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A New Reciprocal System of Burr Type X Densities with Applications in Engineering, Reliability, Economy, and Medicine

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Abstract
Depending on Yousof et al. (2017a), a new one parameter G family of distributions called the reciprocal Burr X-G family is defined and studied. Special member based on the well-known Burr type XII model called the reciprocal Burr X-Burr XII distribution is studied and analyzed. Relevant properties of the new family including ordinary moments, moment of the residual life, moment of the reversed residual life and incomplete moments are derived and some of them are numerically analyzed. Four different applications to real-life data sets are presented to illustrate the applicability and importance of the new family. The new family has proven to be highly capable and flexible in practical applications and statistical modeling of real data.

Keywords: Applications; Burr Type X Family, Reliability; Simulations; Statistical Modeling.

Mathematical Subject Classification: 62N01; 62N02; 62E10.

1. Introduction

The statistical literature contains many new continuous G families of distributions, some of them are have been generated by merging (compounding) such as Aryal and Yousof (2017) for exponentiated generalized-G Poisson family, Alizadeh et al. (2018) for odd log-logistic Poisson G family, Abouelmagd et al. (2019 a, b) for Poisson Burr X family and Topp Leone Poisson family of distributions, El-Morshedy et al. (2021) for the Poisson generalized exponential G family. Many other continuous G families of distributions have been generated by adding one or more parameters to the G family such as Gupta et al. (1998) for the exponentiated-G family, Marshall and Olkin (1997) for the Marshall-Olkin-G family, Eugene et al. (2002) for beta generalized-G family, Yousof et al. (2015) for the transmuted exponentiated generalized-G family, Rezaei et al. (2017) for the Topp Leone generated family, Merovci et al. (2017) for the exponentiated transmuted-G family, Brito et al. (2017) for the Topp-Leone odd log-logistic-G family, Yousof et al. (2017a) for Burr type X G family, Cordeiro et al. (2018) for Burr type XII G family, Nascimento et al. (2019) for the odd Nadarajah-Haghighi family of distributions, Karamikabir et al. (2020) for the Weibull Topp-Leone generated family, Alizadeh et al. (2020 a, b) for flexible Weibull generated family of distributions and transmuted odd log-logistic-G family, Merovci et al. (2020) for Poisson Topp Leone family of distributions and Altun et al. (2021) for the Gudermannian generated family of distributions and among others. These new continuous G families have been employed in modeling of real-life datasets in many applied studies such as econometrics, insurance, medicine, engineering, biology, environmental sciences, and statistical forecasting. As New alternative methodology, we can consider reciprocal distribution for generating a new continuous G family of distributions. Consider the CDF Burr X (BX) distribution where

$$F_{\theta}(y) = [1 - exp(-y^2)]^{-\theta} |y > 0,\ \theta \geq 0.$$

where θ is the shape parameter. For $\theta = 1$, BX distribution reduces to the Rayleigh distribution. Let $X = Y^{-1}$, then the CDF and reciprocal Burr X (RBX) distribution can be written as

$$F_\theta(x) = 1 - [1 - exp(-x^{-2})]^\theta | x > 0,\ \theta \geq 0, \tag{1}$$

where θ is the shape parameter. For $\theta = 1$, RBX distribution reduces to the reciprocal Rayleigh (RR) distribution. The corresponding RBX distribution can be derived

$$f_\theta(x) = 2\theta x^{-3}\, exp(-x^{-2})\, [1 - exp(-x^{-2})]^{\theta-1}, \tag{2}$$

Let $g_{\underline{\Psi}}(x)$ and $G_{\underline{\Psi}}(x)$ denote the density and cumulative functions of the baseline model with parameter vector $\underline{\boldsymbol{\Psi}}$ and consider the CDF of the RBX model with positive parameter θ. Hence, the CDF of the reciprocal Burr X-G (RBX-G) family of distributions can be derived as

$$F_{\theta,\underline{\Psi}}(x) = 1 - \left[1 - exp\left(-\delta_{x;\underline{\Psi}}^{-2}\right)\right]^\theta. \tag{3}$$

where $\delta_{x;\underline{\Psi}}^{-2} = \left[G_{\underline{\Psi}}(x)/\overline{G}_{\underline{\Psi}}(x)\right]^{-2}$ and $\overline{G}_{\underline{\Psi}}(x) = 1 - G_{\underline{\Psi}}(x)$ is the survival function. For $\theta = 1$, RBX-G family of distributions reduces to the reciprocal Rayleigh G (RR) family of distributions. The PDF of the RBX-G is given by

$$f_{\theta,\underline{\Psi}}(x) = 2\theta g_{\underline{\Psi}}(x)\overline{G}_{\underline{\Psi}}(x)G_{\underline{\Psi}}(x)^{-3}\, exp\left(-\delta_{x;\underline{\Psi}}^{-2}\right)\left[1 - exp\left(-\delta_{x;\underline{\Psi}}^{-2}\right)\right]^{\theta-1}. \tag{4}$$

The hazard rate function of the RBX-G family of distributions can be written as

$$h_{\theta,\underline{\Psi}}(x) = 2\theta g_{\underline{\Psi}}(x)\frac{G_{\underline{\Psi}}(x)^{-3}\overline{G}_{\underline{\Psi}}(x)}{1 - exp\left(-\delta_{x;\underline{\Psi}}^{-2}\right)} exp\left(-\delta_{x;\underline{\Psi}}^{-2}\right).$$

For simulating the RBX-G family of distributions, the following formula can be used

$$Q_U = G^{-1}\left(\left\{-ln\left[1 - (1 - U)^{\frac{1}{\theta}}\right]\right\}^{-\frac{1}{2}} + 1\right).$$

The RBX-G family could be useful in modeling the monotonically increasing hazard rate real data sets as illustrated in Figures 4, 5 and 6 (bottom left panels), the bathtub hazard rate real data sets as illustrated in Figure 7 (bottom left panel), the real data sets which have some extreme observations as shown Figures 4, 5 and 6 (top left panels), the real data sets which have no extreme observations as shown Figure 7 (top left panel), the real data sets which their Kernel density is semi-symmetric and bimodal real data as shown in Figure 4 (bottom right panel) and the real data sets which their Kernel density is asymmetric bimodal and left skewed real data as shown in Figure 5, 6 and (bottom right panels). The RBX-G family of distributions proved its wide applicability in mathematical modeling against common G families of distributions as shown below:

I. In modeling engineering data (the breaking stress data), the RBX-G family is compared with many well-known G families of distributions such as the Marshall-Olkin G family, Topp-Leone G family, Zografos-Balakrishnan G family, beta G family, Beta exponentiated G family and Kumaraswamy G family under the Hannan-Quinn information criteria, Bayesian information criteria, consistent-information criteria and Akaike information criteria.

II. In modeling veterinary medicine data (the survival times of guinea pigs' data), the RBX-G family is compared with many well-known G families of distributions such as the Marshall-Olkin G family, Topp-Leone G family, Zografos-Balakrishnan G family, beta G family, Beta exponentiated G family and Kumaraswamy G family under the Hannan-Quinn information criteria, Bayesian information criteria, consistent-information criteria and Akaike information criteria.

III. In modeling econometrics data (the revenue data data), the RBX-G family is compared with many well-known G families of distributions such as the Marshall-Olkin G family, Topp-Leone G family, Zografos-Balakrishnan G family, beta G family, Beta exponentiated G family and Kumaraswamy G family under the Hannan-Quinn information criteria, Bayesian information criteria, consistent-information criteria and Akaike information criteria.

IV. In modeling medicine data (the leukemia data data), the RBX-G family is compared with many well-known G families of distributions such as the Marshall-Olkin family, Topp-Leone family, Zografos-Balakrishnan G family, beta G family, Beta exponentiated G family and Kumaraswamy G family under the Hannan-Quinn information criteria, Bayesian information criteria, consistent-information criteria and Akaike information criteria.

1. Properties

Linear representation

In this section, we provide a very useful linear representation for the RBX-G density function. If $\left|\frac{\varsigma_1}{\varsigma_2}\right| < 1$ and $\varsigma_3 > 0$ is a real non-integer, the power series holds

$$\left(1-\frac{\varsigma_1}{\varsigma_2}\right)^{\varsigma_3} = \sum_{\kappa_0=0}^{\infty} (-1)^{\kappa_0} \frac{\Gamma(1+\varsigma_3)}{\Gamma(1+\kappa_0)\Gamma(1+\varsigma_3-\kappa_0)} \left(\frac{\varsigma_1}{\varsigma_2}\right)^{\kappa_0}. \tag{5}$$

Applying (5) to (4) we have

$$f_{\theta,\underline{\Psi}}(x) = 2\theta g_{\underline{\Psi}}(x)\overline{G}_{\underline{\Psi}}(x)G_{\underline{\Psi}}(x)^{-3}\sum_{\kappa_0=0}^{\infty} \frac{(-1)^{\kappa_0}\Gamma(\theta)}{\kappa_0!\,\Gamma(\theta-\kappa_0)} exp\left[-(\kappa_0+1)\delta_{x;\underline{\Psi}}^{-2}\right]. \tag{6}$$

Applying the power series to the term $exp\left[-(\kappa_0+1)\delta_{x;\underline{\Psi}}^{-2}\right]$, Equation (6) becomes

$$f_{\theta,\underline{\Psi}}(x) = 2\theta g_{\underline{\Psi}}(x) \sum_{\kappa_0,\kappa_1=0}^{\infty} (-1)^{\kappa_0+\kappa_1} \frac{(\kappa_0+1)^{\kappa_1}\Gamma(\theta)}{\kappa_0!\,\kappa_1!\,\Gamma(\theta-\kappa_0)} \frac{G_{\underline{\Psi}}(x)^{-2\kappa_1-3}}{\overline{G}_{\underline{\Psi}}(x)^{-2\kappa_1-1}}. \tag{7}$$

Applying (5) to (7) for the term $\overline{G}_{\underline{\Psi}}(x)^{2\kappa_1+1}$, Equation (7) can be written as

$$f_{\theta,\underline{\Psi}}(x) = \sum_{\kappa_1,\kappa_2=0}^{\infty} C_{\kappa_1,\kappa_2}\pi_\varsigma(x)|_{\varsigma=\kappa_2-2(\kappa_1+1)>0}, \tag{8}$$

where

$$C_{\kappa_1,\kappa_2} = 2\theta(-1)^{\kappa_1+\kappa_2}\frac{\Gamma(\theta)\Gamma(2\kappa_1+2)}{\kappa_1!\,\kappa_2!\,\Gamma(2\kappa_1+2-\kappa_2)\varsigma}\sum_{\kappa_0=0}^{\infty} (-1)^{\kappa_0}\frac{(\kappa_0+1)^{\kappa_1}}{\kappa_0!\,\Gamma(\theta-\kappa_0)},$$

and $\pi_\varsigma(x) = \varsigma g_{\underline{\Psi}}(x)G_{\underline{\Psi}}(x)^{\varsigma-1}$. Based on (8), the density of X can then be expressed as a mixture of the exp-G PDFs and several mathematical properties of the new family can be obtained using the exp-G distribution. Similarly, the CDF of the RBX-G family of distributions can also be expressed as a mixture of exp-G CDFs given by

$$F_{\theta,\underline{\Psi}}(x) = \sum_{\kappa_1,\kappa_2=0}^{\infty} C_{\kappa_1,\kappa_2}\ \Pi_\varsigma(x),$$

where $\Pi_\varsigma(x)$ is the CDF of the exp-G family with power parameter (ς).

