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On estimation and monitoring of population mean using systematic sampling under an exponentially weighted moving average scheme

Amber Karim[1], Hina Khan[1*] Yasar Mahmood[1], Muhammad Riaz[2], Shabbir Ahmad[3]

* Corresponding Author

1. Department of Statistics, Government College University, Lahore, Pakistan, amberkarim522@gmail.com, hinakhan@gcu.edu.pk, syed.yasar@gcu.edu.pk
2. Department of Mathematics, King Fahd University of Petroleum and Minerals, Saudi Arabia. riazm@kfupm.edu.sa
3. Department of Mathematics, COMSATS University, Islamabad. shabbirahmad786@yahoo.com

Abstract

The present study proposes a generalized ratio estimator for estimating the population mean under the systematic sampling technique by considering auxiliary information and auxiliary attribute. Its bias and Mean Square Error (MSE) expressions have been derived. Mathematical comparisons are made by comparing the proposed estimator with the usual mean estimator, Swain (1964) estimator, Bhal and Tuteja (1991) estimator, and Singh and Singh (1998) estimator, and it is shown that the proposed estimator is more efficient than the previous estimators. A numerical comparison is also performed to demonstrate the superiority of the proposed estimator over the traditional estimators. The technique of ratio estimators based on systematic sampling is used to design an Exponentially Weighted Moving Average (EWMA) control chart. The Control chart is a significant industrial tool for monitoring the process mean. To evaluate performance efficiency Average run lengths (ARL) are obtained in this study. The proposed charts are compared based on out-of-control ARLs. A chart based on the proposed estimator is superior as it detects the shifts earlier than charts based on existing estimators. Empirical work is done to support the study. The suggested efficiency is further addressed utilizing real-life examples and simulations using R-Studio.

Key Words: Systematic sampling, Auxiliary attribute, Mean square error, Exponentially weighted moving average control chart, Average run length.

1. Introduction

Ratio estimators are generally used for the estimation of population mean or total by using preceding information on an auxiliary variable x that correlates with y (Shabbir and Yaab 2003). The concept of using auxiliary information is commonly used in estimation and survey sampling techniques. The usage of auxiliary information with ratio estimators causes a significant reduction in the estimator's variance therefore it will raise its precision (Khan 2017). Khan (2017) proposed a ratio type estimator under the Systematic Sampling technique. (Hasel 1942, Griffth 1945, Griffth 1946) studied the efficiency of Systematic Sampling in certain natural populations and found it convenient. Lahiri (1954) discussed in detail the estimation in Systematic Sampling. The auxiliary information in the structure of estimators was used by (Kushwaha and Singh 1989, Banarasi et al. 1993, Singh and Singh 1998).

The terminology of Khan (2017) has been used by considering the finite population N from which a sample of n is taken under the systematic sampling technique. It is presumed that population size and sample size are multiples of each other and k is an interval of a sampling $(N = nk)$. Let the study variable y, auxiliary variable x, and auxiliary attribute ϕ have observations y_i, x_i and ϕ_i correspondingly for the i^{th} unit $(i = 1,2, \dots \dots N)$. Examples of such are.

- If y_i is the production of apples in the year 1998, then x_i will be the area of production of apples in the year 1999 where $N = 204, n = 17$ and $k = \frac{N}{n} = 12$. (Khan 2017)
- If y_i is the profit of a certain cardholder company then x_i will be the income of the cardholder of units and ϕ_i maybe gender where $N = 20, n = 5$ and $k = \frac{N}{n} = 4$. (Ronald E. Walpole et al. 9th ed)

Moreover $(\bar{Y}, \bar{X})$ are the population means and $(\bar{y}, \bar{x})$ are sample means of variables y and x respectively and $P_\phi = \frac{A}{N}$ where $A = \sum_{i=1}^{N} \phi_i$ for population, $p_\phi = \frac{a}{n}$ where $a = \sum_{i=1}^{n} \phi_i$ for the sample are proportion units possessing ϕ_i attribute.

The auxiliary information can be used in control charts (CC) to increase their performance efficiency. The CC is used to monitor the variability in the process mean. Riaz (2008) presented the plotting statistic of a CC using auxiliary information at the time of estimation. He used the regression type estimator in CC to monitor the process mean and shows the dominance of his chart over Shewhart type CC. Roberts (1959) was the first who notices the change in the process mean by introducing EWMA statistic. Abbas et al (2014) has used an EWMA-Type CC to examine the process mean having auxiliary information. Noor-ul-Amin et al (2018) proposed EWMA CC having auxiliary information in the form of dual parametric ratio estimators.

The ratio estimation technique for the mean under Systematic Sampling design can be used in designing the control structure of the proposed charts. The existing studies do not have the estimators based on the Systematic Sampling scheme to propose CC. Thus, in this paper, the structure of EWMA CC's is introduced based on ratio-type estimators under SS such as (Swain 1964, Bahl and Tuteja 1991, Singh, and Singh 1998), and the proposed generalized systematic estimator. The ARL values are used to evaluate the performance of CC when there are slight shifts in the process mean where ARL is the average of samples till the shift is detected. The in-control and out of control ARL's are represented by ARL_0 and ARL_1 respectively.

Section 2 presents notations that we will use in estimators. Section. 3 comprises the existing estimators while section 4 gives details of the proposed estimator. Section 5 consists of mathematical comparisons of proposed estimators with traditional estimators. Section 6 contains a structure of classical EWMA CC. Section 6.1. holds the design of the proposed EWMA CC and 6.2 describes the performance method used for evaluation whereas in section 6.3, an algorithm of the construction process is given. Section 7 has numerical work with sub sections 7.1 containing mean square error and the subsection 7.2 involves percentage relative efficiencies comparing proposed EWMA CCs based on Ratio Estimators. Lastly, the article is concluded in Section 8.

2. Notations

To attain the bias and Mean Square Error (MSE) of different estimators, we need the relative error terms and their expectations which are given below.

