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Statistical Inference on Process Capability Index C_{pyk} for Inverse Rayleigh Distribution under Progressive Censoring

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Abstract

In quality engineering, process capability indices play a crucial role in assessing the capability of a given process. Among the widely recognized indices are C_p, C_{pk}, C_{pm}, and C_{pmk}, all of which presuppose the normality of the product lifetime. However, Maiti et al. (2010) proposed a more versatile process capability index, denoted as C_{pyk}, which does not rely on distributional assumptions. The study is currently investigating statistical inferences for the C_{pyk} index within the context of progressively type-II censored samples, marking the first exploration of this aspect in the research. This paper investigates maximum likelihood and Bayesian inference for the C_{pyk} when the underlying distribution follows the inverse Rayleigh distribution. Additionally, the study explores Bayesian credible intervals and the highest posterior density intervals using the Markov Chain Monte Carlo procedure. Various types of bootstrap confidence intervals are also taken into consideration. To assess the performance of these intervals, a Monte Carlo simulation is executed, comparing their coverage probabilities and mean lengths. The paper concludes with an illustrative example utilizing real data, providing a practical application of the discussed methodologies.

Key Words: Bayesian Estimation; Bootstrap; Capability Index; Monte Carlo Simulation; Progressive Censoring; Confidence Interval.

Mathematical Subject Classification: 60E05, 62E15, 62F10.

1. Introduction

The assessment of a manufacturing process's capability and performance is frequently carried out through the use of the process capability index (PCI). These indices are determined by distribution parameters, lower and upper specification limits, and the target value. The noteworthy characteristic of PCIs is their dimensionless nature. In the industry, various PCIs are employed, including C_p, C_{pk}, C_{pm}, and C_{pmk}. For further details, we refer to the work of Chan et al. (1988), Hsiang (1985), Juran et al. (1974), Kane (1986), Pearn et al. (1992). It is noted that these indices are studied under the normality assumption of the process. However, the normality assumption is violated in most manufacturing processes, according to Gunter (1989). Recently, Maiti et al. (2010) suggested a generalized PCI C_{pyk} based on the arbitrary distribution function. Furthermore, this index can be used in all continuous and discrete processes. Let X be a lifetime of products with cumulative distribution function (cdf) F. Then the C_{pyk} is defined as

$$C_{pyk} = \min\left\{\frac{F(U) - \frac{1}{2}}{\frac{1}{2} - \alpha_2}, \frac{\frac{1}{2} - F(L)}{\frac{1}{2} - \alpha_1}\right\}, \tag{1}$$

where $F(L) + F(U) = 1$, $\alpha_1 = P(X < LDL)$ and $\alpha_2 = P(X > UDL)$. In the α_1 and α_2, the LDL and UDL are the desired lower and upper limits, respectively. It is noted that The limits LDL and UDL can be regarded as the lower and upper tolerance limits, respectively. Specification limits (L, U) represent externally imposed standards, whereas lower and upper desirable limits (LDL, UDL) are internally established goals designed to enhance performance or characteristics. Breaching specification limits could result in regulatory or customer-related repercussions while deviating from desirable limits primarily pertains to internal objectives related to quality or efficiency. The $C_{pyk} > 1$ is interpreted as indicating a capable process. The C_{pyk} can detect the incapability of the process, i.e., the process is not satisfactory from a capability point of view even though the process is in statistical control.

To draw statistical inferences, it is essential to gather data from the distribution of the lifetime. Given the extended lifespan of high-tech products, censoring becomes inevitable in lifetime tests. Among the various censoring schemes, progressive censoring is the most commonly employed. Suppose n identical units are subjected to a lifetime test. Upon the observation of the ith failure $X^{\mathbf{r}}_{i:m:n}$, where r_i surviving units are randomly removed from the test (with $1 \leq i \leq m$), a total of m failures are observed, and $r_1 + \cdots + r_m$ units are progressively censored. Consequently, we have $n = m + r_1 + \cdots + r_m$, and the order statistics $X^{\mathbf{r}}_{1:m:n} < X^{\mathbf{r}}_{2:m:n} < \cdots < X^{\mathbf{r}}_{m:m:n}$ describe the progressively censored type-II sample (PCS). Here, $\mathbf{r} = (r_1, \ldots, r_m)$ represents the censoring scheme in progressive censoring. For more detailed information on progressive censoring, refer to the book authored by Balakrishnan and Aggarwala (2000).

In this paper, we discuss the Bayesian inference of C_{pyk} for the inverse Rayleigh (IR) distribution under progressive censoring. The IR distribution is given by the probability density function (pdf) and cdf

$$f(x) = \frac{2\theta}{x^3}\exp\left(-\frac{\theta}{x^2}\right),\ x > 0, \tag{2}$$

and

$$F(x) = \exp\left(-\frac{\theta}{x^2}\right), \tag{3}$$

respectively, where $\theta > 0$ is a scale parameter. Lately, the IR distribution is being made quite attractive. The IR holds significance as a notable model in the field of reliability studies and finds widespread application in reliability and survival analysis. The latest reference can be found in Dey (2012), Sindhu et al. (2013), Zaki and Jabir (2015), Athirakrishnan and Abdul-Sathar (2022), Kumar and Kumari (2023), Kumar and Gupta (2023).

