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A New Inverse Kumaraswamy Family of Distributions: Properties and Applications

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Abstract

The Kumaraswamy distribution is an important probability distribution used to model several hydrological problems as well as various natural phenomena whose process values are bounded on both sides. In this paper, we have introduced a new family of distributions by using the inverse Kumaraswamy distribution. The statistical properties of the proposed family of distributions have been explored. The maximum likelihood estimation of the new proposed family of distributions have been discussed by using the random sample and the dual generalized order statistics. A particular sub-model of this family; namely, the inverse Kumaraswamy-Weibull distribution, is considered and some of its statistical properties are obtained. Estimation efficiency is numerically evaluated via a simulation study and two real-data applications of the proposed distribution are provided as well.

Key Words: Inverse Kumaraswamy distribution, Weibull distribution, Maximum Likelihood Estimation, Dual Generalized Order Statistics, Least-Square Estimation.

Mathematical Subject Classification: 62E15, 62F10, 62G30

1. Introduction

In statistical literature, the Kumaraswamy distribution, originally called the double-bounded distribution, was proposed by Kumaraswamy (1980) for a double-bounded random processes in hydrological applications as an alternative probabilistic model to the well-known beta distribution. The main feature of the former distribution compared to the latter lies in the fact that the Kumaraswamy distribution has nice algebraic properties. Both distributions are defined on the same support; namely, [0, 1], and have a similar shape. A generic relationship between these two distributions has been obtained by Widemann (2011) using the logistic map. The probability density function (PDF) of the distribution is

$$f\left(x;\alpha,\beta\right)=\alpha\beta x^{\alpha-1}\left(1-x^{\alpha}\right)^{\beta-1}\ ;\ 0<x<1\ ;\ \alpha,\beta>0 \qquad (1)$$

Recently, an Inverted Kumaraswamy distribution was proposed by Al-Fattah et al. (2017) by transforming the distribution (1) $Y=X^{-1}-1$. The density function of the proposed distribution is

$$f\left(y;\alpha,\beta\right)=\alpha\beta\left(1+y\right)^{-(\alpha+1)}\left[1-\left(1+y\right)^{-a}\right]^{\beta-1}\ ;\ y,\alpha,\beta>0 \qquad (2)$$

In hydrology and related areas, the Kumaraswamy distribution has received considerable interest, see for example Cordeiro et al. (2010), Cordeiro and Castro (2011), Fletcher and Ponnambalam (1996), Ganji et al. (2006), Ponnambalam et al. (2001), Sundar and Subbiah (1989) and Seifi et al. (2000). Several papers in the hydrological literature have used this distribution because it is considered to be a ”better alternative” to the beta distribution; see, for example, Koutsoyiannis and Xanthopoulos (1989).

Due to the importance of the Kumaraswamy distribution, researchers have proposed some generalizations for it, while the others have studied some of its related distributions. For instance, Cordeiro and Castro (2011) have studied and derived almost all formulas for probabilistic characteristics of a new class of the Kumaraswamy generalized distributions based on the Kumaraswamy distribution. A transmuted Kumaraswamy distribution has been proposed by Khan et al. (2016). A new inverted Kumaraswamy distribution was proposed by Daghistani et al. (2019) by using the transformation $Y = X^{-1}$. The density function of this inverted Kumaraswamy distribution is

$$f\left(y;\alpha,\beta\right)=\alpha\beta y^{-(\alpha+1)}\left(1-y^{-\alpha}\right)^{\beta-1} \;;\; y>1\;;\; \alpha,\beta>0\,. \tag{3}$$

The corresponding distribution function is

$$F\left(y;\alpha,\beta\right)=\left(1-y^{-\alpha}\right)^{\beta} \;;\; y>1\;;\; \alpha,\beta>0\,. \tag{4}$$

A popular approach to generalize the above inverse Kumaraswamy distribution is to use the transformed-transformer (*T-X*) family of distributions proposed by Alzaatreh et al. (2013). The cumulative distribution function (*cdf*) of a new family of distributions, by using the T-X family, is

$$F\left(x\right)=\int_{a}^{W\left[G(x)\right]} r\left(t\right)dt = R\left[W\left\{G\left(x\right)\right\}\right], \tag{5}$$

where $r(t)$ is the *pdf* of a random variable $T \in [a,b]$ for $-\infty \le a < b \le \infty$ and $W\left[G(x)\right]$ is some function of the *cdf* $G(x)$, of some random variable *X*. The function $W\left[G(x)\right]$ satisfies following conditions

- $W\left[G(x)\right] \in [a,b]$, $W(0) \to a$ and $W(1) \to b$
- $W\left[G(x)\right]$ is differentiable and monotonically non-decreasing.

