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The Marshall–Olkin Pareto Type-I Distribution: Properties, Inference under Complete and Censored Samples with Application to Breast Cancer Data

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Abstract

In this paper, we introduce the Marshall–Olkin Pareto type-I (MOPTI) distribution. Statistical attributes of the MOPTI distribution including the quantile function, mean residual life, and a new theorem for strength-stress measure are introduced. Five methods of estimation for the MOPTI parameters based on complete samples are presented. Furthermore, we explore the estimation of the MOPTI parameters under type-I and type-II censoring. Two Monte Carlo simulation studies are conducted to evaluate the performance of the estimation methods under complete and censored samples. A real-life data set is used to validate the proposed methods.

Key Words: Pareto Type-I distribution; Marshall–Olkin family; strength-stress measure; simulation; type-II censoring.

Mathematical Subject Classification: 60E05, 62F10.

1. Introduction

Many important distributions are widely used in various statistical applications to model several life-time data in applied fields such as engineering, insurance, economics, medicine, and life testing, among others. The limitation of the standard distributions arouses the interest of finding new distributions by extending existing ones. Hence, many authors have been interested in proposing new generalized forms of standard distributions as well as new generated families to overcome these limitations which may include the inability to model non-monotone hazard rate (HR) shapes.

Recent notable families include the Marshall–Olkin-F (MO-F) family which was introduced by Marshall–Olkin (1997). The MO-F family includes the baseline distribution as a special sub-model. This family is one of the most cited families in statistical literature and it has been used to introduce new generalized models or to extend other well-known families. For example, MO Pareto by Alice and Jose (2004), MO Weibull by Ghitany et al. (2005), MO Lomax by Ghitany et al. (2007), MO gamma by Ristic (2007), MO Burr XII by Jayakumar and Mathew (2008), MO uniform by Jose (2011), Marshall-Olkin extended Lindley by Ghitany et al. (2012), MO log-logistic by Gui (2013), MO extended generalized Rayleigh by MirMostafaee et al. (2017), MO power generalized Weibull by Afify et al. (2020), MO alpha power class by Nassar et al. (2019), MO Burr family by Al-Babtain et al. (2021) and Marshall–Olkin–Weibull-H family by Afify et al. (2022), among many others.

In this article, we are motivated to introduce a new flexible extension of the Pareto type-I (PTI) distribution using the MO-F family. The proposed model is called the Marshall–Olkin Pareto type-I (MOPTI) distribution. The MOPTI distribution provides flexible shapes for its HR function (HRF) and probability density function (PDF). It is also more flexible than the PTI distribution. We have also studied its mathematical characteristics in more detail including a new theorem to evaluate strength–stress measure as an important tool for measuring system reliability. We have also show that the exact values of the strength–stress are very close to its approximate values. The MOPTI parameters are estimated using different estimators under complete and censored samples. Comprehensive simulation studies are provided to assess the performance of introduced estimators.

This article consists of six sections. Section 2 introduces the new MOPTI distribution. The structural properties of the MOPTI distribution are derived in Section 3. Five methods of estimation are introduced in Section 4. In Section 5, the MOPTI parameters are estimated under type I and type II censoring schemes. A real-life data set is analyzed in Section 6. Some conclusions are given in Section 7.

2. The MOPTI Distribution

In this section, the new proposed model is defined. Let $\bar{F}(x;\boldsymbol{\psi}) = 1 - F(x;\boldsymbol{\psi})$ be the baseline survival function (SF) of a continuous random variable (RV) x which depends on a parameter vector $\boldsymbol{\psi}$, then the MO-F family with a new shape parameter δ results in another sf defined by

$$\bar{G}(x;\delta,\boldsymbol{\psi}) = \frac{\delta\,\bar{F}(x;\boldsymbol{\psi})}{1-\,\bar{\delta}\,\bar{F}\,(x;\boldsymbol{\psi})}, \qquad x \in \mathbb{R},$$

where $\bar{F}(x;\boldsymbol{\psi}) = 1 - F(x;\boldsymbol{\psi})$ is the baseline sf, $\delta\ > 0$, $\overline{\delta}\ = 1 -\ \delta$.

The corresponding MO-F cumulative distribution function (CDF) takes the form

$$G(x;\delta,\boldsymbol{\psi}) = \frac{F(x;\boldsymbol{\psi})}{1-\ \bar{\delta}\,\bar{F}(x;\boldsymbol{\psi})}, \qquad x \in \mathbb{R}, \tag{1}$$

where $F(x;\boldsymbol{\psi})$ is the baseline CDF. Note that, for $\delta = 1$, then $G(x;\boldsymbol{\psi}) = F\ (x;\boldsymbol{\psi})$.

The PDF of the MO family reduces to

$$g(x;\delta,\boldsymbol{\psi}) = \frac{\delta\, f(x;\boldsymbol{\psi})}{\left(1-\ \bar{\delta}\,\bar{F}(x;\boldsymbol{\psi})\right)^2}, \qquad x \in \mathbb{R}, \tag{2}$$

where $f(x;\boldsymbol{\psi}) = \frac{d}{dx}\,F(x;\boldsymbol{\psi})$ is the baseline PDF.

The CDF and PDF of the PTI distribution are given by

$$F(x) = 1 - \left(\frac{\alpha}{x}\right)^{\beta}, \qquad x \geq \alpha, \qquad \alpha,\beta > 0 \tag{3}$$

and

$$f(x) = \ \beta\,\alpha^{\beta}\,x^{-\beta-1}, \qquad x \geq \alpha, \qquad \alpha,\beta > 0. \tag{4}$$

Then using Equations (3), (4) in (1) and (2), we obtain the CDF and PDF of the MOPTI distribution.

The CDF of the MOPTI distribution takes the form

$$G(x) = \frac{1-\left(\frac{\alpha}{x}\right)^{\beta}}{\left(1-\bar{\delta}\left(\frac{\alpha}{x}\right)^{\beta}\right)}, \quad \alpha, \beta, \delta > 0 \text{ and } x \geq \alpha. \tag{5}$$

Its PDF reduces to

$$g(x) = \frac{\delta\, \beta\, \alpha^{\beta}\, x^{-(\beta+1)}}{\left(1-\bar{\delta}\left(\frac{\alpha}{x}\right)^{\beta}\right)^{2}}, \quad \alpha, \beta, \delta > 0 \text{ and } x \geq \alpha. \tag{6}$$

The HRF of the MOPTI model reduces to

$$h(x) = \frac{\beta\, x^{-1}}{1-\bar{\delta}\left(\frac{\alpha}{x}\right)^{\beta}}, \quad \alpha, \beta, \delta > 0 \text{ and } x \geq \alpha. \tag{7}$$

The graphs of $g(x)$ and $h(x)$ for different values of the parameters are presented in Figures 1 and 2, respectively. These plots show that the MOPTI density provides left-skewed, reversed J-shaped, right-skewed. The failure rate of the MOPTI distribution can be increasing, unimodal and decreasing shaped.

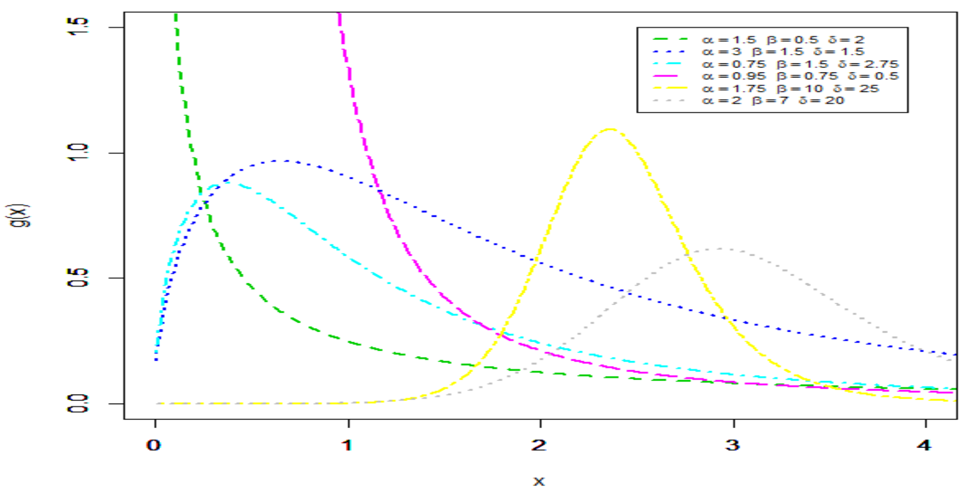

Figure 1: Density plots for various values of the MOPTI parameters.