By substituting Eq. (3) into Eq. (1), the generalized PCI for the IR distribution is written by

$$C_{pyk} = C_{pyk}(\theta) = \min\left\{\frac{\exp\left(-\frac{\theta}{U^2}\right) - \frac{1}{2}}{\frac{1}{2} - \alpha_2}, \frac{\frac{1}{2} - \exp\left(-\frac{\theta}{L^2}\right)}{\frac{1}{2} - \alpha_1}\right\}. \tag{4}$$

By reviewing the literature, we observe that there are several studies on statistical inference for C_{pyk} in various distributions. Please refer to Gedik Balay (2021), Dey and Saha (2019), Dey et al. (2018), and Kumar et al. (2022) for more details. However, statistical inferences for the C_{pyk} index is being examined for the first time in this study under progressive type-II censored samples.

In this paper, we examine progressively Type-II censored samples from the IR distribution, discussing the inference on the capability index C_{pyk}. The rest of the paper is organized as follows: Section 2 considers the maximum likelihood estimation. Section 3 discusses Bayes estimation for the index C_{pyk} under three different loss functions. Additionally, credible (Cr) and highest posterior density (HPD) intervals are outlined. Section 4 covers the discussion of the bootstrap confidence interval (CI) procedure with four methods. Section 5 presents a simulation study aimed at observing the risks of Bayes estimates of the index C_{pyk}. The simulation also evaluates the coverage probability (CP) and mean length (ML) of all constructed intervals. In Section 6, a numerical example illustrates the discussed methodologies in the paper. Finally, Section 7 concludes the paper with some closing remarks.

2. Maximum likelihood estimation

Let $X^{\mathbf{r}}_{1:m:n} < X^{\mathbf{r}}_{2:m:n} < \cdots < X^{\mathbf{r}}_{m:m:n}$ be progressive type-II censored data from the IR distribution with pdf (2), then the log-likelihood function is given by

$$\ell\left(\theta|\mathbf{x}\right) \propto m\log\left(\theta\right) - \theta\sum_{i=1}^{m} x_i^{-2} + \sum_{i=1}^{m} r_i \log\left\{1-\exp\left(-\frac{\theta}{x_i^2}\right)\right\}. \tag{5}$$

Then the likelihood equation is found to be

$$\frac{d\ell\left(\theta|\mathbf{x}\right)}{d\theta} = \frac{m}{\theta} - \sum_{i=1}^{m} x_i^{-2} + \sum_{i=1}^{m} r_i \exp\left(-\frac{\theta}{x_i^2}\right)\left\{x_i^2\left(1-\exp\left(-\frac{\theta}{x_i^2}\right)\right)\right\}^{-1} = 0, \tag{6}$$

where x_i is the realization of $X^{\mathbf{r}}_{i:m:n}$, $i = 1, 2, \ldots, m$ and $\mathbf{x} = \left(x_1, x_2, \ldots, x_m\right)$. The maximum likelihood estimate (MLE) of θ is given by

$$\widehat{\theta} = \arg\max_{\theta} \ell\left(\theta|\mathbf{x}\right).$$

The Eq. (6) does not allow for an explicit solution. A numerical method can be used to attain MLE (See, Ma and Gui (2019)). According to the invariance principle for MLE, by substituting the $\widehat{\theta}$ into Eq. (1), we obtain the MLE of the C_{pyk} as follows:

$$\widehat{C}_{pyk} = \min\left\{\frac{\exp\left(-\frac{\widehat{\theta}}{U^2}\right)-\frac{1}{2}}{\frac{1}{2}-\alpha_2}, \frac{\frac{1}{2}-\exp\left(-\frac{\widehat{\theta}}{L^2}\right)}{\frac{1}{2}-\alpha_1}\right\}. \tag{7}$$

The asymptotic variance of $\widehat{\theta}$ can be estimated by

$$\widehat{Var}\left(\widehat{\theta}\right) = I^{-1}\left(\theta\right)\Big|_{\theta=\widehat{\theta}}, \tag{8}$$

where $I\left(\theta\right)$ is observed Fisher Information which is provided by the negatives of the second derivatives of $\ell\left(\theta|\mathbf{x}\right)$. Using Eq. (4) and Eq. (8) with the well-known delta rule, an estimator of asymptotic variance of the $\widehat{C}_{pyk}$ can be written as

$$\widehat{Var}\left(\widehat{C}_{pyk}\right) = \left(\frac{d}{d\theta}C_{pyk}\right)^2 I^{-1}\left(\theta\right)\Bigg|_{\theta=\widehat{\theta}}.$$

Now, an approximate $100\left(1-\alpha\right)\%$ CI (ACI) of C_{pyk} is given by

$$\left(\widehat{C}_{pyk} - \sqrt{\widehat{Var}\left(\widehat{C}_{pyk}\right)}z_{1-\alpha/2}, \widehat{C}_{pyk} + \sqrt{\widehat{Var}\left(\widehat{C}_{pyk}\right)}z_{1-\alpha/2}\right),$$

where z_a is the a-th quantile of the standard normal distribution. We refer to Lehmann (1999) for information on the asymptotic distribution of MLE and the delta rule.