Let $e_y = \frac{\bar{y}-\bar{Y}}{\bar{Y}}$, $e_x = \frac{\bar{x}-\bar{X}}{\bar{X}}$ and $e_\phi = \frac{p_\phi - P_\phi}{P_\phi}$

Such that $E(e_y) = 0, E(e_x) = 0$ and $E(e_\phi) = 0$

In a first-order approximation, we have

$$E(e_y^2) = \theta q_y C_y^2 \, , E(e_x^2) = \theta q_x C_x^2 \, , E(e_y e_x) = \theta \sqrt{q_y}\sqrt{q_x}\,\rho_{xy}\, C_y\, C_x$$

$$E(e_y e_\phi) = \theta \sqrt{q_y}\sqrt{q_\phi}\,\rho_{\phi y}\, C_y\, C_\phi \, , E(e_\phi e_x) = \theta \sqrt{q_\phi}\sqrt{q_x}\,\rho_{x\phi}\, C_\phi\, C_x$$

where $\theta = \frac{1}{n}\left(\frac{N-1}{N}\right)$, $q_y = \left(1 + (n-1)\rho_y\right)$, $q_x = (1 + (n-1)\rho_x)$

$$q_\phi = \left(1 + (n-1)\rho_\phi\right) \, and \quad C_y^2 = \frac{S_y^2}{\bar{Y}^2} \, , C_x^2 = \frac{S_x^2}{\bar{X}^2} \, , C_\phi^2 = \frac{S_\phi^2}{P_\phi^2}$$

$$\rho_y = \frac{E\{(y_{ij} - \bar{Y})(y_{iv} - \bar{Y})\}}{E(y_{ij} - \bar{Y})^2} \, , \quad \rho_x = \frac{E\{(x_{ij} - \bar{X})(x_{iv} - \bar{X})\}}{E(x_{ij} - \bar{X})^2} \, , \quad \rho_\phi = \frac{E\{(\phi_{ij} - P_\phi)(\phi_{iv} - P_\phi)\}}{E(\phi_{ij} - P_\phi)^2} \, ,$$

$$j \neq v$$

$$\rho_{yx} = \frac{S_{xy}}{S_x S_y};\ \rho_{y\phi} = \frac{S_{y\phi}}{S_\phi S_y}\ ;\ \rho_{x\phi} = \frac{S_{x\phi}}{S_x S_\phi}$$

$S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{Y})^2\ ;\ S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{X})^2\ ;\ S_\phi^2 = \frac{NP_\phi Q_\phi}{N-1}$ where $P_\phi = 1 - Q_\phi$

$$S_{yx} = \frac{1}{N-1}\left(\sum_{i=1}^{N} y_i x_i - N\bar{X}\bar{Y}\right)\ ;\ S_{y\phi} = \frac{1}{N-1}\left(\sum_{i=1}^{N} y_i \phi_i - NP_\phi \bar{Y}\right)$$

$$S_{x\phi} = \frac{1}{N-1}\left(\sum_{i=1}^{N} x_i \phi_i - NP_\phi \bar{X}\right)$$

3. Some Existing Estimators

Now we will discuss some existing estimators that are present in the literature.

i. The classical sample mean ($\bar{y}$) estimator variance under systematic sampling is given as

$$Var(\bar{y}) = \theta \bar{Y}^2 q_y C_y^2 \tag{1}$$

ii. Swain (1964) had used the systematic sampling in usual ratio estimator given as

$$\bar{y}_r = \frac{\bar{y}}{\bar{x}}\bar{X} \tag{2}$$

Bias under systematic sampling was given as

$$B(\bar{y}_r) = \theta \bar{Y} q_x (1 - K_X \rho_x^*) C_x^2 \tag{3}$$

And MSE at first order given as

$$MSE(\bar{y}_r) = \theta \bar{Y}^2 q_x [\rho_x^{2*} C_y^2 + (1 - 2K_X \rho_x^*) C_x^2] \tag{4}$$

where $\rho_x^* = \sqrt{\frac{q_y}{q_x}}$ and $K_X = \rho_{xy}\frac{C_y}{C_x}$

iii. Bahl and Tuteja (1991) had given an exponential ratio type estimator given as

$$\bar{y}_{bt} = \bar{y} exp \frac{\bar{X} - \bar{x}}{\overline{\overline{X + \bar{x}}}} \tag{5}$$

The bias and MSE at first-order approximation under systematic sampling are given as

$$B(\bar{y}_{bt}) = -\theta \bar{Y} q_x C_x^2 \frac{1}{8}(1 + 4K_X \rho_x^*) \tag{6}$$

$$MSE(\bar{y}_{bt}) = \theta\, \bar{Y}^2 q_x [\rho_x^{2*} C_y^2 + \left(\frac{1}{4} - K_X \rho_x^*\right) C_x^2] \tag{7}$$

iv. Singh and Singh (1998) had given the following estimator

$$\bar{y}_{ss} = \sum_{i=1}^{3} w_i\, d_i^{'} \tag{8}$$

where $d_1 = \bar{y}$, $d_2 = \bar{y}\frac{\bar{X}}{\bar{x}}$, $d_3 = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)^2$

and $w_1 = (1 - K_X)^2,\ w_2 = (3 - 2K_X)K_X,\ w_3 = K_X(K_X - 1)$

such that $\sum_{i=1}^{3} w_i = 1$ for $w_i \in$ R and w_i (i = 1,2,3) denotes the constants whose values are to be determined.

MSE of the above estimator under systematic sampling given as

$$MSE(\bar{y}_{ss}) = \theta \bar{Y}^2\, q_y C_y^2 \left[1 - \rho_{xy}{}^2\right] \tag{9}$$

v. Singh et al (2007) had given the Bahl and Tuteja (1991) estimator with the auxiliary attribute as

$$\bar{y}_s = \bar{y} exp \frac{P_\Phi - p_\Phi}{P_\Phi + p_\Phi} \tag{10}$$

The bias and MSE at first-order approximation under systematic sampling are given as

$$B(\bar{y}_s) = -\theta \bar{Y} q_\Phi C_\Phi^2 \frac{1}{8}(1 + 4K_\Phi \rho_\Phi^*) \tag{11}$$

$$MSE(\bar{y}_s) = \theta\, \bar{Y}^2 q_\Phi [\rho_\Phi^{2*} C_y^2 + \left(\frac{1}{4} - K_\Phi \rho_\Phi^*\right) C_\Phi^2] \tag{12}$$

where $\rho_\Phi^* = \sqrt{\frac{q_y}{q_\Phi}}$ and $K_\Phi = \rho_{y\Phi} \frac{C_y}{C_\Phi}$

vi. Noor-ul-Amin et al (2018) gave an EWMA control chart using two parametric ratio estimators. They used the following estimator

$$M_d = \frac{\bar{Y} + k(\mu_X - \bar{X})}{\alpha \bar{X} + \beta}(\alpha \mu_X + \beta) \tag{13}$$

where α and β are parameters of the auxiliary variable whose information is known in advance and k is an optimizing constant.

