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Remarks on the Paper ”On the product and Quotient of Pareto and Rayleigh Random Variables”

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Abstract

Obeid and Kadry (2019) tried to study the product and quotient of the independent Pareto and Rayleigh random variables. The distributions they obtained are incorrect and in fact there is no closed form distributions as discussed here. We will mention the errors made and try to establish the correct versions of the probability density functions of these distributions for the truncated Pareto and Rayleigh random variables.

1. Introduction and Discussion

Obeid and Kadry (2019), take the independent random variables X and Y as Pareto and Rayleigh with the pdf's (probability density functions)

$$f_X\left(x\right) = ca^c x^{-(c+1)}, \quad x > a, \tag{1}$$

and

$$f_Y\left(y\right) = \frac{y}{b^2} e^{-\frac{y^2}{2b^2}}, \quad y > 0, \tag{2}$$

respectively, where $a > 0, b > 0$ and $c > 0$ are parameters.

They defined the distribution of $Z = XY$ for $c + 1 = 2n$ and $c + 1 = 2n + 1$ as

$$F_Z\left(z\right) = 1 - \left(\frac{a}{z}\right)^c \frac{1}{2}\left[c!!b^c\sqrt{2\pi}\right], \quad z \geq 0, \tag{3}$$

and

$$F_Z\left(z\right) = 1 - \left(\frac{a}{z}\right)^c 2^{c/2} b^c\left(c/2\right)!, \quad z \geq 0, \tag{4}$$

where $c!! = \frac{(c+1)!}{2^{(c+1)/2}\left(\frac{c+1}{2}\right)!}$.

Unfortunately, neither (3) nor (4) are cdf's (cumulative distribution functions) since as $z \to 0$, $F_Z\left(z\right) \to -\infty$. Therefore, all of their derivations are incorrect. One way to eliminate this problem is to assume that X and Y have truncated form on finite intervals. Without Loss of Generality, we assume that X and Y have the following distributions, respectively,

$$F_X(x) = 1 - \frac{x^{-\alpha} - \beta^{-\alpha}}{1 - \beta^{-\alpha}}, \quad \alpha > 0, \quad 1 \le x \le \beta, \tag{5}$$

and

$$F_Y(y) = 1 - \frac{e^{-y^2} - e^{-\beta^2}}{e^{-1} - e^{-\beta^2}}, \quad 1 \le y \le \beta, \tag{6}$$

where $\alpha > 0$.

Now, $Z = XY$ would have the range $[1, \beta^2]$, and the pdf of Z has the following form (see (1), page 3; the case $ad = bc$):

$$f_Z(z) = \begin{cases} \int_1^z f_X(\frac{z}{x}) f_Y(y) \frac{1}{y} \mathrm{d}y, & 1 < z < \beta \\ \int_{z/\beta}^{\beta} f_X(\frac{z}{x}) f_Y(y) \frac{1}{y} \mathrm{d}y, & \beta < z < \beta^2 \end{cases}$$

For $1 < z < \beta$, we have

$$\begin{aligned} f_Z(z) &= \frac{\alpha z^{-(\alpha+1)}}{(1-\beta^{-\alpha})\left(e^{-1} - e^{-\beta^2}\right)} \int_1^z 2y^{\alpha+1} e^{-y^2} \mathrm{d}y \\ &= \frac{\alpha z^{-(\alpha+1)}}{(1-\beta^{-\alpha})\left(e^{-1} - e^{-\beta^2}\right)} \int_1^{z^2} u^{\alpha/2} e^{-u} \mathrm{d}u. \end{aligned}$$

Clearly, $\int_1^{z^2} u^{\alpha/2} e^{-u} \mathrm{d}u$ does not have a manageable closed form and consequently F_Z cannot be computed analytically.