3. Bayesian estimation

Let $X^{\mathbf{r}}_{1:m:n} < X^{\mathbf{r}}_{2:m:n} < \cdots < X^{\mathbf{r}}_{m:m:n}$ be a PCS from the IR distribution with cdf (3) and suppose that the unknown parameter θ is a continuous random variable. Let $\widehat{\theta}$ denote the estimator of this parameter. Three loss functions are considered in the Bayesian analysis. These are $L_1\left(\widehat{\theta},\theta\right) = \left(\widehat{\theta}-\theta\right)^2$, $L_2\left(\widehat{\theta},\theta\right) = \left|\widehat{\theta}-\theta\right|$ and $L_3\left(\widehat{\theta},\theta\right) = 1-\delta\left(\widehat{\theta}-\theta\right)$, where δ denotes the Dirac delta function. Let θ has Gamma(λ, ϕ) prior distribution with mean λ/ϕ, where λ and ϕ are shape and scale parameters, respectively. Then, the log-posterior distribution is given by

$$\begin{aligned}\pi(\theta|\mathbf{x}) &\propto \ell(\theta|\mathbf{x}) + \log(\pi(\theta)) \\ &= m\log(\theta) + \theta\sum_{i=1}^{m} x_i^{-2} + \sum_{i=1}^{m} r_i \log\left\{1 - \exp\left(-\frac{\theta}{x_i^2}\right)\right\} \\ &\quad + (\lambda-1)\log(\theta) + \lambda\log(\phi) - \log(\Gamma(\lambda)) - \phi\theta, \end{aligned} \tag{9}$$

where $\pi(\theta)$ is prior pdf. Under the loss functions L_1, L_2 and L_3, the Bayes estimates of θ are given, respectively, by the mean, median, and mode of posterior in (9). The Metropolis-Hastings algorithm can be used to get these Bayes estimates, the Cr and HPD intervals. The steps for the Metropolis-Hastings algorithm are given below:

Algorithm 1.
Step 1: Set $i = 1$ and the initial value $\theta^{(0)}$,
Step 2: Using the Metropolis-Hastings algorithm, generate θ from $\pi\left(\theta^{(i-1)}|\mathbf{x}\right)$ with the normal distribution with mean $\theta^{(i-1)}$ and variance S_θ^2 as a proposal distribution, where S_θ^2 is inverse of the observed Fisher information,
Step 3: Set $i = i + 1$;
Step 4: To produce the Monte Carlo Markov Chain (MCMC) sample of size $M + N : \theta^{(0)}, \theta^{(1)}, \ldots, \theta^{(M+N-1)}$, repeat Steps 3 and 4 $M + N$ times.
Step 5: First M elements of MCMC sample are burned and the our useful sample is $\theta^{(M+1)}, \theta^{(M+2)}, \ldots, \theta^{(M+N)}$. We call the sample $\theta_1, \theta_2, \ldots, \theta_N$ for abbreviation. Under loss functions L_1, L_2, the approximate Bayes estimates of C_{pyk} based on chain $C_{pyk}^1, C_{pyk}^2, \ldots, C_{pyk}^N$ can be calculated as

$$\begin{aligned}\overline{C}_{pyk} &= \frac{1}{N}\sum_{i=1}^{N} C_{pyk}^i, \\ \widetilde{C}_{pyk} &= \operatorname{med}\left(C_{pyk}^1, C_{pyk}^2, \ldots, C_{pyk}^N\right),\end{aligned}$$

where $C_{pyk}^i = C_{pyk}(\theta_i), i = 1, 2, \ldots, N$, $C_{pyk}(\cdot)$ is defined as in Eq. (1) and med$(\cdot)$ is the sample median.
Under loss functions L_3, the approximate bayes estimate of C_{pyk} is defined by

$$\breve{C}_{pyk} = \arg\max_{\theta}(\pi(\theta|\mathbf{x})).$$

The $\breve{C}_{pyk}$ can be achieved by any optimization algorithm such as Nelder-Mead or BFGS which are available in R function **optim**.
It is noted that the credible and HPD intervals can also easily be obtained based on MCMC sample $\theta_1, \theta_2, \ldots, \theta_N$. Based on ordered chain $C_{pyk}^{(1)} < C_{pyk}^{(2)} < \ldots < C_{pyk}^{(N)}$, a $100(1-\alpha)\,\%$ symmetric Cr interval for C_{pyk} is

$$\left(C_{pyk}^{(\lfloor N\alpha/2\rfloor)}, C_{pyk}^{(\lfloor N(1-\alpha/2)\rfloor)}\right),$$

where $\lfloor a\rfloor$ is the floor function of a. To construct the HPD interval, one can utilize the Monte Carlo procedures introduced by Chen and Shao (1999). For a straightforward application of this procedure, we recommend consulting the work of Pasha-Zanoosi et al. (2022). The HPD interval using the Monte Carlo procedure is as follows: First,

$$\left(C_{pyk}^{(\lfloor N^*(1-\alpha)\rfloor+h)}, C_{pyk}^{(h)}\right), 1 \le h \le \lfloor N^*(1-\alpha)\rfloor,$$

intervals are computed. Then the $100(1-\boldsymbol{\alpha})\%$ HPD intervals for C_{pyk} is obtained as

$$\left(C_{pyk}^{(\lfloor N^*(1-\alpha)\rfloor+h^*)}, C_{pyk}^{(h^*)}\right),$$

where

$$h^* = \underset{1\le h\le N^*(1-\alpha)}{\arg\min}\left(C_{pyk}^{(\lfloor N^*(1-\alpha)\rfloor+h)} - C_{pyk}^{(h)}\right).$$