The MSE for the above estimator is

$$MSE(M_d) = \sigma_m^2 = \frac{1}{n}(\sigma_Y^2(1 - \rho_{XY}{}^2)) \tag{14}$$

4. Proposed Estimator

Motivated by Singh and Singh (1998) and Khan (2017) estimators the following generalized class of unbiased exponential type estimators has a qualitative and quantitative variable for the population mean ($\bar{Y}$) under systematic sampling scheme is proposed, which is presented as:

$$\bar{y}_{gs} = \sum_{i=1}^{3} \alpha_i\, w_i' \tag{15}$$

where $w_1 = \bar{y},\ w_2 = \bar{y}\ exp\left(\frac{P_\Phi - p_\Phi}{P_\Phi + p_\Phi}\right),\ \ w_3 = \bar{y}\ exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + (a-1)\bar{x}}\right)$

such that $\sum_{i=1}^{3} \alpha_i = 1$

where $\alpha_i \epsilon\ R\ (i = 1, 2, 3)$ are the constants, whose values are to be determined.

Rewriting Eq. (15), we have

$$\bar{y}_{gs} = \alpha_1 \bar{y} + \alpha_2 \bar{y}\, exp\left(\frac{P_\phi - p_\phi}{P_\phi + p_\phi}\right) + \alpha_3 \bar{y}\, exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + (a-1)\bar{x}}\right) \tag{16}$$

Expressing Eq. (16) in terms of $e's$, to a first-order approximation, we have

$$\bar{y}_{gs} = \left(\bar{Y} e_y + \bar{Y}\right)\left[\alpha_1 + \alpha_2 \exp\left(\frac{-e_\phi}{2}\left(1 + \frac{e_\phi}{2}\right)^{-1}\right) + \alpha_3 \exp\left(\frac{-e_x}{a}\left(1 + \frac{e_x(a-1)}{a}\right)^{-1}\right)\right] \tag{17}$$

$$\bar{y}_{gs} - \bar{Y} = \bar{Y}\left[e_y - \alpha_2 \frac{e_\phi}{2} - \frac{\alpha_3}{a} e_x\right] \tag{18}$$

Squaring both sides of Eq. (17) and taking the expectations, we have

$$E(\bar{y}_{gs} - \bar{Y})^2 = \theta \bar{Y}^2 \left[q_y C_y^2 + \frac{{\alpha_2}^2}{4} q_\phi C_\phi^2 + \frac{{\alpha_3}^2}{4} q_x C_x^2 + \frac{\alpha_2 \alpha_3}{2} \sqrt{q_\phi}\sqrt{q_x} C_\phi C_x \rho_{\phi x} - \alpha_2 \sqrt{q_\phi}\sqrt{q_y} C_y C_\phi \rho_{\phi y} - 2\frac{\alpha_3}{2}\sqrt{q_x}\sqrt{q_y} C_y C_x \rho_{xy}\right] \tag{19}$$

From Eq. (19) the optimum value of α_3 is

$$\alpha_3 = 2\sqrt{\frac{q_y}{q_x}} \frac{C_y}{C_x}\left(\rho_{xy} - \rho_{x\phi}\rho_{\phi y}\right) \Big/ \left(1 - {\rho_{\phi x}}^2\right)$$

From Eq. (19) the optimum value of α_2 is

$$\alpha_2 = 2\frac{\sqrt{q_y}}{\sqrt{q_\phi}} \frac{C_y}{C_\phi}\left[\rho_{\phi y} - \rho_{\phi x}\rho_{xy}\right] \Big/ \left(1 - {\rho_{x\phi}}^2\right)$$

Substituting the value of ‘ α_3’ and ‘α_2’in Eq. (19) and simplifying, we have

$$MSE(\bar{y}_{gs}) = \theta \bar{Y}^2\, q_y C_y^2 \left[1 - R_{y.x\phi}^2\right] \tag{20}$$

where $R_{y.x\phi}^2 = \frac{\rho_{y\phi}^2 + \rho_{x\phi}^2 + 2\rho_{y\phi}\rho_{x\phi}\rho_{yx}}{1 - \rho_{x\phi}^2}$

Now again considering Eq. (17) and taking expectations we have

$$Bias = \theta \bar{Y}\left(\frac{\alpha_2}{2} q_\phi C_\phi^2 \left[\frac{3}{4} - \rho_\phi^* K_\phi\right] + \frac{\alpha_3}{a} q_x C_x^2 \left[a + 1 - \rho_x^* K_x\right]\right) \tag{21}$$

where $K_\phi = \frac{C_y}{C_\phi}\rho_{y\phi}$, $K_x = \rho_{xy}\frac{C_y}{C_x}$ and $\rho_\phi^* = \sqrt{\frac{q_y}{q_\phi}}$, $\rho_x^* = \sqrt{\frac{q_y}{q_x}}$, $a = \frac{\alpha_3}{\frac{\sqrt{q_y} C_y}{\sqrt{q_x} C_x}\rho_{xy} - \frac{\alpha_2}{2}}$

5. Mathematical Comparisons

The generalized systematic proposed estimator is better for practice in real life as compared to existing estimators if it is verified that it has the smallest MSE. So, for this, the mathematical conditions are necessary to be derived. Thus, the generalized systematic estimator will be more efficient if and only if: (Detailed proofs are shown in ‘‘Appendix Section I’’)