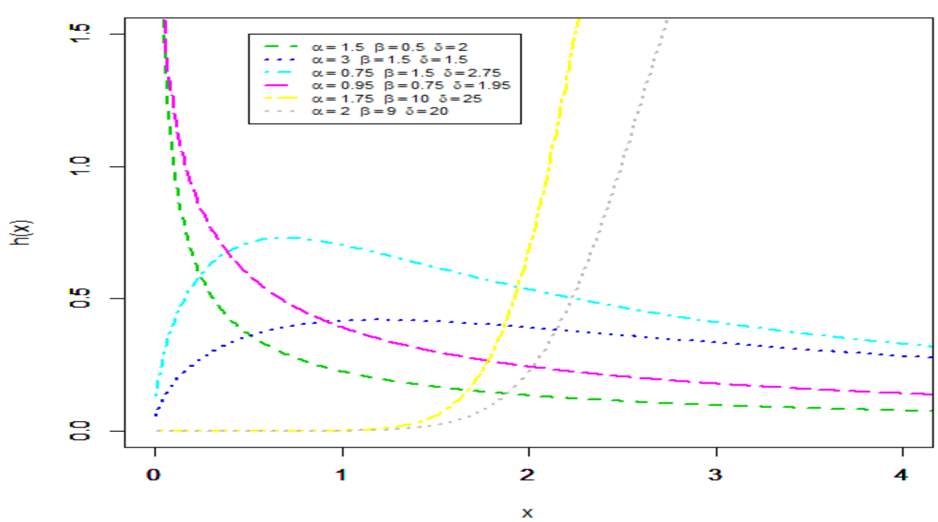

Figure 2: Failure rate plots for various values of the MOPTI parameters.

3. Properties

In this section we derive some statistical attributes of the MOPTI distribution.

3.1 Quantile Function

In heavy tailed distributions, the measures of skewness and kurtosis based on quantiles are better than those based on moments for which the higher moments may not exist. The pth quantile, say Q_p, of the MOPTI model reduces to

$$Q_p = \alpha \left(\frac{1-(1-\delta)P}{1-p}\right)^{1/\beta}, 0 < p < 1. \tag{8}$$

The Bowley skewness (Bsk) measure (Kenney and Keeping, 1962) and the Moor's kurtosis (Mkur) measure (Moors, 1988) are defined by

$$Bsk = \frac{Q_{0.75} - 2Q_{0.5} + Q_{0.25}}{Q_{0.75} - Q_{0.25}}$$

and

$$Mkur = \frac{Q_{0.875} - Q_{0.625} - Q_{0.375} + Q_{0.125}}{Q_{0.75} - Q_{0.25}}.$$

Table 1 gives the values of quantiles, Bsk and Mkur for the MOPTI distribution for some different choices of the parameters. It is noted that skewness and kurtosis decrease with the increase of β.

Table 1: Quantiles, Bsk and Mkur of the MOPTI distribution for $\alpha = 0.2$ and different values of β and δ.

β, δ	$Q_{0.25}$	$Q_{0.5}$	$Q_{0.75}$	$Q_{0.125}$	$Q_{0.375}$	$Q_{0.625}$	$Q_{0.875}$	Bsk	Mkur
2,0.5	0.2160247	0.2449490	0.3162278	0.2070197	0.2280351	0.2708013	0.4242641	0.4226865	1.3217890
4.5,3.5	0.2374928	0.2793755	0.3441443	0.2188575	0.2571707	0.3065518	0.4107660	0.2145877	0.6179097
7,5	0.2300820	0.2583417	0.2971989	0.2160083	0.2438027	0.2751735	0.3337021	0.1578961	0.4579204
9.5,7.5	0.2281919	0.2505322	0.2788386	0.2159343	0.2393111	0.2630344	0.3040619	0.1177984	0.3485064
12,10	0.2259948	0.2442377	0.2662626	0.2153489	0.2352095	0.2540736	0.2852998	0.0939212	0.2822503
16.5,12.5	0.2209301	0.2341721	0.2495276	0.2128119	0.2276976	0.2410953	0.2624379	0.0739051	0.2257855
30,20	0.2140508	0.2213625	0.2293723	0.2092051	0.2178519	0.2250197	0.2358689	0.0455634	0.1437457

Substituting for Q_p and p in Equation (8) by X_u and u, respectively, where $u = u(0,1)$ is a uniform(0,1) random variable or a vector of $u(0,1)$, and X_u is a generating function. If u is a vector of $u(0,1)$ the X_u is a vector of MOPTI distribution with the same length.

3.2 Moments

The rth moment of the MOPTI distribution has the form

$$\mu_r' = E(\chi^r) = \int_\alpha^\infty X^r \, g(x)dx \tag{9}$$

Expanding the denominator of $g(x)$ given in (6) for $0 < \delta < 1$ and for $\delta > 1$. For $\delta > 1$, we can write

$$1-(1-\delta)\left(\frac{\alpha}{x}\right)^\beta = \delta\left[1-\frac{\delta-1}{\delta}\left(1-\left(\frac{\alpha}{x}\right)^\beta\right)\right]. \tag{10}$$

Then (for $0 < \delta < 1, r < \beta$), we have

$$\mu_r' = \sum_{j=0}^{\infty} E_j\ (\alpha, \beta, \delta) = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \beta\,(-1)^k \binom{j}{k}\left(\frac{1}{\delta}\right)\frac{\alpha^r\,(j+1)}{\beta(k+1)-r}\left(1-\frac{1}{\delta}\right)^j, \tag{11}$$

where

$$E_j(\alpha, \beta, \delta) = \alpha^r \beta\ \delta\ \frac{(j+1)(1-\delta)^j}{(\beta(j+1)-r)}. \tag{12}$$

Table 2 presents the exact and approximate values of the first four moments, $(\mu, \mu_2', \mu_3', \mu_4')$, variance (μ_2), coefficient of variation (CV), skewness (SK), and kurtosis (KUR) of the MOPTI distribution. Table 2 shows that the exact and approximate values are approximately the same. The MOPTI distribution can also provide negative and positive skewness.

Table 2: The exact and approximate (in parentheses) values of μ, μ_2, CV, SK and KUR for $\alpha = 2$ and various values of β, δ.