4. Bootstrap CIs for Cpyk

In tandem with the advancement of computer technology, bootstrap-type Confidence Intervals (CIs) are gaining popularity. The appeal of bootstrapping lies in its independence from theoretical results. Notable variations of bootstrap intervals include the normal bootstrap (NB), percentile bootstrap (PB), basic bootstrap (BB), and bias-corrected percentile bootstrap (BCa). The bootstrap method presents itself as an alternative to the approach outlined in the preceding section for constructing an approximate CI for C_{pyk}. Bootstrap CIs have found application in estimating diverse PCIs; for further details, we refer to Rao et al. (2016), Franklin and Gary (1991), and Dey et al. (2018). The parametric bootstrap sampling on a PCS is given by the following algorithm:

Algorithm 2 A1. Consider a PCS from an IR distribution, denoted as $X^{\mathbf{r}}_{1:m:n} < X^{\mathbf{r}}_{2:m:n} < \cdots < X^{\mathbf{r}}_{m:m:n}$. Obtain the MLE of the parameter θ based on this sample, denoted as $\widehat{\theta}$.

A2. Calculate the MLE of C_{pyk} by $\widehat{C}_{pyk} = C_{pyk}\left(\widehat{\theta}\right)$.

A3. Generate the PCS $X^{*}_{1:m:n} < X^{*}_{2:m:n} < \cdots < X^{*}_{m:m:n}$ from the IR distribution with parameter $\widehat{\theta}$.

A4. Obtain the MLE of C_{pyk} based on the bootstrap sample $X^{*}_{1:m:n} < X^{*}_{2:m:n} < \cdots < X^{*}_{m:m:n}$ and denote it by $\widehat{C}^{*}_{pyk}$.

A5. Repeat Steps 2-4 B times, and obtain $\widehat{C}^{*}_{pyk,1}, \widehat{C}^{*}_{pyk,2}, \ldots, \widehat{C}^{*}_{pyk,B}$. These can be treated as a copy of $\widehat{C}^{*}_{pyk}$.

In this section, four parametric bootstrap methods are provided to construct the bootstrap CIs for C_{pyk}. It is noted that, the R function **boot.ci** gives all types of CIs. These methods are adapted from Ugarte et al. (2008) and described below:

The $(1-\alpha)100\%$ NB CI for C_{pyk} is calculated as

$$CI^{1-\alpha}_{Normal} = \left(2\widehat{C}_{pyk} - \overline{\widehat{C}^{*}_{pyk}} - z_{1-\alpha/2}\sqrt{\widehat{Var}\left(\widehat{C}^{*}_{pyk}\right)}, 2\widehat{C}_{pyk} - \overline{\widehat{C}^{*}_{pyk}} + z_{1-\alpha/2}\sqrt{\widehat{Var}\left(\widehat{C}^{*}_{pyk}\right)}\right),$$

where z_p is p-th quantile of standard normal distribution,

$$\overline{\widehat{C}^{*}_{pyk}} = \frac{1}{B}\sum_{i=1}^{B}\widehat{C}^{*}_{pyk,i},$$

and

$$\widehat{Var}\left(\widehat{C}^{*}_{pyk}\right) = \frac{1}{B-1}\sum_{i=1}^{B}\left(\widehat{C}^{*}_{pyk,i} - \overline{\widehat{C}^{*}_{pyk}}\right)^{2}.$$

The $(1-\alpha)100\%$ BB and PB CI for C_{pyk} are given, respectively, by

$$CI^{1-\alpha}_{Basic} = \left(2\widehat{C}_{pyk} - Q_{\widehat{C}^{*}_{pyk}}\left(1-\alpha/2\right), 2\widehat{C}_{pyk} - Q_{\widehat{C}^{*}_{pyk}}\left(\alpha/2\right)\right),$$

and

$$CI^{1-\alpha}_{Percentile} = \left(Q_{\widehat{C}^{*}_{pyk}}\left(\alpha/2\right), Q_{\widehat{C}^{*}_{pyk}}\left(1-\alpha/2\right)\right),$$

where $Q_{\widehat{C}^{*}_{pyk}}(p)$ is p-th sample quantile based on data $\widehat{C}^{*}_{pyk,1}, \widehat{C}^{*}_{pyk,2}, \ldots, \widehat{C}^{*}_{pyk,B}$.

In below, Algorithm 3 is provided to compute the $(1-\alpha)100\%$ BCa bootstrap CI for C_{pyk}.

Algorithm 3

A1. Compute the bias factor:

$$z = \Phi^{-1}\left(\frac{1}{B}\sum_{i=1}^{B} I\left\{\widehat{C}^{*}_{pyk,i} < \widehat{C}_{pyk}\right\}\right),$$

where $\Phi(\cdot)$ is the cdf of standard normal distribution.