Condition (i) The proposed estimator will be more efficient than the classical mean estimator under systematic sampling if

$Var(\bar{y}_{gs}) \leq Var(\bar{y})$ if

$$R^2_{y.x\phi} \geq 0 \tag{22}$$

where $R^2_{y.x\phi} = \frac{\rho^2_{y\phi}+\rho^2_{x\phi}+2\rho_{y\phi}\rho_{x\phi}\rho_{yx}}{1-\rho^2_{x\phi}}$

Condition (ii.) The proposed estimator will be more efficient than Swain (1964) estimator under systematic sampling if

$Var(\bar{y}_{gs}) \leq Var(\bar{y}_r)$ if

$$(1-2K_x\rho_x^*)\frac{C_x^2}{C_y^2\rho_x^{*2}} + R^2_{y.x\phi} \geq 0 \tag{23}$$

Condition (iii.) The proposed estimator will be more efficient than Singh and Singh (1998) estimator under systematic sampling if

$Var(\bar{y}_{gs}) \leq Var(\bar{y}_{ss})$ if

$$R^2_{y.x\phi} - {\rho_{xy}}^2 \geq 0 \tag{24}$$

Condition (iv.) The proposed estimator will be more efficient than Bahl and Tuteja (1991) estimator under systematic sampling if

$Var(\bar{y}_{gs}) \leq Var(\bar{y}_{bt})$ if

$$\left(\frac{1}{4} - K_x\rho_x^*\right)\frac{C_x^2}{C_y^2\rho_x^{*2}} + R^2_{y.x\phi} \geq 0 \tag{25}$$

6. The Design Structure of the EWMA Control Chart

Let us consider X a random variable following a normal distribution with mean μ and standard deviation σ. Then EWMA statistic given by Roberts is defined as

$$Z_i = \lambda x_i + (1-\lambda)Z_{(i-1)} \tag{26}$$

where $0 < \lambda < 1$ is the smoothing constant, $Z_{(i-1)}$ representing past information and Z_0 is the initial value taken as the target mean. Its variance is given as

$$Var(Z_i) = \sigma^2 \frac{\lambda}{(2-\lambda)}[1-(1-\lambda)^{2i}] \ ; i = 1,2, \ldots. \tag{27}$$

From Eq. (27), when $i \to \infty$ then asymptotic variance

$$Var(Z_i) = \sigma^2 \frac{\lambda}{(2-\lambda)} \tag{28}$$

Hence, the time varying and asymptotic control limits of the EWMA control chart are following:

$$UCL/LCL = \mu_0 \pm L\sqrt{\sigma^2 \frac{\lambda}{(2-\lambda)}[1-(1-\lambda)^{2i}]} \tag{29}$$

$$UCL/LCL = \mu_0 \pm L\sqrt{\sigma^2 \frac{\lambda}{(2-\lambda)}} \tag{30}$$

The in-control region is between UCL and LCL and L is the coefficient of control limits.

6.1. The Structure of the Proposed EWMA Control Chart

EWMA is aimed to identify minor shifts in the process mean and detect the process variability and find out the change in the process mean. Here we have proposed the EWMA control chart using different ratio estimators along with $\bar{y}_{gs}$ proposed estimator under systematic sampling.

The sequence $\bar{y}_{ki}$ is based on $\bar{y}_{opt\ ki}$ (where $k = 1,2,3,4$) using the recurrence formula is termed as

$$\bar{y}_{ki} = \lambda \bar{y}_{optki} + (1-\lambda)\bar{y}_{k(i-1)} \tag{31}$$

where $0 < \lambda \leq 1$

So, $\bar{y}_{ki}$ ($k = 1,2,3,4$), is the statistic of the proposed EWMA control chart based on the usual mean estimator, Swain (1964) estimator, Bhal and Tuteja (1991) estimator Singh et al (2007) estimator, and generalized proposed estimator, for mean, and its starting value is $\bar{y}_{k(0)} = \mu_0$. So, the mean and variance of the above statistic is:

$$E(\bar{y}_{ki}) = \mu_y = \bar{Y} \tag{32}$$

$$Var(\bar{y}_{ki}) = \frac{\lambda}{2-\lambda} var(\bar{y}_{opt\ ki}) \tag{33}$$

where $var(\bar{y}_{opt\ ki})$ mentioned in Eq. (1,7,12 and 20) respectively. The $Var(\bar{y}_{ki})$ ($k = 1,2,3,4$) are defined as

$$Var(\bar{y}_{1i}) = \frac{\lambda}{2-\lambda}\theta \bar{Y}^2 q_y C_y^2 \tag{34}$$

$$Var(\bar{y}_{2i}) = \frac{\lambda}{2-\lambda}\theta\, \bar{Y}^2 q_x[\rho_x^{2*} C_y^2 + \left(\frac{1}{4} - K_X \rho_x^*\right) C_x^2] \tag{35}$$

$$Var(\bar{y}_{3i}) = \frac{\lambda}{2-\lambda}\theta\, \bar{Y}^2 q_\Phi[\rho_\Phi^{2*} C_y^2 + \left(\frac{1}{4} - K_\Phi \rho_\Phi^*\right) C_\Phi^2] \tag{36}$$

$$Var(\bar{y}_{4i}) = \frac{\lambda}{2-\lambda}\theta \bar{Y}^2\, q_y C_y^2\left[1 - R_{y.x\Phi}^2\right] \tag{37}$$

where $\rho_\Phi^* = \sqrt{\frac{q_y}{q_\Phi}}$, $K_\Phi = \rho_{y\Phi}\frac{C_y}{C_\Phi}$ and $R_{y.x\Phi}^2 = \frac{\rho_{y\Phi}^2 + \rho_{x\Phi}^2 + 2\rho_{y\Phi}\rho_{x\Phi}\rho_{yx}}{1-\rho_{x\Phi}^2}$

