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A New Cubic Transmuted Inverse Weibull Distribution: Theory and Applications

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Abstract

This paper introduces a new cubic transmutation of the inverse Weibull distribution, known as a cubic transmuted inverse Weibull distribution. The model is thought to be useful for the analysis of complex life data, modeling failure times, accessing product reliability, and many other fields like economics, hydrology, biology, and engineering. Some statistical features of the proposed distribution are explored. These include moments, generating functions, quantile functions, reliability functions, and hazard rate functions. The distribution of order statistics for the proposed cubic transmuted inverse Weibull distribution is also studied. The maximum likelihood estimation approach is used to estimate the model parameters. The effectiveness of the estimation is investigated through extensive simulation study. The suitability of the proposed distribution has been studied by using five real-life datasets. It is found that the proposed distribution is the most suitable fit for the used data sets.

Key Words: Inverse Weibull Distribution, Cubic Transmutation, Maximum Likelihood Estimation, Order Statistics, Reliability Analysis.

Mathematical Subject Classification: 60E05, 62E15.

1. Introduction

The probability distributions have been in use for modeling several real-world phenomena. However, it frequently occurs that some real-world issues cannot be adequately modeled by the standard probability models and hence some extension or generalization is required. The generalization of the probability distribution is usually done to gain more flexibility in modeling of the data and to use them in more challenging real-world issues. In this paper, we have expanded the inverse Weibull (*IW*) distribution so that the resulting distribution will enable us to handle more complicated real-world problems. The inverse Weibull distribution has been considered as a useful probability distribution for modeling dependability data and in characterizing degradation phenomena of mechanical components. The distribution was initially discussed by Keller et al. (1982). The cumulative distribution function (*cdf*) of the *IW* distribution is

$$G\left(x;\alpha,\theta\right)=\exp\left[-\left(\theta/x\right)^{\alpha}\right];\ x\in\mathbb{R}^{+}, \tag{1}$$

where $\alpha>0$ is the shape parameter and $\theta>0$ is the rate parameter.

The *IW* distribution has both increasing and decreasing hazard rates depending on the shape parameter. The inverse exponential distribution appears as a special case of the *IW* distribution for $\alpha=1$. Calabria and Pulcini (1990) have estimated the parameters of the *IW* distribution using maximum likelihood and least square estimation techniques. Mahmoud et al. (2003) have obtained order statistics from the inverse Weibull distribution. Sultan (2008) has derived

Bayes estimates of the parameters by using different loss functions. Kundu and Howaldar (2010) have conducted the Bayesian inference and prediction for the *IW* based on Type-II censored data. De Gusmao (2011) has introduced a generalized inverse Weibull distribution. Aryal and Tsokos (2011) have introduced transmuted Weibull distribution and have also discussed its distributional properties. Khan and King (2012) have introduced a five-parameter generalized *IW* distribution. Khan and King (2013) have proposed a transmuted modified Weibull distribution. Khan and King (2014) have also proposed a three-parameter transmuted *IW* distribution. Recently some researchers have also conducted some other research using the *IW* distribution. These include Alslman and Helu (2022), Haj Ahmed et al. (2023), and Tashkandy et al. (2023), among others.

Shahbaz et al. (2012) have introduced and studied a Kumaraswamy inverse Weibull distribution. Basheer (2019) has introduced a new generalized alpha power *IW* distribution. Kumar and Nair (2021) have introduced a generalization of the log-transformed version of the *IW* distribution and have also applied the proposed distribution in cancer research. Hassan and Nassr (2018) have introduced a new family of univariate distribution called the *IW* generated family. A lot of new models can be generated from this new family.

The transmuted family of distributions has been introduced by Shaw and Buckley (2007) and has since been used by several authors to propose new probability distribution. The *cdf* of this family of distributions is given as

$$F(x) = (1+\lambda)G(x) - \lambda G^2(x)\;;\; x \in \mathbb{R}, \tag{2}$$

where $\lambda \in [-1,1]$ is the transmutation parameter and $G(x)$ is the distribution function of any standard probability model.

Rahman et al. (2019) have proposed a cubic-transmuted family of distributions and have used the proposed family to introduce a cubic-transmuted uniform (*CTU*) distribution. The *CTU* distribution can be used as an alternative to the Beta and Kumaraswamy distributions. Many distributional models have already been generated from this family of distribution and have shown better performance than other competing distributions in handling complex datasets. The *cdf* of the cubic transmuted family of distributions, proposed by Rahman et al. (2019), is

$$F(x) = (1-\lambda)G(x) + 3\lambda G^2(x) - 2\lambda G^3(x)\;;\; x \in \mathbb{R}, \tag{3}$$

where $\lambda \in [-1,1]$ is the transmutation parameter.

This study aims to use the family of distributions given in (3) and to generate a new cubic transmuted inverse Weibull (*CTIW*) distribution. The paper also aims to see the effect of the model parameters on the shape and hazard rate function of the distribution. The structure of the paper follows. A new *CTIW* distribution is proposed in Section 2. Statistical properties including the moments, generating functions, quantile functions, random number generation, and reliability function for the proposed *CTIW* distribution are given in Section 3. The distribution of order statistics from the proposed *CTIW* distribution are given in Section 4. Section 5 contains parameter estimation for the *CTIW* distribution. Section 6 is based upon the consistency of the estimation method on the basis of extensive simulation study. Section 7 is based upon some real data application of the proposed *CTIW* distribution. Some concluding remarks are given in Section 8.