Mathematical and statistical properties

Moments

Let Y_ς be a rv having density $\pi_\varsigma(x)$. The r^{th} ordinary moment of X, say $\mu'_{r,X}$, follows from (8) as

$$\mu'_{r,X} = E(X^r) = \int_{-\infty}^{+\infty} x^r f_{\theta,\underline{\Psi}}(x)dx = \sum_{\kappa_1,\kappa_2=0}^{\infty} C_{\kappa_1,\kappa_2}\ E\left(Y_\varsigma^r\right), \tag{9}$$

Where $E(Y_\varsigma^r) = \varsigma\int_{-\infty}^{\infty} x^r\ g_{\underline{\Psi}}(x)G_{\underline{\Psi}}(x)^{\varsigma-1}\ dx$ can be evaluated numerically in terms of the baseline qf $Q_G(u) = G^{-1}(u)$ as $E(Y_\varsigma^n) = \varsigma\int_0^1 u^{\varsigma-1}\ Q_G(u)^n\ du$. Setting $r=1$ in (9) gives the mean of X.

Incomplete moments

The r^{th} incomplete moment of X is given by $m_{r,X}(y;\underline{\boldsymbol{\Psi}}) = \int_{-\infty}^{y} x^r f_{\theta,\underline{\Psi}}(x)dx$. Using (8), the r^{th} incomplete moment of RBX-G family is

$$m_{r,X}(y;\underline{\boldsymbol{\Psi}}) = \sum_{\kappa_1,\kappa_2=0}^{\infty} C_{\kappa_1,\kappa_2}\ m_{r,\varsigma}(y;\underline{\boldsymbol{\Psi}}),$$

where $m_{r,\varsigma}(y;\underline{\boldsymbol{\Psi}}) = \int_0^{G(y)} u^{\varsigma-1}\ Q_G^r(u)du$. The $m_{r,\varsigma}(y;\underline{\boldsymbol{\Psi}})$ can be calculated numerically by using the software such as Matlab, R, Mathematica etc.

Moment generating function

The moment generating function (MGF) of X, say $M(t) = E(e^{tX})$, is obtained from (8) as

$$M(t) = \sum_{\kappa_1,\kappa_2=0}^{\infty} C_{\kappa_1,\kappa_2}\ M_\varsigma(t),$$

where $M_\varsigma(t)$ is the generating function of Y_ς given by

$$M_\varsigma(t) = \varsigma \int_{-\infty}^{\infty} \exp\,(tx)\, G_{\underline{\Psi}}(x) g_{\underline{\Psi}}(x)^{\varsigma-1} dx = \varsigma \int_0^1 u^{\varsigma-1}\, exp[\, tQ_G(u;\alpha)] du.$$

The last two integrals can be computed numerically for most parent distributions.

Residual and reversed residual life

The n th moment of the residual life, say $m_{n,X}(t) = E[(X-t)^n \;|_{X>t}]$, $n \in N$. The n^{th} moment of the residual life of X is given by

$$m_{n,X}(t) = \frac{1}{1-F_{\theta,\underline{\Psi}}(t)} \int_t^{\infty} (x-t)^n \, dF_{\theta,\underline{\Psi}}(x).$$

Therefore,

$$m_{n,X}(t) = \frac{1}{1-F_{\theta,\underline{\Psi}}(t)} \sum_{\kappa_1,\kappa_2=0}^{\infty} C_{\kappa_1,\kappa_2} \sum_{r=0}^{n} \binom{n}{r} (-t)^{n-r} \int_t^{\infty} \pi_\varsigma(x)\, x^r.$$

The n^{th} moment of the reversed residual life, say $M_{n,X}(t) = E\left[(t-X)^n \;|_{X\le t, t>0}\right]$ and $n \in N$. We obtain

$$M_{n,X}(t) = \frac{1}{F(t)} \int_0^t (t-x)^n \, dF(x).$$

Then, the n th moment of the reversed residual life of X becomes

$$M_{n,X}(t) = \frac{1}{F_{\theta,\underline{\Psi}}(t)} \sum_{\kappa_1,\kappa_2=0}^{\infty} C_{\kappa_1,\kappa_2} \sum_{r=0}^{n} (-1)^r \binom{n}{r} t^{n-r} \int_0^t \pi_\varsigma(x)\, x^r.$$

2. Special RBX-G models

In this Section, we present some new model based on the new family. Table 1 gives some new sub models. Table 2 gives expected value $(E(X))$, variance $(V(X))$, skewness $(S(X))$ and kurtosis $(K(X))$ for the RBXBXII model. Table 3 gives $E(X)$, $V(X)$, $S(X)$ and $K(X)$ for the BXII model. From Tables 2 and 3 we note that the $S_{\text{RBXBXII}}(X)$ of the RBXBXII model can rage in $(-$ 26.0700,51.49377) however $S_{\text{BXII}}(X)$ of the BXII model can rage in $(-$ 0.55325, 4.64758). The $K_{\text{RBXBXII}}(X)$ of the RBXBXII model starts from -602.018 to 2957.861 however the $K_{\text{BXII}}(X)$ of the BXII model starts from 3.070043 to 73.8. Figure1 gives 3-dimensional (3-D) plots for skewness and kurtosis for $\theta = 0.01, 0.05, 0.65$ and 3.5. Figure 2 gives some 3-D plots for skewness and kurtosis for $a = 0.01, 0.25, 5.25$ and 90 . Figure 3 gives 3-D plots for skewness and kurtosis for $b = 0.001, 0.05, 10$ and 50.

Table 1: New sub models based on the new RBXG family.

Baseline model	$\delta_{x;\underline{\Psi}}^{-2}$	Sub model
log-logistic (LL)	$(x/a)^b\|_{a,\,b>0}$	RBXLL
exponential (E)	$\exp(bx) - 1\|_{b>0}$	RBXE
Lomax (Lx)	$(1+x)^b - 1\|_{b>0}$	RBXLx
inverse Lomax (ILx)	$[(1+x^{-1})^b - 1]^{-1}\|_{b>0}$	RBXILx
Burr XII (BXII)	$(1+x^a)^b - 1\|_{a,\,b>0}$	RBXBXII

Table 2: E(X), V(X), S(X) and K(X) for the RBXBXII model.

θ	a	b	E(X)	V(X)	S(X)	K(X)
1	3	2	0.8154317	0.0227300800	1.6074570	8.116445
20			0.6237279	0.0009914400	0.0320100	3.011561
50			0.6009795	0.0005965152	−0.1492178	3.018410
100			0.5872088	0.0004329362	−0.2526406	3.075719
500			0.5623202	0.0002371261	−0.4237477	3.258300
1000			0.5536853	0.0001911473	−0.4730763	3.152824
2000			0.5459456	0.0001571489	−0.5236663	3.417648
5000			0.5368201	0.0001242829	−0.5748409	3.515111
10000			0.5306027	0.0001056807	−0.6078919	3.584151
150	1	5	0.07414598	4.748036×10^{-5}	−0.1336004	2957.861
	2		0.2719993	0.0001623435	−0.2744358	3.084172

	3		0.4196996	0.0001728096	-0.3224319	3.138697
	4		0.5213913	0.0001504779	−0.3466572	3.169352
	5		0.5938978	0.0001251878	−0.3612679	3.188875
3	1	1	0.8674569	0.11056480	2.297332	17.28222
		2	0.3617870	0.01299295	1.599782	8.576392
		5	0.1305731	0.00134601	1.308430	6.467252
		10	0.0631468	0.00029188	1.223986	16.93484
		25	0.02476427	4.290549×10^{-5}	−5.10736	113.5008
		35	0.01762261	2.154636×10^{-5}	−26.0700	285.1315
		45	0.01367801	1.291354×10^{-5}	30.28078	-392.661
		50	0.01230126	1.042869×10^{-5}	51.49377	-602.018

Table 3: E(X), V(X), S(X) and K(X) for the BXII model.

1	5	0.2500000	0.104166700	4.64758000	73.80000
5		0.6824236	0.028995090	0.04014894	3.070043
15		0.8738445	0.005874990	−0.5532521	3.716229
35		0.9429226	0.001308798	−0.7556787	4.268102
50		0.9595446	0.000670762	−0.8044469	4.428862
75		0.9727649	0.000308832	−0.8433321	4.565136
100		0.9794732	0.000176835	−0.8628163	4.608513
150		0.8312523	0.004948929	−0.6735949	3.837231
15	0.5	1.1138790	0.044241730	2.13181600	15.34159
	1	1.0073480	0.015102360	0.59899810	5.108375
	10	0.8738445	0.005874988	−0.5532521	3.716229
	25	0.7802835	0.004183556	−0.7426381	3.929249
	50	0.7445115	0.003757431	−0.7651261	3.962605
	100	0.7106364	0.003400300	−0.7762729	3.979733
	200	0.6784242	0.003088641	−0.7818293	3.988533
	500	0.6381542	0.002727363	−0.7851529	3.993804
	1000	0.6093145	0.002484757	−0.786259	3.995556
	5000	0.5473072	0.002003691	−0.7871443	3.996980
	10000	0.5225898	0.001826675	−0.7872549	3.998318