The density function corresponding to (5) is

$$f\left(x\right)=\left[\frac{d}{dx}W\left\{G\left(x\right)\right\}\right] r\left[W\left\{G\left(x\right)\right\}\right]. \tag{6}$$

The T-X family of distributions have been studied for random variable T having support $(-\infty,\infty)$ and $(0,\infty)$. The *T-X* family of distributions has been explored by various authors when the domain of transformed random variable T is from $(0,\infty)$ or from $(-\infty,\infty)$ for various choices of W[G(x)]. This family has not been explored for the transformed distribution with support from $[1,\infty)$. In this paper, the T-X family of distributions, given in (5), will be used to propose a new family of distributions by using the inverse Kumaraswamy distribution.

The remaining parts of this paper are organized as follows. In section (2), we have proposed a new family of inverse Kumaraswamy distribution and have explored its properties. In section (3), The estimation of parameters of the new inverse Kumaraswamy family of distribution is described. In section (4), the special case: inverse Kumaraswamy-Weibull distribution has been introduced and some of its properties are explored. In section (5), estimation of the parameters of inverse Kumaraswamy-Weibull Distribution is discussed. In section (6), simulation results of the inverse Kumaraswamy-Weibull distribution are presented. Two real financial data sets are analyzed in section (7) to illustrate the application of the proposed distribution. Finally, in section (8), some conclusions are provided.

2. The New Inverse Kumaraswamy Family of Distributions and its General Properties

In this section, we have proposed a new family of inverse Kumaraswamy distributions by using (4) and assuming that *X* is a random variable that follows a probability distribution with *cdf* $G(x)$. The *cdf* of the new inverse Kumaraswamy family of distributions (*NIKFD*)

$$F_{IK-G}\left(x\right)=\left[1-\left\{W\left[G\left(x\right)\right]\right\}^{-\alpha}\right]^{\beta} \;;\; x\in\mathbb{R} \;;\; \alpha,\beta>0 \tag{7}$$

where $W\left[G(x)\right]$ is some function of $G(x)$. The crux of the new family lies in a suitable choice of $W\left[G(x)\right]$ such that $W(0)=1$ and $W(1)=\infty$. One possible function that can be used is

$$W\left[G\left(x\right)\right]=\left[1-G^{a}\left(x\right)\right]^{-b} \tag{8}$$

It can be seen that $W\left[G(-\infty)\right]=1$ and $W\left[G(\infty)\right]=\infty$. Using this function, the *cdf* of the *NIKFD* is

$$F_{IK-G}\left(x\right)=\left[1-\left\{1-G^{a}\left(x\right)\right\}^{\alpha b}\right]^{\beta} \;;\; x\in\mathbb{R} \;;\; a,b,\alpha,\beta>0\,. \tag{9}$$

Also, from (9) we get the density function of the proposed family of distributions is

$$f_{IK-G}(x) = ab\alpha\beta g(x) G^{a-1}(x)\left[1-G^{a}(x)\right]^{\alpha b-1}\left[1-\left\{1-G^{a}(x)\right\}^{\alpha b}\right]^{\beta-1}\ ;\ a,b,\alpha,\beta>0\ . \quad (10)$$

Using Gradshteyn and Ryzhik (2007), the *cdf* (9) of the *NIKFD* can also be written as

$$F_{IK-G}(x) = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} w_{nj} G^{aj}(x)\ , \quad (11)$$

where $w_{nj} = {}^{\beta}C_{n}\,{}^{\alpha bn}C_{j}\,(-1)^{n+j}$. The density function corresponding to (11) is

$$f_{IK-G}(x) = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} w_{nj}\, aj\, g(x) G^{aj-1}(x). \quad (12)$$

The density given in (12) is a weighted sum of the exponentiated–G family of distributions. It can be seen, from (12), that the properties of the *NIKFD* can be studied in context of the exponentiated–G family of distributions. Some special cases of the family of distributions given in (9) are given in Table 1, below

Table 1: Special Cases of the Proposed Family of Distributions

a	***b***	**α**	**β**	**CDF**
-	-	-	1	$1-\left\{1-G^{a}(x)\right\}^{\alpha b}$
-	-	1	-	$\left[1-\left\{1-G^{a}(x)\right\}^{b}\right]^{\beta}$
-	1	-	-	$\left[1-\left\{1-G^{a}(x)\right\}^{\alpha}\right]^{\beta}$
1	-	-	-	$\left[1-\left\{1-G(x)\right\}^{\alpha b}\right]^{\beta}$
-	-	1	1	$1-\left\{1-G^{a}(x)\right\}^{b}$
1	-	-	1	$1-\left\{1-G(x)\right\}^{\alpha b}$
1	1	-	-	$\left[1-\left\{1-G(x)\right\}^{\alpha}\right]^{\beta}$
-	1	1	-	$G^{\alpha\beta}(x)$
1	-	1	1	$1-\left\{1-G(x)\right\}^{\alpha}$
1	1	1	-	$G^{\beta}(x)$
-	1	1	1	$G^{a}(x)$
1	1	1	1	$G(x)$

2.1 Hazard Rate Function

The hazard rate function, Cox and Oakes (2018), for the *NIKFD* is

$$h(x) = \frac{f_{IK-G}(x)}{1-F_{IK-G}(x)} = \frac{ab\alpha\beta g(x) G^{a-1}(x)\left[1-G^{a}(x)\right]^{\alpha b-1}\left[1-\left\{1-G^{a}(x)\right\}^{\alpha b}\right]^{\beta-1}}{1-\left[1-\left\{1-G^{a}(x)\right\}^{\alpha b}\right]^{\beta}}\ . \quad (13)$$

The hazard rate function can be obtained for any baseline distribution, $G(x)$.

