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Modeling Climate data using the Quartic Transmuted Weibull Distribution and Different Estimation Methods

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Abstract

Researchers from various fields of science encounter phenomena of interest, and they seek to model the occurrences scientifically. An important approach of performing modeling is to use probability distributions. Probability distributions are probabilistic models that have many applications in different research areas, including, but not limited to, environmental and financial studies. In this paper, we study a quartic transmuted Weibull distribution from a general quartic transmutation family of distributions as a generalization and an alternative to the well-known Weibull distribution. We also investigate the practical application of this generalization by modeling climate-related data sets and check the goodness-of-fit of the proposed model. The statistical properties of the proposed model, which includes non-central moments, generating functions, survival function, and hazard function, are derived. Different estimation methods to estimate the parameters of the proposed quartic transmuted distribution: the maximum likelihood estimation method, the maximum product of spacings method, two least-squares-based methods, and three goodness-of-fit-based estimation methods. Numerical illustration and an extensive comparative Monte Carlo simulation study is conducted to investigate the performance of the estimators of the considered inferential methods. Regarding estimation methods, simulation outcomes indicated that the maximum likelihood estimation (MLE), Anderson-Darling estimation (ADE) and right Anderson-Darling (RADE) methods in general outperformed the other considered methods in terms of estimation efficiency for large sample size, while all considered estimation methods performed almost same in terms of goodness-of-fit regardless the values of shape and transmuted parameters. Two real-life data sets are used to demonstrate the suggested estimation methods, the applicability and flexibility of the proposed distribution compared to Weibull, transmuted Weibull, and cubic transmuted Weibull distributions. Weighted least squares estimation (WLSE) and least squares estimation (LSE) methods provided best model fitting estimates of the proposed distribution for Wheaton River and rainfall data respectively. The proposed quartic transmuted Weibull distribution provided significantly improved fit for the two datasets as compared with competitive distributions.

Key Words: Weibull Distribution; Quartic Transmuted Family of Distributions; Point Estimation; Simulation & Modeling

1 Introduction

Due to the increasing new challenges in climatology, hydrology, sociology, economics, medicine, biology, and other fields of science that affect human societies, need more flexible statistical models that can capture the asymmetry inherent to specific data (e.g., climate data) has become a necessity. Therefore, numerous generalizations and modifications for well-known skew distributions, such as the Weibull distribution, have been proposed and studied over the last decades. Indeed, statistical literature has been enriched with many flexible models, and we seek to present an additional more flexible model for the scientific community. The Weibull distribution (Weibull, 1951) is a well-known distribution named after W. Weibull, a Swedish physicist. The Weibull distribution has many applications, including, but not limited to, survival analysis, reliability engineering, industrial engineering, climate and weather forecasting, hydrology, and other fields. In reliability and survival analyses, the Weibull distribution is considered as a lifetime model; however, it cannot be used to model practical failure rates which are typically non-monotonic in practice.

The cumulative distribution function (CDF) and probability density function (PDF) of the Weibull distribution with shape parameter α and scale parameter β can be defined as

$$G(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}, \quad x > 0; \quad \alpha, \beta > 0 \tag{1}$$

and

$$g(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}}, \quad x > 0; \quad \alpha, \beta > 0 \tag{2}$$

The shape parameter of the distribution can allow to take characteristics of other types of distributions. It is important to keep in mind that for $\alpha < 1$, indicates the failure rate declines over time, $\alpha = 1$, shows that the failure rate is stable over time and all results go for exponential distribution, and $\alpha > 1$ illustrates the rate of failure increasing overtime. As previously mentioned, the Weibull distribution is a well-known lifetime model that describes constant and monotonic failure rates which are sometimes impractical to assume. Failure rates in practice might be in the form of a bathtub curve or a unimodal curve. Therefore, numerous generalizations and expansions of the Weibull distribution are available in literature. Exponentiated Weibull distribution developed by Mudholkar and Srivastava (1993), Marshall and Olkin (1997) generated extended Weibull distribution, Xie et al. (2002) suggested modified Weibull distribution, beta Weibull distribution presented by Lee et al. (2007), Bebbington et al. (2007) developed a flexible Weibull distribution, Kumaraswamy Weibull distribution developed by Cordeiro et al. (2010), Zhang and Xie (2011) generated the truncated Weibull distribution, Cordeiro and Silva (2014) generated a complementary extended Weibull power series class of distributions and a Weibull distribution with an alpha logarithmic transformation developed by Nassar et al. (2018).

The transmuted Weibull distribution was developed by Aryal and Tsokos (2011) by using the rank transmuted map suggested by Shaw and Buckley (2009). The CDF of transmuted Weibull distribution is defined as

$$F(x) = \left[1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]\left[1 + \lambda\, e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]$$

The cubic transmuted Weibull distributions were generated by different researchers by using different cubic transmuted family of distributions. The CDFs of the cubic transmuted Weibull distributions developed by Granzotto et al. (2017), AL-Kadim and Mohammed (2017) and Rahman et al. (2019) are defined respectively as

$$F(x) = \lambda_1\left[1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right] + 2(\lambda_2 - \lambda_1)\left[1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^2 + 3(1 - \lambda_2)\left[1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^3$$

$$F(x) = (1 + \lambda)\left[1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right] - 2\,\lambda\left[1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^2 + \lambda\left[1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^3$$

$$F(x) = (1 + \lambda_1)\left[1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right] + (\lambda_2 - \lambda_1)\left[1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^2 - \lambda_2\left[1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^3$$

There are some situations where quadratic and cubic transmuted distributions do not fit well to the real-life data properly. Different approaches are available in literature to introduce additional parameters to the distributions for generating quartic transmuted distributions, which will be more flexible, practicable and able to capture more complexity of real-life data. Ali and Athar (2021) suggested a general quartic transmuted family of distributions. The CDF of general quartic transmuted family of distributions with transmuted parameters λ_1, λ_2 and λ_3 defined as

$$F(x) = 2\,\lambda_1 G(x) + 3(\lambda_2 - \lambda_1)G^2(x) + 2(\lambda_1 - 2\lambda_2 + \lambda_3)G^3(x) + (1 - \lambda_1 + \lambda_2 - 2\lambda_3)G^4(x) \tag{3}$$

and the corresponding PDF of quartic transmuted distribution is obtained as

$$f(x) = g(x)[2\,\lambda_1 + 6(\lambda_2 - \lambda_1)G(x) + 6(\lambda_1 - 2\lambda_2 + \lambda_3)G^2(x) + 4(1 - \lambda_1 + \lambda_2 - 2\lambda_3)G^3(x)] \tag{4}$$

where, $\lambda_1 \in [0, 2]$, $\lambda_2 \in [0, 2]$ and $\lambda_3 \in [0, 2]$ and $0 < \lambda_1 + \lambda_2 + \lambda_3 < 2$

In this research, we are interested in proposing a quartic transmuted Weibull distribution by employing Weibull distribution as a baseline distribution from above quartic transmuted family. The statistical properties of the proposed distribution will be discussed. Different techniques will be applied to estimate the QTW distribution's parameters, and a simulation study will be performed to assess how well the estimators for the various techniques work. Real-world data sets will be used to test the proposed distribution's applicability and flexibility in comparison to other competitive distributions.

The outline for the paper is provided below. In Section 2, the proposed QTW distribution is developed. In Section 3, we looked at statistical features of the QTW distribution including moments, moments generating function, characteristic function, mode, reliability function, hazard function, and entropy. The parameter estimation for the QTW distribution using various methods is described in Section 4. In Section 5, Monte Carlo simulation study is carried out to assess the performance of the estimators. An application of real-life data sets in Section 6. We also included some concluded thoughts in Section 7.

2 Quartic Transmuted Weibull Distribution

Here we will derive the CDF and PDF of the quartic transmuted Weibull distribution. The CDF of QTW distribution is obtained by using (1) in equation (3) and on simplifying, the CDF becomes as

$$F(x) = 1 - a_1\, e^{-\left(\frac{x}{\beta}\right)^{\alpha}} - a_2\, e^{-2\left(\frac{x}{\beta}\right)^{\alpha}} - a_3\, e^{-3\left(\frac{x}{\beta}\right)^{\alpha}} - a_4\, e^{-4\left(\frac{x}{\beta}\right)^{\alpha}} \quad (5)$$

where, $a_1 = 4 - 2\lambda_1 - 2\lambda_2 - 2\lambda_3, a_2 = -6 + 3\lambda_1 + 3\lambda_2 + 6\lambda_3, a_3 = 4 - 2\lambda_1 - 6\lambda_3$ and $a_4 = -1 + \lambda_1 - \lambda_2 + 2\lambda_3$

Differentiating above equation with respect to *x* and simplifying, the PDF of QTW distribution can be obtained as

$$f(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-4\left(\frac{x}{\beta}\right)^{\alpha}}\left[4a_4 + 3a_3\, e^{\left(\frac{x}{\beta}\right)^{\alpha}} + 2a_2 e^{2\left(\frac{x}{\beta}\right)^{\alpha}} + a_1 e^{3\left(\frac{x}{\beta}\right)^{\alpha}}\right] \quad (6)$$

It is noted that the quartic transmuted Weibull distribution will be reduced to baseline Weibull distribution when the transmuted parameters $\lambda_1 = \lambda_2 = \lambda_3 = 0.5$.

The potential PDF and CDF shapes for various values of the parameters $\alpha,\ \lambda_1,\ \lambda_2$ and λ_3 with the fixed value of $\beta = 2$ are shown in Figure 1.