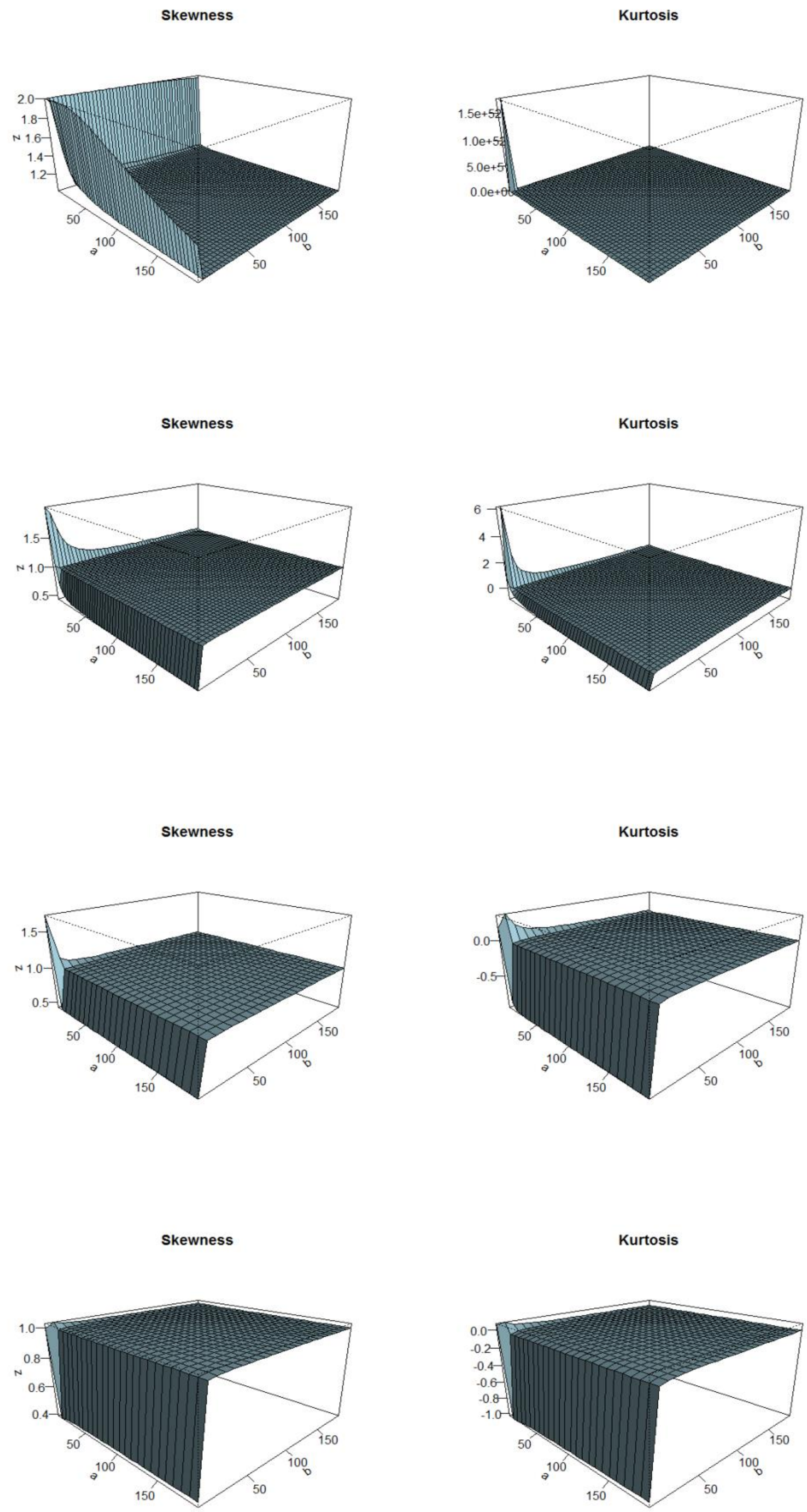

Figure 1: 3-D plots for skewness and kurtosis for θ = 0.01; 0.05; 0.65; 3.5.

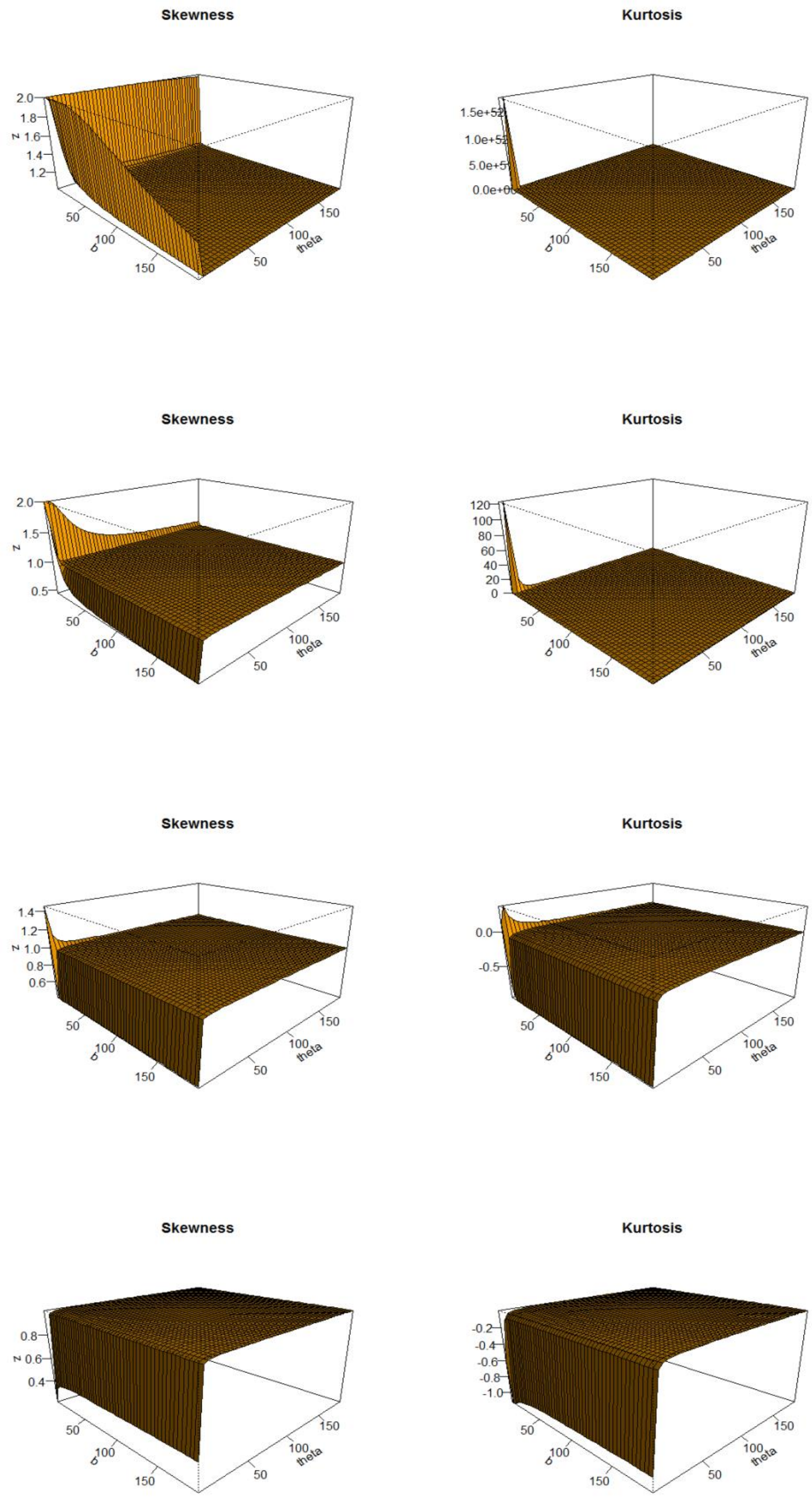

Figure 2: 3-D plots for skewness and kurtosis for a = 0.01; 0.25; 5.25; 90.

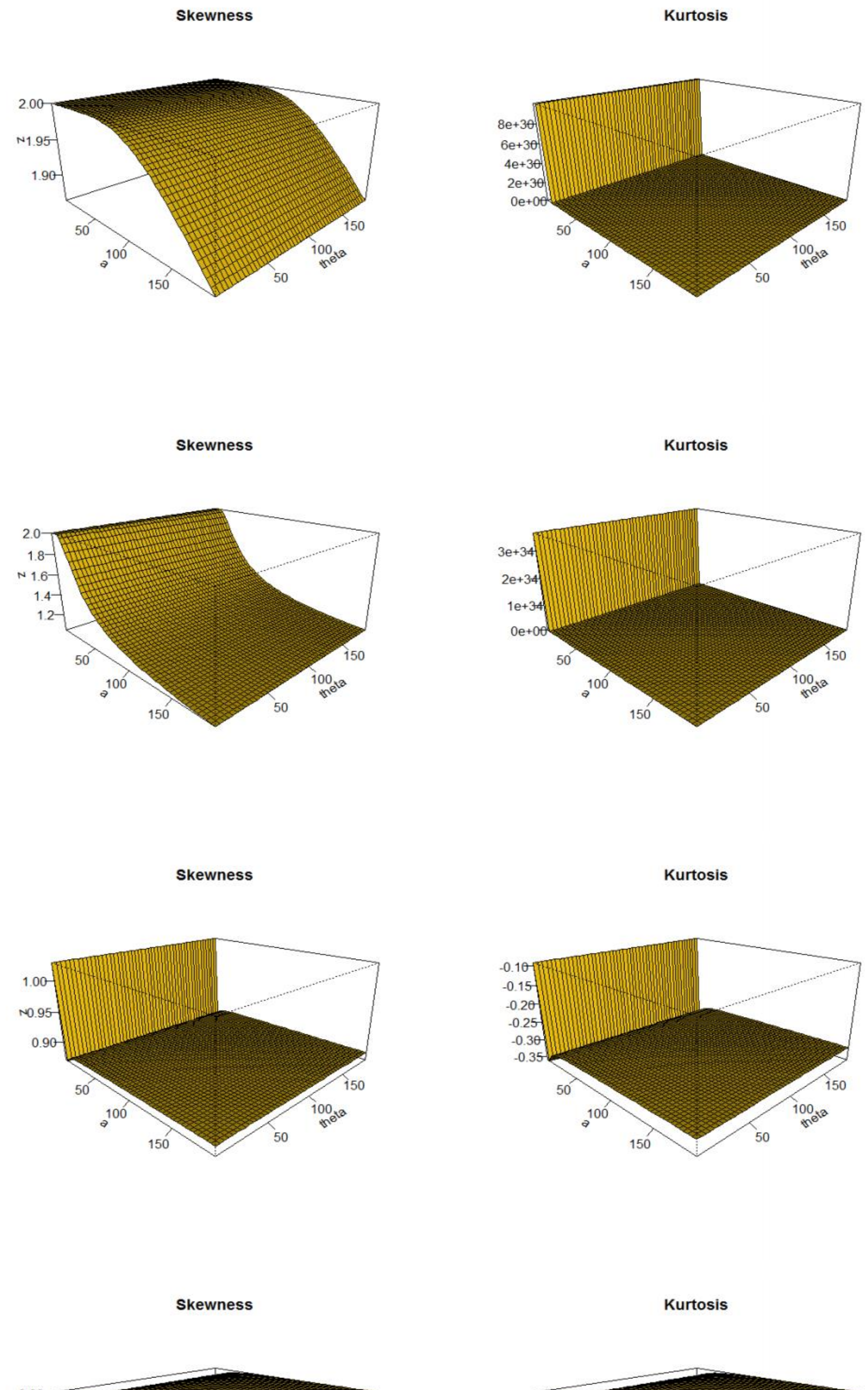

Figure 3: 3-D plots for skewness and kurtosis for b = 0.001; 0.05; 10; 50.