2.2 Moments and Moment Generating Function

The *r*th moment of the *NIKFD* is

$$\mu_r^{/} = \int_{-\infty}^{\infty} x^r f_{IK-G}(x)\,dx = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} w_{nj}\, aj \int_{-\infty}^{\infty} x^r g(x) G^{aj-1}(x)\,dx \quad (14).$$

The moment generating function of the *NIKFD* is

$$M_X(t)=\int_{-\infty}^{\infty}e^{tx}f_{IK-G}(x)dx=\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}w_{nj}aj\int_{-\infty}^{\infty}e^{tx}g(x)G^{aj-1}(x)dx. \tag{15}$$

From above, we can see that the moments and moment generating function of the *NIKFD* are the weighted sum of moments and moment generating function of the exponentiated-G family of distributions.

2.3 Shannon Entropy

The Shannon entropy, see Grebenc (2011), for a random variable *X* having density function $f(x)$ is computed as

$$H(x)=E\left[-\ln f(x)\right]=-\int_{-\infty}^{\infty}\{\ln f(x)\}f(x)dx.$$

Now, using the density function of *NIKFD* in above equation, the Shannon entropy for the *NIKFD* is

$$\begin{aligned}H(x)-\int_{-\infty}^{\infty}&\left[ab\alpha\beta g(x)G^{a-1}(x)\left\{1-G^a(x)\right\}^{\alpha b-1}\left\{1-\left\{1-G^a(x)\right\}^{\alpha b}\right\}^{\beta-1}\right]\left[\ln a+\ln b+\ln\alpha+\ln\beta\right.\\&\left.+\ln g(x)+(a-1)\ln G(x)\right]+\left[(\alpha b-1)\ln\left\{1-G^a(x)\right\}+(\beta-1)\ln\left\{1-\left\{1-G^a(x)\right\}^{\alpha b}\right\}\right]dx.\end{aligned} \tag{16}$$

The Shannon entropy can be computed for any baseline distribution.

3. Estimation of Parameters of the New Inverse Kumaraswamy Family of Distributions

In this section estimation of the parameters of the *NIKFD* is discussed. We have discussed maximum likelihood and least square estimation of the parameters.

3.1 The Maximum Likelihood Estimation

In this section, the maximum likelihood estimation of the parameters of the *NIKFD* is discussed. The maximum likelihood estimation has been discussed for random and for ordered sample. These are discussed below.

3.1.1 Using the Random Sample

The maximum likelihood estimation is the most popular method to estimate the parameters of any distribution. In this part, we have given the maximum likelihood estimation of the *NIKFD* by using the random sample. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the new inverse Kumaraswamy family of distributions. By using (10), the likelihood function is,

$$\begin{aligned}L(\mathbf{x})&=\prod_{i=1}^{n}f_{IK-G}(x_i;a,b,\alpha,\beta)\\&=a^nb^n\alpha^n\beta^n\prod_{i=1}^{n}\left[g(x_i)G^{a-1}(x_i)\left\{1-G^a(x_i)\right\}^{\alpha b-1}\left\{1-\left\{1-G^a(x_i)\right\}^{\alpha b}\right\}^{\beta-1}\right].\end{aligned}$$

The log of likelihood function is

$$\begin{aligned}\ell(\mathbf{x})=\ln L(\mathbf{x})&=n\ln a+n\ln b+n\ln\alpha+n\ln\beta+\sum_{i=1}^{n}\ln g(x_i)+(a-1)\sum_{i=1}^{n}\ln G(x_i)\\&+(\alpha b-1)\sum_{i=1}^{n}\ln\left\{1-G^a(x_i)\right\}+(\beta-1)\sum_{i=1}^{n}\ln\left[1-\left\{1-G^a(x_i)\right\}^{\alpha b}\right].\end{aligned} \tag{17}$$

The derivatives of the log-likelihood function with respect to the unknown parameters are

$$\begin{aligned}\frac{\partial}{\partial a}\ell(\mathbf{x})&=\frac{n}{a}+\sum_{i=1}^{n}\ln G(x_i)-(\alpha b-1)\sum_{i=1}^{n}\frac{G^a(x_i)\ln G(x_i)}{1-G^a(x_i)}\\&+\alpha b(\beta-1)\sum_{i=1}^{n}\frac{G^a(x_i)\left\{1-G^a(x_i)\right\}^{\alpha b-1}\ln G(x_i)}{1-\left\{1-G^a(x_i)\right\}^{\alpha b}},\end{aligned} \tag{18}$$