For $\beta < z < \beta^2$, we have

$$\begin{aligned} f_Z(z) &= \frac{\alpha z^{-(\alpha+1)}}{(1-\beta^{-\alpha})\left(e^{-1} - e^{-\beta^2}\right)} \int_{z/\beta}^{\beta} 2y^{\alpha+1} e^{-y^2} \mathrm{d}y \\ &= \frac{\alpha z^{-(\alpha+1)}}{(1-\beta^{-\alpha})\left(e^{-1} - e^{-\beta^2}\right)} \int_{z^2/\beta^2}^{\beta^2} u^{\alpha/2} e^{-u} \mathrm{d}u. \end{aligned}$$

Again, $\int_{z^2/\beta^2}^{\beta^2} u^{\alpha/2} e^{-u} \mathrm{d}u$ does not have a manageable closed form and consequently F_Z cannot be computed analytically in this case either.

The pdf of Z is

$$f_Z(z) = \begin{cases} \frac{\alpha z^{-(\alpha+1)}}{(1-\beta^{-\alpha})\left(e^{-1} - e^{-\beta^2}\right)} \int_1^{z^2} u^{\alpha/2} e^{-u} \mathrm{d}u, & 1 < z < \beta \\ \frac{\alpha z^{-(\alpha+1)}}{(1-\beta^{-\alpha})\left(e^{-1} - e^{-\beta^2}\right)} \int_{z^2/\beta^2}^{\beta^2} u^{\alpha/2} e^{-u} \mathrm{d}u, & \beta < z < \beta^2 \end{cases}$$

Using the package MAPLE 15, we have

$$\int_1^{z^2} u^{\alpha/2} e^{-u} du = 2 \frac{z^{\alpha/2} e^{-z^2/2} Whitta\ker M\left(\frac{\alpha}{4}, \frac{\alpha}{4} + \frac{1}{2}, z^2\right) - e^{-1/2} Whitta\ker M\left(\frac{\alpha}{4}, \frac{\alpha}{4} + \frac{1}{2}, 1\right)}{\alpha + 2}$$

and

$$\int_{z^2/\beta^2}^{\beta^2} u^{\alpha/2} e^{-u} du = 2\frac{\beta^{\alpha/2} e^{-\beta^2/2} Whitta\ker M\left(\frac{\alpha}{4}, \frac{\alpha}{4}+\frac{1}{2}, \beta^2\right) - \left(\frac{z}{\beta}\right)^{\alpha/2} e^{-z^2/\left(2\beta^2\right)} Whitta\ker M\left(\frac{\alpha}{4}, \frac{\alpha}{4}+\frac{1}{2}, \frac{z^2}{\beta^2}\right)}{\alpha+2},$$

where $Whitta\ker M(\mu,\nu,z) = e^{-z/2} z^{\nu+1/2}\, {}_1F_1\left(\left[\frac{1}{2}+\nu-\mu\right], [1+2\nu], z\right)$ is the WhittakerM function, ${}_pF_q([n],[d],\lambda) = \sum_{j=0}^{\infty} \frac{\lambda^j \Pi_{i=1}^{p} \Gamma(n_i+j)\Gamma^{-1}(n_i)}{\Gamma(j+1)\Pi_{i=1}^{q}\Gamma(d_i+j)\Gamma^{-1}(d_i)}$ is the generalized hypergeometric function, $\mathbf{n} = [n_1, n_2, ..., n_p]$, p is the number of operands of $\mathbf{n}, \mathbf{d} = [d_1, d_2, ..., d_q]$, q is the number of operands of $\mathbf{d}$ and $\Gamma(\cdot)$ is the complete gamma function. The WhittakerM and generalized hypergeometric functions are quickly evaluated and readily available in the standard software such as MAPLE.

Remark 1.1. *Similar derivations can be obtained for* $V = \frac{X}{Y}$.

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References

1. Glen, A. G., Leemis, L. M., and Drew, J. H. (2004). Computing the distribution of the product of two continuous random variables. *Computational statistics & data analysis*, 44(3):451–464.
2. Obeid, N. and Kadry, S. (2019). On the product and ratio of pareto and rayleigh random variables. *Pak. J. Stat*, 35(4):285–300.