2. A New Cubic Transmuted Inverse Weibull Distribution

A random variable X is said to have an inverse Weibull (*IW*) distribution if it has *cdf* as given in (1). The density function corresponding to (1) is given as

$$g(x;\alpha,\theta) = \alpha\theta^{\alpha} x^{-(\alpha+1)} e^{-(\theta/x)^{\alpha}}\;;\; x \in \mathbb{R}^+, \tag{4}$$

where $\alpha > 0$ is the shape parameter and $\theta > 0$ is the rate parameter. Khan and King (2014) have introduced a transmuted inverse Weibull (*TIW*) distribution by using (1) in (2). The *cdf* of *TIW* distribution is

$$F\left(x\right)=\mathrm{e}^{-(\theta/x)^{\alpha}}\left[1+\lambda-\lambda\mathrm{e}^{-(\theta/x)^{\alpha}}\right];\ x\in\mathbb{R}^{+}\ ,$$

where $\lambda\in\left[-1,1\right]$ is the transmutation parameter. A new cubic transmuted inverse Weibull (*CTIW*) is obtained by using (1) in (3). The *cdf* of the proposed *CTIW* distribution is

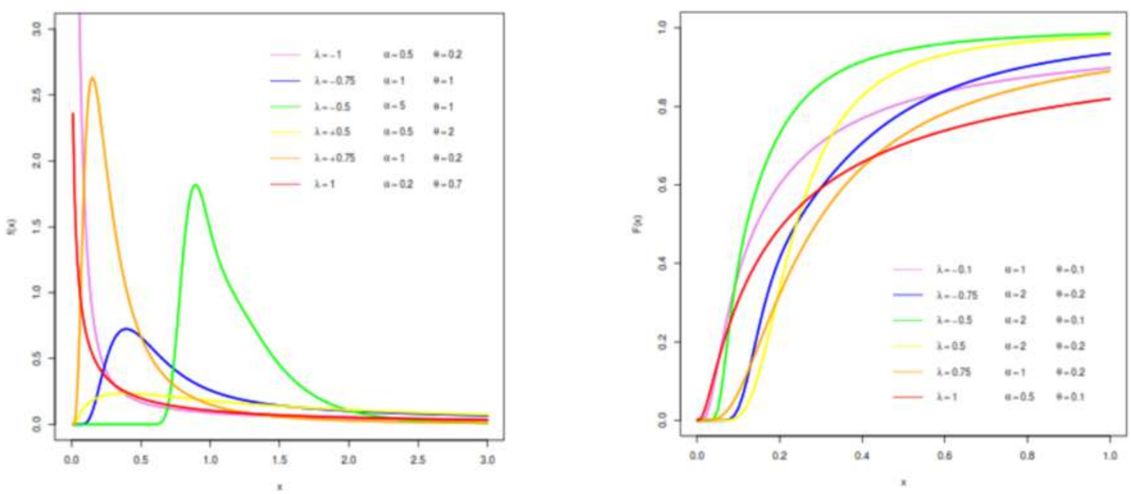
$$F\left(x;\alpha,\theta,\lambda\right)=\mathrm{e}^{-(\theta/x)^{\alpha}}\left[1-\lambda+\lambda\mathrm{e}^{-(\theta/x)^{\alpha}}\left\{3-2\mathrm{e}^{-(\theta/x)^{\alpha}}\right\}\right];\ x\in\mathbb{R}^{+}, \tag{5}$$

where $\alpha>0$ is the shape parameter, $\theta>0$ is the rate parameter, and $\lambda\in\left[-1,1\right]$ is the transmutation parameter. The probability density function (*pdf*) of *CTIW* distribution is readily written, from (5), as

$$f\left(x;\alpha,\theta,\lambda\right)=\alpha\theta^{\alpha}x^{-(\alpha+1)}\mathrm{e}^{-(\theta/x)^{\alpha}}\left[1-\lambda+6\lambda\mathrm{e}^{-(\theta/x)^{\alpha}}\left\{1-\mathrm{e}^{-(\theta/x)^{\alpha}}\right\}\right];\ x\in\mathbb{R}^{+}. \tag{6}$$

It is easy to see that the *CTIW* distribution reduces to the inverse Weibull distribution for $\lambda=0$. A cubic transmuted inverse exponential distribution appears as a special case of *CTIW* distribution for $\alpha=1$, Also, for $\alpha=2$, a cubic transmuted inverse Rayleigh distribution appears as a special case of *CTIW* distribution. Some plots of the *pdf* and *cdf* of *CTIW* distribution for different values of the parameters are given in Figure-1 below.

Figure 1: The Density and Distribution Functions of Cubic Transmuted Inverse Weibull Distribution



(a) The Density Function (b) The Distribution Function

The plot of the density function shows that the distribution is unimodal. We will, now, discuss some useful distributional properties of the proposed *CTIW* distribution.

3. Distributional Properties

In this section, we have discussed some useful properties of *CTIW* distribution. These properties are discussed in the following sub-sections.

3.1 The Moments

The moments are very important in studying certain properties of a distribution. The *r*th moment of a random variable *X* is obtained as

$$\mu_{r}^{/}=E\left(X^{r}\right)=\int_{-\infty}^{\infty}x^{r}dF\left(x\right)=\int_{-\infty}^{\infty}x^{r}f\left(x\right)dx\ .$$

The *r*th moment for *CTIW* distribution is given in the following Theorem.

Theorem 1: The *r*th moment of a random variable *X* having a *CTIW* distribution is given as

$$\mu_r^{/} = \theta^r \Gamma\left(1-\frac{r}{\alpha}\right)\left[1-\lambda\left\{1-3\times 2^{r/\alpha}+2\times 3^{r/\alpha}\right\}\right];\ r<\alpha. \tag{7}$$

Proof: The *r*th moment of *CTIW* distribution is given as

$$\mu_r^{/} = E\left(X^r\right) = \int_0^\infty x^r \left[\alpha\theta^\alpha x^{-(\alpha+1)} \mathrm{e}^{-(\theta/x)^\alpha}\left\{1-\lambda+6\lambda \mathrm{e}^{-(\theta/x)^\alpha}\left(1-\mathrm{e}^{-(\theta/x)^\alpha}\right)\right\}\right]dx.$$

Solving the integral we have (7) and this completes the proof.