β, δ	μ	μ_2'	μ_3'	μ_4'	μ_2	CV	SK	KUR
6,0.5	2.2693242 (2.269313)	5.3006204 (5.300322)	12.985798 (12.9778)	34.9029806 (34.66299)	0.1507879 (0.1505405)	0.1711144 (0.1709749)	4.6555832 (4.562022)	51.9129152 (44.32418)
8,0.7	2.2350860 (2.23508600)	5.0828745 (5.08287100)	11.840232 (11.84014)	28.5782047 (28.575960)	0.0872652 (0.0872616)	0.1321680 (0.13216525)	3.4710717 (3.46865424)	24.1106503 (23.91210787)
10,2	2.3036363 (2.31418600)	5.4161436 (5.44516600)	12.9962772 (13.07792)	32.0039697 (32.2400500)	0.1094035 (0.0897092)	0.1435826 (0.12942558)	0.4255912 (2.29205417)	15.1394784 (10.22703082)
12,4	2.0878056 (2.3479740)	4.9058125 (5.5906650)	11.6996748 (13.529470)	28.3979627 (33.3735000)	0.4568801 (0.0776831)	0.3542061 (0.1187053)	-2.043065 (1.7522114)	3.6709389 (6.0520811)

3.3 Strength–Stress Measure of System Reliability

In mechanical reliability of a system, the relation between the strength and stress of a device can be measured using statistical methods. If X is a RV represents the strength which is subject to stress Y, where Y is another RV independent of X, then the probability $SSM = Pr\ (X \geq y)$ is called the strength–stress measure (SSM) in statistical literature and it is considered as a measure of system reliability. When stress exceeds strength the system fails, so SSM is the probability of the system failure. The measure SSM has been introduced when X and Y belong to different families (see, Church and Harris (1970), Kotz et al, (2003) and Ali et al. (2010)).

More specifically, the case of X and Y are generalized inverted exponential distribution is studied by Abouammoh and Alshingiti (2009). Inference of SSM for logistic distribution is given in Babayi et al. (2014). Inference of SSM for Poisson distribution is given in Barbiero (2013). Estimation of SSM for generalized exponential and Weibull distributions are given in Kundu and Gupta (2005) and (2006), respectively. The SSM model for Burr type III distribution is introduced in Mokhlis (2005). The estimation of SSM based on progressively type II censored samples when X and Y are two independent generalized Pareto is studied by Rezaei et al. (2015). The estimation of SSM for two independent PTI RVs is introduced by El Damsesy et al. (2014).

Now we study the estimation of the measure SSM when X and Y belong to MOPTI(α, β, δ) without any constraints on the parameters. Simply, the estimation of SSM when X and Y belong to Pareto type-I distribution with four parameters $\alpha_1, \beta_1, \alpha_2$ and β_2 is a special case when $\delta_1 = \delta_2 = 1$.

Let X and Y be two independent MOPTI$(\alpha_1, \beta_1, \delta_1)$ and MOPTI$(\alpha_2, \beta_2, \delta_2)$ respectively, $\alpha_1, \beta_1, \delta_1, \alpha_2, \beta_2$ and δ_2 are greater than zero, then SSM is given by

$$SSM = Pr\ (X \geq y) = \int\int_{x > y} g(x) g(y) dx\, dy,$$

where $g(\,.\,)$ is the PDF of the MOPTI model given by (2), $x > \alpha_1,\ y > \alpha_2, x > y > \max(\alpha_1, \alpha_2)$, then

$$SSM = \int_{\max(\alpha_1, \alpha_2)}^{\infty} g_X(x)\, G_Y(x)\ dx,$$

where $g_X(x)$ is the PDF of X and $G_Y(x)$ is the cdf of Y.

Using (5) and (6), we have (for $\alpha_1, \beta_1, \alpha_2, \beta_2, \delta_1, \delta_2 > 0$)

$$SSM = \int_{\max(\alpha_1,\alpha_2)}^{\infty} \frac{\delta_1\beta_1\alpha_1^{\beta_1}x^{-(\beta_1+1)}}{\left[1-(1-\delta 1)\left(\frac{\alpha_1}{x}\right)^{\beta_1}\right]^2} \frac{1-\left(\frac{\alpha_2}{x}\right)^{\beta_2}}{\left[1-(1-\delta_2)\left(\frac{\alpha_2}{x}\right)^{\beta_2}\right]}\, dx. \tag{13}$$

Now we introduce a new theorem to evaluate the SSM. The theorem has four formulae according to the values of δ_1 and δ_2 when both X and Y are MOPTI (α, β, δ), without any constraints on the parameters, except that they are all positive real numbers.

Theorem 1:

If X and Y are two independent RVs having MOPTI$(\alpha_1, \beta_1, \delta_1)$ and MOPTI$(\alpha_2, \beta_2, \delta_2)$ distributions respectively, then SSM is given by

$$\begin{aligned} SSM &= c1 \sum_{k=0}^{\infty} c(k) \sum_{j=0}^{\infty} c(j)\, h(k,j) \text{ for } 0<\delta_1<1,\ 0<\delta_2<1, \\ &= c2 \sum_{k=0}^{\infty} \sum_{s=0}^{k} c(k,s) \sum_{j=0}^{\infty} \sum_{r=0}^{j} c(j,r)\, h(s,r), \text{ for } \delta_1>1,\ \delta_2>1, \\ &= c3 \sum_{k=0}^{\infty} c(k) \sum_{j=0}^{\infty} \sum_{r=0}^{j} c(j,r)\, h(k,r), \text{ for } 0<\delta_1<1,\ \delta_2>1, \\ &= c4 \sum_{k=0}^{\infty} \sum_{s=0}^{k} c(k,s) \sum_{j=0}^{\infty} c(j) h(s,j), \text{ for } \delta_1>1,\ 0<\delta_2<1, \end{aligned} \tag{14}$$

where

$$h(u,v) = \max(\alpha_1,\alpha_2)^{-(u+1)\beta_1 - v\beta_2} \left[\frac{1}{(u+1)\,\beta_1 + v\ \beta_2} - \frac{(\alpha_2/\max(\alpha_1,\alpha_2)^{\beta 2}}{(u+1)\,\beta_1 + (v+1)\,\beta_2}\right]$$

and

$c = \beta_1(\alpha_1)^{\beta_1},\ c1 = \delta_1 c,\ c2 = \frac{c}{\delta_1\delta_2},\ c3 = \left(\frac{\delta_1}{\delta_2}\right)c,\ c4 = \frac{c}{\delta_1},$

$c(k) = (k+1)(1-\delta_1)^k (\alpha_1)^{k\beta_1},$

$c(j) = (1-\delta_2)^j (\alpha_2)^{j\beta_2},$

$c(k,s) = (k+1)\left(\frac{\delta_1-1}{\delta_1}\right)^k (-1)^s \binom{k}{s} (\alpha_1)^{s\ \beta_1},$

$c(j,r) = \left(\frac{\delta_2-1}{\delta_2}\right)^j (-1)^r \binom{j}{r} (\alpha_2)^{r\ \beta_2}.$

Proof:

If $0<\delta_1<1$ and $0<\delta_2<1$, then we can write

$$\left(1-(1-\delta_1)\left(\frac{\alpha_1}{x}\right)^{\beta 1}\right)^{-2} = \sum_{k=0}^{\infty} (k+1)(1-\delta_1)^k (\alpha 1)^{k\beta_1}(x)^{-k\beta_1}, \tag{15}$$

for $0<1-\delta_1<1$, $x > \alpha_1$, then $0<(1-\delta_1)\left(\frac{\alpha_1}{x}\right)^{k\beta_1}<1$ and

$$\left(1-(1-\delta_2)\left(\frac{\alpha_2}{x}\right)^{\beta_2}\right)^{-1} = \sum_{j=0}^{\infty} (1-\delta_2)^j(\alpha 2)^{j\beta_2}(x)^{-j\beta_2}, \tag{16}$$

$0<1-\delta_2<1, x > \alpha_2$, then $0<(1-\delta_2)\left(\frac{\alpha_2}{x}\right)^{j\beta_2}<1.$

Using (15) and (16) in (13), we get

$$SSM = \delta_1\,\beta_1\,(\alpha_1)^{\beta_1} \sum_{k=0}^{\infty} (k+1)(1-\delta_1)^k (\alpha_1)^{k\beta_1} \sum_{j=0}^{\infty} (1-\delta_2)^j (\alpha_2)^{j\beta_2}$$

$$\int_{\max(\alpha_1,\alpha_2)}^{\infty} x^{-(k+1)\,\beta_1 - j\,\beta_2 - 1} \left(1-(\alpha_2)^{\beta_2}x^{-\beta_2}\right)dx,$$

where the above integral reduces to

$$\max(\alpha_1,\alpha_2)^{-(k+1)\beta_1-j\beta_2}\left[\frac{1}{(k+1)\beta_1+j\beta_2}-\frac{(\alpha_2/\max(\alpha_1,\alpha_2))^{\beta_2}}{(k+1)\beta_1+(j+1)\beta_2}\right].$$