$$\frac{\partial}{\partial b}\ell(\mathbf{x})=\frac{n}{b}+\alpha\sum_{i=1}^{n}\ln\left\{1-G^a(x_i)\right\}-\alpha(\beta-1)\sum_{i=1}^{n}\frac{\left\{1-G^a(x_i)\right\}^{\alpha b}\ln\left\{1-G^a(x_i)\right\}}{1-\left\{1-G^a(x_i)\right\}^{\alpha b}}, \tag{19}$$

$$\frac{\partial}{\partial\alpha}\ell(\mathbf{x})=\frac{n}{\alpha}+b\sum_{i=1}^{n}\ln\left\{1-G^{a}(x_i)\right\}-b(\beta-1)\sum_{i=1}^{n}\frac{\left\{1-G^{a}(x_i)\right\}^{\alpha b}\ln\left\{1-G^{a}(x_i)\right\}}{1-\left\{1-G^{a}(x_i)\right\}^{\alpha b}}, \tag{20}$$

$$\frac{\partial}{\partial\beta}\ell(\mathbf{x})=\frac{n}{\beta}+\sum_{i=1}^{n}\ln\left[1-\left\{1-G^{a}(x_i)\right\}^{\alpha b}\right], \tag{21}$$

and

$$\begin{aligned}\frac{\partial}{\partial\theta}\ell(\mathbf{x})&=\sum_{i=1}^{n}\frac{\partial}{\partial\theta}\ln g(x_i)+ab\alpha(\beta-1)\sum_{i=1}^{n}\left[\frac{\left\{1-G^{a}(x_i)\right\}^{\alpha b-1}G^{a-1}(x_i)}{1-\left\{1-G^{a}(x_i)\right\}^{\alpha b}}\right]\frac{\partial}{\partial\theta}G(x_i)\\&-a(\alpha\beta-1)\sum_{i=1}^{n}\left[\frac{G^{a-1}(x_i)}{1-G^{a}(x_i)}\right]\frac{\partial}{\partial\theta}G(x_i)+(a-1)\sum_{i=1}^{n}\frac{1}{G(x_i)}\frac{\partial}{\partial\theta}G(x_i),\end{aligned} \tag{22}$$

where θ is parameter of the baseline distribution $G(x)$. The maximum likelihood estimates are obtained by equating the above derivatives to zero and numerically solving the resulting equations.

3.1.2 Using the Dual Generalized Order Statistics

In this part, we have given the maximum likelihood estimation of the parameters of the *NIKFD* by using the dual generalized order statistics *dgos.* For this, we know that the joint density function of *dgos* some distribution $F(x)$ is given as; see Kamps et al. (2003) and Shahbaz et al. (2016);

$$f_{1,2,\ldots,n:n,m,k}(x_1,\ldots,x_n)=k\left(\prod\nolimits_{j=1}^{n}\gamma_j\right)\left[F(x_n)\right]^{k-1}f(x_n)\prod\nolimits_{i=1}^{n-1}f(x_i)\left[F(x_i)\right]^{m}, \tag{23}$$

where $\gamma_j=k+(n-j)(m+1)$. Using (9) and (10) in (23), the joint density function of *dgos* for the *NIKFD*; which is also the likelihood function for *n dgos*; is

$$\begin{aligned}L_{dgos}(\mathbf{x})&=ka^{n}b^{n}\alpha^{n}\beta^{n}\left(\prod\nolimits_{j=1}^{n}\gamma_j\right)g(x_n)G^{a-1}(x_n)\left[1-G^{a}(x_n)\right]^{\alpha b-1}\left[1-\left\{1-G^{a}(x_n)\right\}^{\alpha b}\right]^{\beta k-1}\\&\times\prod_{i=1}^{n}\left[g(x_i)G^{a-1}(x_i)\left\{1-G^{a}(x_i)\right\}^{\alpha b-1}\left\{1-\left\{1-G^{a}(x_i)\right\}^{\alpha b}\right\}^{\beta(m+1)-1}\right].\end{aligned} \tag{24}$$

The log of likelihood function, (24), is

$$\begin{aligned}\ell_{dgos}(\mathbf{x})&=\ln k+n\ln a+n\ln b+n\ln\alpha+n\ln\beta+\sum\nolimits_{j=1}^{n-1}\ln\gamma_j+\ln g(x_n)+(a-1)\ln G(x_n)+(\alpha b-1)\\&\times\ln\left[1-G^{a}(x_n)\right]+(\beta k-1)\ln\left[1-\left\{1-G^{a}(x_n)\right\}^{\alpha b}\right]+\sum\nolimits_{i=1}^{n-1}\ln g(x_i)+(a-1)\sum\nolimits_{i=1}^{n-1}\ln G(x_i)\\&+(\alpha b-1)\sum\nolimits_{i=1}^{n-1}\ln\left[1-G^{a}(x_i)\right]+\left[\beta(m+1)-1\right]\sum\nolimits_{i=1}^{n-1}\ln\left[1-\left\{1-G^{a}(x_i)\right\}^{\alpha b}\right].\end{aligned} \tag{25}$$