3. Parameter Estimation

Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. So, we consider the estimation of the unknown parameters of this family from complete samples only by maximum likelihood. Let $x_1, x_2, \dots, x_n$ be a random sample from the RBX-G family with parameters θ and ξ. For determining the MLE of $\underline{\boldsymbol{\Psi}}$, we have the log-likelihood function

$$\ell_{\theta,\underline{\boldsymbol{\Psi}}}(x_i) = n\,log\,2 + n\,log\,\theta + \sum_{i=1}^{n} log\,g_{\underline{\boldsymbol{\Psi}}}(x_i) - 3\sum_{i=1}^{n} log\,G_{\underline{\boldsymbol{\Psi}}}(x_i)$$
$$+\sum_{i=1}^{n} log\,\overline{G}_{\underline{\boldsymbol{\Psi}}}(x_i) - \sum_{i=1}^{n} \delta_{x_i;\underline{\boldsymbol{\Psi}}}^{-2} + (\theta-1)\sum_{i=1}^{n} log\left[1-exp\left(-\delta_{x_i;\underline{\boldsymbol{\Psi}}}^{-2}\right)\right],$$

The components of the score vector, $U(\theta,\underline{\boldsymbol{\Psi}}) = \frac{\partial}{\partial \underline{\Theta}}\ell_{\theta,\underline{\boldsymbol{\Psi}}}(x_i) = \left(U_\theta = \frac{\partial}{\partial\theta}\ell_{\theta,\underline{\boldsymbol{\Psi}}}(x_i), U_{\underline{\boldsymbol{\Psi}}} = \frac{\partial}{\partial\underline{\boldsymbol{\Psi}}}\ell_{\theta,\underline{\boldsymbol{\Psi}}}(x_i)\right)^{\mathrm{T}}$. Setting the nonlinear system of equations $U_\theta(x_i) = 0$ and $U_{\underline{\boldsymbol{\Psi}}}(x_i) = 0$ and solving them simultaneously yields the MLEs. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize $\ell_{\theta,\underline{\boldsymbol{\Psi}}}(x_i)$.

4. Four examples for comparing models

Based on Table 1, we will consider the Burr XII distribution as base line model for deriving the reciprocal Burr X-Burr XII (RBXBXII) distribution. To illustrate the flexibility of the RBXBXII model, we provide four applications to four real-life data sets. For the four real-life data sets, we compare the RBXBXII distribution, with the standard Burr XII (BXII), Marshall-Olkin Burr XII (MOBXII), Topp-Leone Burr XII (TLBXII), Zografos-Balakrishnan Burr XII (ZBBXII), Five Parameters beta Burr XII (FBBXII), Beta Burr XII, Beta exponentiated Burr XII (BEBXII), Five Parameters Kumaraswamy Burr XII (FKmBXII) and Kumaraswamy Burr XII distributions.

All competitive Burr XII models and data sets are given in Afify et al. (2018), Altun et al. (2018 a, b) and Elsayed and Yousof (2019). We consider the following well-known (G-O-F) statistic tests: the Akaike Information Criterion $(AI-Cr)$, Bayesian Information Criterion $(Bayes-Cr)$, Hannan-Quinn Information Criterion $(HQ-Cr)$, Consistent Akaike Information Criterion $(CA-Cr)$. The data set **I** is the breaking stress data (see Nichols, Padgett (2006)). The data set **II** presents survival times of guinea pigs see (Bjerkedal (1960)). The data set **III** are taxes revenue data see (Altun et al. (2018 a, b)). The data set **IV** called leukemia data see (Elsayed and Yousof (2019)).

Many useful graphical tolls are used such as the plots and the box plots, Quantile-Quantile (Q-Q) plots, the total time in test (TTT) plots and the nonparametric Kernel density estimation (NKDE). Figures 4, 5, 6 and 7 give the box plots, the Q-Q, TTT plot and NKDE plots for the four real data sets. The outliers (extreme observations) are checked ad spotted by the box plot (see Figures 4, 5, 6, 7 (top left panels)). The normality of the four real-life data sets is checked using the Q-Q plot (see Figures 4, 5, 6, 7 (top right panels)). The initial HRF shapea are explored by using the TTT tool (see Figures 4, 5, 6, 7 (bottom left panels)). The NKDE tool is used for exploring the initial PDF shape (see Figures 4, 5, 6, 7 (bottom right panels)). Based on Figures 4, 5, 6, 7 (top left panels), it is proved that no extreme values were spotted in data set **IV**. However, data sets **I**, **II** and **III** have some extreme values. Based on Figures 4, 5, 6, 7 (top right panels), it is seen that the normality is exists for the data sets **I** and **III.** Based on Figures 4, 5, 6, 7 (bottom left panels), it is seen that the HRF of the two real data sets is monotonically increasing for data sets **I**, **II** and **III** and U-HRF for data set **IV.** Based on Figures 4, 5, 6, 7 (bottom right panels), it is seen that the NKDE of the four data sets are bimodal and right skewed density for data sets **I**, bimodal and right skewed density with heavy tail for data sets **II**, **III** and **IV.** Tables 4, 5, 6 and 7 give the estimates of the maximum likelihood method (MLEs), standard errors (SEs), 95%-confidence interval (95%-CL) with AI-Cr, Bayes-Cr, HQ-Cr and CA-Cr values for the data set **I**, **II**, **III** and **IV** respectively. Figures 8, 9, 10 and 11give estimated PDF (EPDF), estimated CDF (ECDF), Probability-Probability (P-P) plot, Kaplan-Meier survival plot and estimated HRF (EHRF) for data sets **I**, **II**, **III** and **IV.** Based on the values in Tables 8, 9, 10 and 11, the RBXBXII model has the best fits as compared to BXII extensions in the four applications with small values of AI-Cr, Bayes-Cr, HQ-Cr and CA-Cr where for data set **I** AI-Cr=291.102, Bayes-Cr=298.917, HQ-Cr=294.265 and CA-Cr=291.352, for data set **II** AI-Cr=206.371, Bayes-Cr=213.561, HQ-Cr=209.450 and CA-Cr=207.084, for data set **III** AI-Cr=382.752, Bayes-Cr=388.985, HQ-Cr=383.188 and CA-Cr=385.185 and finally for data set **IV** AI-Cr=312.988, Bayes-Cr=317.478, HQ-Cr=313.816 and CA-Cr=314.499.

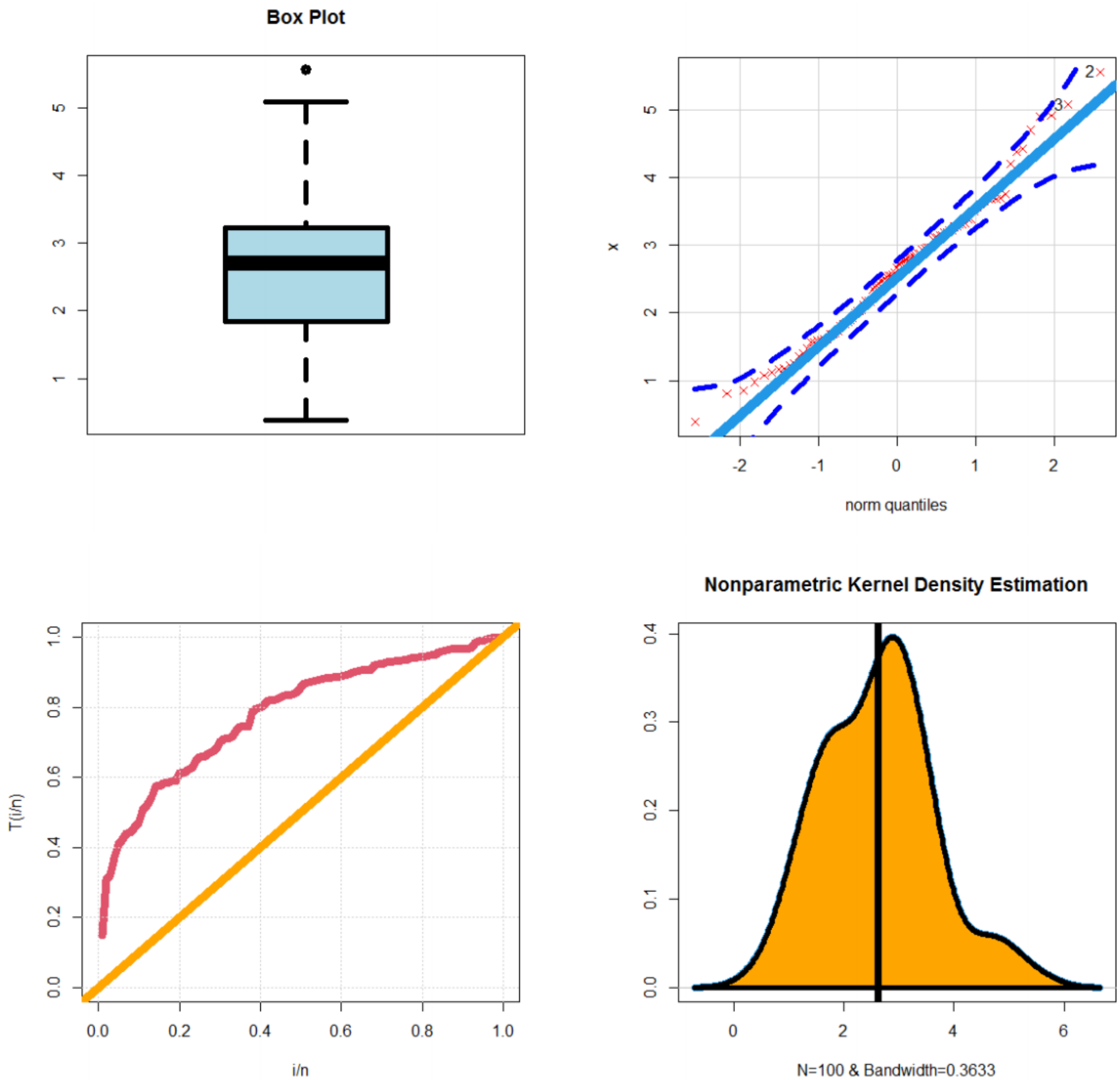

Figure 4: Plots for describing and exploring data set **I**.