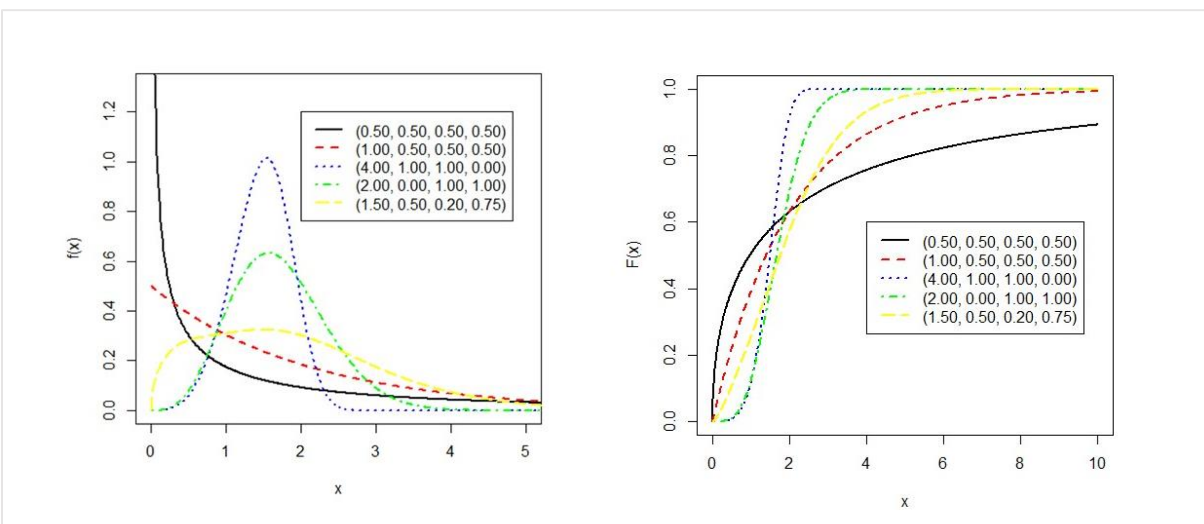

Figure 1: The PDFs and CDFs of the QTW for the different values of $\alpha, \lambda_1, \lambda_2,$ and λ_2 with fixed value of $\beta = 2$

3 Distributional Properties

In this part, we have covered the QTW distribution's distributional features, including moments, moment generating function, characteristic functions, mode, reliability function, hazard rate function, and entropy.

3.1 Moments

Moments have a crucial role in characterizing a distribution. The expression for the r^{th} raw moments of QTW distribution is defined as

$$E(X^r) = \frac{\beta^r}{24^{\frac{r}{\alpha}}}\,\Gamma\left(\frac{r+\alpha}{\alpha}\right)\left[a_4\, 6^{\frac{r}{\alpha}} + a_3 8^{\frac{r}{\alpha}} + a_2 12^{\frac{r}{\alpha}} + a_1\, 24^{\frac{r}{\alpha}}\right] \tag{7}$$

Using equation (7), the suggested distribution's mean and variance can be calculated as

$$\text{Mean} = E(X) = \frac{\beta}{24^{\frac{1}{\alpha}}}\,\Gamma\left(\frac{1+\alpha}{\alpha}\right)\left[a_4\, 6^{\frac{1}{\alpha}} + a_3 8^{\frac{1}{\alpha}} + a_2 12^{\frac{1}{\alpha}} + a_1\, 24^{\frac{1}{\alpha}}\right]$$

$$\text{Varinace} = E(X^2) - \{E(X)\}^2$$

$$= \frac{\beta^2}{24^{\frac{2}{\alpha}}}\,\Gamma\left(\frac{2+\alpha}{\alpha}\right)\left[a_4\, 6^{\frac{2}{\alpha}} + a_3 8^{\frac{2}{\alpha}} + a_2 12^{\frac{2}{\alpha}} + a_1\, 24^{\frac{2}{\alpha}}\right] - \left\{\frac{\beta}{24^{\frac{1}{\alpha}}}\,\Gamma\left(\frac{1+\alpha}{\alpha}\right)\left[a_4\, 6^{\frac{1}{\alpha}} + a_3 8^{\frac{1}{\alpha}} + a_2 12^{\frac{1}{\alpha}} + a_1\, 24^{\frac{1}{\alpha}}\right]\right\}^2$$

Table 1 displayed the means and variances of QTW distribution for different combinations of parameters.

Table 1: Mean and variance of QTW distribution for different combinations of parameters.

Parameters		$\lambda_1 = 0.50$, $\lambda_2 = 0.50$, $\lambda_3 = 0.50$		$\lambda_1 = 1.00$, $\lambda_2 = 1.00$, $\lambda_3 = 0.00$		$\lambda_1 = 0.00$, $\lambda_2 = 0.00$, $\lambda_3 = 2.00$		$\lambda_1 = 1.20$, $\lambda_2 = 0.50$, $\lambda_3 = 0.25$	
		Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance
$\beta = 1$	$\alpha = 0.50$	2.00	20.00	0.319	0.397	1.597	4.360	0.547	3.049
	$\alpha = 1.00$	1.00	1.00	0.417	0.146	1.083	0.424	0.483	0.313
	$\alpha = 2.00$	0.886	0.215	0.580	0.080	0.996	0.091	0.604	0.118
	$\alpha = 4.00$	0.906	0.065	0.737	0.038	0.986	0.023	0.745	0.049
$\beta = 2$	$\alpha = 0.50$	4.00	80.00	0.639	1.587	3.194	17.439	1.094	12.196
	$\alpha = 1.00$	2.00	4.00	0.833	0.583	2.167	1.694	0.967	1.254
	$\alpha = 2.00$	1.772	0.858	1.160	0.320	1.992	0.365	1.209	0.472
	$\alpha = 4.00$	1.813	0.259	1.473	0.151	1.972	0.094	1.490	0.197
$\beta = 3$	$\alpha = 0.50$	6.00	180.00	0.958	3.571	4.792	39.238	1.642	27.440
	$\alpha = 1.00$	3.00	9.00	1.250	1.312	3.250	3.813	1.450	2.823
	$\alpha = 2.00$	2.659	1.931	1.741	0.720	2.988	0.822	1.813	1.062
	$\alpha = 4.00$	2.719	0.582	2.210	0.340	2.958	0.211	2.235	0.444

For the various values of the parameters, Table 2 exhibited the skewness and kurtosis of the QTW distribution.

Table 2: Skewness and kurtosis of QTW distribution for the different values of $\alpha, \lambda_1, \lambda_2,$ and λ_2 with fixed value of $\beta = 2$.

Parameters	$\lambda_1 = 0.50$, $\lambda_2 = 0.50$, $\lambda_3 = 0.50$		$\lambda_1 = 1.00$, $\lambda_2 = 1.00$, $\lambda_3 = 0.00$		$\lambda_1 = 0.00$, $\lambda_2 = 0.00$, $\lambda_3 = 2.00$		$\lambda_1 = 1.20$, $\lambda_2 = 0.50$, $\lambda_3 = 0.25$	
	Skewness	Kurtosis	Skewness	Kurtosis	Skewness	Kurtosis	Skewness	Kurtosis
$\alpha = 0.50$	6.619	87.720	5.548	61.292	3.862	31.556	13.846	426.025
$\alpha = 1.00$	2.000	9.000	1.725	7.367	1.288	5.633	2.980	18.733
$\alpha = 1.50$	1.072	4.390	0.891	3.857	0.685	3.641	1.543	6.862
$\alpha = 2.00$	0.631	3.245	0.481	3.002	0.392	3.146	0.960	4.373
$\alpha = 2.50$	0.359	2.857	0.222	2.747	0.215	2.991	0.627	3.503
$\alpha = 3.00$	0.168	2.729	0.039	2.699	0.094	2.984	0.403	3.138
$\alpha = 3.50$	0.025	2.712	-0.099	2.737	0.006	2.050	0.240	2.981
$\alpha = 4.00$	-0.087	2.748	-0.209	2.813	-0.061	2.970	0.114	2.923
$\alpha = 5.00$	-0.254	2.880	-0.372	3.004	-0.157	3.027	-0.069	2.940

3.2 Generating Functions

In this part, the moment generating function and characteristic function are discussed.

3.2.1 Moment Generating Functions

A moment generating function (MGF) generates moments from a distribution and is useful to characterize the distribution. The MGF of the QTW distribution is obtained from the theorem given below.

Theorem 1: Suppose X follows QTW distribution, then the moment generating function $M_X(t)$ is given as

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{\beta^r}{24^{\frac{r}{\alpha}}} \Gamma\left(\frac{r+\alpha}{\alpha}\right)\left[a_4\, 6^{\frac{r}{\alpha}} + a_3\, 8^{\frac{r}{\alpha}} + a_2 12^{\frac{r}{\alpha}} + a_1\, 24^{\frac{r}{\alpha}}\right] \quad (8)$$

where, $t \in \mathbb{R}$

Proof:
The MGF of the distribution of X is defined as

$$M_X(t) = E(e^{Xt}) = \int_0^{\infty} e^{xt} f(x)dx$$

Where, $f(x)$ is given in (6). Using the series expansion of e^{tx}, we have

$$M_X(t) = \int_0^{\infty} \sum_{r=0}^{\infty} \frac{t^r}{r!}\, x^r\, f(x)dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f(x)dx = \sum_{r=0}^{\infty} \frac{t^r}{r!}\, E(X^r) \quad (9)$$

Using $E(X^r)$ from (7) in (9), we have (8).
Moments can be obtained by differentiating $M_X(t)$ with respect to *t* and setting $t = 0$ in (8).

3.2.2 Characteristic Function

Characteristic function (CF) is a function which is always exist and used to completely define probability density function. The following theorem states QTW distribution's the Characteristic function.

Theorem 2: If the random variable *X* having the QTW distribution, then characteristic function, $\emptyset_X(t)$ is obtained as

$$\emptyset_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \frac{\beta^r}{24^{\frac{r}{\alpha}}} \Gamma\left(\frac{r+\alpha}{\alpha}\right)\left[a_4\, 6^{\frac{r}{\alpha}} + a_3\, 8^{\frac{r}{\alpha}} + a_2 12^{\frac{r}{\alpha}} + a_1\, 24^{\frac{r}{\alpha}}\right] \quad (10)$$

where, $t \in \mathbb{R}$ and $i = \sqrt{-1}$ is an imaginary number.

Proof: The proof is same as MGF.