A2. Next, compute the skewness correction factor:

$$a = \frac{\sum_{i=1}^{n} \left(\overline{\widehat{C}_{pyk}}_{(-i)} - \widehat{C}_{pyk(-i)} \right)^3}{6 \left[\sum_{i=1}^{n} \left(\overline{\widehat{C}_{pyk}}_{(-i)} - \widehat{C}_{pyk(-i)} \right)^2 \right]^{3/2}},$$

where $\widehat{C}_{pyk(-i)}$ is the value of $\widehat{C}_{pyk}$ when the i^{th} value is deleted from the sample of n values and

$$\overline{\widehat{C}_{pyk}}_{(-i)} = \frac{1}{n} \sum_{i=1}^{n} \widehat{C}_{pyk(-i)}.$$

A3. Using z and a, compute

$$\alpha_1 = \Phi \left[z + \frac{z + z_{\alpha/2}}{1 - a \left(z + z_{\alpha/2} \right)} \right] \text{ and } \alpha_2 = \Phi \left[z + \frac{z + z_{1-\alpha/2}}{1 - a \left(z + z_{1-\alpha/2} \right)} \right].$$

A4. The $(1-\alpha)100\%$ BCa CI for C_{pyk} is given by

$$CI_{\text{BC}a}^{1-\alpha} = \left(Q_{\widehat{C}^*_{pyk}} (\alpha_1), Q_{\widehat{C}^*_{pyk}} (\alpha_2) \right),$$

where $Q_{\widehat{C}^*_{pyk}} (p)$ is p-th sample quantile based on data $\widehat{C}^*_{pyk,1}, \widehat{C}^*_{pyk,2}, \ldots, \widehat{C}^*_{pyk,B}$.

5. Simulation Study

In the simulation study, 1000 trials are performed. The prior distribution is fixed as the Gamma distribution with shape parameter $\lambda = 2$ and scale parameter $\phi = 0.5$. Different 18 censoring schemes are considered. In the bootstrap sampling re-sample size is fixed $B = 1000$. The nominal level is fixed at 0.95 for the CIs. The MCMC method with the Metropolis-Hasting algorithm is used to get the Bayes estimates. 10000 iteration is used for the chain and the first 1000 samples are removed from the chain to give the Markov Chain time to reach its equilibrium distribution. From a Bayesian point of view, the simulated risks of MLE and Bayes estimates are given in Table 1 under different loss functions. The simulated CPs and MLs are presented in Table 2 and Table 3, respectively.

Table 1: Estimated risks of the estimators of C_{pyk}

Schemes	$\overline{C}_{pyk}$	$\widetilde{C}_{pyk}$	$\breve{C}_{pyk}$	$\widehat{C}_{pyk}$
(0^{10})	0.0273	0.0233	0.0217	0.0389
(1^{10})	0.0170	0.0154	0.0161	0.0217
$(5, 0^8, 5)$	0.0177	0.0159	0.0163	0.0219
$(5, 5, 0^8)$	0.0187	0.0163	0.0166	0.0228
$(0^8, 5, 5)$	0.0149	0.0134	0.0145	0.0182
$(0^4, 5, 5, 0^4)$	0.0159	0.0143	0.0150	0.0195
(0^{20})	0.0163	0.0149	0.0154	0.0205
(1^{20})	0.0092	0.0087	0.0096	0.0108
$(10, 0^{18}, 10)$	0.0103	0.0097	0.0102	0.0120
$(10, 10, 0^{18})$	0.0122	0.0114	0.0123	0.0147
$(0^{18}, 10, 10)$	0.0083	0.0079	0.0086	0.0098
$(0^9, 10, 10, 0^9)$	0.0082	0.0077	0.0085	0.0096
(0^{50})	0.0067	0.0065	0.0072	0.0078
(1^{50})	0.0044	0.0044	0.0047	0.0050
$(25, 0^{48}, 25)$	0.0044	0.0044	0.0049	0.0051
$(25, 25, 0^{48})$	0.0057	0.0056	0.0063	0.0066
$(0^{48}, 25, 25)$	0.0038	0.0038	0.0042	0.0044
$(0^{24}, 25, 25, 0^{24})$	0.0037	0.0037	0.0041	0.0043

Table 2: CPs of CIs of the C_{pyk}

Schemes	NB	BB	PB	BCa	HPD	Cr	ACI
(0^{10})	0.9610	0.8990	0.9210	0.9480	0.9800	0.9270	0.9580
(1^{10})	0.9450	0.9270	0.9380	0.9490	0.9780	0.9280	0.9440
$(5, 0^8, 5)$	0.9460	0.9050	0.9530	0.9480	0.9800	0.9500	0.9440
$(5, 5, 0^8)$	0.9420	0.9130	0.9400	0.9440	0.9850	0.9340	0.9450
$(0^8, 5, 5)$	0.9540	0.9140	0.9340	0.9270	0.9820	0.9450	0.9470
$(0^4, 5, 5, 0^4)$	0.9700	0.9250	0.9440	0.9640	0.9840	0.9380	0.9670
(0^{20})	0.9510	0.9240	0.9340	0.9440	0.9730	0.9360	0.9450
(1^{20})	0.9490	0.9320	0.9450	0.9540	0.9730	0.9350	0.9450
$(10, 0^{18}, 10)$	0.9560	0.9400	0.9410	0.9360	0.9790	0.9460	0.9530
$(10, 10, 0^{18})$	0.9460	0.9350	0.9460	0.9460	0.9710	0.9260	0.9450
$(0^{18}, 10, 10)$	0.9450	0.9310	0.9460	0.9410	0.9770	0.9470	0.9430
$(0^9, 10, 10, 0^9)$	0.9630	0.9410	0.9350	0.9660	0.9810	0.9420	0.9560
(0^{50})	0.9600	0.9470	0.9320	0.9400	0.9800	0.9540	0.9520
(1^{50})	0.9560	0.9480	0.9500	0.9550	0.9550	0.9420	0.9530
$(25, 0^{48}, 25)$	0.9550	0.9460	0.9510	0.9480	0.9670	0.9600	0.9540
$(25, 25, 0^{48})$	0.9500	0.9360	0.9500	0.9490	0.9680	0.9430	0.9490
$(0^{48}, 25, 25)$	0.9620	0.9520	0.9460	0.9440	0.9620	0.9470	0.9590
$(0^{24}, 25, 25, 0^{24})$	0.9500	0.9500	0.9350	0.9440	0.9710	0.9560	0.9400

