

$$1-\beta_2 s=\left[\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 x}{\alpha_2\beta_2 y}\right)^{\alpha_1}\right]^{1/(\alpha_1+\alpha_2)} \Rightarrow \beta_2 s = 1-\left[\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 x}{\alpha_2\beta_2 y}\right)^{\alpha_1}\right]^{1/(\alpha_1+\alpha_2)},$$

or
$$s=\frac{1}{\beta_2}\left[1-\left\{\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 x}{\alpha_2\beta_2 y}\right)^{\alpha_1}\right\}\right]^{1/(\alpha_1+\alpha_2)}. \tag{B.9}$$

with a caution of a double sign if $\alpha_1+\alpha_2$ is even.

Appendix C

To find the value of $\hat{t}_0$, we solve $\frac{\partial}{\partial t}K_1(t)=x.$ (C.1)

or $$\frac{\lambda\alpha_1\beta_1\exp\left[\lambda(1-\beta_1 t)^{-\alpha_1}-\lambda\right]}{(1-\beta_1 t)^{(\alpha_1+1)}\left[\exp\left\{\lambda(1-\beta_1 t)^{-\alpha_1}-\lambda\right\}-\mathrm{e}^{-\lambda}\right]}-x=0. \quad \text{(C.2)}$$

Making the substitution $z=\lambda(1-\beta_1 t)^{-\alpha_1}$, we have $(1-\beta_1 t)=(\lambda/z)^{1/\alpha_1}$, with the caution of a double sign before the power if α_1 is even. The equation (C.2) is, then, transformed into

$$x=\frac{z\alpha_1\beta_1\exp(z-\lambda)}{(\lambda/z)^{1/\alpha_1}\left[\exp(z-\lambda)-\mathrm{e}^{-\lambda}\right]}.$$

Hence,

$$z^{(\alpha_1+1)/\alpha_1}\alpha_1\beta_1\mathrm{e}^z=\lambda^{1/\alpha_1}x\left(\mathrm{e}^z-1\right) \text{ or } \left(\alpha_1\beta_1 z^{(\alpha_1+1)/\alpha_1}-\lambda^{1/\alpha_1}x\right)\mathrm{e}^z+\lambda^{1/\alpha_1}x=0,$$

which is similar to equation (B.7), therefore, the solution of the original equation is

$$\hat{t}_0=\frac{1}{\beta_1}\left[1-\left(\frac{\lambda}{z}\right)^{1/\alpha_1}\right],$$

where, z satisfies $\left(\alpha_1\beta_1 z^{(\alpha_1+1)/\alpha_1}-\lambda^{1/\alpha_1}x\right)\mathrm{e}^z+\lambda^{1/\alpha_1}x=0.$

In addition, if α_1 is even, then the double-sign versions $\left(\pm\alpha_1\beta_1 z^{(\alpha_1+1)/\alpha_1}-\lambda^{1/\alpha_1}x\right)\mathrm{e}^z+\lambda^{1/\alpha_1}x=0,$

and $\hat{t}_0=\frac{1}{\beta_1}\left[1\pm\left(\frac{\lambda}{z}\right)^{1/\alpha_1}\right].$ should be considered.

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