3.3 Reliability and Hazard Functions

Reliability and hazard functions are very important for distribution. The reliability function generates the likelihood that a device will perform properly for time *t* without failing. The reliability function is defined as $R(t) = 1 - F(t)$ and for the QTW distribution it become as

$$R(t) = a_1\, e^{-\left(\frac{t}{\beta}\right)^{\alpha}} + a_2 e^{-2\left(\frac{t}{\beta}\right)^{\alpha}} + a_3 e^{-3\left(\frac{t}{\beta}\right)^{\alpha}} + a_4 e^{-4\left(\frac{t}{\beta}\right)^{\alpha}} \quad (11)$$

where $t \in \mathbb{R}$

The hazard function is defined as $h(t) = \frac{f(t)}{R(t)}$ and further obtained for QTW distribution as

$$h(t) = \frac{\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-4\left(\frac{t}{\beta}\right)^{\alpha}}\left[4a_4 + 3a_3\, e^{\left(\frac{t}{\beta}\right)^{\alpha}} + 2a_2\, e^{2\left(\frac{t}{\beta}\right)^{\alpha}} + a_1 e^{3\left(\frac{t}{\beta}\right)^{\alpha}}\right]}{a_1\, e^{-\left(\frac{t}{\beta}\right)^{\alpha}} + a_2\, e^{-2\left(\frac{t}{\beta}\right)^{\alpha}} + a_3\, e^{-3\left(\frac{t}{\beta}\right)^{\alpha}} + a_4\, e^{-4\left(\frac{t}{\beta}\right)^{\alpha}}} \quad (12)$$

where, $t \in \mathbb{R}^+$

The reliability functions (left) and hazard rate functions (right) of the QTW distribution are illustrated in Figure 2 for several combinations of the model parameters α, λ_1, λ_2 and λ_3 with the fixed value of $\beta = 2$.

It is observed from the Figure 2 that reliability and distribution functions acted complementary to each other, and hazard rate functions showed upward, constant, and downward shapes. Different directions of the curves of hazard rate function indicated the more flexibility and applicability of the proposed distribution.

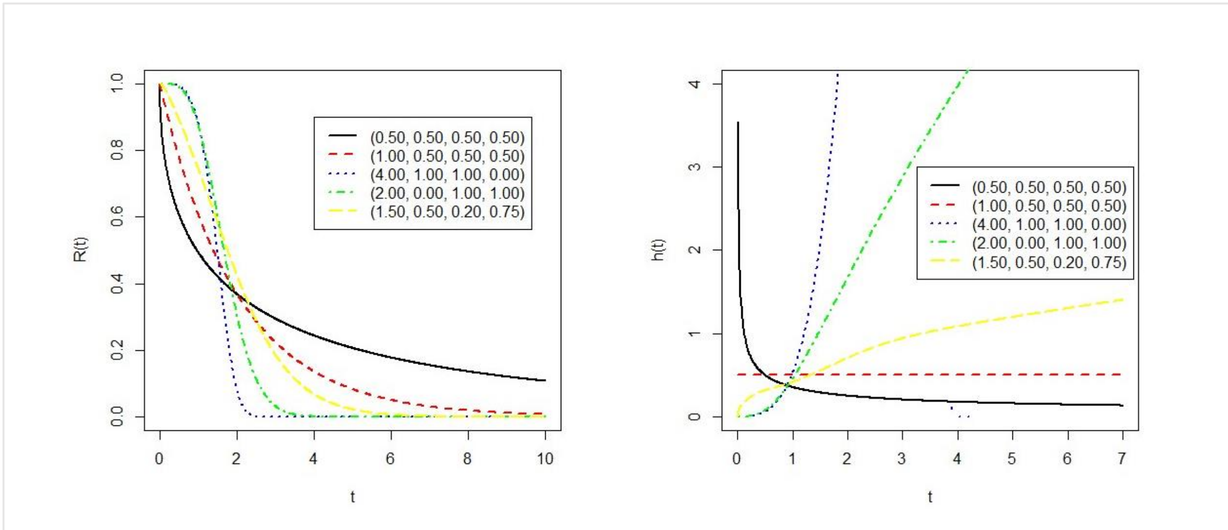

Figure 2: The graphs of reliability and hazard functions of QTW for different values of $\alpha, \lambda_1, \lambda_2$ and λ_3 for fixed value of $\beta = 2$.

3.4 Mode

The mode or modal value of a random variable *X* with the PDF $f(x)$ is the value of *x* for which $f(x)$ has maximum value.

We have $f(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-4\left(\frac{x}{\beta}\right)^{\alpha}}\left[4a_4 + 3a_3\, e^{\left(\frac{x}{\beta}\right)^{\alpha}} + 2a_2 e^{2\left(\frac{x}{\beta}\right)^{\alpha}} + a_1 e^{3\left(\frac{x}{\beta}\right)^{\alpha}}\right]$

Differentiate $f(x)$ to get $f'(x)$

$$f'(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-4\left(\frac{x}{\beta}\right)^{\alpha}} \frac{d}{dx}\left[4a_4 + 3a_3\, e^{\left(\frac{x}{\beta}\right)^{\alpha}} + 2a_2 e^{2\left(\frac{x}{\beta}\right)^{\alpha}} + a_1 e^{3\left(\frac{x}{\beta}\right)^{\alpha}}\right] + \left[4a_4 + 3a_3\, e^{\left(\frac{x}{\beta}\right)^{\alpha}} + 2a_2 e^{2\left(\frac{x}{\beta}\right)^{\alpha}} + a_1 e^{3\left(\frac{x}{\beta}\right)^{\alpha}}\right] \frac{d}{dx}\left\{\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-4\left(\frac{x}{\beta}\right)^{\alpha}}\right\}$$

$$= \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-4\left(\frac{x}{\beta}\right)^{\alpha}}\left[3\alpha\, a_3\, e^{\left(\frac{x}{\beta}\right)^{\alpha}}\left(\frac{x}{\beta}\right)^{\alpha-1}\frac{1}{\beta} + 4\,\alpha\, a_2 e^{2\left(\frac{x}{\beta}\right)^{\alpha}}\left(\frac{x}{\beta}\right)^{\alpha-1}\frac{1}{\beta} + 3\,\alpha\, a_1 e^{3\left(\frac{x}{\beta}\right)^{\alpha}}\left(\frac{x}{\beta}\right)^{\alpha-1}\frac{1}{\beta}\right] + \left[4a_4 + 3a_3\, e^{\left(\frac{x}{\beta}\right)^{\alpha}} + 2a_2 e^{2\left(\frac{x}{\beta}\right)^{\alpha}} + a_1 e^{3\left(\frac{x}{\beta}\right)^{\alpha}}\right]\left\{\frac{\alpha}{\beta^2}(\alpha-1)e^{-4\left(\frac{x}{\beta}\right)^{\alpha}}\left(\frac{x}{\beta}\right)^{\alpha-2} - 4\frac{\alpha^2}{\beta^2} e^{-4\left(\frac{x}{\beta}\right)^{\alpha}}\left(\frac{x}{\beta}\right)^{2\alpha-2}\right\} \quad (13)$$

Mode of the QTW distribution can be obtained by equating $f'(x) = 0$ and solving the equation numerically.

3.5 Entropy

Entropy, which is defined for a probability distribution over a finite sample space, or a finite number of outcomes, can be thought of as a measure of the probability distribution's uncertainty.

3.5.1 Renyi Entropy

The Rényi entropy is important in statistics as an index of diversity. The Rényi entropy is also important in quantum information, where it can be used as a measure of entanglement. If X is a non-negative random variable with PDF $f(x)$, then Renyi entropy of order δ of X is defined as

$$H_\delta(x) = \frac{1}{1-\delta} log \int_0^\infty [f(x)]^\delta\, dx, \delta > 0, \delta \neq 1 \quad (14)$$

Theorem 3: If X follows the QTW distribution and $\lambda_1 \neq \lambda_2 \neq \lambda_3$, then Renyi entropy of the QTW distribution can be obtained as

$$H_\delta(x) = \frac{1}{1-\delta} log\left[\beta^{-\delta+1}\alpha^{\delta-1}\Gamma\left(\frac{\delta\alpha-\delta+1}{\alpha}\right)\sum_{j=0}^{\delta}\sum_{k=0}^{j}\sum_{r=0}^{k}\binom{\delta}{j}\binom{j}{k}\binom{k}{r} 4^{\delta-j}3^{j-k}2^{k-r}\frac{a_4^{\delta-j}a_3^{j-k}a_2^{k-r}a_1^{r}}{(4\delta-j-k-r)^{\frac{\alpha\delta-\delta+1}{\alpha}}}\right]$$

Proof:

The PDF of X is given in equation (6). We would like to calculate the term

$$[f(x)]^{\delta} = \frac{\alpha^{\delta}}{\beta^{\alpha\delta}} x^{\delta(\alpha-1)} e^{-4w\delta}[4a_4 + 3a_3e^{w} + 2a_2e^{2w} + a_1e^{3w}]^{\delta} \tag{15}$$

where $\left(\frac{x}{\beta}\right)^{\alpha} = w$

By binomial expansion, we have

$$\begin{aligned}[4a_4 + 3a_3e^{w} + 2a_2e^{2w} + a_1e^{3w}]^{\delta} &= \sum_{j=0}^{\delta} \binom{\delta}{j} (4a_4)^{\delta-j}[3a_3e^{w} + 2a_2e^{2w} + a_1e^{3W}]^{j} \\ &= \sum_{j=0}^{\delta} \sum_{k=0}^{j} \binom{\delta}{j}\binom{j}{k}(4a_4)^{\delta-j}(3a_3e^{w})^{j-k}[2a_2e^{2w} + a_1e^{3W}]^{k} \\ &= \sum_{j=0}^{\delta} \sum_{k=0}^{j} \sum_{r=0}^{k} \binom{\delta}{j}\binom{j}{k}\binom{k}{r}(4a_4)^{\delta-j}(3a_3e^{w})^{j-k}(2a_2e^{2w})^{k-r}(a_1e^{3W})^{r} \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{j} \sum_{r=0}^{k} \binom{\delta}{j}\binom{j}{k}\binom{k}{r} 4^{\delta-j}a_4^{\delta-j}3^{j-k}a_3^{j-k}e^{w(j-k)}\, 2^{k-r}a_2^{k-r}e^{2w(k-r)}a_1^{r}e^{3Wr} \\ &= \sum_{j=0}^{\delta} \sum_{k=0}^{j} \sum_{r=0}^{k} \binom{\delta}{j}\binom{j}{k}\binom{k}{r} 4^{\delta-j}3^{j-k}2^{k-r}a_4^{\delta-j}\, a_3^{j-k}a_1^{r}\, a_2^{k-r}e^{w(j+k+r)}\end{aligned} \tag{16}$$