The mean of the distribution is easily obtained by using $r = 1$ in (7) and is

$$\mu_1 = \theta\Gamma\left(1-\frac{1}{\alpha}\right)\left[1-\lambda\left\{1-3\times 2^{1/\alpha}+2\times 3^{1/\alpha}\right\}\right],\ \alpha>1.$$

The variance of the *CTIW* distribution can be easily obtained by using $V(X) = \mu_2^{/} - \mu_1^{/2}$, where $\mu_2^{/}$ is obtained by using $r = 2$ in (7). The coefficients of skewness and kurtosis can also be obtained by computing higher order moments from (7).

3.2 The Moment Generating Function

The moment generating function of a random variable is very useful in computing its moments. The moment generating function is defined as

$$M_X(t) = E\left(\mathrm{e}^{tX}\right) = \int_{-\infty}^\infty \mathrm{e}^{tx} dF(x) = \int_{-\infty}^\infty \mathrm{e}^{tx} f(x)dx.$$

The moment generating function of the proposed *CTIW* distribution is given in the following theorem.

Theorem 2: The moment generating function of *CTIW* distribution is

$$M_X(t) = \sum_{r=0}^\infty \frac{t^r}{r!}\theta^r\Gamma\left(1-\frac{r}{\alpha}\right)\left[1-\lambda\left\{1-3\times 2^{r/\alpha}+2\times 3^{r/\alpha}\right\}\right]\ ;\ r<\alpha. \tag{8}$$

Proof: The proof is simple.

A more useful function is the characteristic function and is defined as

$$\phi_X(t) = E\left(\mathrm{e}^{\mathrm{i}tX}\right) = \int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}\,tx} dF(x) = \int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}\,tx} f(x)dx.$$

The characteristic function of *CTIW* distribution is readily written from (8) as

$$\phi_X(t) = \sum_{r=0}^\infty \frac{(\mathrm{i}t)^r}{r!}\theta^r\Gamma\left(1-\frac{r}{\alpha}\right)\left[1-\lambda\left\{1-3\times 2^{r/\alpha}+2\times 3^{r/\alpha}\right\}\right]\ ;\ r<\alpha,$$

where $\mathrm{i} = \sqrt{-1}$.

3.3 Quantile Function and Median

In this section, we will derive the quantile function of *CTIW* distribution. The quantile function of a random variable *X*, having *cdf* $F(x)$, is obtained as a solution of $F(x) = p$ for *x*. Now, for *CTIW* distribution the quantile function is obtained as a solution of

$$\mathrm{e}^{-(\theta/x)^{\alpha}}\left[1-\lambda+\lambda\mathrm{e}^{-(\theta/x)^{\alpha}}\left\{3-2\mathrm{e}^{-(\theta/x)^{\alpha}}\right\}\right]=p\,, \tag{9}$$

for *x*. Writing $\mathrm{e}^{-(\theta/x)^{\alpha}} = w$, the above equation can be written as

$$w\left[1-\lambda+\lambda w(3-2w)\right]=p \text{ or } (1-\lambda)w+3\lambda w^2-2\lambda w^3=p \text{ or } 2\lambda w^3-3\lambda w^2-(1-\lambda)w+p=0$$

or $c_1w^3+c_2w^2+c_3w+p=0$ where $c_1=2\lambda$; $c_2=-3\lambda$ and $c_3=\lambda-1$.

The real solution of the above cubic equation is

$$w=-\frac{c_2}{3c_1}-\frac{2^{1/2}\xi_1}{3c_1\xi_3^{1/3}}+\frac{\xi_3^{1/3}}{3\times2^{1/3}c_1}=\frac{1}{2}\left[1-\frac{2^{1/2}\xi_1}{3\lambda\xi_3^{1/3}}+\frac{\xi_3^{1/3}}{3\lambda\times2^{1/3}}\right]. \tag{10}$$

Using $\mathrm{e}^{-(\theta/x)^{\alpha}} = w$ in (10), and solving for *x*, the quantile function of *CTIW* distribution is

$$\mathrm{e}^{-(\theta/x)^{\alpha}}=\frac{1}{2}\left[1-\frac{2^{1/2}\xi_1}{3\lambda\xi_3^{1/3}}+\frac{\xi_3^{1/3}}{3\lambda\times2^{1/3}}\right] \text{ or } Q_X(p)=-\frac{1}{\theta}\left[\ln\left\{\frac{1}{2}\left(1-\frac{2^{1/2}\xi_1}{3\lambda\xi_3^{1/3}}+\frac{\xi_3^{1/3}}{3\lambda\times2^{1/3}}\right)\right\}\right]^{-1/\alpha}, \tag{11}$$

where $\xi_1=\left|-c_2^2+3c_1c_3\right|=\left|-3\lambda(\lambda+2)\right|$; $\xi_2=-2c_2^3+9c_1c_2c_3-27c_1^2p=54\lambda^2(1-2p)$,

and $\xi_3=\xi_2+\sqrt{4\xi_1^3+\xi_2^2}=54\lambda^2(1-2p)+\sqrt{\left|108\lambda^3\left\{27\lambda(1-2p)^2-(\lambda+2)^3\right\}\right|}$.

The quantiles of *CTIW* can be obtained by using $0<p<1$ in (11). Specifically, the median can be obtained by using *p* = 0.5 in (11) and then solving for *x*. It is to be noted that for *p* = 0.5, the quantity $\xi_2 = 0$ and ξ_3 reduces to $\xi_3=6\sqrt{\left|-3\lambda^3(\lambda+2)^3\right|}$.