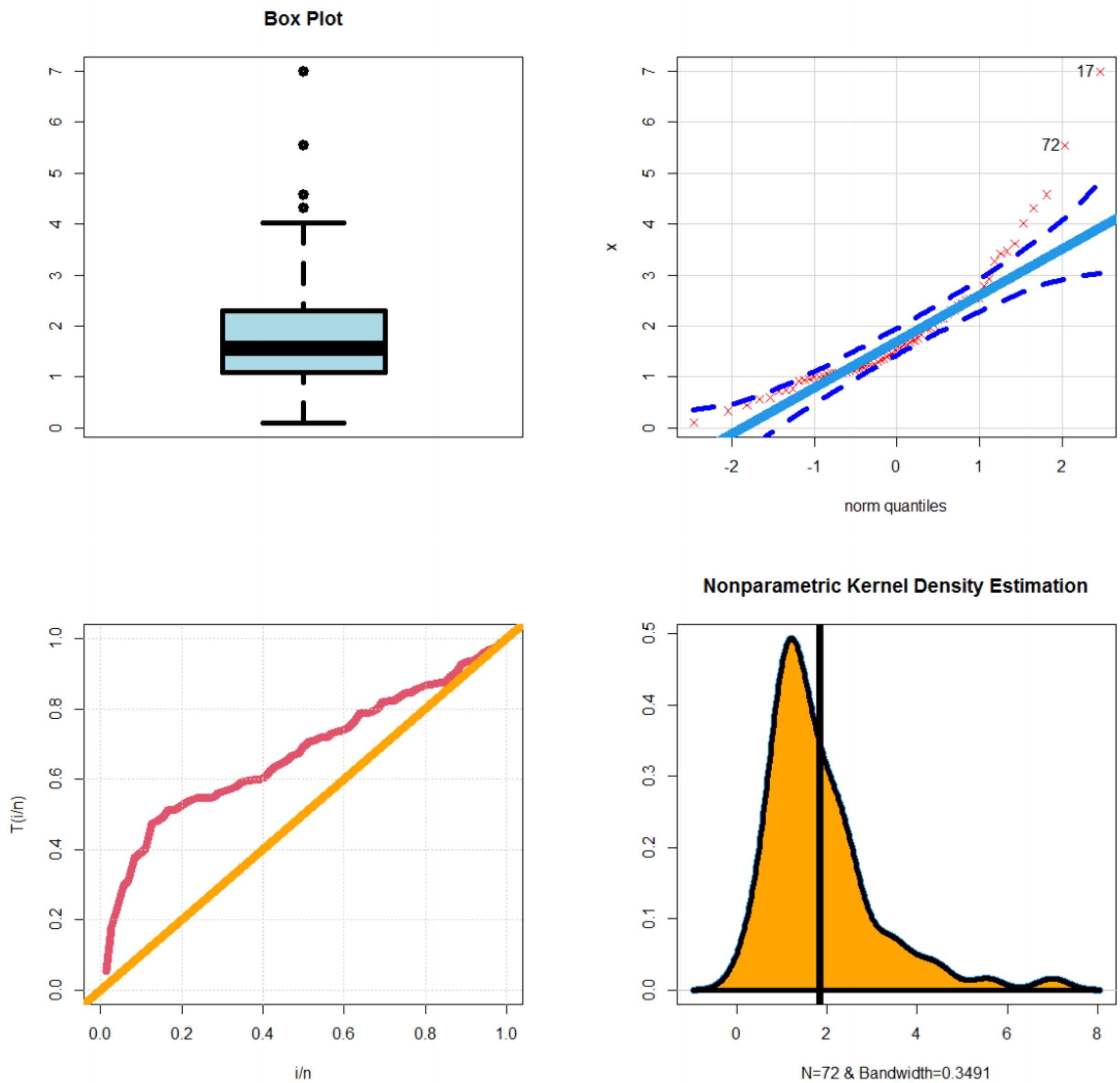

Figure 5: Plots for describing and exploring data set **II**.

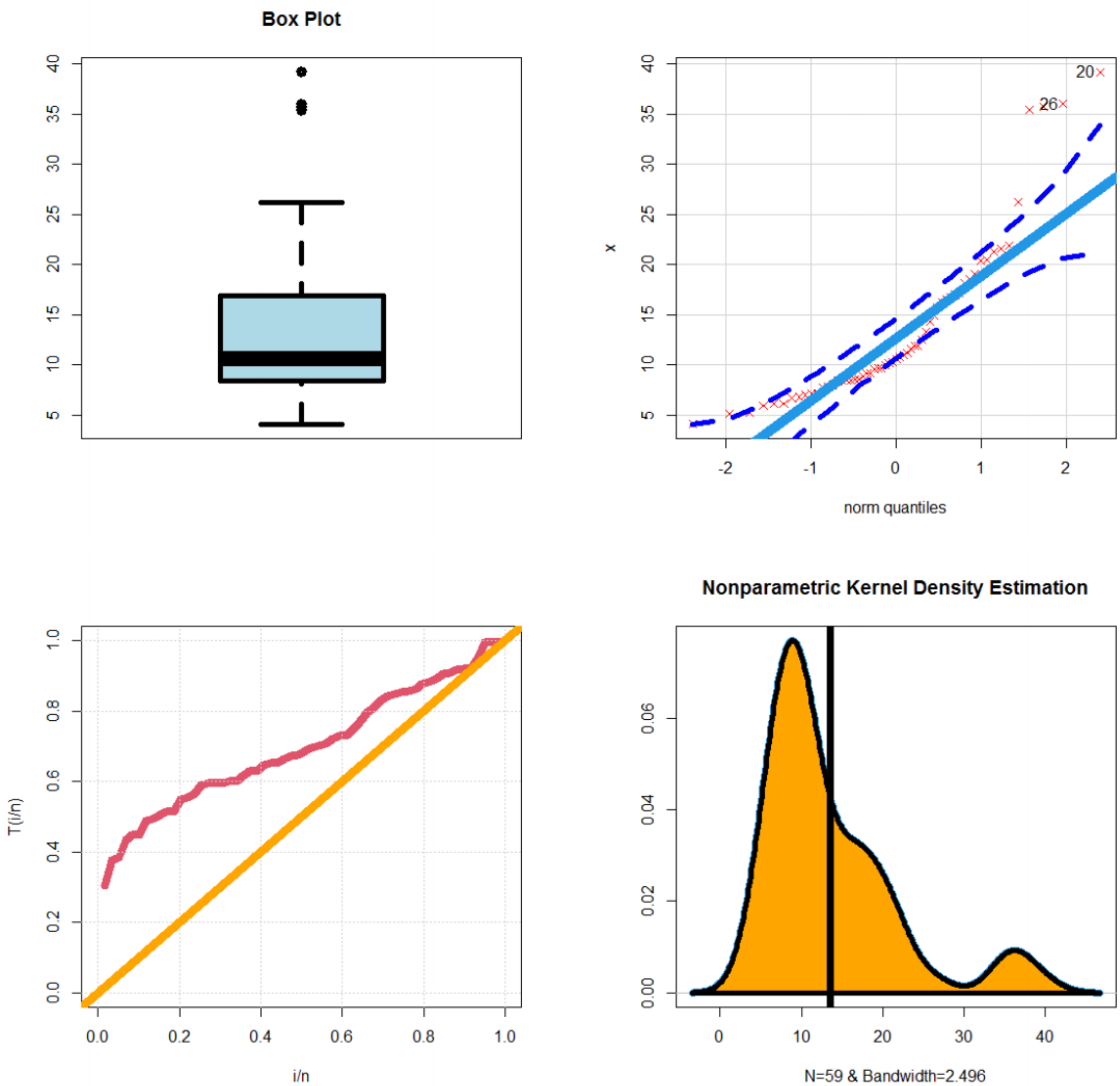

Figure 6: Plots for describing and exploring data set **III**.