Table 3: MLs of CIs of the of C_{pyk}

Schemes	NB	BB	PB	BCa	HPD	Cr	ACI
(0^{10})	0.5613	0.5583	0.5583	0.5000	0.5196	0.5668	0.5088
(1^{10})	0.3886	0.3878	0.3878	0.3634	0.4184	0.4546	0.3661
$(5, 0^8, 5)$	0.4008	0.3995	0.3995	0.4101	0.4324	0.4695	0.3782
$(5, 5, 0^8)$	0.4258	0.4249	0.4249	0.3983	0.4450	0.4841	0.4035
$(0^8, 5, 5)$	0.3755	0.3753	0.3753	0.3856	0.4088	0.4437	0.3532
$(0^4, 5, 5, 0^4)$	0.3870	0.3865	0.3865	0.3610	0.4150	0.4503	0.3642
(0^{20})	0.3722	0.3715	0.3715	0.3483	0.4083	0.4419	0.3488
(1^{20})	0.2652	0.2652	0.2652	0.2570	0.3231	0.3480	0.2569
$(10, 0^{18}, 10)$	0.2796	0.2798	0.2798	0.2831	0.3351	0.3612	0.2714
$(10, 10, 0^{18})$	0.3020	0.3022	0.3022	0.2923	0.3557	0.3839	0.2949
$(0^{18}, 10, 10)$	0.2543	0.2542	0.2542	0.2571	0.3134	0.3369	0.2460
$(0^9, 10, 10, 0^9)$	0.2620	0.2619	0.2619	0.2532	0.3160	0.3414	0.2539
(0^{50})	0.2240	0.2243	0.2243	0.2183	0.2828	0.3035	0.2184
(1^{50})	0.1628	0.1634	0.1634	0.1610	0.2293	0.2423	0.1603
$(25, 0^{48}, 25)$	0.1739	0.1742	0.1742	0.1748	0.2360	0.2511	0.1717
$(25, 25, 0^{48})$	0.1949	0.1953	0.1953	0.1926	0.2600	0.2773	0.1926
$(0^{48}, 25, 25)$	0.1567	0.1572	0.1572	0.1576	0.2197	0.2319	0.1547
$(0^{24}, 25, 25, 0^{24})$	0.1618	0.1622	0.1622	0.1601	0.2229	0.2362	0.1595

According to simulation results given in Tables 1-3, some concluding remarks are given as follows:

1. All CIs are nearly at the nominal level $1 - \alpha = 0.95$.
2. For small sample sizes, the CIs of bootstrap methods slightly fall below the nominal level. However, for moderate sample sizes, the CIs tend to converge to the nominal level.
3. The MLs of CIs decrease to zero as the sample size increases in all cases.
4. All risks decrease to zero as the sample size increases in all cases.
5. In small sample cases, the Bayes estimate with loss function L_2 exhibits better risk than other estimators. As the sample size increases, the performance of MLE and Bayes estimates becomes equivalent.
6. HPD intervals demonstrate better CP and ML than the Cr intervals.
7. The Bias-corrected and accelerated (BCa) Bootstrap CIs outperform other Bootstrap CIs in terms of CPs and compete with the ACI. Both the ACI and BCa Bootstrap CIs exhibit the same CPs and MLs for all discussed censoring schemes.

6. An Application

In this section, a real data analysis is conducted to evaluate the performance of the proposed confidence intervals. The data consists of the failure times of 20 mechanical components reported by Murthy et al. (2004). The complete data are: 0.067, 0.068, 0.076, 0.081, 0.084, 0.085, 0.085, 0.086, 0.089, 0.098, 0.098, 0.114, 0.114, 0.115, 0.121, 0.125, 0.131, 0.149, 0.160, and 0.485. The p-value for the Kolmogorov-Smirnov test for the IR distribution on this data is 0.1159>0.05 and we conclude that the IR distribution fits this data. According to Akdoğan (2022), it is considered desirable for a failure to occur between times t_1 and t_2 from the manufacturer's perspective. In this context, the C_{pyk} index will be utilized for lifetime data in this example. To apply the results obtained through progressive type censoring, a synthetic PCS is generated from the complete data as shown below with censoring scheme $\mathbf{r} = (5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$: 0.067, 0.076, 0.081, 0.084, 0.085, 0.086, 0.089, 0.098, 0.098, 0.114, 0.115, 0.125, 0.131, 0.149, 0.160. In real data analysis, the lower and upper specification limits ($L = 0.0574$ and $U = 0.4387$, respectively) and lower and upper tolerance limits ($LDL = 0.0721$ and $UDL = 0.3061$, respectively) are determined based on the selected values $F(L) = 0.05$, $F(U) = 0.95$, $\alpha_1 = 0.15$, and $\alpha_2 = 0.10$ to create a scenario suitable. This is done to configure a scenario that ensures the process is capable. Based on PCS data,