3.4 Random Number Generation

The random data can be generated from *CTIW* distribution on the lines of the quantiles. Specifically, a random observation can be generated from any distribution, with *cdf* $F(x)$, by solving $F(x) = u$ for *x* with *u* be a random observation from $U[0,1]$. Now, the random observation from *CTIW* distribution can be generated by solving

$$\mathrm{e}^{-(\theta/x)^{\alpha}}\left[1-\lambda+\lambda\mathrm{e}^{-(\theta/x)^{\alpha}}\left\{3-2\mathrm{e}^{-(\theta/x)^{\alpha}}\right\}\right]=u$$

for *x* where *u* is a random observation from $U[0,1]$. Specifically, a random observation can be generated by using (11) where *p* is replaced by *u*, a random observation from $U[0,1]$.

3.5 Reliability Analysis

The reliability function is a function of time that gives the probability of an item operating for a certain period without failure. It is the complement of a distribution function. The reliability function for the proposed *CTIW* distribution is given as

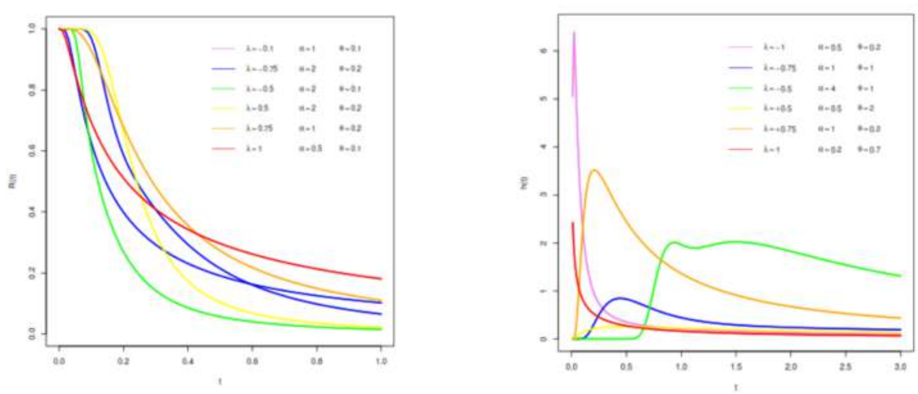
$$R(t)=1-F(t)=\left[1-e^{-(\theta/t)^{\alpha}}\right]\left[1+\lambda e^{-(\theta/t)^{\alpha}}\left\{1-2e^{-(\theta/t)^{\alpha}}\right\}\right];\ t\in\mathbb{R}^{+}. \tag{12}$$

The hazard rate function $h(t)$, also known as the force of mortality rate or failure rate. The hazard rate is the ratio of the density function to the reliability function. The hazard rate function for the *CTIW* distribution is obtained as the ratio of (6) to (12) and is

$$h(t)=\frac{\alpha\theta^{\alpha}x^{-(\alpha+1)}e^{-(\theta/x)^{\alpha}}\left[1-\lambda+6\lambda e^{-(\theta/x)^{\alpha}}\left\{1-e^{-(\theta/x)^{\alpha}}\right\}\right]}{\left[1-e^{-(\theta/t)^{\alpha}}\right]\left[1+\lambda e^{-(\theta/t)^{\alpha}}\left\{1-2e^{-(\theta/t)^{\alpha}}\right\}\right]};\ t\in\mathbb{R}^{+}. \tag{13}$$

The plots of reliability and hazard rate functions, for different values of the parameters, are given in Figure 2 below

Figure 2: The Reliability and Hazard Rate Functions of Cubic Transmuted Inverse Weibull Distribution



(a) The Reliability Function (b) The Hazard Rate Function

The hazard rate plots show that the distribution has increasing as well as decreasing hazard rates.

4. Order Statistics

Order statistics is a useful branch of statistics and helps study the behavior of extremes in random sampling from some probability distribution. In this section we will discuss the distribution of order statistics when a random sample is available from *CTIW* distribution. Suppose that $X_1, X_2, \ldots, X_n$ is a random sample of size *n* from some distribution, then $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ are the order statistics. The distribution of *r*th order statistics is

$$f_{r:n}(x)=\frac{n!}{(r-1)!(n-r)!}f(x)\left[F(x)\right]^{r-1}\left[1-F(x)\right]^{n-r}\ ;\ x\in\mathbb{R}.$$

Now, using the density and distribution functions of *CTIW* distribution, the distribution of *r*th order statistics is

$$\begin{aligned}f_{r:n}(x)=&\frac{n!}{(r-1)!(n-r)!}\alpha\theta^{\alpha}x^{-(\alpha+1)}e^{-(\theta/x)^{\alpha}}\left[1-\lambda+6\lambda e^{-(\theta/x)^{\alpha}}\left\{1-e^{-(\theta/x)^{\alpha}}\right\}\right]\\&\times\left[e^{-(\theta/x)^{\alpha}}\left\{1-\lambda+\lambda e^{-(\theta/x)^{\alpha}}\left(3-2e^{-(\theta/x)^{\alpha}}\right)\right\}\right]^{r-1}\left[\left\{1-e^{-(\theta/t)^{\alpha}}\right\}\left\{1+\lambda e^{-(\theta/t)^{\alpha}}\left(1-2e^{-(\theta/t)^{\alpha}}\right)\right\}\right]^{n-r},\end{aligned} \tag{14}$$

for $x > 0$ and $r = 1, 2, \ldots, n$. The distribution of the smallest and largest observations can be easily obtained by using r = 1 and $r = n$, respectively in (14). It is to be noted that for $\lambda = 0$, in (14), we can obtain the distribution of the rth order statistics from the inverse Weibull distribution.