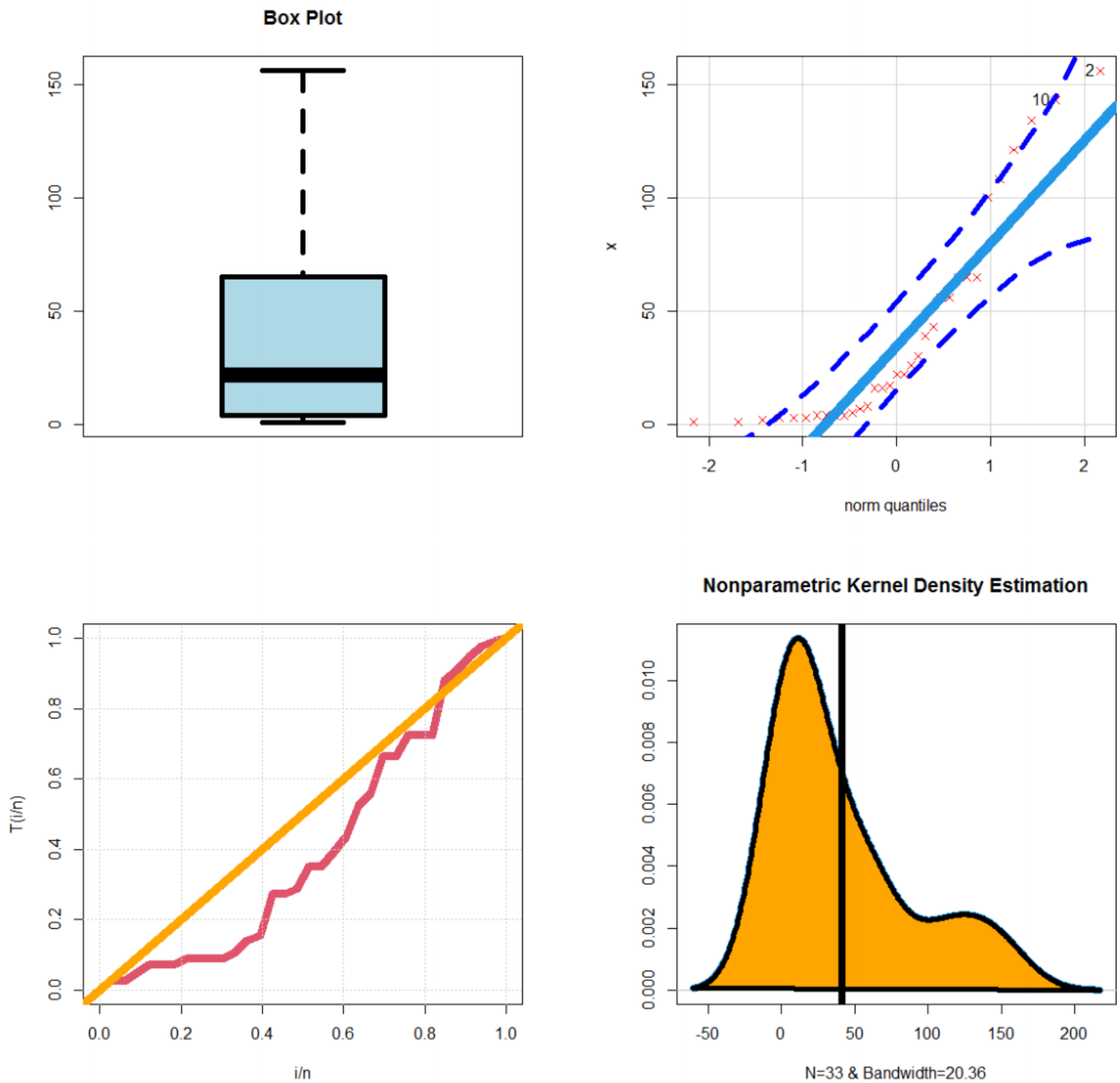

Figure 7: Plots for describing and exploring data set **IV**.

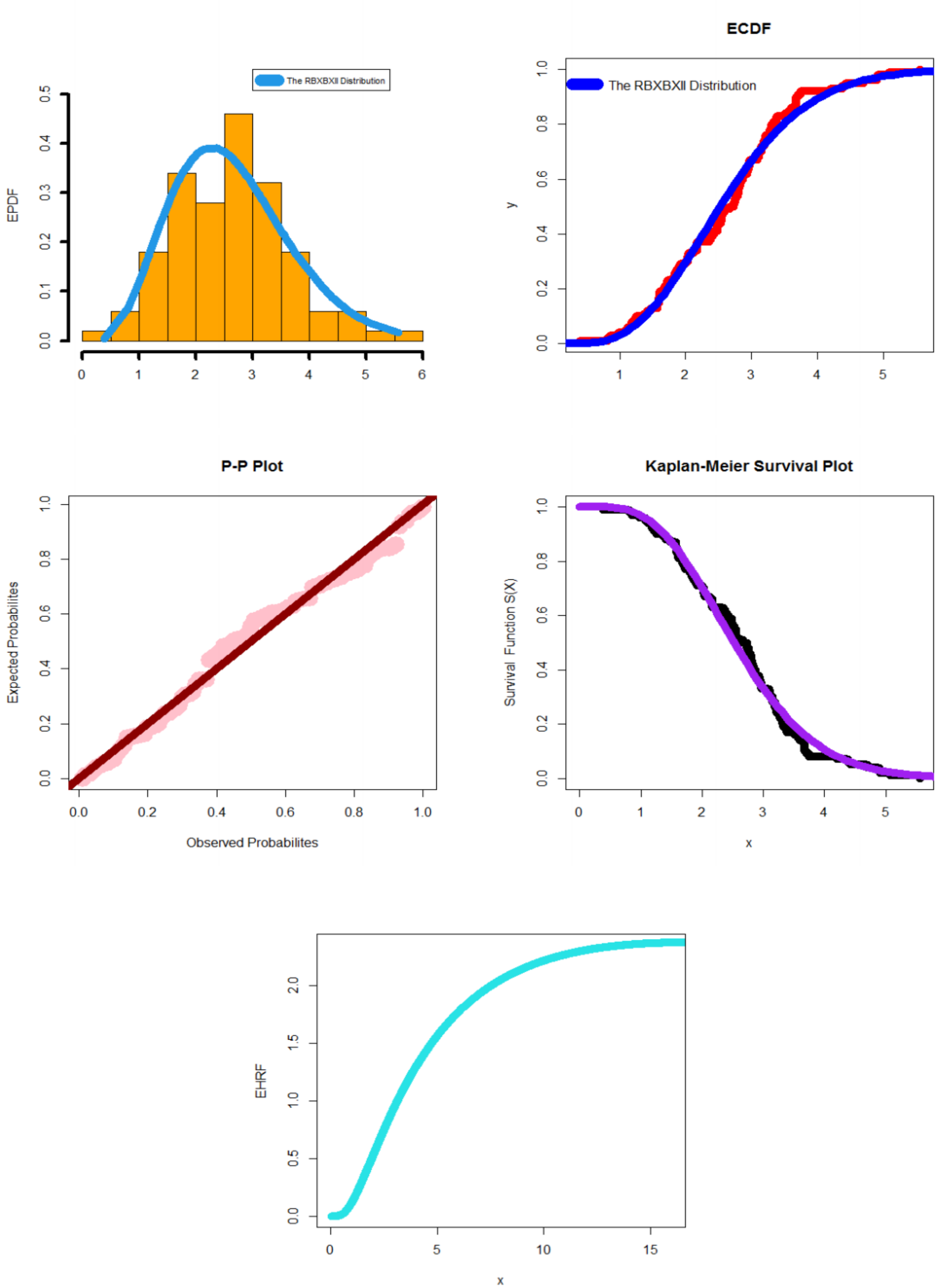

Figure 8: EPDF, ECDF, P-P plot, Kaplan-Meier survival plot and EHRF for data set **I**.

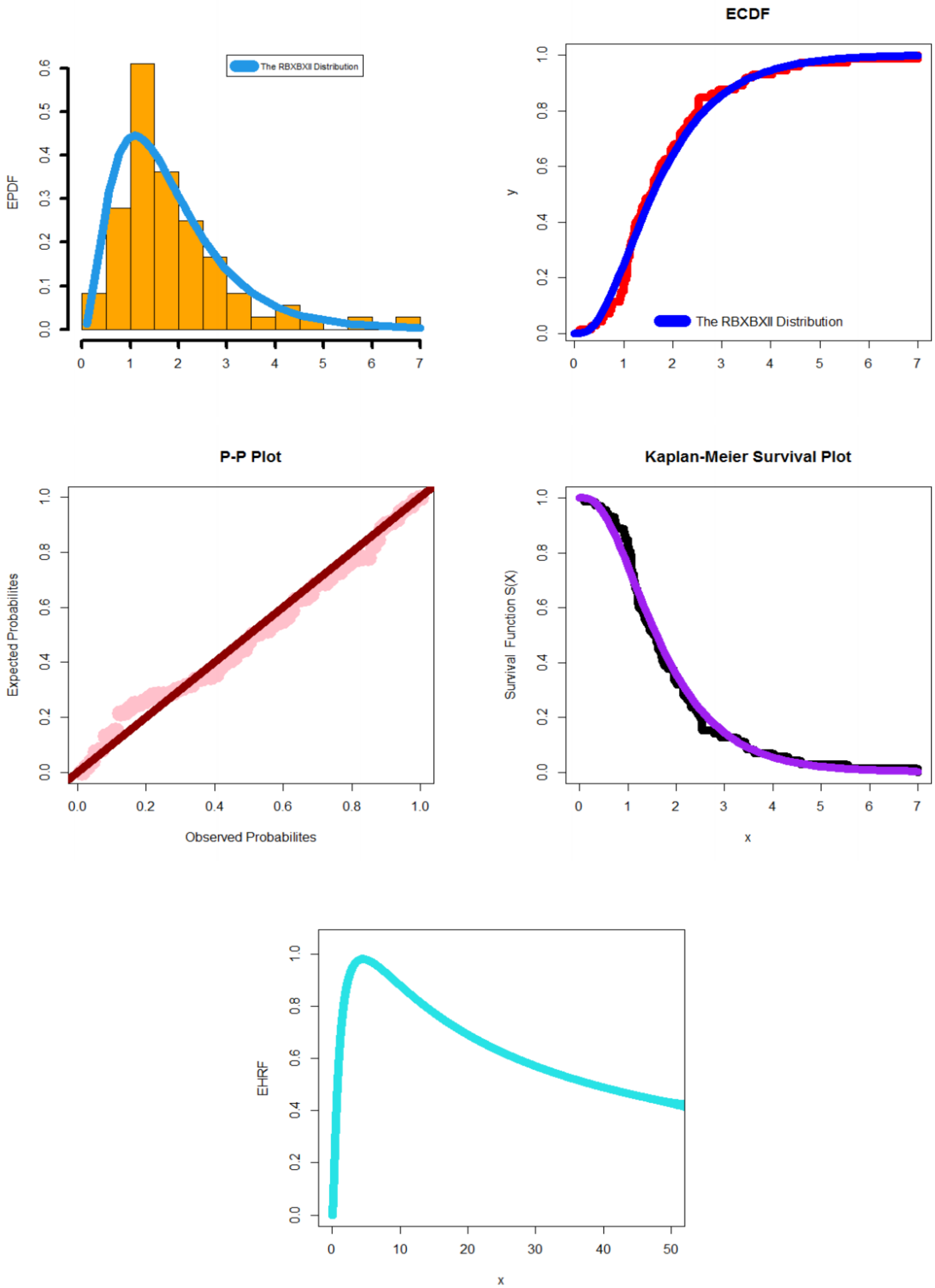

Figure 9: EPDF, ECDF, P-P plot, Kaplan-Meier survival plot and EHRF for data set **II**.