Then, we obtain the SSM (for $0<\delta_1<1$ and $0<\delta_2<1$) as given in (14)

Similarly, for $\delta_1>1$ and $\delta_2>1$, we can use (10) to obtain:

$$(1-(1-\delta_1)\left(\frac{\alpha_1}{x}\right)^{\beta_1})^{-2}=\delta_1^{-2}\sum_{k=0}^{\infty}(k+1)\left(\frac{\delta_1-1}{\delta_1}\right)^k\sum_{s=0}^{k}\binom{k}{s}(\alpha_1)^{s\beta_1}(-1)^s(x)^{-s\beta_1} \quad (17)$$

$\delta_1>1,\ 0<\left(\frac{\delta_1-1}{\delta_1}\right)<1, x>\alpha_1$.

Similarly, for $\delta_2>1, 0<\left(\frac{\delta_2-1}{\delta_2}\right)<1$ and $x>\alpha_2$, we have

$$\left(1-(1-\delta_2)\left(\frac{\alpha_2}{x}\right)^{\beta_2}\right)^{-1}=\delta_2^{-1}\sum_{j=0}^{\infty}\left(\frac{\delta_2-1}{\delta_2}\right)^j\sum_{r=0}^{j}\binom{j}{r}(-1)^r(\alpha_2)^{r\beta_2}(x)^{-r\beta_2}. \quad (18)$$

Using (17) and (18) in (13), we get the SSM for $\delta_1>1$ and $\delta_2>1$, as given in (14).

Similarly, using (15) and (18) in (13), we obtain the SSM for $0<\delta_1<1$ and $\delta_2>1$ as given in (14). Further, using (16) and (17) in (13), we get the SSM for $\delta_1>1$ and $0<\delta_2<1$ as given in (14).

Remark 1:

The values of the SSM given by Equation (13) can be calculated numerically using the R function integrate (). Like all numerical integration routines, the R integrate function gives the approximate values and can fail if misused.

The exact values of the SSM using the above new theorem are given in Table 3.

Table 3: The values of SSM for $\alpha_1=0.5, \alpha_2=0.7$ and different values for β_1,β_2,δ_1 and δ_2.

β_1,β_2	δ_1,δ_2				
	0.7,0.5	1.5,1.7	0.5,1.7	1.5,0.7	1,1
2,1	0.18133020	0.4708448	0.07471165	0.2597749	0.17006800
3,3	0.17046460	0.5628357	0.08460033	0.2702072	0.18221570
5,3	0.06726691	0.2116794	0.02832857	0.1143934	0.06972541
3,5	0.20124190	0.7357544	0.11348160	0.3232251	0.22776970

Corollary 1:

If X and Y are two independent RVs having PTI(α_1,β_1) and PTI(α_2,β_2) distributions, respectively, the SSM measure can be obtained for all cases of Theorem 1 by taking $\delta_1=\delta_2=1$. We obtain the first term in each case which is equal to $\beta_1(\alpha_1)^{\beta_1}h(0,0)$. Then, for $\delta_1=\delta_2=1$, we obtain

$$SSM=\left(\frac{\alpha_1}{\max(\alpha1,\alpha2)}\right)^{\beta_1}-\frac{\beta_1(\alpha1)^{\beta_1}(\alpha2)^{\beta_2}(\max(\alpha1,\alpha2))^{-(\beta_1+\beta_2)}}{\beta_1+\beta_2}. \quad (19)$$

We provide the exact values of the SSM using (19) in Corollary 1 and the corresponding approximate values using the R integrate function in evaluating (13). These values are reported in Table 4. It is noted that the approximate values are very close to the exact values.

Table 4: The values of exact and approximate SSM for $\alpha_1 = 0.5, \alpha_2 = 0.7, \delta_1 = \delta_2 = 1$ and different values of β_1 and β_2.

β_1, β_2	Exact SSM	Approximate SSM
2,1	0.17006800	0.1694576
3,3	0.18221570	0.1822001
5,3	0.06972541	0.0697254
3,5	0.22776970	0.2277541

Corollary 2:

If $\alpha_1 = \alpha_2$ in (19), we obtain

$$SSM = \frac{\beta_2}{\beta_1 + \beta_2}, \tag{20}$$

which is the same result given in [27].

The exact values of the SSM using (20) in Corollary 2 and the corresponding approximate values using the R function integrate() in evaluating (13) are given in Table 5.

Table 5: The values of exact and approximate SSM for $\alpha_1 = \alpha_2 = 0.5$ and $\delta_1 = \delta_2 = 1$, and different values of β_1 and β_2.

β_1, β_2	Exact SSM	Approximate SSM
2,1	0.33333333	0.332718700
3,3	0.500	0.499999999
5,3	0.370	0.375000000
3,5	0.625	0.624984400

3.4 Mean Residual Life (MRL)

In life testing, the expected remaining lifetime given that a unit has survived until time t is a function of t called the MRL (see, Bryson and Siddiqui (1969), Muth (1977), Hollander and Proschan (1975), Greenwood et al. (1979), Bradley and Gupta (2003), and Chaubey and Sen (1999)). Specifically, if the RV X represents the life of a unit, then the MRL takes the form $\text{MRL}(t) = E\,(X - t\,|x > t)$ (Balkema and de-Hann, 1974).

If X has the MOPTI(α, β, δ) distribution, then the MRL(t) reduces to (for $t \geq \max(x)$)

$$\text{MRL}(t) = \frac{1}{1 - G(t)} \int_t^\infty (x - t) dG(x), t > 0 = \frac{1}{1 - G(t)} \int_t^\infty x\, g(x) dx - t = 0.$$

The empirical MRL function EMRL(t) can be calculated generally for any model using the order statistics of a random sample of size n, say $x_{1:n} < x_{2:n} < \cdots < x_{n:n}$. It follows as

$$\text{EMRL}(t) = \frac{1}{n-k} \sum_{i=k+1}^{n} (x_{i:n} - t), \text{for } t \in \left(x_{k:n}, x_{(k+1):n}\right) \text{ and } k = 0, 1, 2, \dots, n-1,$$

where $x_{i:n}$ is the random life of unit i. It is noted that $\text{EMRL}(t) = 0$, for $t \geq x_{n:n}$ and for $x_{0:n} = 0$, EMRL(0) =mean(x) $= \frac{1}{n} \sum_{i=1}^{n} x_i$.

Simply, the EMRL of the MOPTI model can be obtained using a random sample of size n from the MOPTI(α, β, δ).

4. Methods of Estimation

In this section we estimate the parameters α, β and δ of the MOPTI distribution by five different methods of estimation from complete samples. These methods are: maximum likelihood (ML), least–squares (LS) and weighted least–squares (WLS), percentile (PC), and maximum product of spacings (MPS) methods.