The control limits under the proposed estimator using SS are given as

$$UCL_1/LCL_1 = \mu_y \pm L\sqrt{\frac{\lambda}{2-\lambda}\theta \bar{Y}^2 q_y C_y^2} \tag{38}$$

$$UCL_2/LCL_2 = \mu_y \pm L\sqrt{\frac{\lambda}{2-\lambda}\theta\,\bar{Y}^2 q_x[\rho_x^{2*}C_y^2 + \left(\frac{1}{4} - K_X\rho_x^*\right)C_x^2]} \quad (39)$$

$$UCL_3/LCL_3 = \mu_y \pm L\sqrt{\frac{\lambda}{2-\lambda}\theta\,\bar{Y}^2 q_\phi[\rho_\phi^{2*}C_y^2 + \left(\frac{1}{4} - K_\phi\rho_\phi^*\right)C_\phi^2]} \quad (40)$$

$$UCL_4/LCL_4 = \mu_y \pm L\sqrt{\frac{\lambda}{2-\lambda}\theta\bar{Y}^2\, q_y C_y^2\left[1 - R_{y.x\phi}^2\right]} \quad (41)$$

where L is the control constant and its values are selected in such a manner that the proposed EWMA control chart in-control ARL_s obtain a definite level of certain value of target ARL of process.

6.2. Performance Method

There are different methods to check the performance efficiency of the control chart but here the ARL values are intended to use for this purpose. It is a common measure used to judge the performance of a chart under some specific shifts. The in-control status is symbolized by ARL_0 while the out-of-control byARL_1. A chart will be considered more effective compared to others if it has a smaller ARL_1 at fixed choices of ARL_0. It may be defined as:

$$ARL = \frac{\sum_m (RL)_m}{m} \quad (42)$$

The ARL_1.are attained by using the following algorithm through simulation:

6.3. Algorithm

a) Let us consider three variables taken from multivariate normal distribution which are Y, X and ϕ. These variables can be expressed in matrix form as $\begin{pmatrix} Y \\ X \\ \phi \end{pmatrix} \sim N_3\left(\begin{pmatrix} \mu_Y \\ \mu_X \\ \mu_\phi \end{pmatrix}, \begin{pmatrix} \sigma_{yy} & \sigma_{yx} & \sigma_{y\phi} \\ \sigma_{xy} & \sigma_{xx} & \sigma_{x\phi} \\ \sigma_{\phi y} & \sigma_{\phi x} & \sigma_{\phi\phi} \end{pmatrix}\right)$

b) A sample of 100000 is generated from a multivariate normal distribution having a subgroup sample $n = 10$.

c) Calculate $\bar{y}_{optki}$ for each subgroup.

d) Calculate mean $E(\bar{y}_{ki})$ and variance $V(\bar{y}_{ki})$ from subgroup sample.

e) Calculate $\bar{y}_{ki}$ based on the above information.

f) Select the values for λ and choose L according to the selected ARL_0 and calculate the control limits.

g) Computing out of control ARL of size n from a shifted multivariate distribution.

h) Repeat the steps (c) to (d) and compute $ARL_1's$ under different mean shifts using the same value of L obtained in (f).

7. Numerical Study

In this section mean squares of the systematic estimator have been numerically compared with the MSE's of the existing ratio estimators. For the comparisons, eight populations are used from real-life data. The details of the populations are given below:

Table .1 Summary Statistics

	Source Kadilar and Cingi (2017) X= production of apples in year 1998, Y= production of apples in year 1999 P_ϕ : ϕ = 1 if number of apple trees < 50,000 in 1999 and 0 otherwise	Source Government of Pakistan (2016-2017; 2017-2018) X= Area in hectares (2016-2017, Y= Production of crops 2016-2017, P_ϕ: ϕ= 1 for area in hectares >100 (‘000) and 0 otherwise (2017-2018)	Source female Walpole 9th edition Pg. 476 X= Income of card holder, Y= Profit of a certain credit card company, P_ϕ: ϕ = 1 for male and 0 for female
Pop	1	2	3
N	204	30	20
n	17	6	5
K	12	5	4
$\bar{X}$	1014.853	659.4933	43282.5
$\bar{Y}$	966.956	4375.583	48.45
P_ϕ	0.1078	0.4333	0.5
C_y^2	6.0784	10.3495	26.8203
C_x^2	6.1425	6.8014	0.2596
C_ϕ^2	8.2727	1.353	1.05263
ρ_{xy}	0.9434	0.4233	0.2034
$\rho_{y\phi}$	0.6040	0.3585	-0.3676
$\rho_{x\phi}$	0.6159	0.4277	0.3702
ρ_y	-0.0141	-0.0561	-0.2135
ρ_x	-0.0159	-0.0784	-0.1512
ρ_ϕ	-0.0381	-0.1674	-0.2

Table .1 Continued

	Source: Applied Linear Statistical Models 2004, Pg 1350 data set 3 X= Average monthly price of product (dollars), Y= Average monthly market share for product (percent), P_ϕ Presence or absence of discount price during period: ϕ = 1 if discount, and 0 otherwise				Source: Applied Linear Statistical Models 2004, Pg 1351-52 data set 5 X= Serum prostate-specific antigen level (mg/ml), Y= Prostate weight (gm), P_ϕ= Presence or absence of seminal vesicle invasion: ϕ = 1 if yes and 0 otherwise		
Pop	4				5		
N	36				96		
n	4	6	9	12	12	16	32
k	9	6	4	3	8	6	3
$\bar{X}$	2.3244				21.2162		
$\bar{Y}$	2.6639				45.4133		
P_ϕ	0.5833				0.21		
C_y^2	0.0098				1.0233		
C_x^2	0.0049				2.3577		
C_ϕ^2	0.7347				3.7747		
ρ_{xy}	0.1885				0.0201		

$\boldsymbol{\rho_{y\phi}}$	0.7907				-0.0058		
$\boldsymbol{\rho_{x\phi}}$	-0.0085				0.5269		
$\boldsymbol{\rho_{y}}$	0.1182	0.3472	-0.0485	0.1265	-0.0217	-0.0317	-0.0151
$\boldsymbol{\rho_{x}}$	-0.0833	0.059	0.098	0.0534	-0.0503	-0.0394	-0.0259
$\boldsymbol{\rho_{\phi}}$	-0.1048	0.0629	-0.0286	-0.0286	-0.0335	-0.0358	-0.0146