The moments of order statistics from *CTIW* distribution can be numerically computed from (14) for specific values of the parameters. Specifically, the mean and variance of order statistics for $\theta = 2$, $\alpha = 4$ and different values are given in Table 1 and Table 2 below.

Table 1: Mean of Order Statistics from Cubic Transmuted Inverse Weibull Distribution

λ	*n*	*r* = 1	*r* = 2	*r* = 3	*r* = 4	*r* = 5	*r* = 6	*r* = 7	*r* = 8	*r* = 9	*r* = 10
	1	2.569									
	2	1.978	3.161								
	3	1.786	2.361	3.561							
	4	1.690	2.077	2.644	3.867						
–0.75	**5**	1.631	1.925	2.304	2.872	4.116					
	6	1.591	1.831	2.114	2.494	3.061	4.327				
	7	1.561	1.767	1.992	2.276	2.657	3.222	4.511			
	8	1.539	1.721	1.907	2.133	2.420	2.798	3.364	4.674		
	9	1.520	1.685	1.845	2.031	2.260	2.548	2.924	3.490	4.823	
	10	1.505	1.657	1.797	1.955	2.144	2.376	2.662	3.036	3.603	4.958
	1	2.490									
	2	1.982	2.999								
	3	1.817	2.313	3.341							
	4	1.729	2.078	2.548	3.606						
–0.25	**5**	1.674	1.952	2.268	2.735	3.824					
	6	1.634	1.871	2.114	2.422	2.891	4.010				
	7	1.604	1.813	2.014	2.247	2.553	3.027	4.174			
	8	1.581	1.770	1.943	2.133	2.362	2.667	3.147	4.321		
	9	1.561	1.736	1.889	2.051	2.236	2.463	2.769	3.255	4.454	
	10	1.545	1.708	1.847	1.988	2.144	2.327	2.554	2.861	3.353	4.576
	1	2.411									
	2	1.994	2.829								
	3	1.854	2.274	3.106							
	4	1.777	2.086	2.463	3.320						
0.25	**5**	1.725	1.982	2.242	2.610	3.497					
	6	1.688	1.913	2.120	2.364	2.733	3.650				
	7	1.659	1.862	2.038	2.228	2.467	2.840	3.785			
	8	1.635	1.823	1.979	2.137	2.318	2.556	2.935	3.906		
	9	1.616	1.792	1.933	2.071	2.220	2.397	2.635	3.020	4.017	
	10	1.599	1.766	1.896	2.020	2.148	2.292	2.468	2.707	3.099	4.119
	1	2.332									
	2	2.014	2.651								
	3	1.898	2.245	2.853							
	4	1.832	2.097	2.394	3.007						
0.75	**5**	1.787	2.011	2.225	2.506	3.132					
	6	1.754	1.954	2.127	2.322	2.598	3.239				
	7	1.728	1.911	2.061	2.215	2.402	2.677	3.332			
	8	1.706	1.877	2.011	2.143	2.288	2.471	2.746	3.416		
	9	1.688	1.850	1.973	2.089	2.210	2.351	2.531	2.807	3.492	
	10	1.673	1.827	1.941	2.046	2.152	2.268	2.405	2.584	2.863	3.562

Table 1 above contains the mean of order statistics for different values of *n*, *r,* and λ. From this table we can see that for fixed *n* and λ, the mean of order statistics increases with an increase in *r*. Further, for fixed *r* and λ, the mean of order statistics decreases with an increase in *n*. We can also see, from Table 1, that for fixed *n* and *r*, the mean increases with an increase in λ when $r \le n/2$ and decreases with an increase in λ for $r > n/2$. The variance also shows a trend similar to the mean except for $r = 1$ when n and λ are fixed.

Table 2: Variance of Order Statistics from Cubic Transmuted Inverse Weibull Distribution

λ	n	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6	r = 7	r = 8	r = 9	r = 10
	1	1.665									
	2	0.329	2.302								
	3	0.149	0.469	2.738							
	4	0.087	0.224	0.553	3.092						
–0.75	5	0.058	0.132	0.277	0.608	3.404					
	6	0.043	0.086	0.170	0.311	0.649	3.688				
	7	0.034	0.060	0.114	0.198	0.334	0.683	3.952			
	8	0.028	0.045	0.081	0.137	0.218	0.350	0.715	4.199		
	9	0.024	0.035	0.060	0.100	0.155	0.231	0.362	0.744	4.434	
	10	0.022	0.029	0.046	0.075	0.116	0.168	0.240	0.372	0.773	4.657
	1	1.280									
	2	0.242	1.802								
	3	0.118	0.325	2.188							
	4	0.076	0.154	0.386	2.509						
–0.25	5	0.056	0.095	0.182	0.436	2.790					
	6	0.044	0.067	0.112	0.205	0.478	3.044				
	7	0.037	0.051	0.078	0.126	0.224	0.515	3.277			
	8	0.032	0.041	0.059	0.088	0.138	0.240	0.549	3.495		
	9	0.028	0.034	0.047	0.066	0.096	0.148	0.255	0.581	3.699	
	10	0.025	0.029	0.038	0.052	0.072	0.104	0.157	0.269	0.611	3.893
	1	0.883									
	2	0.170	1.247								
	3	0.093	0.209	1.536							
	4	0.065	0.103	0.243	1.784						
0.25	5	0.051	0.068	0.114	0.275	2.004					
	6	0.043	0.051	0.073	0.125	0.304	2.204				
	7	0.037	0.041	0.053	0.078	0.136	0.332	2.388			
	8	0.033	0.035	0.042	0.056	0.084	0.147	0.358	2.561		
	9	0.030	0.031	0.035	0.044	0.059	0.089	0.157	0.382	2.722	
	10	0.028	0.027	0.030	0.036	0.046	0.063	0.094	0.166	0.405	2.876
	1	0.473									
	2	0.113	0.630								
	3	0.068	0.123	0.760							
	4	0.051	0.068	0.135	0.874						
0.75	5	0.042	0.047	0.071	0.146	0.978					
	6	0.036	0.037	0.048	0.074	0.156	1.074				
	7	0.032	0.031	0.037	0.049	0.078	0.166	1.164			
	8	0.029	0.027	0.030	0.037	0.051	0.081	0.176	1.249		
	9	0.027	0.024	0.025	0.030	0.038	0.053	0.084	0.185	1.330	
	10	0.025	0.021	0.022	0.025	0.030	0.039	0.054	0.088	0.193	1.407