4.1 Maximum Likelihood Estimation

Let $x = (x_1, \ldots, x_n)^T$ be a random sample of size n from the pdf (6) and unknown vector parameter $\Theta = (\alpha, \beta, \delta)^T$. Define the log-likelihood function by ℓ, hence the partial derivatives of ℓ with respect to α, β and δ follows as

$$\frac{\partial \ell}{\partial \alpha} = \frac{n\,\beta}{\alpha} + 2\,(1-\delta)\beta\alpha^{(\beta-1)} \sum_{i=1}^{n} \left(\frac{1}{x_i^{\beta} - (1-\delta)\alpha^{\beta}} \right),$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + n\,\ln(\alpha) - \sum_{i=1}^{n} \ln(x_i) + 2\,(1-\delta)\,\alpha^{\beta} \sum_{i=1}^{n} \left(\frac{\ln(\alpha/x_i)}{x_i^{\beta} - (1-\delta)\,\alpha^{\beta}} \right)$$

and

$$\frac{\partial \ell}{\partial \delta} = \frac{n}{\delta} - 2\alpha^{\beta} \sum_{i=1}^{n} \left(\frac{1}{x_i^{\beta} - (1-\delta)\,\alpha^{\beta}} \right).$$

Setting the above equations to zero and solving them simultaneously yields the ML estimates (MLEs), say $\widehat{\Theta} = \left(\hat{\alpha}, \hat{\beta}, \hat{\delta}\right)^T$. These equations may not be solved mathematically; hence we used the R software to solve them numerically.

4.2 Least–Squares and Weighted Least-Squares

We use the LS method to estimate the MOPTI parameters α, β and δ. The LS estimates (LSEs) are obtained by minimizing the following quantity:

$$Q_1 = \sum_{i=1}^{n} \left(G\left(x_{(i)}\right) - \frac{i}{n+1} \right)^2,$$

with respect to α, β and δ, where $G(x)$ is the MOPTI CDF in (5) and $x_{(i)}$ is the ith order statistic of the MOPTI model.

The WLS estimates (WLSEs) of α, β and δ of the MOPTI distribution are obtained by minimizing the following quantity:

$$Q_2 = \sum_{i=1}^{n} \frac{(n+1)^2\,(n+2)}{n-i+1} \left(G\left(x_{(i)}\right) - \frac{i}{n+1} \right)^2,$$

with respect to α, β and δ, where $G(x)$ is as given by (5).

Charnes et. al. [3] showed that the WLSEs are identical to those obtained using the MLE method in the exponential family under wild conditions. In this paper, we showed that the LSEs and WLSEs of α, β and δ are better than those of MLEs.

4.3 Percentile Estimation

This method was originally introduced by Kao (1985) and Kao (1959). The PC estimates (PCEs) of α, β and δ can be determined by minimizing the quantity:

$$\sum_{i=1}^{n} \left[X_{(i)} - \alpha \left(\frac{1-(1-\delta)\left(\frac{i}{n+1}\right)}{1-\left(\frac{i}{n+1}\right)} \right)^{1/\beta} \right]^2,$$

where $X_{(i)}$ is the ith order statistic of the MOPTI distribution.

4.4 Maximum Product of Spacings

The MPS method is introduced by Cheng and Amin (1979). Consider the quantity $D_i\,(\alpha, \beta, \delta)$ which refers to the uniform spacings of a random sample from the MOPTI distribution and it takes the form

$$D_i\,(\alpha, \beta, \delta) = G\left(x_{(i)}|\alpha, \beta, \delta\right) - G\left(x_{(i-1)}|\alpha, \beta, \delta\right), i = 1, 2, \ldots, n,$$

where G(.) as given by (5), $G\left(x_{(0)}|\alpha, \beta, \delta\right) = 0$ and $G\left(x_{(n+1)}|\alpha, \beta, \delta\right) = 1$.

The MPS estimates (MPSEs) of α , β and δ are determined by maximizing the following quantity:

$$\frac{1}{n+1}\sum_{i=1}^{n+1} \ell n\, D_i\,(\alpha\,,\beta\,,\delta),$$

with respect to α , β and δ.

4.5 Simulation Results

In this subsection, a Monte Carlo simulation study is provided to explore the performance of different estimation methods. We apply the above estimation methods to estimate the parameters α , β and δ of the MOPTI distribution using the R software with number of replications, $N = 10000$. The results are obtained from samples of size $n = (20, 200)$ and by choosing $\alpha = (1,2)$, $\beta = (\ 0.5\ , 1.5)$ and $\delta = (0.5, 1.5)$. The average values of the MLEs, LSEs, WLSEs, PCEs and MPSEs as well as the mean square errors (MSEs) are calculated and reported in Tables 6 and 7. The results in these tables illustrate that the MSEs decrease as sample size increases, which shows the consistency of all estimates. The results show also that the LS and WLS methods provide the best estimates for α, β and δ, in terms of their MSEs in most cases. Then, the performance ordering of the estimates from best to worst is the LSEs, WLSEs, MLEs, PCEs and MPSEs, in terms of their MSEs.

Table 6: The numerical results of estimates and their (MSEs) for $n = 20$.

Parameters	MLEs	LSEs	WLSEs	PCEs	MPSEs
$\alpha = 1$	1.299(0.304)	0.987(0.006)	0.995(0.010)	0.928(1.238)	1.559(0.339)
$\beta = 0.5$	0.878(0.593)	0.459(0.085)	0.410(0.149)	0.411(0.237)	0.883(0.278)
$\delta = 0.5$	0.451(0.780)	0.517(0.243)	0.556(0.554)	0.508(0.582)	0.315(0.048)
$\alpha = 1$	1.409(0.891)	1.029(0.042)	1.052(0.066)	0.986(3.838)	1.534(0.966)
$\beta = 0.5$	0.571(0.096)	0.441(0.044)	0.447(0.047)	0.314(0.241)	0.660(0.149)
$\delta = 1.5$	1.518(5.402)	1.444(1.381)	1.476(1.433)	1.545(8.613)	1.630(9.001)
$\alpha = 1$	1.529(0.429)	0.994(0.001)	0.989(0.002)	0.967(0.049)	1.284(0.145)
$\beta = 1.5$	1.576(0.672)	1.442(1.213)	1.415(1.136)	1.311(1.024)	1.696(3.521)
$\delta = 0.5$	0.357(0.300)	0.649(0.643)	0.724(1.049)	0.571(0.330)	0.302(0.975)
$\alpha = 1$	1.229(0.180)	1.007(0.005)	1.007(0.007)	0.975(0.104)	1.243(0.130)
$\beta = 1.5$	1.779(1.400)	1.285(0.511)	1.364(0.437)	1.277(0.574)	2.124(2.381)
$\delta = 1.5$	1.506(17.97)	1.495(2.067)	1.570(1.987)	1.361(1.526)	1.481(16.18)
$\alpha = 2$	2.297(1.349)	1.987(0.024)	1.995(0.033)	1.899(5.232)	2.613(0.958)
$\beta = 0.5$	0.755(0.545)	0.439(0.110)	0.451(0.133)	0.405(0.220)	0.993(1.076)
$\delta = 0.5$	0.478(0.766)	0.507(0.292)	0.559(0.461)	0.448(0.476)	0.663(4.267)
$\alpha = 2$	2.572(3.084)	2.051(0.170)	2.102(0.241)	1.976(11.58)	2.966(2.868)
$\beta = 0.5$	0.599(0.075)	0.458(0.048)	0.466(0.048)	0.366(0.212)	0.666(0.158)
$\delta = 1.5$	1.951(6.065)	1.517(1.469)	1.529(1.598)	1.417(5.824)	1.791(9.312)
$\alpha = 2$	1.443(1.732)	1.985(0.003)	1.980(0.005)	1.854(0.221)	2.966(1.495)
$\beta = 1.5$	1.623(3.373)	1.472(1.156)	1.482(1.002)	1.361(0.915)	1.967(1.622)
$\delta = 0.5$	0.383(3.106)	0.681(0.695)	0.700(0.714)	0.636(0.387)	0.455(0.105)
$\alpha = 2$	1.665(1.621)	2.004(0.021)	2.003(0.028)	2.028(0.405)	2.495(0.495)
$\beta = 1.5$	1.336(1.443)	1.328(0.511)	1.384(0.432)	1.258(0.575)	2.179(1.949)
$\delta = 1.5$	1.148(6.878)	1.624(2.570)	1.678(2.708)	1.248(1.297)	2.100(20.28)

Table 7: The numerical results of estimates and their (MSEs) for $n = 200$.