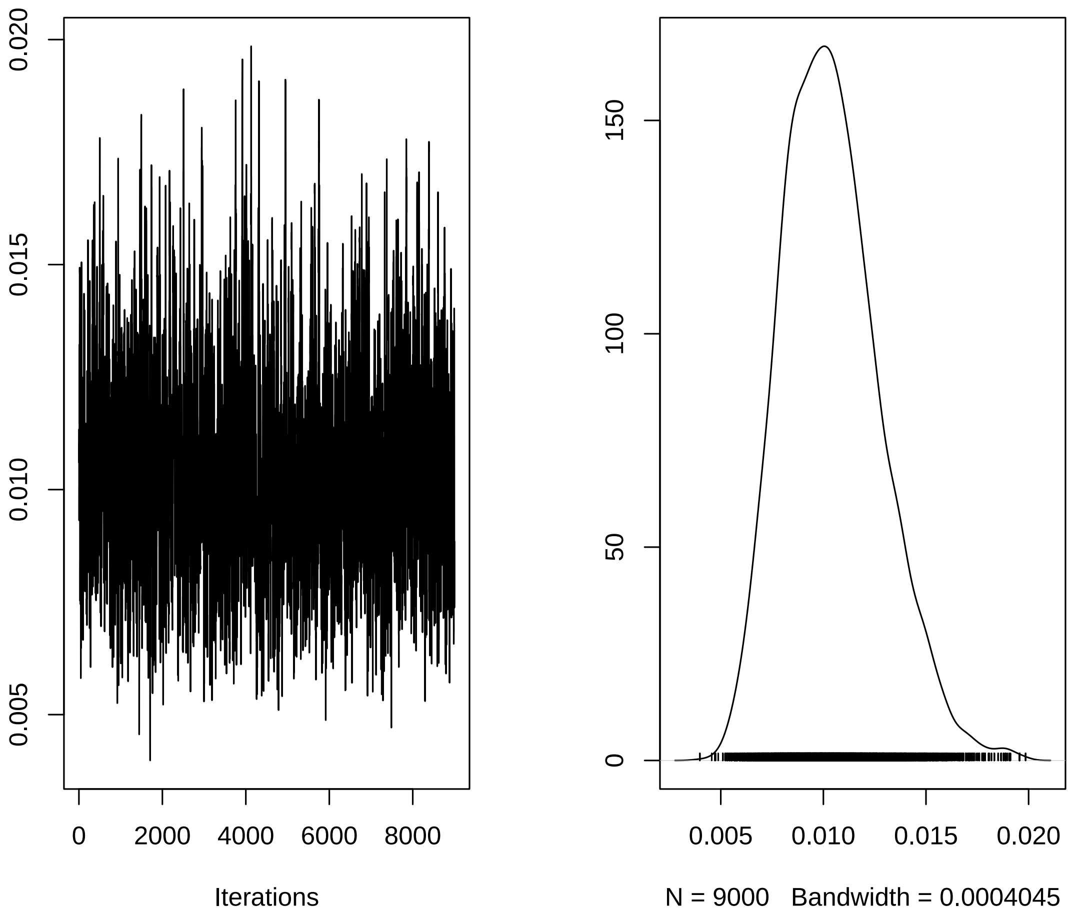

Figure 1: Trace plot of MCMC sample and kernel estimate of the posterior pdf

the $\widehat{C}_{pyk}, \overline{C}_{pyk}, \widetilde{C}_{pyk}$ and $\breve{C}_{pyk}$ found to be 1.1250, 1.1080, 1.1151, and 1.1250. A trace plot of the MCMC sample and density estimate of the posterior pdf are given in Figure 1, and it is satisfactory for Bayes estimation. Note that in Bayes estimation, 10000 MCMC sample is generated, and the first 1000 the sample is removed as burn-in. Uniform(Improper) prior is used in the analysis. The Bootstrap CIs, ACI, HPD, and Cr CI are given in Table 4.

Table 4: CIs of C_{pyk} based on real data

Methods	CIs
NB	$(1.064, 1.221)$
BB	$(1.099, 1.258)$
PB	$(0.992, 1.151)$
BCa	$(1.062, 1.152)$
HPD	$(1.050, 1.151)$
Cr	$(1.016, 1.149)$
ACI	$(1.069, 1.180)$

From Table 4, all intervals, except for PB, having a lower limit greater than one, it can be said that the process is under control.