The maximum likelihood estimates of unknown parameters, using *dgos,* are obtained by differentiating the above log-likelihood function with respect to unknown parameters, equating the resulting derivatives to zero and then solving the equations. Now, the derivatives of (25) with respect to the unknown parameters are

$$\begin{aligned}\frac{\partial}{\partial a}\ell_{dgos}(\mathbf{x})&=\frac{n}{a}+\ln G(x_n)+(\alpha b-1)\frac{G^{a}(x_n)\ln G(x_n)}{1-G^{a}(x_n)}+\alpha b(\beta k-1)\\&\times\frac{G^{a}(x_n)\ln G(x_n)\left\{1-G^{a}(x_n)\right\}^{\alpha b-1}}{1-\left\{1-G^{a}(x_n)\right\}^{\alpha b}}+\sum_{i=1}^{n-1}\ln G(x_i)-(\alpha b-1)\sum_{i=1}^{n-1}\frac{G^{a}(x_i)\ln G(x_i)}{1-G^{a}(x_i)}\\&+\alpha b\left[\beta(m+1)-1\right]\sum_{i=1}^{n-1}\frac{G^{a}(x_i)\ln G(x_i)\left\{1-G^{a}(x_i)\right\}^{\alpha b-1}}{1-\left\{1-G^{a}(x_i)\right\}^{\alpha b}},\end{aligned} \tag{26}$$

$$\frac{\partial}{\partial b}\ell_{dgos}(\mathbf{x})=\frac{n}{b}+\alpha\ln\left\{1-G^{a}(x_n)\right\}+\alpha(\beta k-1)\frac{\left\{1-G^{a}(x_n)\right\}^{\alpha b}\ln\left\{1-G^{a}(x_n)\right\}}{1-\left\{1-G^{a}(x_n)\right\}^{\alpha b}}$$
$$+\alpha\sum_{i=1}^{n-1}\ln\left\{1-G^{a}(x_i)\right\}^{\alpha b}+\alpha b\left[\beta(m+1)-1\right]\sum_{i=1}^{n-1}\frac{\left\{1-G^{a}(x_i)\right\}^{\alpha b}\ln\left\{1-G^{a}(x_i)\right\}}{1-\left\{1-G^{a}(x_i)\right\}^{\alpha b}}, \tag{27}$$

$$\frac{\partial}{\partial \alpha}\ell_{dgos}(\mathbf{x})=\frac{n}{\alpha}+b\ln\left\{1-G^{a}(x_n)\right\}-b(\beta k-1)\frac{\left\{1-G^{a}(x_n)\right\}^{\alpha b}\ln\left\{1-G^{a}(x_n)\right\}}{1-\left\{1-G^{a}(x_n)\right\}^{\alpha b}}$$
$$-b\sum_{i=1}^{n-1}\ln\left\{1-G^{a}(x_i)\right\}-b\left[\beta(m+1)-1\right]\sum_{i=1}^{n-1}\frac{\left\{1-G^{a}(x_i)\right\}^{\alpha b}\ln\left\{1-G^{a}(x_i)\right\}}{1-\left\{1-G^{a}(x_i)\right\}^{\alpha b}}, \tag{28}$$

$$\frac{\partial}{\partial \beta}\ell_{dgos}(\mathbf{x})=\frac{n}{\beta}+k\ln\left[1-\left\{1-G^{a}(x_n)\right\}^{\alpha b}\right]+(m+1)\sum_{i=1}^{n-1}\ln\left[1-\left\{1-G^{a}(x_i)\right\}^{\alpha b}\right], \tag{29}$$

and

$$\frac{\partial}{\partial \theta}\ell(\mathbf{x})=\frac{\partial}{\partial \theta}\ln g(x_n)+\frac{(a-1)}{G(x_n)}\frac{\partial}{\partial \theta}G(x_n)-a(\alpha b-1)\frac{G^{a-1}(x_n)}{1-G^{a}(x_n)}\frac{\partial}{\partial \theta}G(x_n)$$
$$+ab\alpha(\beta k-1)\frac{\left\{1-G^{a}(x_n)\right\}^{\alpha b-1}G^{a-1}(x_n)}{1-\left\{1-G^{a}(x_n)\right\}^{\alpha b}}\frac{\partial}{\partial \theta}G(x_n)+\sum_{i=1}^{n-1}\frac{\partial}{\partial \theta}\ln g(x_i)$$
$$+(a-1)\sum_{i=1}^{n-1}\frac{1}{G(x_i)}\frac{\partial}{\partial \theta}G(x_i)-a(\alpha b-1)\sum_{i=1}^{n-1}\frac{G^{a-1}(x_i)}{1-G^{a}(x_i)}\frac{\partial}{\partial \theta}G(x_i)$$
$$+ab\alpha\left[\beta(m+1)-1\right]\sum_{i=1}^{n-1}\left[\frac{\left\{1-G^{a}(x_i)\right\}^{\alpha b-1}G^{a-1}(x_i)}{1-\left\{1-G^{a}(x_i)\right\}^{\alpha b}}\right]\frac{\partial}{\partial \theta}G(x_i), \tag{30}$$

where θ is parameter of the baseline distribution $G(x)$. The maximum likelihood estimates can be obtained by equating the above derivatives to zero and solving the resulting equations.