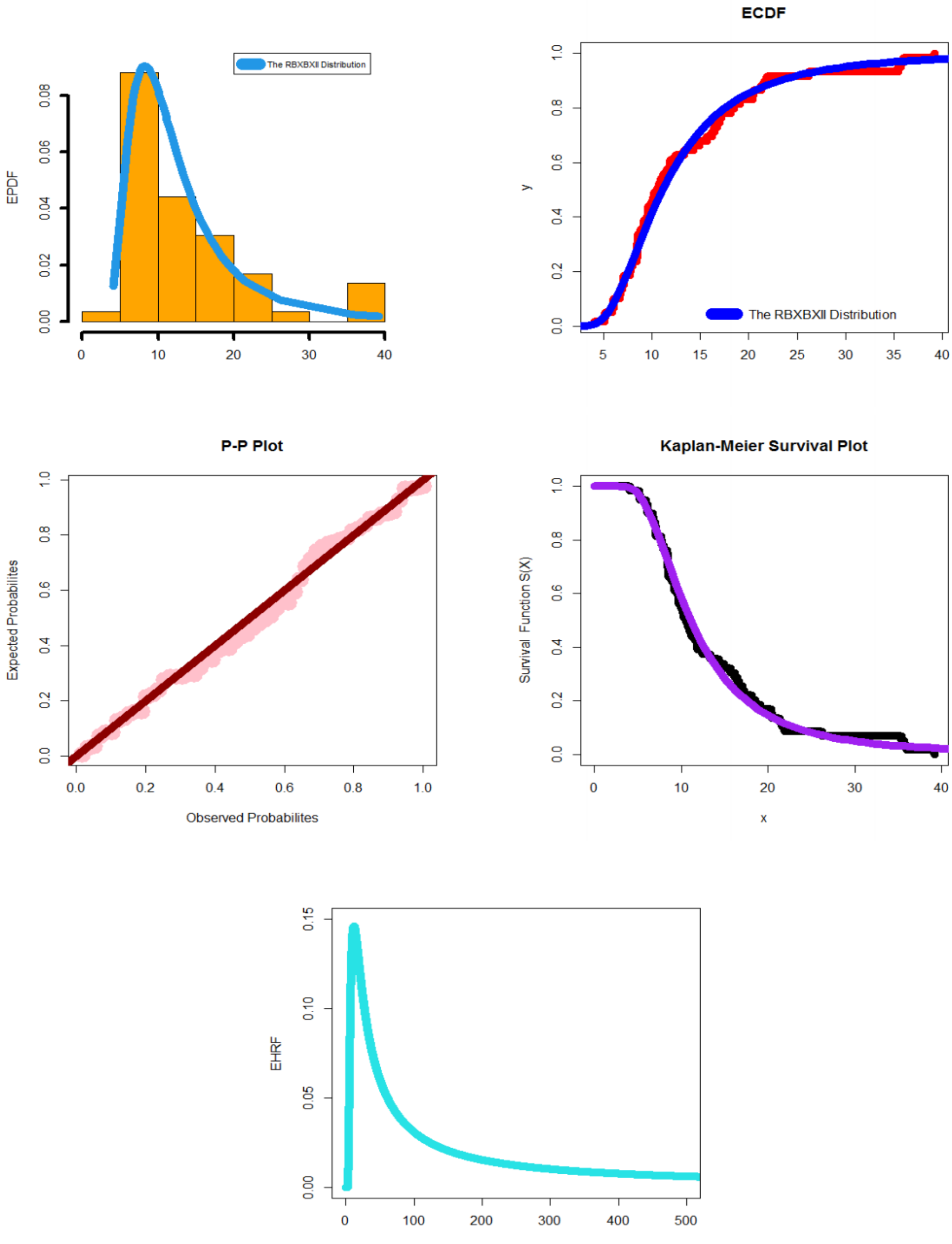

Figure 10: EPDF, ECDF, P-P plot, Kaplan-Meier survival plot and EHRF for data set **III**.

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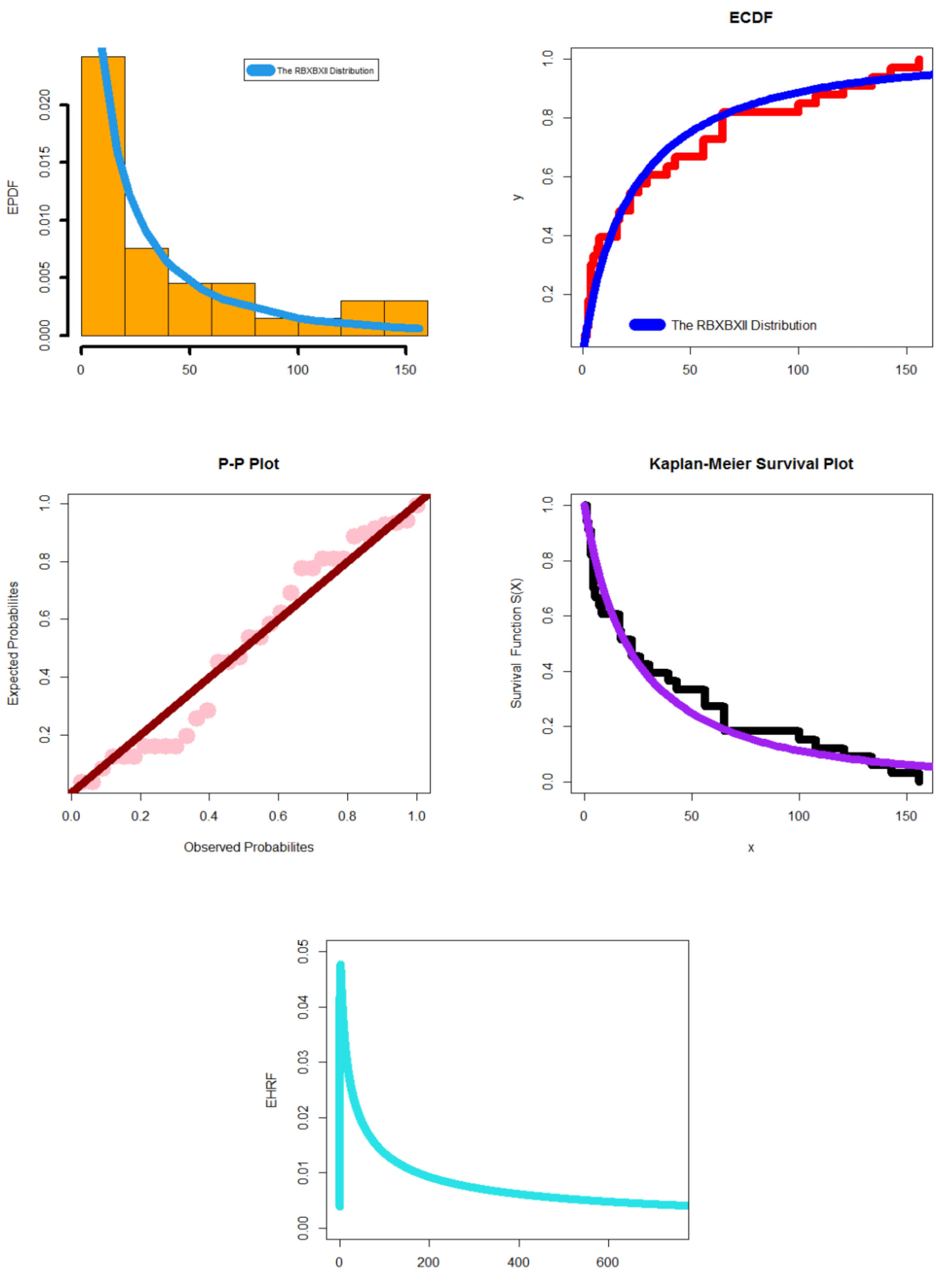

Figure 11: EPDF, ECDF, P-P plot, Kaplan-Meier survival plot and EHRF for data set **IV**.

Table 4: MLEs, SEs and CL with AI-Cr, Bayes-Cr, HQ-Cr and CA-Cr values for the data set **I**.

Model	θ,a,b,α,β	AI-Cr, Bayes-Cr, CA-Cr, HQ-Cr
BXII	___, 5.9415, 0.1878, ___	382.94, 388.15, 383.06, 385.05
	___, (1.2792), (0.0443), ___	
	___, (3.43,8.455), (0.10,0.272), ___	
MOBXII	___, 1.1924,4.8345,838.7341, ___	305.78, 313.61, 306.03, 308.96
	___, (0.9524), (4.8965), (229.344), ___	
	___, 0, 3.06), (0, 14.43), (389.22,1288.24), ___	
TLBXII	___, 1.3502,1.0611,13.72773, ___	323.52, 331.35, 323.77, 326.70
	___, (0.3783), (0.384), (8.4003), ___	
	___, (0.613, 2.09), (0.31,1.81), (0, 30.19), ___	
KmBXII	48.1033 ,79.516 ,0.351 ,2.730, ___	303.76, 314.20, 304.18, 308.00
	(19.3483), (58.1864), (0.098), (1.0773), ___	
	(10.18,86.03), (0,193.56), (0.16,0.54), (0.62,4.84), ___	
BBXII	359.683 ,260.097 ,0.17534 ,1.1235, ___	305.64, 316.06, 306.06, 309.85
	(57.944), (132.213), (0.013), (0.243), ___	
	(246.1,473.2), (0.96,519.2), (0.14,0.20), (0.65,1.6), ___	
BEBXII	0.381, 11.949, 0.937, 33.402, 1.705	305.82, 318.84, 306.46, 311.09
	(0.078), (4.635), (0.267), (6.287), (0.478)	
	(0.23,0.533), (2.86,21), (0.41,1.5), (21,45), (0.8,2.6)	
FBBXII	0.4214, 0.8343, 6.115, 1.674, 3.450	304.26, 317.31, 304.89, 309.56
	(0.011), (0.943), (2.314), (0.226), (1.957)	
	(0.4,0.44), (0. 2.7), (1.57, 10.7), (1.23, 2.1), (0, 7)	
FKmBXII	0.5424,4.2232, 5.31332, 0.4112, 4.1523	305.50, 318.55, 306.14, 310.80
	(0.137), (1.882), (2.318), (0.497), (1.995)	
	(0.3, 0.8), (0.53,7.9), (0.9,9), (0, 1.7), (0.2,8)	
ZBBXII	123.101,0.3683, 139.24725, ___, ___	302.96, 310.78, 303.21, 306.13
	(243.011), (0.343), (318.546), ___, ___	
	(0, 599.40), (0, 1.04), (0, 763.59), ___, ___	
RBXBXII	1565.5, 0.2252, 0.383, ___, ___	291.102, 298.917, 291.352,294.265
	(8.169), (0.0025), (0.00945), ___, ___	
	(1549.3,1581.7), (0.2212,0.229), (0.365,0.40), ___, ___	

Table 5: MLEs, SEs and CL with AI-Cr, Bayes-Cr, HQ-Cr and CA-Cr values for the data set **II**.