7. Concluding Remark

The novelty of this study lies in the estimation of the C_{pyk} index for the first time under progressively Type-II censored samples. The point and interval estimation of C_{pyk} index deals with classical and Bayesian perspectives for IR distribution. We hope that quality engineers will utilize the methods discussed in the paper. In subsequent studies, the performances of the Tierney-Kadane and Lindley approximation for Bayesian estimation can be examined. Additionally, uncorrected likelihood ratio confidence intervals can be proposed, and their performances can be compared with the methods presented in this paper.

References

1. Akdoğan, Y. (2022). On the confidence intervals of process capability index cpm based on a progressive type-ii censored sample. *Quality and Reliability Engineering International*, 38(5):2845–2861.
2. Athirakrishnan, R. and Abdul-Sathar, E. (2022). E-bayesian and hierarchical bayesian estimation of inverse rayleigh distribution. *American Journal of Mathematical and Management Sciences*, 41(1):70–87.
3. Balakrishnan, N. and Aggarwala, R. (2000). *Progressive censoring: theory, methods, and applications*. Springer Science & Business Media.
4. Chan, L. K., Cheng, S. W., and Spiring, F. A. (1988). A new measure of process capability: Cpm. *Journal of Quality Technology*, 20(3):162–175.
5. Chen, M.-H. and Shao, Q.-M. (1999). Monte carlo estimation of bayesian credible and hpd intervals. *Journal of computational and Graphical Statistics*, 8(1):69–92.
6. Dey, S. (2012). Bayesian estimation of the parameter and reliability function of an inverse rayleigh distribution. *Malaysian Journal of Mathematical Sciences*, 6(1):113–124.
7. Dey, S. and Saha, M. (2019). Bootstrap confidence intervals of generalized process capability index cpyk using different methods of estimation. *Journal of Applied Statistics*, 46(10):1843–1869.
8. Dey, S., Saha, M., Maiti, S. S., and Jun, C.-H. (2018). Bootstrap confidence intervals of generalized process capability index cpyk for lindley and power lindley distributions. *Communications in Statistics-Simulation and Computation*, 47(1):249–262.
9. Franklin, L. A. and Gary, W. (1991). Bootstrap confidence interval estimates of cpk: an introduction. *Communications in Statistics-Simulation and Computation*, 20(1):231–242.
10. Gedik Balay, İ. (2021). Estimation of the generalized process capability index c pyk based on bias-corrected maximum-likelihood estimators for the generalized inverse lindley distribution and bootstrap confidence intervals. *Journal of Statistical Computation and Simulation*, 91(10):1960–1979.
11. Gunter, B. H. (1989). The use and abuse of cpk. *Quality progress*, 22(1):72–73.
12. Hsiang, T. C. (1985). A tutorial on quality control and assurance-the taguchi methods. In *ASA Annual Meeting LA, 1985*.
13. Juran, J. M., Gryna, F. M., and Bingham, R. S. (1974). *Quality control handbook*, volume 3. McGraw-Hill New York.
14. Kane, V. E. (1986). Process capability indices. *Journal of quality technology*, 18(1):41–52.
15. Kumar, K. and Kumari, A. (2023). Inferences for two inverse rayleigh populations based on joint progressively type-ii censored data. *Computational Intelligence in Sustainable Reliability Engineering*, pages 159–179.
16. Kumar, R. and Gupta, R. (2023). Bayesian analysis of inverse rayleigh distribution under non-informative prior for different loss functions. *Thailand Statistician*, 21(1):76–92.
17. Kumar, S., Yadav, A. S., Dey, S., and Saha, M. (2022). Parametric inference of generalized process capability index cpyk for the power lindley distribution. *Quality Technology & Quantitative Management*, 19(2):153–186.
18. Lehmann, E. L. (1999). *Elements of large-sample theory*. Springer.
19. Ma, Y. and Gui, W. (2019). Pivotal inference for the inverse rayleigh distribution based on general progressively type-ii censored samples. *Journal of Applied Statistics*, 46(5):771–797.
20. Maiti, S. S., Saha, M., and Nanda, A. K. (2010). On generalizing process capability indices. *Quality Technology & Quantitative Management*, 7(3):279–300.
21. Murthy, D. P., Xie, M., and Jiang, R. (2004). *Weibull models*. John Wiley & Sons.
22. Pasha-Zanoosi, H., Pourdarvish, A., and Asgharzadeh, A. (2022). Multicomponent stress-strength reliability

with exponentiated teissier distribution. *Austrian Journal of Statistics*, 51(4):35–59.
23. Pearn, W. L., Kotz, S., and Johnson, N. L. (1992). Distributional and inferential properties of process capability indices. *Journal of Quality Technology*, 24(4):216–231.
24. Rao, G. S., Aslam, M., and Kantam, R. (2016). Bootstrap confidence intervals of cnpk for inverse rayleigh and log-logistic distributions. *Journal of Statistical Computation and Simulation*, 86(5):862–873.
25. Sindhu, T. N., Aslam, M., and Feroze, N. (2013). Bayes estimation of the parameters of the inverse rayleigh distribution for left censored data. In *ProbStat Forum*, volume 6, pages 42–59.
26. Ugarte, M. D., Militino, A. F., and Arnholt, A. T. (2008). *Probability and Statistics with R*. CRC press.
27. Zaki, H. A. R. D. S. and Jabir, I. D. A. G. (2015). A comparison of the classical estimators with the bayes estimators of one parameter inverse rayleigh distribution. *International Journal*, 3(8):738–749.