3.2 The Least Square Estimation

In this part, we have discussed the least square estimation of the parameters of the *NIKFD*. For this suppose that $X_{(1)}\le X_{(2)}\le\cdots\le X_{(n)}$ be the order statistics of a random sample of size n from distribution that has cumulative distribution function $F(x)$. The mean is $E(X)=i/(n+1)$. The least square function is

$$SS(\theta\,|\,\mathbf{x})=\sum_{i=1}^{n}\left[F\left(x_{(i)};\theta\right)-\frac{i}{n+1}\right]^{2}. \tag{31}$$

Using (9) in (31), the least square function for *NIKFD* is

$$S=SS(a,b,\alpha,\beta\,|\,\mathbf{x})=\sum_{i=1}^{n}\left[\left\{1-\left\{1-G^{a}\left(x_{(i)}\right)\right\}^{\alpha b}\right\}^{\beta}-\frac{i}{n+1}\right]^{2}. \tag{32}$$

The derivative of (32) with respect to the unknown parameters are

$$\frac{\partial}{\partial a}S=2b\alpha\beta\sum_{i=1}^{n}\left[G^{a}\left(x_{(i)}\right)\ln G\left(x_{(i)}\right)\left\{1-G^{a}\left(x_{(i)}\right)\right\}\left\{1-\left\{1-G^{a}\left(x_{(i)}\right)\right\}^{\alpha b}\right\}^{\beta-1}\right.$$
$$\left.\left(\left\{1-\left\{1-G^{a}(x)\right\}^{\alpha b}\right\}^{\beta}-\frac{i}{n+1}\right)\right], \tag{33}$$

$$\frac{\partial}{\partial b}S=-2\alpha\beta\sum_{i=1}^{n}\left[\left\{1-G^{a}\left(x_{(i)}\right)\right\}^{\alpha b}\ln\left\{1-G^{a}\left(x_{(i)}\right)\right\}\left\{1-\left\{1-G^{a}\left(x_{i}\right)\right\}^{\alpha b}\right\}^{\beta-1}\right.$$
$$\left.\left(\left\{1-\left\{1-G^{a}\left(x_{(i)}\right)\right\}^{\alpha b}\right\}^{\beta}-\frac{i}{n+1}\right)\right], \tag{34}$$

$$\frac{\partial}{\partial \alpha}S=-2b\beta\sum_{i=1}^{n}\left[\left\{1-G^{a}\left(x_{(i)}\right)\right\}^{\alpha b}\ln\left\{1-G^{a}\left(x_{(i)}\right)\right\}\left\{1-\left\{1-G^{a}\left(x_{(i)}\right)\right\}^{\alpha b}\right\}^{\beta-1}\right.$$
$$\left.\left(\left\{1-\left\{1-G^{a}\left(x\right)\right\}^{\alpha b}\right\}^{\beta}-\frac{i}{n+1}\right)\right], \tag{35}$$

$$\frac{\partial}{\partial \beta}S=2\sum_{i=1}^{n}\left[\ln\left\{1-\left\{1-G^{a}\left(x_{(i)}\right)\right\}^{\alpha b}\right\}\left\{1-\left\{1-G^{a}\left(x_{(i)}\right)\right\}^{\alpha b}\right\}^{\beta}\left(\left\{1-\left\{1-G^{a}\left(x\right)\right\}^{\alpha b}\right\}^{\beta}-\frac{i}{n+1}\right)\right], \tag{36}$$

and

$$\frac{\partial}{\partial \theta}S=2ab\alpha\beta\sum_{i=1}^{n}\left[G^{a-1}\left(x_{(i)}\right)\left\{1-G^{a}\left(x_{(i)}\right)\right\}^{\alpha b-1}\left\{1-\left\{1-G^{a}\left(x_{(i)}\right)\right\}^{\alpha b}\right\}^{\beta-1}\right.$$
$$\left.\left(\left\{1-\left\{1-G^{a}\left(x\right)\right\}^{\alpha b}\right\}^{\beta}-\frac{i}{n+1}\right)\right]\frac{\partial}{\partial \theta}G\left(x_{(i)}\right), \tag{37}$$

The least square estimates of the unknown parameters are obtained by equating the above derivatives to zero and then solving the resulting equations. We will now discuss a member of the *NIKFD* by using the Weibull baseline distribution.