Now, substitute (16) in (15), to get

$$[f(x)]^{\delta} = \frac{\alpha^{\delta}}{\beta^{\alpha\delta}} x^{\delta(\alpha-1)} \sum_{j=0}^{\delta} \sum_{k=0}^{j} \sum_{r=0}^{k} \binom{\delta}{j}\binom{j}{k}\binom{k}{r} 4^{\delta-j}3^{j-k}2^{k-r}\, a_4^{\delta-j}a_3^{j-k}a_1^{r}\, a_2^{k-r}e^{-w(4\delta-j-k-r)}$$

We have to find

$$\int_0^{\infty}[f(x)]^{\delta}\, dx = \frac{\alpha^{\delta}}{\beta^{\alpha\delta}} \sum_{j=0}^{\delta} \sum_{k=0}^{j} \sum_{r=0}^{k} \binom{\delta}{j}\binom{j}{k}\binom{k}{r} 4^{\delta-j}3^{j-k}2^{k-r}a_4^{\delta-j}\, a_3^{j-k}a_1^{r}\, a_2^{k-r} \int_0^{\infty} x^{\delta(\alpha-1)}e^{-w(4\delta-j-k-r)}dx \tag{17}$$

Now evaluate the integral term $\int_0^{\infty} x^{\delta(\alpha-1)}e^{-(4\delta-j-k-r)w}dx$

Let $\left(\frac{x}{\beta}\right)^{\alpha} = w$, then $\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} dx = dw$ and $0 < w < \infty$, the integral term becomes as

$$\begin{aligned}\int_0^{\infty} x^{\delta(\alpha-1)}e^{-(4\delta-k-r)\left(\frac{x}{\beta}\right)^{\alpha}} dx &= \frac{\beta^{\delta\alpha-\delta+1}}{\alpha} \int_0^{\infty} w^{\delta-\frac{\delta}{\alpha}+\frac{1}{\alpha}-1}e^{-(4\delta-j-k-r)w}dw \\ &= \frac{\beta^{\delta\alpha-\delta+1}}{\alpha} \frac{\Gamma\left(\delta-\frac{\delta}{\alpha}+\frac{1}{\alpha}\right)}{(4\delta-j-k-r)^{\frac{\alpha\delta-\delta+1}{\alpha}}} \\ &= \frac{\beta^{\delta\alpha-\delta+1}}{\alpha} \frac{\Gamma\left(\frac{\delta\alpha-\delta+1}{\alpha}\right)}{(4\delta-j-k-r)^{\frac{\alpha\delta-\delta+1}{\alpha}}}\end{aligned} \tag{18}$$

Substitute from (18) in (17), we get

$$\begin{aligned}\int_0^{\infty}[f(x)]^{\delta}\, dx &= \frac{\alpha^{\delta}}{\beta^{\alpha\delta}} \sum_{j=0}^{\delta} \sum_{k=0}^{j} \sum_{r=0}^{k} \binom{\delta}{j}\binom{j}{k}\binom{k}{r} 4^{\delta-j}3^{j-k}2^{k-r}a_4^{\delta-j}\, a_3^{j-k}a_1^{r}\, a_2^{k-r} \frac{\beta^{\delta\alpha-\delta+1}}{\alpha} \frac{\Gamma\left(\frac{\delta\alpha-\delta+1}{\alpha}\right)}{(4\delta-j-k-r)^{\frac{\alpha\delta-\delta+1}{\alpha}}} \\ &= \beta^{-\delta+1}\alpha^{\delta-1}\Gamma\left(\frac{\delta\alpha-\delta+1}{\alpha}\right) \sum_{j=0}^{\delta} \sum_{k=0}^{j} \sum_{r=0}^{k} \binom{\delta}{j}\binom{j}{k}\binom{k}{r} 4^{\delta-j}3^{j-k}2^{k-r} \frac{a_4^{\delta-j}a_3^{j-k}\, a_1^{r}\, a_2^{k-r}}{(4\delta-j-k-r)^{\frac{\alpha\delta-\delta+1}{\alpha}}}\end{aligned} \tag{19}$$

Substitute (19) in (14), the Renyi entropy is finally obtained as

$$H_{\delta}(x) = \frac{1}{1-\delta} log\left[\beta^{-\delta+1}\alpha^{\delta-1}\Gamma\left(\frac{\delta\alpha-\delta+1}{\alpha}\right) \sum_{j=0}^{\delta} \sum_{k=0}^{j} \sum_{r=0}^{k} \binom{\delta}{j}\binom{j}{k}\binom{k}{r} 4^{\delta-j}3^{j-k}2^{k-r} \frac{a_4^{\delta-j}a_3^{j-k}a_1^{r}\, a_2^{k-r}}{(4\delta-j-k-r)^{\frac{\alpha\delta-\delta+1}{\alpha}}}\right]$$

Hence the proof.

3.5.2 Shannon Entropy

Shannon's entropy is the unique measure of uncertainty of a random variable X. Shannon (1948) defined the Shannon entropy of a random variable and is obtained for QTW distribution as

$$\begin{aligned}H &= -E[log\{f(x)\}] \\ &= -E\left[log\left\{\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-4\left(\frac{x}{\beta}\right)^{\alpha}}\left[4a_4 + 3a_3e^{\left(\frac{x}{\beta}\right)^{\alpha}} + 2a_2e^{2\left(\frac{x}{\beta}\right)^{\alpha}} + a_1e^{3\left(\frac{x}{\beta}\right)^{\alpha}}\right]\right\}\right] \\ &= -(l_1 + l_2)\end{aligned} \tag{20}$$

Where, $I_1 = E\left[\log\left\{\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-4\left(\frac{x}{\beta}\right)^{\alpha}}\right\}\right]$ and $I_2 = E\left[log\left\{4a_4 + 3a_3e^{\left(\frac{x}{\beta}\right)^{\alpha}} + 2a_2e^{2\left(\frac{x}{\beta}\right)^{\alpha}} + a_1e^{3\left(\frac{x}{\beta}\right)^{\alpha}}\right\}\right]$

On simplification and using the expressions of l_1 and l_2 in (20), the Shannon entropy can be computed numerically.

4 Estimation of Parameters

There are different methods available in the literature to estimate the parameters of the suggested distribution. Here we applied seven different methods to estimate the parameters.

4.1 Maximum Likelihood Estimation Method

The most popular conventional method to estimate parameters is the maximum likelihood estimation (MLE) method. Consistency, asymptotic efficiency, and invariance property are only a few of its many appealing qualities.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n so that the likelihood function is given by

$$L = \prod_{i=0}^{n} f(x_i; \boldsymbol{\theta})$$

where, $\boldsymbol{\theta}$ is the parameter space.

The log-likelihood function of the QTW distribution becomes as

$$l = n\log(\alpha) - n\,\alpha\log(\beta) + (\alpha-1)\sum_{i=1}^{n}\log(x_i) - 4\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha} + \sum_{i=1}^{n} log\left[4\,a_4 + 3a_3 e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} + 2a_2 e^{2\left(\frac{x_i}{\beta}\right)^{\alpha}} + a_1 e^{3\left(\frac{x_i}{\beta}\right)^{\alpha}}\right] \quad (21)$$

The maximum likelihood estimates (MLEs) of $\alpha, \beta, \lambda_1, \lambda_2$ and λ_3, maximizes the log-likelihood function and must satisfy the following normal equations:

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - n\log(\beta) + \sum_{i=1}^{n}\log(x_i) - 4\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha} log\left(\frac{x_i}{\beta}\right) + \sum_{i=1}^{n}\frac{3\,a_3\,e^{\left(\frac{x}{\beta}\right)^{\alpha}}\left(\frac{x_i}{\beta}\right)^{\alpha} log\left(\frac{x_i}{\beta}\right) + 4\,a_2\,e^{2\left(\frac{x}{\beta}\right)^{\alpha}}\left(\frac{x_i}{\beta}\right)^{\alpha} log\left(\frac{x_i}{\beta}\right) + 6\,a_1\,e^{\left(\frac{x}{\beta}\right)^{\alpha}}\left(\frac{x_i}{\beta}\right)^{\alpha} log\left(\frac{x_i}{\beta}\right)}{4\,a_4 + 3a_3 e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} + 2a_2 e^{2\left(\frac{x_i}{\beta}\right)^{\alpha}} + a_1 e^{3\left(\frac{x_i}{\beta}\right)^{\alpha}}} = 0$$

$$\frac{\partial l}{\partial \beta} = -\frac{n\,\alpha}{\beta} + 4\,\alpha\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha-1}\frac{x_i}{\beta^2} + \sum_{i=1}^{n}\frac{-3\,a_3\,e^{\left(\frac{x_i}{\beta}\right)^{\alpha}}\left(\frac{x_i}{\beta}\right)^{\alpha-1}\frac{x_i}{\beta^2} - 4a_2 e^{2\left(\frac{x_i}{\beta}\right)^{\alpha}}\left(\frac{x_i}{\beta}\right)^{\alpha-1}\frac{x_i}{\beta^2} - 6\,a_1 e^{3\left(\frac{x_i}{\beta}\right)^{\alpha}}\left(\frac{x_i}{\beta}\right)^{\alpha-1}\frac{x_i}{\beta^2}}{4\,a_4 + 3a_3 e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} + 2a_2 e^{2\left(\frac{x_i}{\beta}\right)^{\alpha}} + a_1 e^{3\left(\frac{x_i}{\beta}\right)^{\alpha}}} = 0$$

$$\frac{\partial l}{\partial \lambda_1} = \sum_{i=1}^{n}\frac{4 - 6e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} + 6e^{2\left(\frac{x_i}{\beta}\right)^{\alpha}} - 2e^{3\left(\frac{x_i}{\beta}\right)^{\alpha}}}{4\,a_4 + 3a_3 e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} + 2a_2 e^{2\left(\frac{x_i}{\beta}\right)^{\alpha}} + a_1 e^{3\left(\frac{x_i}{\beta}\right)^{\alpha}}} = 0$$

$$\frac{\partial l}{\partial \lambda_2} = \sum_{i=1}^{n}\frac{-4 + 6e^{2\left(\frac{x_i}{\beta}\right)^{\alpha}} - 2e^{3\left(\frac{x_i}{\beta}\right)^{\alpha}}}{4\,a_4 + 3a_3 e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} + 2a_2 e^{2\left(\frac{x_i}{\beta}\right)^{\alpha}} + a_1 e^{3\left(\frac{x_i}{\beta}\right)^{\alpha}}} = 0$$

$$\frac{\partial l}{\partial \lambda_3} = \sum_{i=1}^{n}\frac{8 - 18e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} - 12e^{2\left(\frac{x_i}{\beta}\right)^{\alpha}} - 2e^{3\left(\frac{x_i}{\beta}\right)^{\alpha}}}{4\,a_4 + 3a_3 e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} + 2a_2 e^{2\left(\frac{x_i}{\beta}\right)^{\alpha}} + a_1 e^{3\left(\frac{x_i}{\beta}\right)^{\alpha}}} = 0$$

The above-mentioned nonlinear system of equations is solved to obtain the MLE $\hat{\boldsymbol{\theta}} = \left(\hat{\alpha}, \hat{\beta}, \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3\right)$. For numerically maximizing the log-likelihood function in equation (21), it is typically more practical to employ nonlinear optimization techniques like quasi-Newton or Newton-Raphson.