Parameters	MLEs	LSEs	WLSEs	PCEs	MPSEs
$\alpha = 1$	1.173(0.235)	1.000(0.000)	1.001(0.001)	1.081(3.355)	1.668(0.480)
$\beta = 0.5$	0.703(0.257)	0.488(0.014)	0.488(0.011)	0.313(0.171)	0.808(0.104)
$\delta = 0.5$	0.586(0.245)	0.493(0.023)	0.493(0.026)	0.641(1.161)	0.334(0.033)
$\alpha = 1$	1.244(0.370)	1.008(0.003)	1.012(0.006)	1.101(6.129)	1.493(0.707)
$\beta = 0.5$	0.481(0.023)	0.493(0.005)	0.494(0.005)	0.243(0.144)	0.548(0.054)
$\delta = 1.5$	1.470(1.799)	1.496(0.164)	1.500(0.199)	1.407(9.255)	1.332(3.095)
$\alpha = 1$	1.615(0.529)	1.000(0.000)	0.999(0.000)	0.922(0.073)	1.319(0.196)
$\beta = 1.5$	1.624(0.806)	1.466(0.120)	1.500(0.091)	1.251(0.355)	1.438(2.355)
$\delta = 0.5$	0.467(0.055)	0.501(0.030)	0.520(0.032)	0.497(0.149)	0.402(2.308)
$\alpha = 1$	1.230(0.186)	1.003(0.000)	1.003(0.001)	0.881(0.106)	1.207(0.136)
$\beta = 1.5$	1.608(0.725)	1.441(0.064)	1.459(0.040)	1.304(0.267)	1.761(0.922)
$\delta = 1.5$	1.574(12.31)	1.434(0.215)	1.447(0.163)	1.483(1.275)	1.518(9.253)
$\alpha = 2$	2.334(1.423)	1.998(0.002)	2.008(0.003)	1.919(0.100)	2.643(0.914)
$\beta = 0.5$	0.545(0.135)	0.492(0.016)	0.489(0.011)	0.351(0.140)	0.646(0.309)
$\delta = 0.5$	0.435(0.090)	0.507(0.032)	0.493(0.026)	0.583(0.800)	0.534(2.929)
$\alpha = 2$	2.556(4.560)	2.000(0.019)	2.033(0.024)	2.228(13.74)	2.610(2.232)
$\beta = 0.5$	0.528(0.026)	0.499(0.005)	0.494(0.004)	0.354(0.104)	0.622(0.042)
$\delta = 1.5$	1.728(2.213)	1.537(0.164)	1.489(0.173)	1.649(6.447)	1.969(5.245)
$\alpha = 2$	2.051(0.756)	1.999(0.000)	1.997(0.000)	1.660(0.398)	3.260(2.411)
$\beta = 1.5$	2.546(3.869)	1.429(0.292)	1.509(0.089)	1.287(0.345)	1.367(1.132)
$\delta = 0.5$	0.436(1.437)	0.502(0.060)	0.524(0.031)	0.647(0.297)	0.399(0.089)
$\alpha = 2$	2.051(0.959)	2.002(0.001)	2.002(0.003)	1.727(0.467)	2.295(0.192)
$\beta = 1.5$	1.475(0.918)	1.478(0.056)	1.488(0.039)	1.295(0.268)	1.337(0.425)
$\delta = 1.5$	1.116(3.899)	1.515(0.211)	1.523(0.183)	1.500(1.296)	1.011(12.33)

5. Estimation under Type-I and Type-II Censoring

In this section, we use censoring types I and II to estimate α, β and δ of the MOPTI using the ML method.

5.1 The ML under Censoring Type-I

In censoring of type-I, the unit i is observed for a fixed time $x\,c_i$, $i = 1, 2, \dots, n$, where n is the sample size and the number of failures in the sample is random and it is defined by $s = \sum_{i=1}^{n} d_i$, where d_i refers to the death indicator such that, $d_i = 1$ if the unit i dies and $d_i = 0$ otherwise. Hence, the likelihood function takes the form

$$L = \prod_{i=1}^{n} h(x_i)^{d_i}\, S(x_i),$$

where $d_i = 1$ if $x_i \le x\,c_i$, $d_i = 0$ if $x_i > x\,c_i$, $S(x)$ and $h(x)$ are given by (7) and (8).

Then, the log likelihood function is

$$LL1 = \sum_{i=1}^{n} d_i \ln g(x_i) + \sum_{i=1}^{n} (1 - d_i) \ln S(x\,c_i).$$

If the fixed observation times for all units are equal to a constant value, $x\,c$ the log likelihood function reduces to

$$LL1 = \sum_{i=1}^{n} d_i \ln g(x_i) + (n - s) \ln S(x\,c),$$

where $s = \sum_{i=1}^{n} d_i$ and $x_i \le x\,c$.

By setting the partial derivatives, of the log likelihood function with respect to α, β and δ, to zero and solving them simultaneously yields the MLEs $\widehat{\Theta} = \left(\hat{\alpha}, \hat{\beta}, \hat{\delta}\right)^T$. But these partial derivatives cannot be solved analytically, hence the R statistical package is used to solve them numerically.

5.2 The ML under Censoring Type-II

In type-II censoring, the sample is followed until r units have failed. The number of failures r determines the precision of the study and is fixed in advance. Hence, the likelihood function under censoring type-II reduces to

$$L = \prod_{i=1}^{n} g_{(x_i)}^{d_i} \, S_{(x_i)}^{1-d_i}.$$

Let $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ be the sample order statistics, then the log-likelihood function takes the form

$$LL2 = \sum_{i=1}^{n} d_i \ln g\left(x_{(i)}\right) + (n-r)\ \ln S\left(x_{(r)}\right),$$

where $d_i = 1$, for $i \le r$ and $d_i = 0$, for $i > r$.

The partial derivatives of the above function with respect to α , β and δ can be calculated and solving them simultaneously yields the MLEs. These equations cannot be solved analytically; hence the R software can be used to solve them numerically.

5.3 Simulation Results for Censored Samples

In this section, a Monte Carlo simulation study is employed to explore the performance of proposed methods. We generate censored data from the MOPTI distribution under Type-I and Type-II censoring schemes for different choice of n, where $n = 20, 50,\ 100, 200$ with number of replications 1000. The initial parameters of the MOPTI are $\delta = 0.5, \beta = 1.5,\ \alpha = 0.01$.

1. Supposed schemes for Type-I censoring: $p = 30\%, 60\%, 90\%, 100\%$
2. Supposed schemes for Type-II censoring: $q = 30\%, 60\%, 90\%, 100\%$

where p represent a fraction (in %) of time censoring and q represent a fraction of number of failure items. Clearly, $p = 100\%$ and $q = 100\%$ represent the complete sampling case in both censoring types.

The average estimate (AVE), interval estimate, means square error (MSE), lower (L) and upper (U) limits of asymptotic (ASY) confidence intervals (CI), and average interval length (AIL) with coverage percentage (CP) are reported for both types of censoring schemes (Type-I and Type-II) in Tables 8 and 9, respectively.

The results of the MLE for any sample size n and complete sampling can be shown from the corresponding rows of the following tables. For example, the results of $n =\ 200$ and complete sampling are given in the last three rows.