Table .1 Continued

	Source: Applied Linear Statistical Models 2004, Pg 1352 data set 6 X=Number of months team has been together, Y= Number of website orders in backlog at the close of the quarter, P_ϕ=A change in the website development process occurred during the second quarter of 2002: ϕ = 1 if quarter 2 or 3, 2002 and 0 otherwise			Source: Applied Linear Statistical Models 2004, Pg 1353 data set 7 X= Finished area of residence (square feet), Y= Sales price of residence (dollars), P_ϕ= Presence or absence of air conditioning: ϕ = 1 if yes and 0 otherwise		Source: Applied Linear Statistical Models 2004, Pg 1354-55 data set 9 X= Total number of interventions or procedures carried out, Y= Number of emergency room visits, P_ϕ=Gender of subscriber: ϕ = 1 if male and 0 otherwise	
Pop	6			7		8	
N	72			520		780	
n	9	12	18	26	52	39	78
k	8	6	4	20	10	20	10
$\bar{\mathbf{X}}$	10.7222			2264.1981		4.7038	
$\bar{\mathbf{Y}}$	27.8472			278540.86		3.4231	
$\mathbf{P_\phi}$	0.3472			0.8327		0.2282	
$\mathbf{C_y^2}$	0.0831			0.2447		0.5889	
$\mathbf{C_x^2}$	0.2725			0.0983		1.4253	
$\mathbf{C_\phi^2}$	1.9031			0.2013		3.3864	
$\boldsymbol{\rho_{xy}}$	0.7835			0.8184		0.3681	
$\boldsymbol{\rho_{y\phi}}$	0.4457			0.2861		0.1079	
$\boldsymbol{\rho_{x\phi}}$	0.6242			0.2646		0.033	
$\boldsymbol{\rho_{y}}$	-0.0789	-0.0613	-0.0424	-0.0274	-0.0161	0.0003	-0.00462
$\boldsymbol{\rho_{x}}$	-0.0975	-0.079	-0.0522	-0.0255	0.0147	-0.0121	-0.0046
$\boldsymbol{\rho_{\phi}}$	-0.103	-0.0751	-0.0489	-0.0066	-0.0028	0.0066	0.0019

7.1 Performance Efficiency Comparison of Proposed EWMA Control Charts based on Ratio Estimators

This section gives the evaluation of proposed chart using $ARL's$ at different shifts ($\boldsymbol{\delta}$) i-e 0-3. The purpose of a proposed CC is to monitor the shifts in the process mean under $\bar{y}$, $\bar{y}_{bt}$, $\bar{y}_s$ and $\bar{y}_{gs}$ estimator. Furthermore, for the evaluation of performance the Monte Carlo simulations with 5000 iterations are used. In this study, Pearson correlation is using to investigate the relationship between numeric and binary variables. The ratio estimators follow a normal distribution of $N = 10^4$ and the sample size of $n = 10$ is taken to attain the ARL values, at 0.05 and 0.2 values of λ, and compare the performance of the proposed repetitive control chart.

These results of ARL's are provided for ARL_0 370 from various specific shifts 0-3 at $\rho = 0.90$, ρ=0.50 and ρ = 0.25 in the form of tables (Table 2-4). The values of L is determined for ARL_0 370 is also stated in tables 2-4.

Table 1: ARL values for Proposed EWMA CC when $ARL_0 = 370$ and $\rho_{xy} = 0.9$

	$\lambda = 0.05$				$\lambda = 0.2$			
	L=2.5082	L=2.48981	L=2.4874	L=2.5142	L=2.8790	L=2.8521	L=2.8429	L=2.8684
δ	$\bar{y}$	$\bar{y}_{bt}$	$\bar{y}_s$	$\bar{y}_{gs}$	$\bar{y}$	$\bar{y}_{bt}$	$\bar{y}_s$	$\bar{y}_{gs}$
0	370.168	370.528	371.160	371.672	369.374	369.520	369.150	370.872
0.25	13.996	7.206	15.528	5.528	14.226	5.934	16.256	4.220
0.5	6.252	3.468	6.986	2.780	5.016	2.472	5.534	1.952
0.75	4.068	2.404	4.564	2.044	2.846	1.714	3.298	1.302
1	3.106	1.996	3.474	1.748	2.142	1.226	2.382	1.012
2	1.902	1.000	1.984	1.000	1.062	1.000	1.146	1.000
2.5	1.402	1.000	1.770	1.000	1.002	1.000	1.008	1.000
3	1.050	1.000	1.234	1.000	1.000	1.000	1.000	1.000

Table 2: ARL Values for Proposed EWMA CC when $ARL_0 = 370$ and $\rho_{xy} = 0.5$

	$\lambda = 0.05$				$\lambda = 0.2$			
	L=2.5106	L=2.493	L=2.4892	L=2.5338	L=2.8794	L=2.8552	L=2.8598	L=2.930
δ	$\bar{y}$	$\bar{y}_{bt}$	$\bar{y}_s$	$\bar{y}_{gs}$	$\bar{y}$	$\bar{y}_{bt}$	$\bar{y}_s$	$\bar{y}_{gs}$
0	371.436	370.528	370.010	371.668	371.208	370.522	370.522	371.878
0.25	14.120	11.794	15.926	11.882	14.442	10.552	17.000	11.182
0.5	6.230	5.374	7.140	5.424	4.936	3.908	5.568	4.080
0.75	4.080	3.654	4.598	3.676	2.928	2.470	3.336	2.488
1	3.166	2.804	3.476	2.876	2.176	1.888	2.390	1.934
2	1.934	1.666	2.002	1.796	1.048	1.004	1.182	1.012
2.5	1.450	1.118	1.818	1.170	1.000	1.000	1.004	1.000
3	1.050	1.002	1.314	1.004	1.000	1.000	1.000	1.000

Table 3: ARL Values for Proposed EWMA CC when $ARL_0 = 370$ and $\rho_{xy} = 0.25$