4.2 Least Squares Estimation Method

For estimating the beta distribution's parameter values, Swain et al. (1988) suggested the least-squares estimation (LSE) approach. Let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ be an ordered sample of size n from the proposed distribution given in (6). According to Arnold et al. (2008), it is given that

$$E[F(x_{(i)})] = \frac{i}{n+1}$$

Given that $F(x_{(1)}), F(x_{(2)}),\ldots, F(x_{(n)})$ are order statistics, a standard uniform distribution is formed. The least squares estimators (LSEs) of the unknown parameters $\alpha, \beta, \lambda_1, \lambda_2,$ and λ_3 of the QTW distribution can be attained by minimizing the succeeding function

$$LS = \sum_{i=1}^{n} \left[F(x_{(i)}) - \frac{i}{n+1}\right]^2$$

4.3 Weighted Least Squares Method

Alongside the LSE method, the weighted least-squares estimation (WLSE) method was used to estimate the beta distribution's parameters (Swain et al. (1988); Alam (2022)). Recall that $x_{(1)}, x_{(2)},\ldots, x_{(n)}$ is a sample of order statistics of size *n*. Since $F(x_{(1)}), F(x_{(2)}),\ldots, F(x_{(n)})$ are order statistics, it creates a standard uniform distribution. It is known from Arnold et al. (2008) that

$$Var[F(x_{(i)})] = \frac{i(n-i+1)}{(n+1)^2(n+2)}$$

By minimizing the following function, the weighted least squares estimators (WLSEs) of the parameters $\alpha, \beta, \lambda_1, \lambda_2,$ and λ_3 of the QTW distribution can be produced.

$$WS = \sum_{i=1}^{n} w_i \left[F(x_{(i)}) - \frac{i}{n+1}\right]^2$$

where, $w_i = \frac{(n+1)^2(n+2)}{i(n-i+1)}$ is the weight of $x_{(i)}$.

4.4 Maximum Product of Spacing Estimation Method

Cheng and Amin (1979) and Ranneby (1984) presented the maximum product of spacings (MPS) approach as a substitute for the MLE. The MPS method is a member of a class of more versatile estimate techniques that use spacings. A sample of order statistics of size *n* are given as $x_{(1)}, x_{(2)},\ldots, x_{(n)}$. The maximum product of spacing estimators (MPSEs) of the unknown parameters $\alpha, \beta, \lambda_1, \lambda_2,$ and λ_3 of the QTW distribution can be determined by maximizing the function given below.

$$PS = \frac{1}{n+1} \sum_{i=1}^{n+1} log d_i , \quad i = 1, 2, 3, \ldots, n+1$$

where, $d_i = F(x_{(i)}) - F(x_{(i-1)})$ is the uniform spacings of a random sample from the QTW distribution with $F(x_{(0)}) = 0$ and $F(x_{(n+1)}) = 1$.

4.5 Cramer-von Mises Estimation Method

The Cramer-von Mises method belongs to the category of minimal distance techniques. The Cramer-von Mises Estimators (CVMEs) of $\alpha, \beta, \lambda_1, \lambda_2,$ and λ_3 are found by minimizing the function

$$CS = \frac{1}{12n} + \sum_{i=1}^{n} \left[F(x_{(i)}) - \frac{2i-1}{2n}\right]^2$$

4.6 Anderson- Darling Estimation Method

The class of minimum distance approaches also includes the Anderson-Darling estimation method. Based on the ordered sample $x_{(1)} < x_{(2)} <,\ldots,< x_{(n)}$, the Anderson-Darling estimators (ADEs) can be attained by minimizing the below function

$$AD = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \log \left\{F(x_{(n-i+1)})\, F(x_{(i)})\right\}$$

4.7 Right Anderson- Darling Estimation Method

Using an n-piece random sample and the associated observed order sample $x_{(1)} < x_{(2)} <,\ldots,< x_{(n)}$, The Right Anderson-Darling Estimators (RADEs) of $\alpha, \beta, \lambda_1, \lambda_2,$ and λ_3 are attained by minimizing the function

$$RAD = \frac{n}{2} - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \log\left\{F(x_{(n-i+1)})\right\} - 2 \sum_{i=1}^{n} F(x_{(i)})$$

DOI: http://dx.doi.org/10.18187/pjsor.v19i4.4423

5 Simulation

An extensive Monte Carlo simulation study is carried out to numerically investigate and compare the performances of the proposed estimators. The simulation findings are divided into two sections: the first looks at estimators' efficiency, and the second emphasize on goodness of fit analysis. For simulation, the sample size is considered as n = 10(10)100. The values of the parameters were taken as $\alpha = 0.5, 1.5\ , 2.5$; $\lambda_1 = 1.1, 1.2$; $\lambda_2 = 0.15, 0.30$; $\lambda_3 = 0.15, 0.30$ and the scale parameter keeps fixed at $\beta = 1$. The simulation study is based on M = 10,000 simulation runs. The results of simulation analysis are represented by heat maps.

5.1 Estimation efficiency

The simulated root mean-squared-error (RMSE) is obtained for each estimator as

$$RMSE\ (\hat{\theta}) = \sqrt{\frac{1}{M}\sum_{i=1}^{M}\left(\theta - \hat{\theta}\right)^2}$$

The following is a discussion of the simulation's first section's findings. Overall, the performance of the estimators was excellent because, regardless of the parameter values, RMSEs start to decline as sample sizes rise. Figures 3 to 6 showed the simulation results of the efficiency for all the estimators.

In practice, when increasing the values of the parameters, one should expect an increase in the RMSE. A solution to decrease it is by increasing the sample size. It is depicted from the Figure 3 that no matter what the values of the transmuted parameters were, all approaches gave the lowest RMSEs at $\alpha = 0.5$. With an increase in the values of the shape parameter, the RMSEs of shape for all estimators rise. However, regardless of the shape values and transmuted parameter values, the RMSEs of shape for all estimators are dropping as the sample size grows. In comparison to other estimators for the high sample size, ADE and MLE offered the smallest RMSEs of shape at $\alpha = 2.5$.

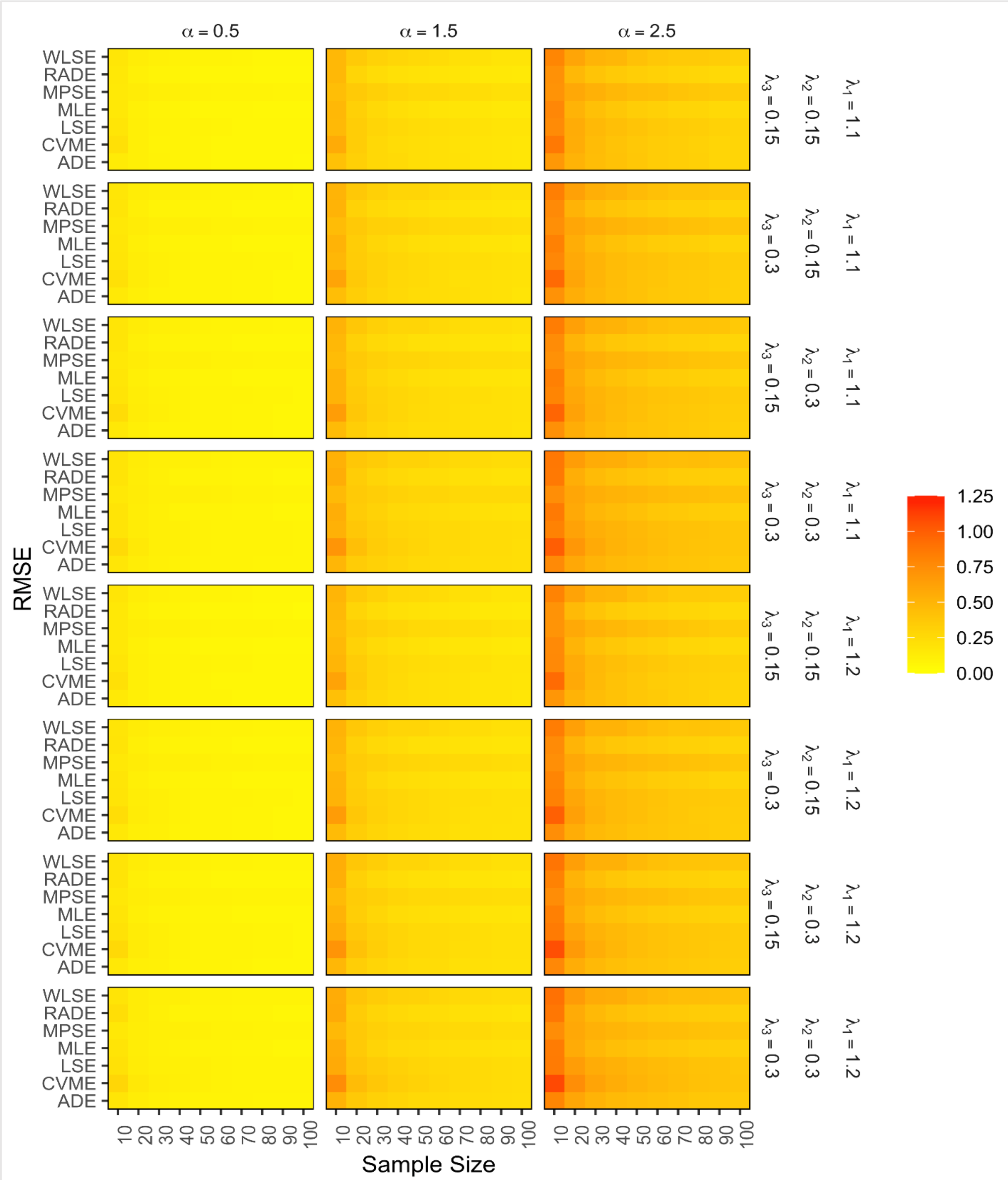

Figure 3: The simulated RMSEs of the shape parameter α.