4. The Inverse Kumaraswamy Weibull Distribution and its Properties

In this section we have proposed a new inverse Kumaraswamy Weibull (*NIKW*) distribution by using the following *cdf* and *pdf* of the Weibull distribution as a baseline distribution in the *NIKFD*

$$g\left(x\right)=\frac{c}{\lambda}\left(\frac{x}{\lambda}\right)^{c-1}\mathrm{e}^{-\left(x/\lambda\right)^{c}}\text{ and }G\left(x\right)=1-\mathrm{e}^{-\left(x/\lambda\right)^{c}}\ ;\ x,c,\lambda>0\ . \tag{38}$$

Using (38) in (9), the *cdf* of the *NIKW* distribution is

$$F_{IK-W}\left(x\right)=\left[1-\left\{1-\left\{1-e^{-\left(x/\lambda\right)^{c}}\right\}^{a}\right\}^{\alpha b}\right]^{\beta}\ ;\ x,a,b,\alpha,\beta,\lambda,c>0\ . \tag{39}$$

The density function corresponding to (39) is

$$f_{IK-W}\left(x\right)=ab\alpha\beta\frac{c}{\lambda}\left(\frac{x}{\lambda}\right)^{c-1}\mathrm{e}^{-\left(x/\lambda\right)^{c}}\left[1-\mathrm{e}^{-\left(x/\lambda\right)^{c}}\right]^{a-1}\left[1-\left\{1-\mathrm{e}^{-\left(x/\lambda\right)^{c}}\right\}^{a}\right]^{\alpha b-1}\left[1-\left\{1-\left\{1-e^{-\left(x/\lambda\right)^{c}}\right\}^{a}\right\}^{\alpha b}\right]^{\beta-1}$$
$$=ab\alpha\beta\frac{c}{\lambda}\left(\frac{x}{\lambda}\right)^{c-1}\mathrm{e}^{-\left(x/\lambda\right)^{c}}\Delta_{1}^{a-1}\left(x\right)\left\{1-\Delta_{1}^{a}\left(x\right)\right\}^{\alpha b-1}\Delta_{3}^{\beta-1}\left(x\right),$$

or

$$f_{IK-W}\left(x\right)=ab\alpha\beta\left(c/\lambda\right)\left(x/\lambda\right)^{c-1}\mathrm{e}^{-\left(x/\lambda\right)^{c}}\Delta_{1}^{a-1}\left(x\right)\Delta_{2}^{\alpha b-1}\left(x\right)\Delta_{3}^{\beta}\left(x\right)\ ;\ x>0,\ \boldsymbol{\theta}=\left(a,b,\alpha,\beta,c,\lambda\right)>0\ , \tag{40}$$

where $\Delta_{1}\left(x\right)=\left\{1-\mathrm{e}^{-\left(x/\lambda\right)^{c}}\right\}$; $\Delta_{2}\left(x\right)=1-\Delta_{1}^{a}\left(x\right)$ and $\Delta_{3}\left(x\right)=1-\left\{1-\left\{1-e^{-\left(x/\lambda\right)^{c}}\right\}^{a}\right\}^{\alpha b}=1-\Delta_{2}^{\alpha b}\left(x\right)$. The density and distribution functions of the *NIKW* distribution can also be written as the weighted sums of the density and distribution functions of the exponentiated-Weibull distribution as

$$F_{IK-W}\left(x\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}w_{nj}\Delta_{1}^{aj}\left(x\right),$$

and

$$f_{IK-W}(x)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}aj\,w_{nj}\left[(x/\lambda)^{c-1}\,\mathrm{e}^{-(x/\lambda)^c}\,\Delta_1^{aj-1}(x)\right],$$

where w_{nj} is defined earlier. The above representations can be used to study the properties of the *NIKW* distribution by using the properties of exponentiated-Weibull distribution. We will, now, discuss certain properties of the *NIKW* distribution in the following sections.

4.1 The Hazard Rate Function

The hazard rate function of the *NIKW* distribution is given as

$$h(x)=\left[ab\alpha\beta(c/\lambda)(x/\lambda)^{c-1}\,\mathrm{e}^{-(x/\lambda)^c}\,\Delta_1^{a-1}(x)\Delta_2^{\alpha b-1}(x)\Delta_3^{\beta}(x)\right]\Big/\left[1-\Delta_3^{\beta}(x)\right]. \tag{41}$$

The hazard rate function can be plotted for different combinations of the parameters.

4.2 The Moments and Moment Generating Function

The rth moment of the *NIKW* distribution is

$$\mu_r^{/}=\int_0^{\infty}x^r f_{IK-W}(x)\,dx=\int_0^{\infty}x^r ab\alpha\beta(c/\lambda)(x/\lambda)^{c-1}\,\mathrm{e}^{-(x/\lambda)^c}\,\Delta_1^{a-1}(x)\Delta_2^{\alpha b-1}(x)\Delta_3^{\beta}(x)\,dx. \tag{42}$$

The rth moment can also be computed as

$$\mu_r^{/}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}aj\,w_{nj}\left[\int_0^{\infty}x^r(x/\lambda)^{c-1}\,\mathrm{e}^{-(x/\lambda)^c}\,\Delta_1^{aj-1}(x)\,dx\right]. \tag{43}$$

The moment generating function can also be easily written as

$$M_X(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}aj\,w_{nj}\left[\int_0^{\infty}\mathrm{e}^{tx}(x/\lambda)^{c-1}\,\mathrm{e}^{-(x/\lambda)^c}\,\Delta_1^{aj-1}(x)\,dx\right]. \tag{44}$$

The moments can be numerically computed by using (42) or (43).