Table 8: Average estimated values of the MLE, MSE and associated CI estimates, AILs and CPs (in %) of the MOPTI distribution under Type-I censoring scheme for different values of q_i

q_i	n	Par.	Initial: $\delta = 0.5,\ \beta = 1.5,\ \alpha = 0.01$					
			AVE	MSE	ASY CI-L	ASY CI-U	AIL	CP
30%	20	δ	0.5604	0.0090	0.4176	0.7033	0.2857	98.60
		β	1.5635	0.0088	1.4282	1.6987	0.2705	98.60
		α	0.0196	0.0001	0.0141	0.0252	0.0111	99.90
	50	δ	0.5470	0.0063	0.4212	0.6729	0.2517	94.20
		β	1.5607	0.0082	1.4286	1.6928	0.2642	98.60
		α	0.0192	0.0001	0.0145	0.0239	0.0094	93.50
	100	δ	0.5344	0.0037	0.4371	0.6318	0.1948	91.90

		β	1.5579	0.0077	1.4284	1.6875	0.2591	98.60
		α	0.0188	0.0001	0.0153	0.0223	0.0071	92.20
	200	δ	0.5300	0.0025	0.4508	0.6091	0.1583	94.10
		β	1.5504	0.0067	1.4247	1.6762	0.2515	98.60
		α	0.0187	0.0001	0.0159	0.0215	0.0056	94.60
60%	20	δ	0.5577	0.0094	0.4045	0.7109	0.3064	97.80
		β	1.5630	0.0097	1.4149	1.7110	0.2961	98.60
		α	0.0219	0.0002	0.0110	0.0328	0.0218	95.00
	50	δ	0.5471	0.0070	0.4109	0.6833	0.2724	96.10
		β	1.5541	0.0079	1.4160	1.6922	0.2762	98.60
		α	0.0216	0.0002	0.0115	0.0317	0.0202	96.20
	100	δ	0.5387	0.0055	0.4152	0.6621	0.2470	93.00
		β	1.5505	0.0074	1.4144	1.6866	0.2722	98.60
		α	0.0211	0.0001	0.0124	0.0299	0.0175	93.80
	200	δ	0.5332	0.0049	0.4121	0.6544	0.2423	94.30
		β	1.5410	0.0065	1.4055	1.6765	0.2710	98.60
		α	0.0211	0.0001	0.0117	0.0305	0.0188	97.00
90%	20	δ	0.5796	0.0127	0.4234	0.7358	0.3124	97.10
		β	1.5212	0.0094	1.3352	1.7073	0.3721	98.60
		α	0.0224	0.0002	0.0078	0.0371	0.0293	97.10
	50	δ	0.5871	0.0140	0.4302	0.7440	0.3138	94.90
		β	1.5089	0.0086	1.3281	1.6897	0.3615	98.60
		α	0.0216	0.0002	0.0101	0.0332	0.0231	95.30
	100	δ	0.5900	0.0149	0.4277	0.7523	0.3246	92.80
		β	1.4962	0.0097	1.3035	1.6889	0.3854	99.90
		α	0.0216	0.0002	0.0087	0.0345	0.0258	94.90
	200	δ	0.6054	0.0202	0.4185	0.7923	0.3738	92.80
		β	1.4810	0.0139	1.2525	1.7094	0.4569	99.70
		α	0.0229	0.0003	0.0028	0.0429	0.0401	95.90
100%	20	δ	0.6597	0.0464	0.3765	0.9429	0.5664	99.30
		β	1.3809	0.0624	0.9504	1.8113	0.8609	98.60
		α	0.0370	0.0015	0.0000	0.0910	0.0910	93.90

	50	δ	0.6630	0.0467	0.3851	0.9410	0.5559	99.30
		β	1.3685	0.0644	0.9428	1.7942	0.8514	98.60
		α	0.0364	0.0014	0.0000	0.0881	0.0881	95.20
	100	δ	0.6892	0.0536	0.4275	0.9509	0.5235	99.10
		β	1.3455	0.0785	0.8871	1.8039	0.9168	99.40
		α	0.0377	0.0014	0.0000	0.0883	0.0883	95.40
	200	δ	0.7093	0.0595	0.4636	0.9550	0.4914	99.20
		β	1.2933	0.0938	0.8499	1.7368	0.8869	98.80
		α	0.0392	0.0015	0.0000	0.0891	0.0891	94.90

Table 9: Average estimated values of the MLE, MSE and associated CI estimates, AILs and CPs (in %) of the MOPTI distribution under type-II censoring scheme for different values of r_i

r_i	n	Param	Initial: $\delta = 0.5,\ \beta = 1.5,\ \alpha = 0.01$					
			AVE	MSE	ASY CI-L	ASY CI-U	AIL	CP
30%	20	δ	0.5529	0.0071	0.4235	0.6822	0.2587	99.50
		β	1.5734	0.0109	1.4272	1.7195	0.2923	98.60
		α	0.0194	0.0001	0.0124	0.0265	0.0140	96.90
	50	δ	0.5469	0.0059	0.4278	0.6660	0.2382	98.10
		β	1.5636	0.0092	1.4232	1.7040	0.2808	98.60
		α	0.0193	0.0001	0.0135	0.0251	0.0116	95.00
	100	δ	0.5353	0.0039	0.4337	0.6369	0.2032	88.80
		β	1.5611	0.0085	1.4261	1.6962	0.2701	98.60
		α	0.0188	0.0001	0.0143	0.0234	0.0091	92.30
	200	δ	0.5314	0.0030	0.4438	0.6190	0.1752	92.10
		β	1.5532	0.0072	1.4242	1.6822	0.2581	98.60
		α	0.0187	0.0001	0.0150	0.0225	0.0075	92.40
60%	20	δ	0.5477	0.0068	0.4152	0.6802	0.2650	97.60
		β	1.5709	0.0109	1.4205	1.7213	0.3008	98.60
		α	0.0217	0.0002	0.0112	0.0323	0.0212	95.10
	50	δ	0.5434	0.0059	0.4190	0.6678	0.2488	96.70
		β	1.5585	0.0089	1.4131	1.7038	0.2908	98.60
		α	0.0218	0.0002	0.0124	0.0311	0.0187	95.70
	100	δ	0.5365	0.0053	0.4132	0.6598	0.2466	96.40
		β	1.5505	0.0082	1.4036	1.6975	0.2940	98.60
		α	0.0214	0.0002	0.0115	0.0314	0.0199	96.30
	200	δ	0.5333	0.0049	0.4133	0.6534	0.2401	95.70
		β	1.5453	0.0072	1.4044	1.6862	0.2818	98.60
		α	0.0212	0.0002	0.0117	0.0308	0.0191	97.10
90%	20	δ	0.5680	0.0144	0.3738	0.7623	0.3885	95.80
		β	1.5355	0.0173	1.2872	1.7837	0.4965	98.60
		α	0.0231	0.0003	0.0000	0.0479	0.0479	97.00

	50	δ	0.5660	0.0130	0.3834	0.7485	0.3651	94.10
		β	1.5299	0.0145	1.3008	1.7590	0.4582	98.60
		α	0.0215	0.0002	0.0045	0.0386	0.0341	96.60
	100	δ	0.5779	0.0155	0.3871	0.7686	0.3815	93.00
		β	1.5073	0.0152	1.2662	1.7485	0.4823	98.60
		α	0.0217	0.0002	0.0065	0.0369	0.0304	95.50
	200	δ	0.5979	0.0215	0.3838	0.8121	0.4283	94.30
		β	1.4855	0.0166	1.2347	1.7363	0.5015	99.90
		α	0.0228	0.0003	0.0041	0.0416	0.0374	94.80
100%	20	δ	0.6327	0.0381	0.3522	0.9132	0.5611	96.70
		β	1.4357	0.0524	1.0049	1.8664	0.8615	98.60
		α	0.0334	0.0012	0.0000	0.0836	0.0836	92.40
	50	δ	0.6586	0.0448	0.3839	0.9334	0.5494	98.90
		β	1.3762	0.0609	0.9575	1.7948	0.8373	98.60
		α	0.0357	0.0014	0.0000	0.0874	0.0874	95.30
	100	δ	0.6885	0.0533	0.4272	0.9498	0.5225	99.10
		β	1.3461	0.0783	0.8877	1.8045	0.9168	99.40
		α	0.0376	0.0014	0.0000	0.0882	0.0882	95.20
	200	δ	0.7093	0.0595	0.4636	0.9550	0.4914	99.20
		β	1.2933	0.0938	0.8499	1.7368	0.8869	98.80
		α	0.0392	0.0015	0.0000	0.0891	0.0891	94.90