5. Estimation of the Model Parameters

In this section, we have discussed the maximum likelihood estimation of the parameters of *CTIW* distribution. Suppose that a random sample of n observations is available from the *CTIW* distribution. The likelihood function is then

$$L(\alpha,\theta,\lambda;\mathbf{x})=\prod_{i=1}^{n}\alpha\theta^{\alpha}x_i^{-(\alpha+1)}\mathrm{e}^{-(\theta/x_i)^{\alpha}}\left[1-\lambda+6\lambda\mathrm{e}^{-(\theta/x_i)^{\alpha}}\left\{1-\mathrm{e}^{-(\theta/x_i)^{\alpha}}\right\}\right].$$

The log-likelihood function is

$$\begin{aligned}\ell(\alpha,\theta,\lambda;\mathbf{x})&=n\ln\alpha+n\alpha\ln\theta-(\alpha+1)\sum\nolimits_{i=1}^{n}\ln x_i-\sum\nolimits_{i=1}^{n}(\theta/x_i)^{\alpha}\\&+\sum\nolimits_{i=1}^{n}\ln\left[1-\lambda+6\lambda\mathrm{e}^{-(\theta/x_i)^{\alpha}}\left\{1-\mathrm{e}^{-(\theta/x_i)^{\alpha}}\right\}\right].\end{aligned}\tag{15}$$

The maximum likelihood estimates of the parameters are obtained by differentiating (15) with respect to the unknown parameters, equating the derivatives to zero, and simultaneously solving the resulting equations. Now, the derivatives of the log-likelihood function with respect to the unknown parameters are

$$\begin{aligned}\frac{\partial}{\partial\alpha}\ell(\alpha,\theta,\lambda;\mathbf{x})&=\frac{n}{\alpha}+n\ln\theta-\sum\nolimits_{i=1}^{n}\ln x_i-\sum\nolimits_{i=1}^{n}(\theta/x_i)^{\alpha}\ln(\theta/x_i)\\&+6\lambda\sum_{i=1}^{n}\frac{(\theta/x_i)^{\alpha}\ln(\theta/x_i)\mathrm{e}^{-(\theta/x)^{\alpha}}\left(1-2\mathrm{e}^{-(\theta/x)^{\alpha}}\right)}{1-\lambda+6\lambda\mathrm{e}^{-(\theta/x_i)^{\alpha}}\left(1-\mathrm{e}^{-(\theta/x_i)^{\alpha}}\right)},\end{aligned}\tag{16}$$

$$\frac{\partial}{\partial\theta}\ell(\alpha,\theta,\lambda;\mathbf{x})=\frac{n\alpha}{\theta}-\sum_{i=1}^{n}\frac{\alpha}{x_i}\left(\frac{\theta}{x_i}\right)^{\alpha-1}-\frac{6\alpha\lambda}{\theta}\sum_{i=1}^{n}\frac{(\theta/x_i)^{\alpha}\mathrm{e}^{-(\theta/x)^{\alpha}}\left(1-2\mathrm{e}^{-(\theta/x)^{\alpha}}\right)}{1-\lambda+6\lambda\mathrm{e}^{-(\theta/x_i)^{\alpha}}\left(1-\mathrm{e}^{-(\theta/x_i)^{\alpha}}\right)},\tag{17}$$

and

$$\frac{\partial}{\partial\lambda}\ell(\alpha,\theta,\lambda;\mathbf{x})=\sum_{i=1}^{n}\frac{6\mathrm{e}^{-(\theta/x_i)^{\alpha}}\left(1-\mathrm{e}^{-(\theta/x_i)^{\alpha}}\right)-1}{1-\lambda+6\lambda\mathrm{e}^{-(\theta/x_i)^{\alpha}}\left(1-\mathrm{e}^{-(\theta/x_i)^{\alpha}}\right)}.\tag{18}$$

The maximum likelihood estimators of α,θ and λ can be obtained by equating the derivatives in (16)–(18) to zero and numerically solving the resulting equations. The asymptotic distribution of maximum likelihood estimators is given as; see for example Rehman et al. (2018a,2018b) and Sarhan and Zaindin (2009);

$$\begin{bmatrix}\hat{\alpha}\\\hat{\theta}\\\hat{\lambda}\end{bmatrix}\ is\ N_3\left(\begin{bmatrix}\alpha\\\theta\\\lambda\end{bmatrix};\begin{bmatrix}\hat{V}_{11}&\hat{V}_{12}&\hat{V}_{13}\\&\hat{V}_{22}&\hat{V}_{23}\\&&\hat{V}_{33}\end{bmatrix}\right),$$

where entries of the variance–covariance matrix can be obtained by inverting the Fisher information matrix; whose entries are given as $\left[I_{ij}\right]=\left[-nE\left(\partial^2\ell/\partial\theta_i\partial\theta_j\right)\right]$, where (θ_i,θ_j) is (*i,j*)th element of $\left[\alpha\quad\theta\quad\lambda\right]'$.