	$\lambda = 0.05$				$\lambda = 0.2$			
	L=2.509	L=2.489	L=2.471	L=2.532	L=2.880	L=2.856	L=2.851	L=2.916
δ	$\bar{y}$	$\bar{y}_{bt}$	$\bar{y}_s$	$\bar{y}_{gs}$	$\bar{y}$	$\bar{y}_{bt}$	$\bar{y}_s$	$\bar{y}_{gs}$
0	370.168	370.528	370.010	370.870	371.208	370.522	370.010	369.822
0.25	14.012	14.254	15.940	13.504	14.442	13.132	17.636	13.302
0.5	6.296	6.186	7.222	6.200	4.936	4.590	5.480	4.746
0.75	4.102	4.118	4.600	4.056	2.928	2.848	3.262	2.822
1	3.090	3.192	3.458	3.116	2.176	2.186	2.404	2.152
2	1.938	1.934	1.984	1.934	1.048	1.060	1.142	1.034

2.5	1.470	1.544	1.828	1.404	1.000	1.000	1.006	1.002
3	1.032	1.052	1.258	1.026	1.000	1.000	1.000	1.000

In Tables 2-4, it can be understood that the EWMA control chart was constructed on the proposed systematic estimator ($\bar{y}_{gs}$) can detect the shift in the process mean before mean estimator, Bahl and Tuteja (1991), and Singh et al. (2007) estimator when ρ=0.9, ρ=0.50, and ρ=0.25**.** For example, in Table 2 when the shift is 0.25 and λ=0.05, the value of ARL at $\bar{y}_{gs}$ is 5.528 and similarly for $\bar{y}$ is 13.996 and for $\bar{y}_{bt}$ is 7.206 and similarly, at shift 0.5 and λ=0.2, the value of ARL is 1.952 at $\bar{y}_{gs}$ and for $\bar{y}_{s}$ is 5.534. At higher λ the detection of shifts is slightly earlier.

Tables 2-4 show that there is a decrease in ARL values for all ratio estimators but ARL's based on the proposed systematic estimator have minimum ARL values which mean that perform better in detecting the shifts earlier in the proposed repetitive EWMA chart than other ratio estimators at considered λ's.

7.1. Percentage Relative Efficiency

The percent relative efficiency in Table 5 is calculated by comparing the estimators with the usual mean estimator given as efficiency $= \frac{Var(\bar{y})}{var(\bar{y}_{estimator})} * 100$.

Table 4: Percent Relative Efficiencies of Estimators Based on Different Population

Pop	N	n	k	$\bar{y}$	$\bar{y}_r$	$\bar{y}_{bt}$	$\bar{y}_{ss}$	$\bar{y}_{gs}$
1	204	17	12	100	894.0985	319.8143	909.1203	916.1948
2	30	6	5	100	108.1761	121.4501	121.8307	127.8305
3	20	5	4	100	104.1275	102.7091	104.3137	136.7302
4	36	4	9	100	67.8134	85.6008	103.6842	292.6628
		6	6	100	70.4462	86.9045	103.6853	292.6659
		9	4	100	34.3772	62.858	103.6844	292.6633
		12	3	100	64.5885	83.9409	103.6844	292.6633
5	96	12	8	100	43.3852	76.0731	100.0406	100.0779
		16	6	100	36.4443	70.3	100.0406	100.0779
		32	3	100	55.0735	83.7073	100.0406	100.0779
6	72	9	8	100	130.6331	254.3111	258.9342	261.0183
		12	6	100	192.6217	232.5298	258.9342	261.0183
		18	4	100	192.0655	232.7622	258.9342	261.0183
7	520	26	20	100	286.5016	179.0667	302.782	307.6316
		52	10	100	299.2382	190.7408	302.782	307.6316
8	780	39	20	100	68.7305	110.5532	115.6762	116.9158
		78	10	100	43.9331	96.84361	115.6762	116.9158

The numerical values are shown in Table 5 and it can be easily seen that the proposed generalized systematic estimator ($\bar{y}_{gs}$) is more efficient than Swain ($\bar{y}$) estimator, Bhal and Tuteja ($\bar{y}_{bt}$) estimator and Singh and Singh ($\bar{y}_{ss}$)estimator for all the populations 1-8 at high moderate and low correlations between auxiliary and study variables. In population 5 the Percent Relative Efficiencies of $\bar{y}_{gs}$ is close to $\bar{y}_{ss}$ this may be because of the history of exponential estimators made on medium correlation.

8. Conclusion

In this study, the generalized estimator has been proposed under systematic sampling and its properties are derived. Mathematical comparisons are made. Furthermore, a numerical study is done to compare existing estimators with the proposed estimator at different correlations i-e high, low and medium having systematic sampling scheme and it is observed that the proposed systematic estimator is more efficient than Swain (1964), Bahl and Tuteja (1991) and Singh and Singh (1998) estimator.

Also, it is shown that at low correlations between the study and auxiliary variable the generalized proposed systematic estimator is slightly more efficient than existing estimators at different values of n and k and at medium correlations the results of $\bar{y}_{ss}$ and $\bar{y}_{gs}$ are almost similar this may be because of the history of exponential estimators made on medium correlation.

A proposed EWMA control chart modifies the EWMA chart by using four ratio estimators based on SS scheme and is examined exclusively and mutually to observe their performance efficiencies based on ARLs. Though comparing the proposed control charts, based on $\boldsymbol{\bar{y}_r}$,$\boldsymbol{\bar{y}_{bt}}$,$\boldsymbol{\bar{y}_s}$ and $\boldsymbol{\bar{y}_{gs}}$ estimator, collectively, it is shown that in all cases, the EWMA control chart for the proposed estimator outperforms other competitors by detecting small shifts in the process mean much prior than others by having small values of ARL's at different values of λ. Hence these control charts can be used for works in the future to get such results that would give assurance of quality products.

Disclosure statement

The authors report no potential conflict. The authors on their own are answerable for the content of an article.

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Appendix

Appendix A. Mathematical Comparison

The proposed estimator is more efficient than existing estimators under systematic sampling if following conditions holds true.