6. Real-Life Data Analysis

The real-life data represents the survival times of 121 patients suffering from breast cancer. The data is reported by a large hospital in a period 1929-1938 [13]. The data are: 0.3, 4.0, 0.3, 5.0, 6.2, 5.6, 6.3, 6.8, 6.6, 7.4, 8.4, 7.5, 8.4, 11.0, 10.3, 11.8, 12.3, 12.2, 14.4, 13.5, 14.8, 14.4, 15.5, 16.2, 15.7, 16.3, 16.8, 16.5, 17.2, 17.5, 17.3, 17.9, 20.4, 19.8, 20.9, 21.0, 21.0, 21.1, 23.4, 23.0, 23.6, 27.9, 24.0, 28.2, 24.0, 29.1, 31.0, 30.0, 31.0, 35.0, 32.0, 37.0, 35.0, 37.0, 38.0, 37.0, 38.0, 45.0, 39.0, 41.0, 39.0, 40.0, 38.0, 40.0, 43.0, 41.0, 43.0, 41.0, 40.0, 42.0, 44.0, 43.0, 45.0, 48.0, 46.0, 47.0, 46.0, 51.0, 49.0, 51.0, 52.0, 51.0, 55.0, 54.0, 56.0, 58.0, 57.0, 59.0, 62.0, 60.0, 65.0, 60.0, 67.0, 60.0, 61.0, 65.0, 68.0, 67.0, 69.0, 80.0, 78.0, 83.0, 89.0, 88.0, 90.0, 96.0, 93.0, 105.0, 103.0, 109.0, 111.0, 109.0, 115.0, 125.0, 117.0, 126.0, 129.0, 129.0, 154.0, 127.0, 139.0. This data is analyzed by Ramos et al. (2013).

We first check whether the MOPTI distribution is suitable for analyzing this data set. Now, we obtain the results from real-life data using the studied methods in the previous section. The MLEs of the parameters and the value of Kolmogorov–Smirnov (K–S) test statistic is reported to judge the goodness-of-fit. The calculated K-S distance between the empirical and the fitted MOPTI distribution is 0.1061 and its $p-value = 0.1310$ where $\hat{\delta} = 6859.49$, $\hat{\beta} = 1.8541$, and $\hat{\alpha} = 0.2914$. This shows that the MOPTI distribution can be considered as an adequate model for the given data.

Now, one can suppose the following schemes for Type-I and Type-II censoring with censoring fractions for Type-I and Type-II (30%,60%,90%,100%) and numbers of failures s and r, respectively, as follows:

Type-I Censoring			**Type-II Censoring**		
Scheme	Censoring fraction	(n, s)	Scheme	Censoring fraction	(n, r)
I	30%	(121, 21)	I	30%	(121, 37)
II	60%	(121, 45)	II	60%	(121, 73)
III	90%	(121, 105)	III	90%	(121, 109)
Complete	100%	$n = s = 121$	Complete	100%	$n = r = 121$

The MOPTI distribution is fitted to the given data under the considered schemes. The Akaike's information criterion (AIC), negative log-likelihood criterion (NLC) and Bayesian information criterion (BIC) are calculated to compare these methods. The MLEs of the MOPTI parameters along with their standard errors (SEs), AIC, BIC and NLC are respectively listed in Tables 10 and 12 for type-I and type-II censoring schemes. It is shown that EMOIP distribution provides a better fit under the two types of censoring.

Table 10: The MLEs, SEs, AIC, BIC and NLC of MOPTI distribution using real data under Type-I censoring.

Scheme	Par.	MLEs	SEs	Goodness-of-fit		
				AIC	BIC	NLC
I	δ	1424.3064	2.42e-05	379.6744	384.5072	186.8372
	β	1.5072	2.29e-01			
	α	0.2958	1.91e-01			
II	δ	2026.9402	1.29e-05	728.9003	735.7717	361.4501
	β	1.5804	1.60e-01			
	α	0.2960	1.43e-01			
III	δ	3964.5991	5.09e-06	1070.8981	1078.9720	532.4491
	β	1.7361	1.39e-01			
	α	0.2977	1.18e-01			
Complete	δ	6859.4946	2.63e-06	1181.1640	1189.5512	587.5820
	β	1.8541	1.40e-01			
	α	0.2914	1.12e-01			

Furthermore, the MRL and EMRL are calculated empirically using the real-life data for the first and the last 10 values of $x, n = 121, \alpha = 0.2914, \beta = 1.8541$ and $\delta = 8659.4946$. The results are reported in Table 12. Table 12 shows that the values of MRL and EMRL are approximately the same for small values of x and both decrease with increasing the survival time x.

Table 11: The MLEs, SEs, AIC, BIC and NLC of MOPTI distribution using real data under Type-II censoring

Scheme	Par.	MLE	St. Error	Goodness-of-fit		
				AIC	BIC	NLC
I	δ	1423.3064	2.05e-05	379.6043	384.4923	186.8332
	β	1.5170	2.32e-01			

	α	0.2950	1.95e-01			
II	δ	2024.0851	1.25e-05	728.9182	735.7065	361.4521
	β	1.5803	1.63e-01			
	α	0.2962	1.48e-01			
III	δ	3964.5908	5.19e-06	1070.8901	1078.9801	532.4452
	β	1.7368	1.16e-01			
	α	0.2979	1.08e-01			
Complete	δ	6859.4946	2.63e-06	1181.1640	1189.5512	587.5820
	β	1.8541	1.40e-01			
	α	0.2914	1.12e-01			

Table 12: The values of the MRL and EMRL for the real data set

No	X	MRL	EMRL	No	x	MRL	EMRL
1	0.3	45.53989	46.41250	102	83	11.9902689	30.36482
2	0.3	45.53989	46.80252	103	88	11.4643226	26.77778
3	4.0	44.30625	43.46780	104	89	10.9388729	27.29412
4	5.0	43.43936	42.83077	105	90	10.4139056	27.93750
5	5.6	42.79920	42.59483	106	93	9.8705343	26.60000
6	6.2	42.27240	42.36000	107	96	9.3092750	25.28571
7	6.3	41.88183	42.63070	108	103	8.6935783	19.69231
8	6.6	41.5513	42.70531	109	105	8.0708255	19.16667
9	6.8	41.27215	42.88482	110	109	7.4230227	16.54545
10	7.4	40.98875	42.66577	111	109	6.7868921	18.20000

7. Conclusion

In this paper we introduced a three-parameter extended Marshall–Olkin Pareto type-I (MOPTI) distribution. Structural properties including moments, quantile function, mean residual life, and a new theorem for the strength–stress measure is provided. The MOPTI parameters are estimated using five methods of estimation based on complete sampling. Furthermore, the MOPTI parameters are also estimated using the maximum likelihood under censoring schemes of types I and II. A real data set on breast cancer is analyzed to validate our results and to illustrate the usefulness of the newly MOPTI distribution in applications.

The work in this paper can be extended in some ways. For example, entropy estimation based on the MOPTI distribution considering the works of Zamanzade and Mahdizadeh (2016 and 2017) and Mahdizadeh and Zamanzade (2017 and 2019). Bayesian inference the MOPTI distribution based on lower k-record values considering the work of MirMostafaee et al. (2016). Another extended versions of the PTI distribution can be established considering the works of Mead (2014), Fatima and Roohi (2015), and Tamandi et al. (2019).

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