6. Simulation Study

In this section, we have presented a simulation study to see the performance of the estimates. The simulation study is conducted by drawing random samples of sizes 50, 100, 200, 500, and 1000 from *CTIW* distribution with $\alpha=1$, $\theta=3$ and $\lambda=-1$. For each generated sample of a specific size, the maximum likelihood estimates of the unknown parameters are computed; by using the "*bbmle*" package of *R*, see Bolker and Bolker (2017); and then the average estimate is computed by using

$$\bar{\hat{\alpha}}=\frac{1}{10000}\sum_{j=1}^{10000}\hat{\alpha}_j\ ;\ \bar{\hat{\theta}}=\frac{1}{10000}\sum_{j=1}^{10000}\hat{\theta}_j\ \text{ and }\ \bar{\hat{\lambda}}=\frac{1}{10000}\sum_{j=1}^{10000}\hat{\lambda}_j\ .$$

The mean square errors of the estimates are computed as

$$MSE\left(\hat{\alpha}\right)=\frac{1}{10000}\sum_{j=1}^{10000}\left(\hat{\alpha}_j-\bar{\hat{\alpha}}\right)^2 ;MSE\left(\hat{\theta}\right)=\frac{1}{10000}\sum_{j=1}^{10000}\left(\hat{\theta}_j-\bar{\hat{\theta}}\right)^2$$

and $$MSE\left(\hat{\lambda}\right)=\frac{1}{10000}\sum_{j=1}^{10000}\left(\hat{\lambda}_j-\bar{\hat{\lambda}}\right)^2 .$$

The results of the simulation study are given in Table 3 below

Table 3: Average and Mean Square Error of Estimates for Different Sample Sizes

Sample Size	Estimate			Mean Square Error		
	α	ϴ	λ	α	ϴ	λ
50	0.990	3.030	–0.818	0.017	0.252	0.262
100	0.996	3.014	–0.895	0.007	0.115	0.102
200	0.994	3.008	–0.927	0.003	0.055	0.046
500	0.996	3.000	–0.952	0.001	0.021	0.017
1000	0.996	3.002	–0.967	0.001	0.010	0.008

The results of Table 3 show that the estimates converges to the population parameters as the sample size increases. We can also see, from the above table, that the mean square error of the estimates decreases with an increase in the sample size.

7. Real Data Applications

In this section, we have used some real data sets to see the suitability of the proposed *CTIW* distribution. We have used the following five data sets for the analysis

1. Carbon Fiber Data used by Badar and Priest (1982), Abu El Azm, et al. (2021)
2. Breast Cancer Data used by Mansour et al. (2015) and Pobočíková et al. (2018)
3. Repairable Items Data used by Murthy et al. (2004)
4. Glass Fiber Data used by Afify et al. (2017)
5. Turbocharger Data used by Xu et al. (2003) and by Afify et al. (2017)

Some useful summary measures of these data sets are given in Table 4 below

Table 4: Summary Measures of Various Data Sets

Data	Min	Q_1	Median	Q_3	Max	Mean	Skewness	Kurtosis
Carbon Fiber	1.312	2.098	2.478	2.773	3.585	2.451	–0.027	–0.144
Breast Cancer	12.000	36.000	42.000	50.000	90.000	43.650	0.678	2.096
Repairable Items	0.110	0.717	1.235	1.942	4.730	1.542	1.231	1.036
Glass Fiber	0.550	1.375	1.590	1.958	2.240	1.507	–0.878	0.800
Turbocharger	1.600	5.075	6.500	7.825	9.000	6.253	–0.638	–0.489

We have fitted the *CTIW* distribution alongside the transmuted inverse Weibull (TIW) and inverse Weibull (IW) distributions on these data sets. The goodness of fit of the distributions has been determined by computing Akike information criteria (*AIC*), corrected *AIC* (*AICc*), and Bayesian information criteria (*BIC*); see Brewer et al. (2016) for these criteria. The maximum likelihood estimates of the model parameters of *CTIW*, *TIW,* and *IW* distributions for different data sets are given in Table 5 below. From this table, we can see that the *CTIW* distribution seems a reasonable fit for all the data sets as it has the smallest value of the log–likelihood function in comparison with the other two distributions. The values of various fitted criteria for different distributions and for different data sets are given in

Table 6. The results of this table also indicate that the *CTIW* is the best fit for all five data sets. We have, also, constructed the plots of empirical distribution functions for the five data sets alongside the fitted distribution functions for different distributions. These plots are given in Figure 3 below. The figures also indicate that the *CTIW* distribution is the best fit for all of the data sets.