Figure 4 demonstrated that RADE, MLE and CVME have the smallest amount of RMSE of $\lambda_1$for the large sample size whereas MPSE gives a fixed amount of RMSE of λ_1 for the values of shape and transmuted parameters.

Figure 4: The simulated RMSEs of the transmuted parameter λ_1.

It can be seen in Figure 5, the RMSEs of λ_2 of all the methods except MPSE decline as the sample size increases. Regardless of the values of shape and transmuted parameters, MLE provided the least RMSE for large samples.

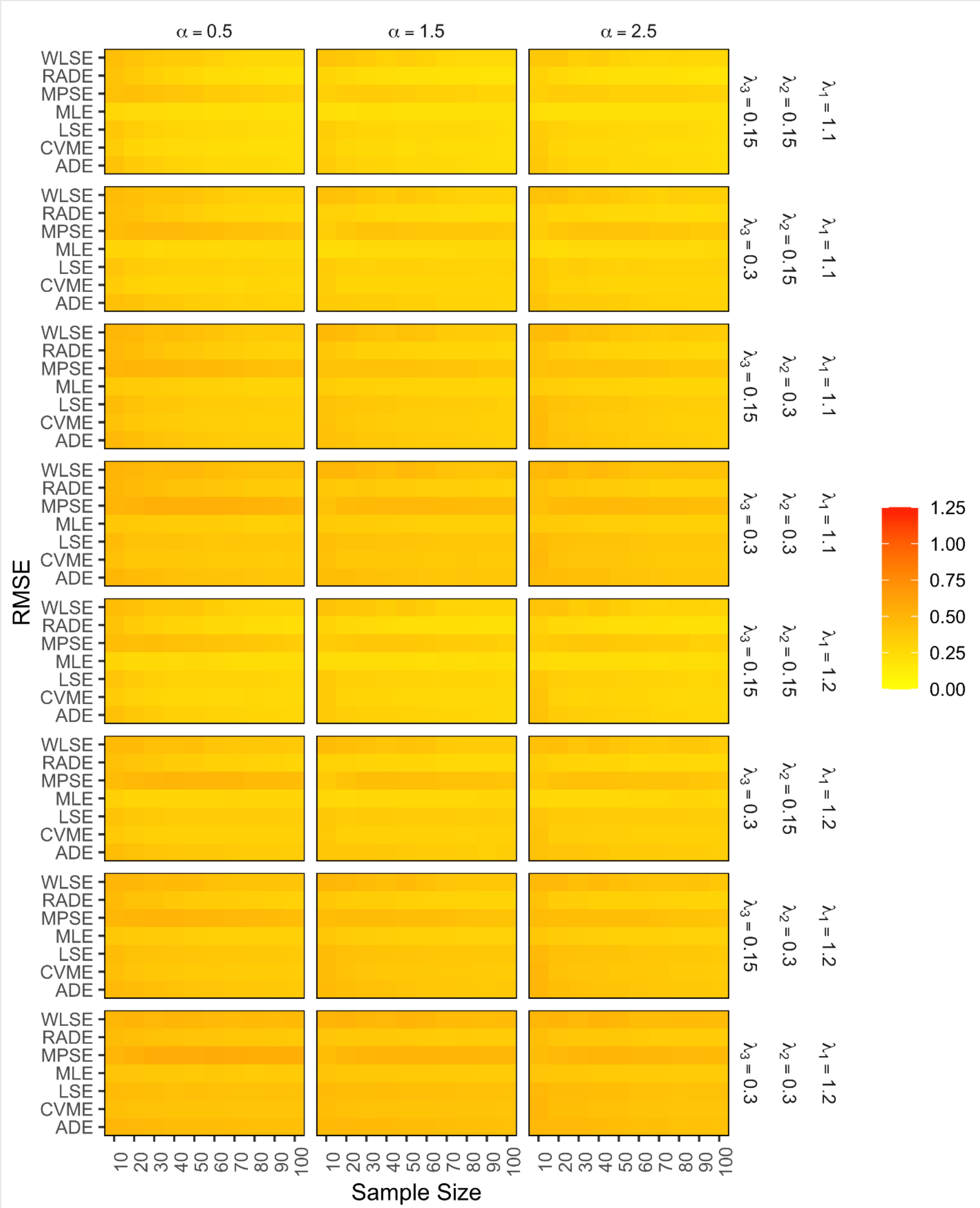

Figure 5: The simulated RMSEs of the transmitted parameter λ_2.

From Figure 6, it is observed that the RMSEs of λ_3 for all the methods are increased for the larger value of λ_3. On the other hand, RMSEs of λ_3 for all the estimators decreases with the increase of sample size. RADE provided the smallest RMSE for $\lambda_3 = 0.15$ and for different values of shape parameters.

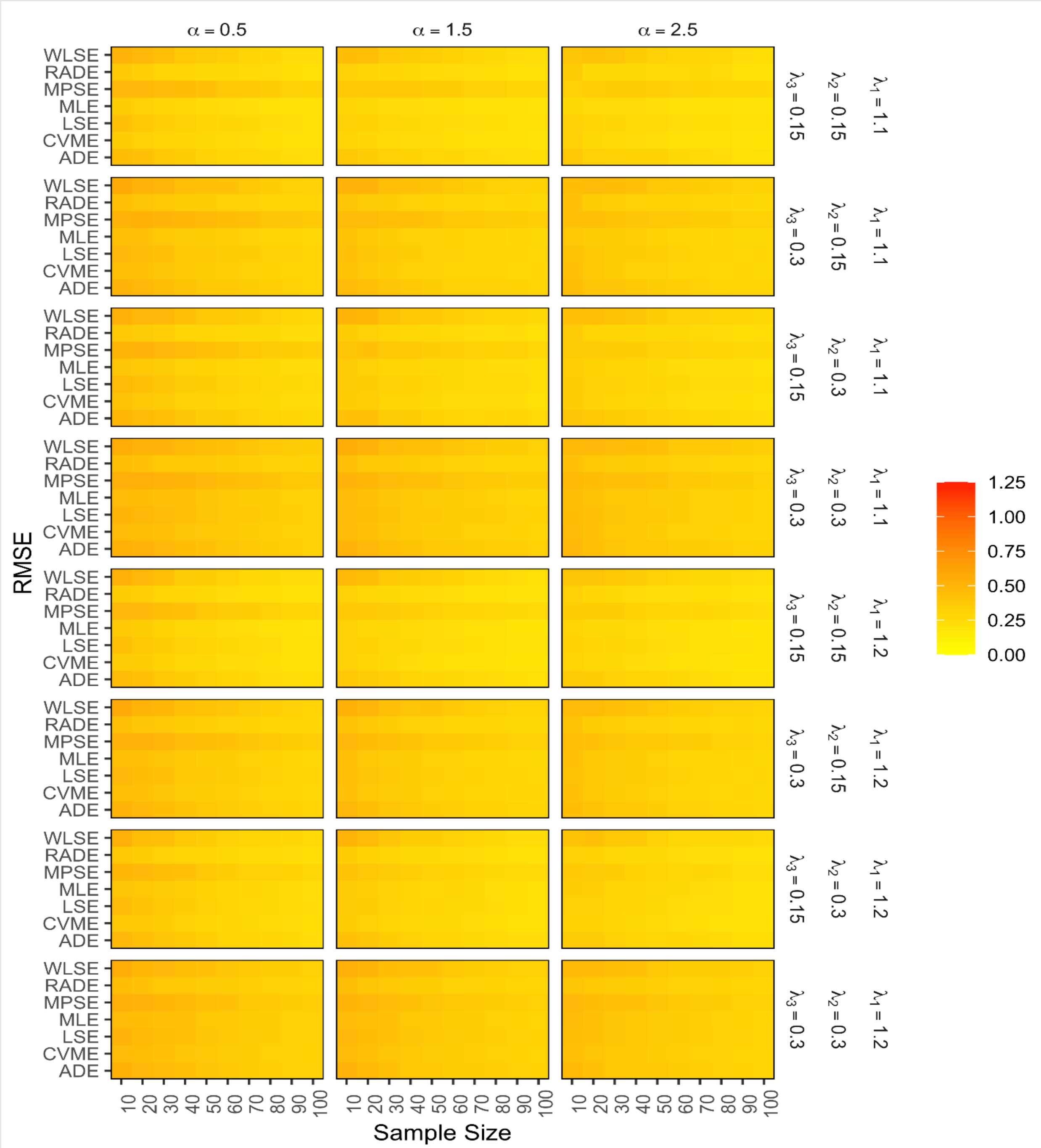

Figure 6: The simulated RMSEs of the transmitted parameter λ_3.