Model	θ,a,b,α,β	AI-Cr, Bayes-Cr, CA-Cr, HQ-Cr
BXII	___, 3.1025, 0.4654, ___, ___	209.601, 214.155, 209.774, 211.402
	___, (0.5384), (0.0777), ___, ___	
	___, (2.053,4.16), (0.31,0.62), ___, ___	
MOBXII	___, 2.2593,1.5333, 6.7603, ___	209.74, 216.56, 210.09, 212.444
	___, (0.864), (0.907), (4.587), ___	
	___, (0.57,3.95), (0,3.31), (0, 15.75), ___	
TLBXII	___, 2.393,0.458,1.796, ___	211.801, 218.638, 212.153, 214.524
	___, (0.907), (0.244), (0.915), ___	
	___, (0.62,4.17), (0, 0.94), (0.002,3.59), ___	
KmBXII	14.105,7.424, 0.525, 2.274, ___	208.764, 217.858, 209.362, 212.384
	(10.805), (11.850), (0.279), (0.990), ___	
	(0, 35.28), (0.30.65), (0, 1.07), (0.33, 4.21), ___	
BBXII	2.555, 6.058,1.800,0.294, ___	210.444, 219.544, 211.034, 214.064
	(1.859), (10.391), (0.955), (0.466), ___	
	(0, 6.28), (0, 26.42), (0, 3.67), (0, 1.21), ___	
BEBXII	1.876, 2.991, 1.780, 1.341, 0.572	212.103, 223.503, 213.001, 216.604
	(0.094), (1.731), (0.702), (0.816), (0.325)	
	(1.7,2.06), (0, 6.4), (0.40, 3.2), (0, 2.9), (0, 1.21)	
FBBXII	0.621, 0.549,3.838, 1.381, 1.665	206.804, 218.243, 207.711, 211.302
	(0.541), (1.011), (2.785), (2.3121), (0.436)	

	(0, 1.7), (0, 2.54), (0, 9.34), (0, 5.9), (0.8, 4.5)	
FKmBXII	0.558,0.308, 3.999, 2.131, 1.475	206.50, 217.90, 207.41, 211.00
	(0.442), (0.314), (2.082), (1.833), (0.361)	
	(0, 1.4), (0, 0.9), (0, 3.1), (0, 5.7), (0.76, 2.2)	
RBXBXII	317.363, 0.1743, 0.4623, ___, ___	206.371, 213.561, 207.084, 209.449
	428.088, 0.0389, 0.039, ___, ___	
	(0,1177.36), (0,0.574), (0,1.262), ___, ___	

Table 6: MLEs, SEs and CL with AI-Cr, Bayes-Cr, HQ-Cr and CA-Cr values for the data set **III**.

Model	θ,a,b,α,β	AI-Cr, Bayes-Cr, CA-Cr, HQ-Cr
BXII	___, 5.6153, 0.0725, ___, ___	518.458, 522.622, 518.671, 520.082
	___, (15.048), (0.194), ___, ___	
	___, (0, 35.11), (0, 0.45), ___, ___	
MOBXII	___, 8.0173, 0.419, 70.359, ___	387.220, 389.388, 387.663, 389.683
	___, (22.083), (0.312), (63.831), ___	
	___, (0, 51.29), (0, 1.035), (0, 195.47), ___	
TLBXII	___, 91.324, 0.012, 141.073, ___	385.943, 392.184, 386.384, 388.40
	___, (15.071), (0.002), (70.028), ___	
	___, (61.78,120.86) (0.008, 0.02) (3.82,278.33), ___	
KmBXII	18.130, 6.857, 10.694, 0.081, ___	385.588, 393.901, 386.323, 388.86
	(3.689), (1.035), (1.166), (0.012), ___	
	(10.89,25.4), (4.83,8.89), (8.41,12.98), (0.06,0.10), ___	
BBXII	26.725, 9.756, 27.364, 0.020, ___	385.563, 394.10, 386.30, 389.105
	(9.465), (2.781), (12.351), (0.007), ___	
	(8.17,45.27), (4.31,15.21), (3.16,51.57), (0.01,0.03), ___	
BEBXII	2.9245, 2.911, 3.270, 12.486, 0.3741	387.05, 397.424, 388.175, 391.094
	(0.564), (0.549), (1.2515), (6.938), (0.788)	
	(1.82,4.03), (1.83,3.99), (0.82,5.72), (0, 26.08), (0, 1.92)	
FBBXII	30.4412, 0.5845, 1.0888, 5.166, 7.8624	386.740, 397.143, 387.872, 390.841
	(91.745), (1.064), (1.0213), (8.268), (15.036)	
	(0, 210.26), (0, 2.67), (0, 3.09), (0, 21.37), (0, 37.33)	
FKmBXII	12.878, 1.225, 1.665, 1.411, 3.732	386.966, 397.365, 388.093, 391.064
	(3.442), (0.131), (0.034), (0.088), (1.172)	
	(6.13,19.6), (0.97,2), (1.56,1.7), (1.2,1.58), (1.43,6), ___	
DBXII	5.376, 1.449, 0.149, ___, ___	382.752, 388.985, 383.188, 385.19
	2.6429, 1.422, 0.129, ___, ___	
	(0.176,10.57), (0,3.249), (0,0.389), ___, ___	

Table 7: MLEs, SEs and CL with AI-Cr, Bayes-Cr, HQ-Cr and CA-Cr values for the data set **IV**.

Model	θ,a,b,α,β	AI-Cr, Bayes-Cr, CA-Cr, HQ-Cr
BXII	___, 58.711,0.006, ___, ___	328.20, 331.19, 328.60, 329.19
	___, (42.382), (0.0044), ___, ___	
	___, (0, 141.78), (0, 0.01), ___, ___	
MOBXII	___, 11.838, 0.078, 12.251, ___	315.54, 320.01, 316.37, 317.04
	___, (4.368), (0.013), (7.770), ___	
	___, (0, 141.78), (0, 0.01), (0, 27.48), ___	
TLBXII	___,0.2815, 1.8824 ,50.215, ___	316.26, 320.75, 317.09, 317.76
	___, (0.288), (2.402), (176.50), ___	
	___, (0, 0.85), (0, 6.59), (0, 396.16), ___	
KmBXII	9.201, 36.428, 0.242,0.941, ___	317.36, 323.30, 318.79, 319.34
	(10.060), (35.650), (0.167), (1.045), ___	
	(0, 28.912), (0, 106.30), (0, 0.57), (0, 2.99), ___	
BBXII	96.104, 52.121, 0.104, 1.227, ___	316.46, 322.45, 317.89, 318.47
	(41.201), (33.490), (0.023), (0.326), ___	
	(15.4,176.8), (0, 117.8), (0.6, 0.15), (0.59,1.9), ___	
BEBXII	0.087, 5.007, 1.561, 31.270, 0.318	317.58, 325.06, 319.80, 320.09

	(0.077), (3.851), (0.012), (12.940), (0.034)	
	(0, 0.3), (0, 12.6), (1.5, 1.6), (5.9, 56.6), (0.3,0.4)	
FBBXII	15.194, 32.048, 0.233, 0.581, 21.855	317.86, 325.34, 320.08, 320.36
	(11.58), (9.867), (0.091), (0.067), (35.548)	
	(0, 37.8), (12.7,51.4), (0.05,0.4), (0.45,0.7), (0, 91.5)	
FKmBXII	14.732, 15.285, 0.293, 0.839, 0.034	317.76, 325.216, 319.98, 320.26
	(12.390), (18.868), (0.215), (0.854), (0.075)	
	(0, 39.02), (0, 52.27), (0, 0.71), (0, 2.51), (0, 0.18)	
DBXII	253.235, 0.0815, 0.4201, ___, ___	312.99, 317.4778, 313.82, 314.499
	642.76, 0.036, 0.0507, ___, ___	
	(0,1533.3), (0 0.0015,0.162), (0.5201,0.3), ___, ___	

5. Conclusions

A new one parameter G family of distributions called the Reciprocal Burr X-G Family (RBX-G) family is defined and studied. The RBX-G family is constructed by investing the well-known Burr X family (Yousof et al. (2017a)). Special member based on the Burr XII model called the reciprocal Burr X-Burr XII (RBXBXII) distribution is studied and analyzed. Relevant properties of the new family including ordinary moments, moment of the residual life, Moment of the reversed residual lif and incomplete moments are derived and some of them are numerically analyzed. The expected value, variance, skewness, and kurtosis for the RBXBXII model are numerically analyzed. It is noted that noted that the skewness of the RBXBXII model can rage in $(-26.0700, 51.49377)$ however skewness of the base line Burr XII model can rage in $(-0.55325, 4.64758)$. The kurtosis of the RBXBXII model starts from -602.018 to 2957.861 however the kurtosis of the Burr XII model starts from 3.070043 to 73.8. Four different applications to real-life datasets are presented to illustrate the applicability and importance of the RBXBXII model. Based on the four applications, the RBXBXII distribution gives the lowest values for all statistic tests where for data set **I** AI-Cr=291.102, Bayes-Cr=298.917, HQ-Cr=294.265 and CA-Cr=291.352, for data set **II** AI-Cr=206.371, Bayes-Cr=213.561, HQ-Cr=209.450 and CA-Cr=207.084, for data set **III** AI-Cr=382.752, Bayes-Cr=388.985, HQ-Cr=383.188 and CA-Cr=385.185 and finally for data set **IV** AI-Cr=312.988, Bayes-Cr=317.478, HQ-Cr=313.816 and CA-Cr=314.499.

The new family has high flexibility in statistical modeling operations, and therefore we hope that it will receive the expected interest from the statistics, as the new family can be used in many applications, including:

- **I.** Introducing a new discrete family based on the new argument (see Aboraya et al. (2020), see Ibrahim et al. (2021b) and see Eliwa et al. (2022)).
- **II.** Many new continuous distributions are derived based on it.
- **III.** Applying many statistical hypothesis tests in the case of complete data (see Ibrahim et al. (2020), Ibrahim et al. (2021a) and Khalil et al. (2023)).
- **IV.** Apply a lot of statistical hypothesis tests in the case of censored data.
- **V.** Presenting applications in the field of validity and engineering based on the stress-strength models and their extensions with Bayesian and non-Bayesian estimations.
- **VI.** Presenting applications in the field of validity and engineering based on the pressure and resistance model (follow Mohamed et al. (2022a, b, c) and Hamed et al. (2022)).

We hope that the new family will attract the attention of researchers in the fields of mathematical statistics, applied statistics, and others. We hope that researchers will find sufficient importance in the fields of mathematical and statistical modeling and practical applications in the fields of engineering, medicine, industry, insurance, and actuarial sciences. We recommend employing the new family also in the field of statistical hypothesis tests on various data (complete and censored). The mathematical characteristics of the new family also make us recommend its application in the areas of actuarial risk analysis and evaluation in relation to insurance and reinsurance companies. Finally, the new family can also be used in acceptance sampling planes and stress-strength applications. In other possible future work, we may present a bivariate version of the new family with some applications to bivariate data.

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