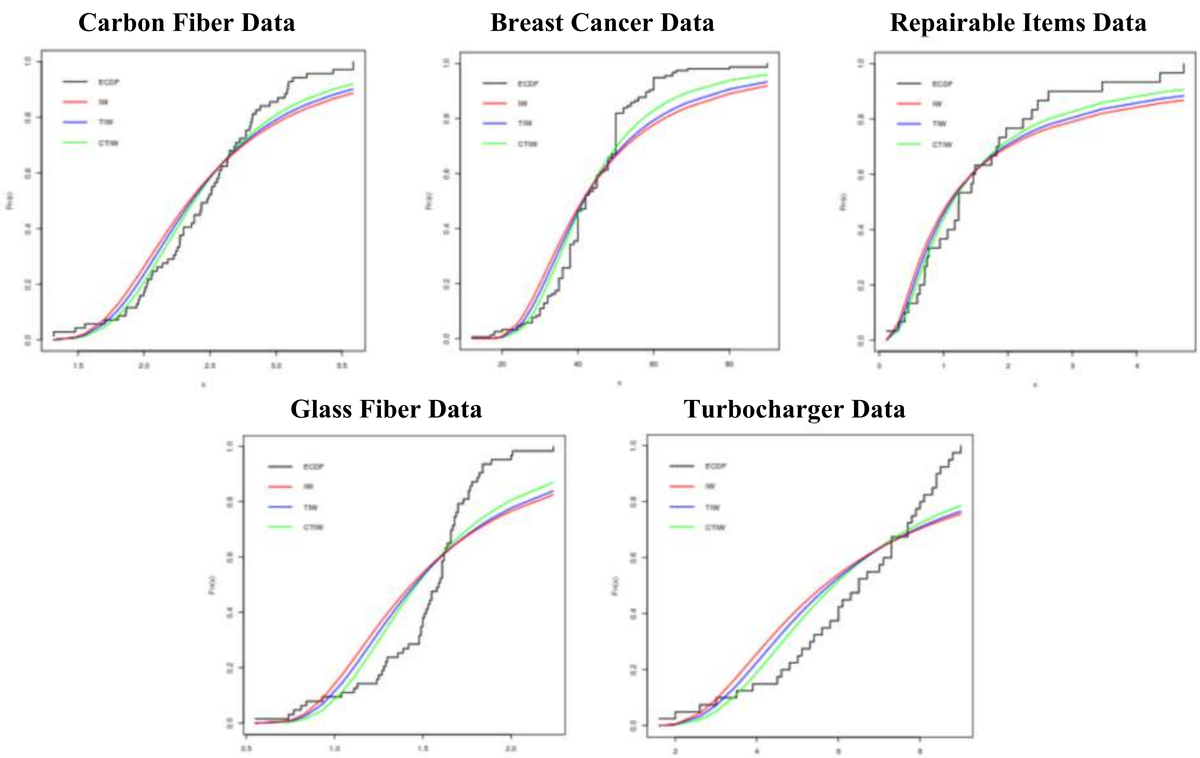
Table 5: Maximum Likelihood Estimates of Model Parameters of Different Distributions

Data Sets	Distribution	Parameter	Estimate	SE	Log-likelihood
Carbon Fiber	*CTIW*	α	3.500	0.357	–57.852
		ϴ	2.144	0.061	
		λ	0.803	0.158	
	TIW	α	4.588	0.374	–60.609
		ϴ	1.939	0.067	
		λ	–0.751	0.160	
	IW	α	4.127	0.338	–63.624
		ϴ	2.144	0.066	
Breast Cancer	*CTIW*	α	2.369	0.137	–624.96
		ϴ	35.638	0.949	
		λ	0.876	0.023	
	TIW	α	2.990	0.149	–636.91
		ϴ	29.989	0.961	
		λ	–0.854	0.078	
	IW	α	2.678	0.135	–646.52
		ϴ	35.855	1.144	
Repairable Items	*CTIW*	α	0.921	0.129	–43.483
		ϴ	0.765	0.120	
		λ	0.824	0.198	
	TIW	α	1.197	0.145	–44.893
		ϴ	0.507	0.095	
		λ	–0.799	0.212	
	IW	α	1.073	0.131	–46.376
		ϴ	0.766	0.139	
Glass Fiber	*CTIW*	α	2.537	0.236	–38.487
		ϴ	1.268	0.052	
		λ	0.840	0.136	
	TIW	α	3.221	0.256	–43.151
		ϴ	1.094	0.056	
		λ	–0.774	0.156	
	IW	α	2.887	0.234	–46.853
		ϴ	1.264	0.058	
Turbocharger	*CTIW*	α	1.708	0.203	–97.012
		ϴ	4.738	0.361	
		λ	0.797	0.176	
	TIW	α	2.176	0.223	–99.393
		ϴ	3.815	0.347	
		λ	–0.743	0.194	
	IW	α	1.944	0.203	–101.591
		ϴ	4.672	0.405	

Table 6: Selection Criteria for Different Distributions

Data Set	Distribution	–2LL	AIC	AICc	BIC
Carbon Fiber	*CTIW*	115.705	121.705	122.074	128.407
	TIW	121.219	127.219	127.588	133.921
	IW	127.247	131.247	131.429	135.715
Breast Cancer	*CTIW*	1249.927	1255.927	1256.086	1265.057
	TIW	1273.820	1279.820	1279.979	1288.950
	IW	1293.037	1297.037	1297.116	1303.124
Repairable Items	*CTIW*	86.965	92.965	93.888	97.169
	TIW	89.786	95.786	96.709	99.989
	IW	92.751	96.751	97.196	99.554
Glass Fiber	*CTIW*	76.975	82.975	83.382	97.169
	TIW	86.303	92.303	92.710	99.989
	IW	93.707	97.707	97.907	99.554
Turbocharger	*CTIW*	194.025	200.025	200.691	205.091
	TIW	198.786	204.786	205.453	209.853
	IW	203.184	207.184	207.508	210.561

Figure 3: Empirical and Fitted Distribution Functions



8. Conclusions

In this paper, we have proposed a new cubic transmuted inverse Weibull (*CTIW*) distribution by adding an extra parameter to the well-known inverse Weibull (*IW*) distribution. Some useful properties of the proposed *CTIW* distribution are discussed. They include explicit expansions of moments, the quantile and the generating functions, reliability analysis, etc. The distribution of order statistics for *CTIW* distribution is also obtained alongside numerical computation of mean and variance of different order statistics for different sample sizes and for different values of the parameters. The model parameters have been estimated by using the maximum likelihood estimation method. A simulation study has been conducted to see the consistency of the estimation process. It has been shown, by using five real data sets that the proposed model is a better fit than other competitive models. We hope that the proposed *CTIW* distribution may be highly competitive in real-life data sets and will be extensively used in different areas of statistics.

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