5.2 Goodness-of-fit analysis

In this segment of the simulation results, two simulated goodness of fit criteria, the average absolute difference between the actual and estimated CDFs (D_{abs}) and the maximum absolute difference between the real and estimated CDFs (D_{max}) are used to compare the estimation approaches. These metrics are defined as

$$D_{abs} = \frac{1}{M \times n} \sum_{i=1}^{M} \sum_{j=1}^{n} \left| F\left(x_j, \alpha, \beta, \lambda_1, \lambda_2, \lambda_3\right) - F\left(x_j, \hat{\alpha}, \hat{\beta}, \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3\right) \right|$$

$$D_{max} = \frac{1}{M} \sum_{i=}^{M} \max_{j=1,2,\ldots,n} \left| F\left(x_j, \alpha, \beta, \lambda_1, \lambda_2, \lambda_3\right) - F\left(x_j, \hat{\alpha}, \hat{\beta}, \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3\right) \right|$$

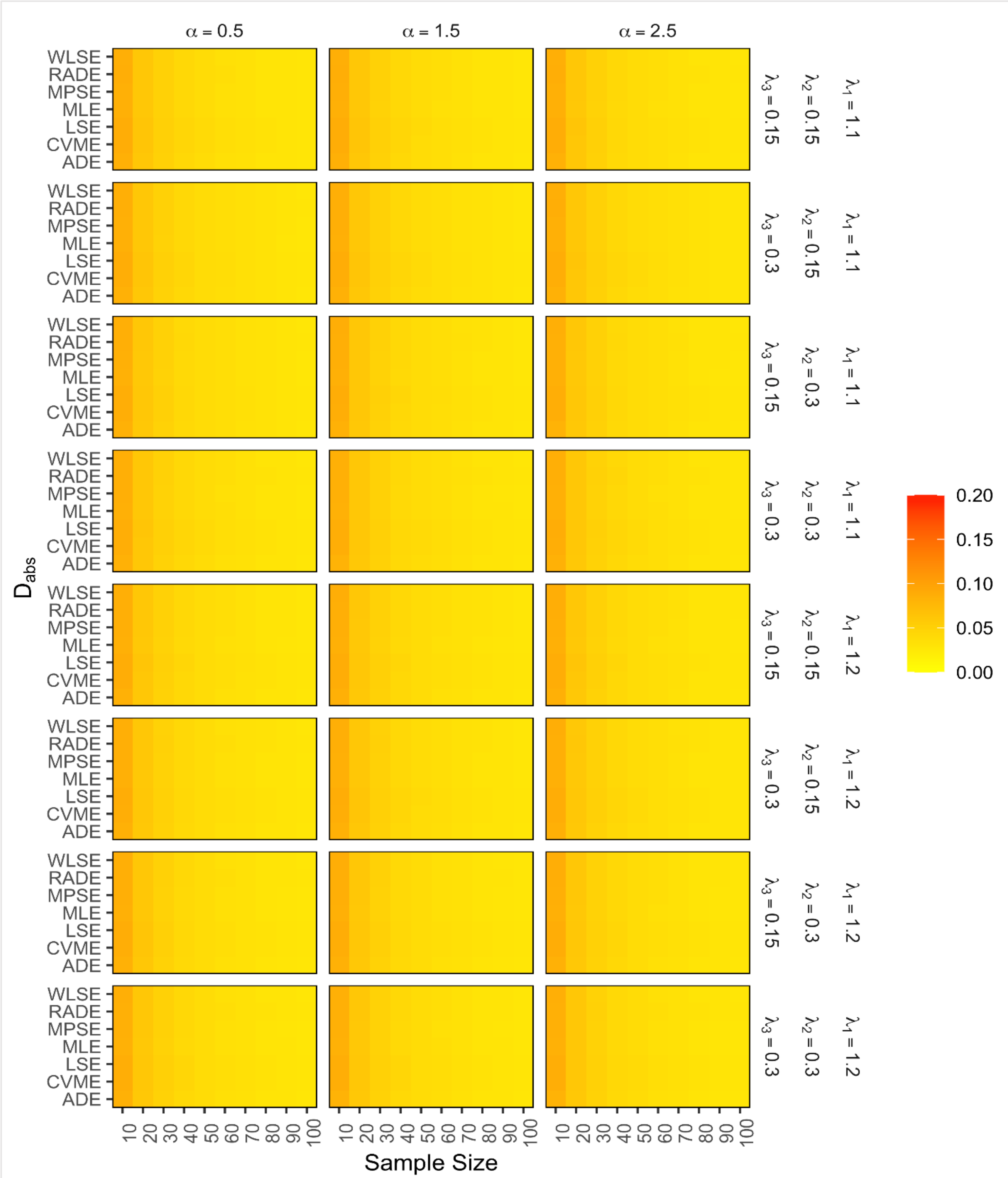

Figure 7: The simulated average absolute difference between the true and estimated CDFs

The simulated results of the goodness of fit are depicted in Figure 7 and Figure 8. The values of D_{abs} and D_{max} are decreasing with the increase in the sample size regardless of the values shape and transmuted parameters. The amount of D_{abs} is same for all the estimators for whatever the values of shape and transmuted parameters. However, all the approaches provided the same amount of D_{max} irrespective of the values of shape and transmuted parameters.

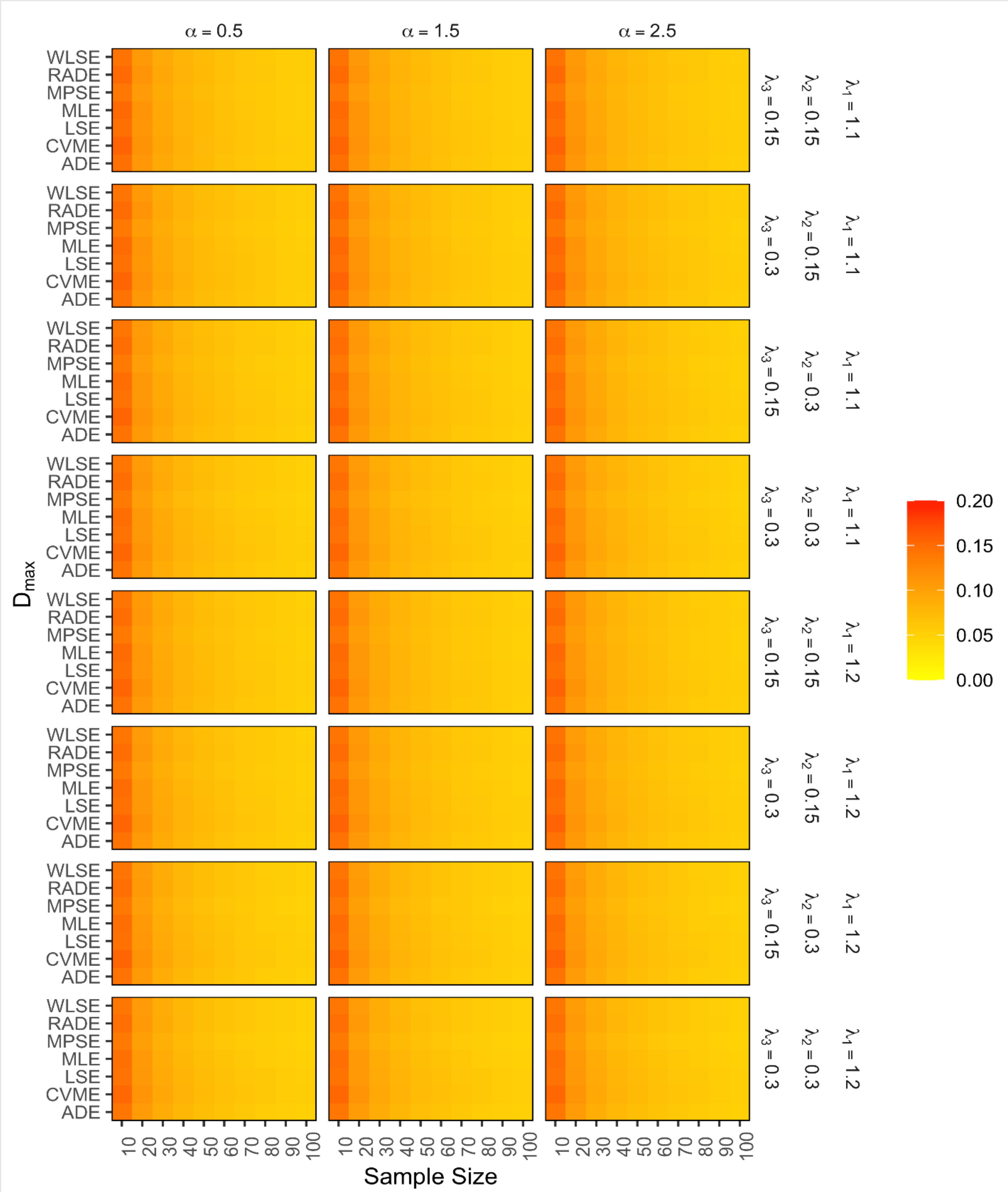

Figure 8: The simulated maximum absolute difference between the true and estimated CDFs

6. Applications

In this section, we provided an application of the quartic transmuted Weibull distribution. Wheaton river data and rainfall data sets were used to demonstrate the suggested estimation methods and compare the applicability and flexibility of the QTW distribution with some selected distributions.

6.1 Wheaton River Flood Data

The data consists of 72 exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada for the years 1958–1984. This data was previously studied by Choulakian and Stephens (2001); Akinsete et al. (2008). Summary statistics of the data set are presented in Table 3.

Table 3: Summary statistics for Wheaton River flood data set.

Minimum	Q1	Median	Mean	Q3	Maximum	SD
0.100	2.125	9.500	12.204	20.125	64.000	12.297

The estimated model parameters employing the seven approaches together with the Kolmogorov-Smirnov (KS) distance statistics and p-values are listed in Table 4. From Table 4, we observed that all the estimation methods delivered close estimates as well as close RMSEs for all parameters. However, the WLSE offered the best-fitted estimates based on the KS test statistic and its p-value for Wheaton River data.

Table 4: Estimates values of the parameters of several methods together with related goodness of fit statistics using Wheaton River flood data.

Methods	$\hat{\alpha}$	RMSE ($\hat{\alpha}$)	$\hat{\beta}$	RMSE ($\hat{\beta}$)	$\hat{\lambda}_1$	RMSE ($\hat{\lambda}_1$)	$\hat{\lambda}_2$	RMSE ($\hat{\lambda}_2$)
MLE	1.050	4.050	9.988	8.459	0.896	4.124	5.14E-06	4.488
LSE	1.037	4.167	10.205	8.678	0.926	4.221	4.64E-07	4.581
WLSE	1.067	4.309	10.560	8.912	0.954	4.356	5.23E-08	4.740
MPSE	0.964	4.050	9.897	8.333	0.660	4.167	4.43E-06	4.423
CvME	1.054	4.155	10.203	8.730	0.927	4.210	1.82E-07	4.587
ADE	1.083	4.330	10.615	9.217	0.960	4.382	2.74E-08	4.766
RADE	1.047	4.131	10.157	8.761	0.923	4.188	2.12E-07	4.566

Methods	$\hat{\lambda}_3$	RMSE ($\hat{\lambda}_3$)	KS	P-values
MLE	1.12E-06	4.485	0.053806	0.709
LSE	2.85E-08	4.571	0.053309	0.545
WLSE	1.54E-08	4.747	**0.050424**	**0.739**
MPSE	0.156	4.282	0.111528	0.013
CvME	1.83E-08	4.573	0.050928	0.606
ADE	1.01E-06	4.753	0.052845	0.584
RADE	1.35E-06	4.553	0.051506	0.673

We made a comparison of the proposed (QTW) distribution with the existing Weibull (W) distribution (Weibull, 1951), transmuted Weibull (TW) distribution (Aryal & Tsokos, 2011) and cubic transmuted Weibull (CTW) distribution (Granzotto et al., 2017). Table 5 showed the MLEs of the model parameters and the -log-likelihood (-logL), Akaike's information criterion (AIC), corrected Akaike's information criterion (AICc), Schwarz's Bayesian information criterion (BIC), Anderson-Darling statistic (A^2), Kolmogorov-Smirnov (KS) distance statistics and its corresponding p-values. On the basis of all the model selection criteria (the smaller the better), it is revealed that the proposed QTW distribution is the most appropriate model for the Wheaton River data set.