Condition (i)

$$\mathrm{Var}(\bar{y}_{gs}) \leq \mathrm{Var}(\bar{y})\text{if}$$

$$\mathrm{Var}(\bar{y}) - \mathrm{Var}(\bar{y}_{gs}) \geq 0, \text{or if}$$

$$\theta\bar{Y}^2\, q_y C_y^2 - \theta\bar{Y}^2\, q_y C_y^2\left[1 - R^2_{y.x\phi}\right] \geq 0$$

$$\theta\bar{Y}^2\, q_y C_y^2[1 - (1 - R^2_{y.x\phi})] \geq 0$$

$$1 - 1 + R^2_{y.x\phi} \geq 0$$

$$R^2_{y.x\phi} \geq 0 \qquad \text{Eq. (A.1)}$$

where $R^2_{y.x\phi} = \frac{\rho^2_{y\phi}+\rho^2_{x\phi}+2\rho_{y\phi}\rho_{x\phi}\rho_{yx}}{1-\rho^2_{x\phi}}$

Condition (ii)

$$\mathrm{Var}(\bar{y}_{gs}) \leq \mathrm{Var}(\bar{y}_r) \text{ if}$$

$$\mathrm{Var}(\bar{y}_r) - \mathrm{Var}(\bar{y}_{gs}) \geq 0$$

$$\theta\bar{Y}^2\, q_x[\rho_x^{*2} C_y^2 + (1 - 2K_x\rho_x^*)C_x^2] - \theta\bar{Y}^2\, q_y C_y^2\left[1 - R^2_{y.x\phi}\right] \geq 0$$

$$\theta\bar{Y}^2\, q_x C_y^2[\rho_x^{*2} + (1 - 2K_x\rho_x^*)\frac{C_x^2}{C_y^2} - \frac{q_y}{q_x}(1 - R^2_{y.x\phi})] \geq 0$$

$$\rho_x^{*2} + (1 - 2K_x\rho_x^*)\frac{C_x^2}{C_y^2} - \frac{q_y}{q_x}(1 - R^2_{y.x\phi}) \geq 0$$

$$\frac{q_y}{q_x} + (1 - 2K_x\rho_x^*)\frac{C_x^2}{C_y^2} - \frac{q_y}{q_x}\left(1 - R_{y.x\phi}^2\right) \geq 0$$

$$\frac{q_y}{q_x}[1 + (1 - 2K_x\rho_x^*)\frac{C_x^2}{C_y^2\rho_x^{*2}} - \left(1 - R_{y.x\phi}^2\right)] \geq 0$$

$$1 + (1 - 2K_x\rho_x^*)\frac{C_x^2}{C_y^2\rho_x^{*2}} - 1 + R_{y.x\phi}^2 \geq 0$$

$$(1 - 2K_x\rho_x^*)\frac{C_x^2}{C_y^2\rho_x^{*2}} + R_{y.x\phi}^2 \geq 0$$

$$(1 - 2K_x\rho_x^*)\frac{C_x^2}{C_y^2\rho_x^{*2}} + R_{y.x\phi}^2 \geq 0 \qquad \text{Eq. (A.2)}$$

Condition (iii)

$$\mathrm{Var}\left(\bar{y}_{gs}\right) \leq \mathrm{Var}(\bar{y}_{ss})\text{if}$$

$$\mathrm{Var}(\bar{y}_{ss}) - \mathrm{Var}\left(\bar{y}_{gs}\right) \geq 0, \text{or if}$$

$$\theta\bar{Y}^2\, q_y C_y^2\left[1 - {\rho_{xy}}^2\right] - \theta\bar{Y}^2\, q_y C_y^2\left[1 - R_{y.x\phi}^2\right] \geq 0$$

$$\theta\bar{Y}^2\, q_y C_y^2(1 - {\rho_{xy}}^2 - 1 + R_{y.x\phi}^2) \geq 0$$

$$1 - {\rho_{xy}}^2 - 1 + R_{y.x\phi}^2 \geq 0$$

$$R_{y.x\phi}^2 - {\rho_{xy}}^2 \geq 0$$

$$R_{y.x\phi}^2 - {\rho_{xy}}^2 \geq 0 \qquad \text{Eq. (A.3)}$$

Condition (iv)

$$\mathrm{Var}\left(\bar{y}_{gs}\right) \leq \mathrm{Var}(\bar{y}_{bt})\text{ if}$$

$$\mathrm{Var}(\bar{y}_{bt}) - \mathrm{Var}\left(\bar{y}_{gs}\right) \geq 0$$

$$\theta\bar{Y}^2\, q_x[\rho_x^{*2}C_y^2 + \left(\frac{1}{4} - K_x\rho_x^*\right)C_x^2] - \theta\bar{Y}^2\, q_y C_y^2\left[1 - R_{y.x\phi}^2\right] \geq 0$$

$$\theta\bar{Y}^2\, q_x C_y^2[\rho_x^{*2} + \left(\frac{1}{4} - K_x\rho_x^*\right)\frac{C_x^2}{C_y^2} - \frac{q_y}{q_x}\left(1 - R_{y.x\phi}^2\right)] \geq 0$$

$$\rho_x^{*2} + \left(\frac{1}{4} - K_x\rho_x^*\right)\frac{C_x^2}{C_y^2} - \frac{q_y}{q_x}\left(1 - R_{y.x\phi}^2\right) \geq 0$$

$$\frac{q_y}{q_x} + \left(\frac{1}{4} - K_x\rho_x^*\right)\frac{C_x^2}{C_y^2} - \frac{q_y}{q_x}\left(1 - R_{y.x\phi}^2\right) \geq 0$$

$$\frac{q_y}{q_x}[1 + \left(\frac{1}{4} - K_x\rho_x^*\right)\frac{C_x^2}{C_y^2\rho_x^{*2}} - \left(1 - R_{y.x\phi}^2\right)] \geq 0$$

$$1 + \left(\frac{1}{4} - K_x\rho_x^*\right)\frac{C_x^2}{C_y^2\rho_x^{*2}} - 1 + R_{y.x\phi}^2 \geq 0$$

$$\left(\frac{1}{4} - K_x\rho_x^*\right)\frac{C_x^2}{C_y^2\rho_x^{*2}} + R_{y.x\phi}^2 \geq 0 \qquad \text{Eq. (A.4)}$$