Table 05: MLEs of the parameters and values of the model selection criteria for Wheaton River data.

Distributions	Parameters	Estimates	-logL	AIC	AICc	BIC	A^2	KS	P-value
QTW	α	1.050	247.31	504.621	505.524	516.004	0.228	0.0538	0.709
	β	9.988							
	λ_1	0.896							
	λ_2	5.14E-06							
	λ_3	1.12E-06							
CTW	α	0.924	249.93	507.887	508.464	516.933	0.694	0.1159	0.288
	β	9.245							
	λ_1	0.999							

	λ_2	0.108							
	α	0.895							
TW	β	11.406	251.50	508.997	511.605	518.082	0.839	0.1056	0.397
	λ_1	-0.032							
W	α	0.901	251.50	506.997	507.171	516.390	0.844	0.1052	0.403
	β	11.634							

The estimated PDF and CDF of the Wheaton River data set are plotted over empirical density and distribution functions presented in Figure 9.

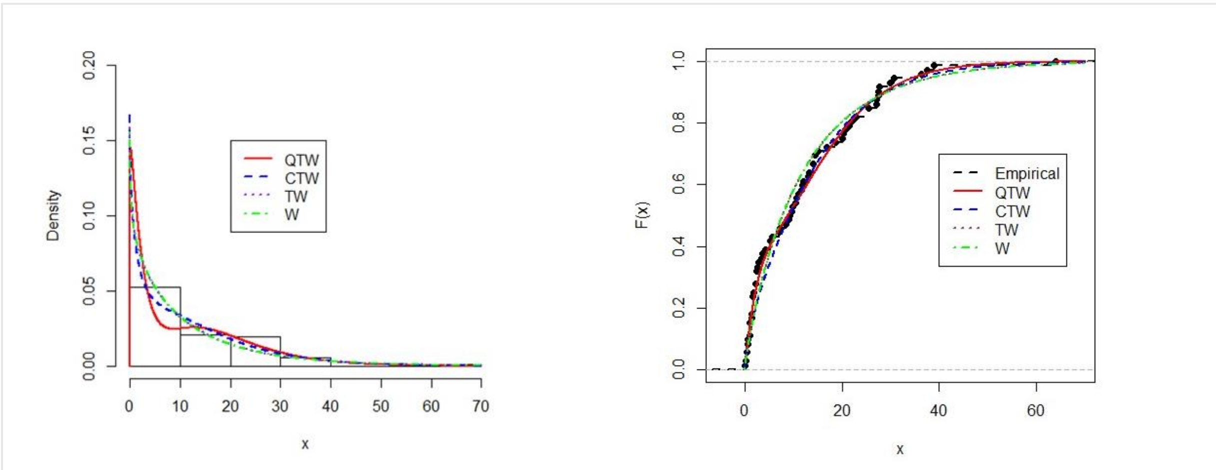

Figure 9: Estimated PDF and CDF of QTW, CTW, TW and W for Wheaton River dataset.

6.2 Rainfall Data

The rainfall data consists of the annual maximum daily precipitation (unit: mm) at Busan, Korea in the period 1904-2011. The data set has been previously used by Jeong et al. (2014). It consists of 108 observations because only one maximum daily precipitation is used for each year. The summary statistics of the data set are given in Table 6.

Table 6: Summary statistics of rainfall data

Minimum	Q1	Median	Mean	Q3	Maximum	SD
20.700	101.600	131.600	144.600	165.500	354.700	66.187

Table 7 included the calculated values of the model's parameters obtained using the previously stated approaches together with the KS statistics and their accompanying p values.

From the outcomes of Table 7, we observed that estimated values of a parameters of all the estimation methods are close to each other. According to the value of KS test statistic and accompanying p-value, the LSE provided the best estimates.

Table 7: The estimated values of the parameters with their RMSEs (in parenthesis) and KS statistic and corresponding p-values based on rainfall data.

Methods	$\hat{\alpha}$ (RMSE)	$\hat{\beta}$ (RMSE)	$\hat{\lambda}_1$ (RMSE)	$\hat{\lambda}_2$ (RMSE)	$\hat{\lambda}_3$ (RMSE)	KS	P-values
MLE	2.310	175.994	0.243	1.414	3.74E-07	0.0728	0.161
	(77.652)	(156.302)	(78.480)	(78.450)	(78.545)		
LSE	2.272	175.999	0.155	1.550	1.00E-07	**0.0655**	**0.998**
	(77.706)	(156.520)	(78.648)	(78.022)	(78.719)		
WLSE	2.138	176.816	0.117	1.570	2.10E-08	0.0722	0.011
	(78.170)	(155.014)	(78.873)	(78.861)	(78.871)		
MPSE	2.157	176.001	0.206	1.411	1.86E-07	0.0819	0.492
	(77.800)	(156.360)	(78.585)	(78.291)	(78.564)		

CVME	2.303 (77.681)	175.972 (156.487)	0.158 (78.635)	1.558 (78.007)	1.01E-08 (78.707)	0.0680	0.998
ADE	2.149 (77.764)	176.002 (156.468)	0.116 (78.652)	1.554 (78.076	1.12E-08 (78.690)	0.0741	0.812
RADE	2.013 (77.584)	175.456 (156.321)	0.001 (78.436)	1.682 (78.813)	1.10E-08 (78.426)	0.0717	0.829

The Weibull (W) distribution (Weibull, 1951), transmuted Weibull (TW) distribution (Aryal & Tsokos, 2011) and cubic transmuted Weibull (CTW) distribution (Granzotto et al., 2017) have been considered as an alternative to the proposed QTW distribution for comparison purposes. Table 8 provided the -log-likelihood (-logL), AIC, AICc, BIC, A^2 and KS statistics with corresponding p-values for the selected models. From Table 8, it is easily observed that our proposed QTW distribution performed better than other selected distributions for rainfall data.

Table 8: MLEs of the parameters and values of the model selection criteria of the selected models for rainfall data.

Distributions	Parameter	Estimates	-logL	AIC	AICc	BIC	A^2	KS	P-value
QTW	α	2.301	578.50	1167.002	1167.608	1175.272	0.504	0.0728	0.542
	β	175.994							
	λ_1	0.243							
	λ_2	1.414							
	λ_3	3.74E-07							
CTW	α	1.777	583.00	1175.560	1175.907	1186.123	23.405	0.1253	0.074
	β	163.667							
	λ_1	0.042							
	λ_2	0.874							
TW	α	2.427	582.30	1170.592	1170.830	1178.554	3.889	0.1116	0.146
	β	171.057							
	λ_1	0.622							
W	α	2.319	583.75	1171.507	1171.815	1176.815	1.586	0.1257	0.072
	β	163.421							

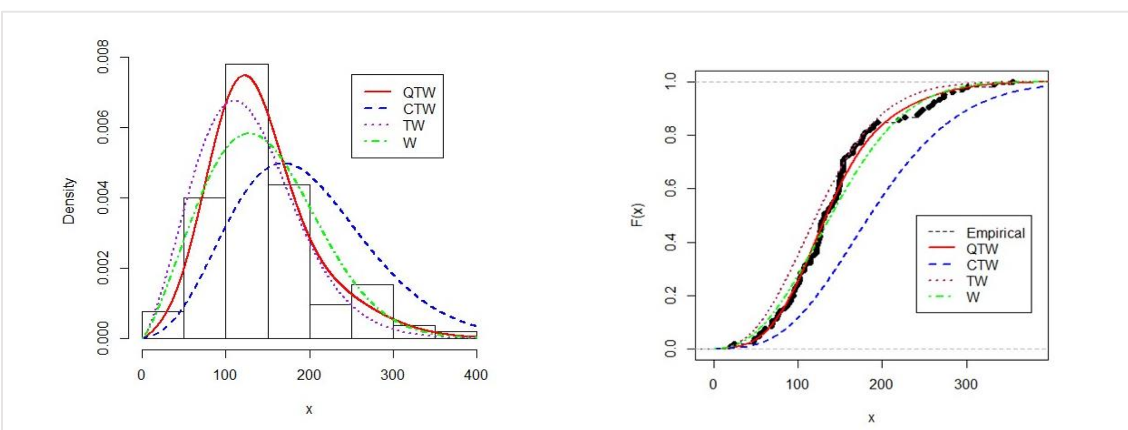

Figure 10: The fitted PDF and CDF of QTW, CTW, TW and W distribution for rainfall data.

The estimated PDF and CDF of the rainfall data set are plotted over empirical density and distribution functions presented in Figure 10. From Figure 9 and Figure 10, it is also depicted that the proposed distribution is a good fit to both the data sets.

7 Conclusion

A new quartic transmuted Weibull distribution is suggested in this article. Distributional properties such as moments, generating functions, reliability and hazard rate functions were discussed. Seven methods have been used to estimate the parameters of the proposed distribution, and the performance of the estimators has been investigated using simulations. With the exception, all the methods performed better with a large sample size. Two real data sets were used to compare the estimation methods and to test the applicability of the proposed distribution. WLSE and LSE methods provided best model fitting estimates of the proposed distribution for Wheaton River and rainfall data respectively. Our proposed distribution provided a better fit compared with Weibull, transmuted Weibull, and cubic transmuted Weibull distributions for both the real-life data sets.

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