4.3 The Shannon Entropy

The Shannon entropy for the *NIKW* distribution is given as

$$\begin{aligned}H(x)=&-\int_0^{\infty}ab\alpha\beta(c/\lambda)(x/\lambda)^{c-1}\,\mathrm{e}^{-(x/\lambda)^c}\,\Delta_1^{a-1}(x)\Delta_2^{\alpha b-1}(x)\Delta_3^{\beta-1}(x)\big[\ln a+\ln b\\&+\ln\alpha+\ln\beta+\ln(c/\lambda)+(c-1)\ln(x/\lambda)-(x/\lambda)^c+(a-1)\ln\Delta_1(x)\\&+(\alpha b-1)\ln\Delta_2(x)+(\beta-1)\ln\Delta_3(x)\big]dx.\end{aligned} \tag{45}$$

The Shannon entropy can be numerically computed for given values of the parameters.

5. Parameter Estimation for the New Inverse Kumaraswamy Weibull Distribution

In the following, we will discuss estimation of the parameters of the *NIKW* distribution. We have discussed maximum likelihood and least square estimation of the parameters.

5.1 The Maximum Likelihood Estimation

In the following, the maximum likelihood estimation of the parameters of the *NIKW* distribution is discussed. The maximum likelihood estimation has been discussed for random sample and for *dgos*.

5.1.1 Using the Random Sample

In this part, we have given the maximum likelihood estimation of the parameters of *NIKW* distribution. For this, suppose $X_1, X_2, \ldots, X_n$ be a random sample of size n from the *NIKW* distribution. By using (39) and (40) in (17), the log-likelihood function

$$\begin{aligned}\ell(\mathbf{x})=\ln L(\mathbf{x})=&\,n\ln a+n\ln b+n\ln\alpha+n\ln\beta+n\ln(c/\lambda)+(c-1)\sum_{i=1}^{n}\ln(x_i/\lambda)-(x/\lambda)^c\\&+(a-1)\sum_{i=1}^{n}\Delta_1(x_i)+(\alpha b-1)\sum_{i=1}^{n}\ln\Delta_2(x_i)+(\beta-1)\sum_{i=1}^{n}\ln\Delta_3(x_i).\end{aligned} \tag{45}$$

The derivatives of the log-likelihood function with respect to the unknown parameters are

$$\frac{\partial}{\partial a}\ell(\mathbf{x})=\frac{n}{a}+\sum_{i=1}^{n}\ln\Delta_1(x_i)-\sum_{i=1}^{n}\Delta_1^{a}(x_i)\ln\Delta_1(x_i)\left[\frac{(\alpha b-1)}{\Delta_2(x_i)}-\frac{\alpha b(\beta-1)\Delta_2^{\alpha b-1}(x_i)}{\Delta_3(x_i)}\right], \tag{46}$$

$$\frac{\partial}{\partial b}\ell(\mathbf{x})=\frac{n}{b}+\alpha\sum_{i=1}^{n}\ln\Delta_2(x_i)\left[1-\frac{(\beta-1)\Delta_2^{\alpha b}(x_i)}{\Delta_3(x_i)}\right], \tag{47}$$

$$\frac{\partial}{\partial \alpha}\ell(\mathbf{x})=\frac{n}{\alpha}+b\sum_{i=1}^{n}\ln\Delta_2(x_i)\left[1-\frac{(\beta-1)\Delta_2^{\alpha b}(x_i)}{\Delta_3(x_i)}\right], \tag{48}$$

$$\frac{\partial}{\partial \beta}\ell(\mathbf{x})=\frac{n}{\beta}+\sum_{i=1}^{n}\ln\Delta_3(x_i). \tag{49}$$

The derivatives of the *pdf* and *cdf* of the Weibull distribution with respect to λ and c are

$$\frac{\partial}{\partial c}\ln g(x_i)=\frac{1}{c}+\ln\left(\frac{x_i}{\lambda}\right)\left\{1-\left(\frac{x_i}{\lambda}\right)^{c}\right\};\ \frac{\partial}{\partial c}G(x_i)=\left(\frac{x_i}{\lambda}\right)^{c}\ln\left(\frac{x_i}{\lambda}\right)e^{-(x_i/\lambda)^{c}}$$

and $$\frac{\partial}{\partial \lambda}\ln g(x_i)=-\frac{c}{\lambda}+\left(\frac{c}{\lambda}\right)\left(\frac{x_i}{\lambda}\right)^{c};\ \frac{\partial}{\partial \lambda}G(x_i)=-\frac{c}{\lambda}\left(\frac{x_i}{\lambda}\right)^{c}e^{-(x_i/\lambda)^{c}}.$$

Using above, the derivatives of the log-likelihood function with respect to c and λ are

$$\begin{aligned}\frac{\partial}{\partial c}\ell(\mathbf{x})=&\frac{n}{c}+\sum_{i=1}^{n}\ln\left(\frac{x_i}{\lambda}\right)\left\{1-\left(\frac{x_i}{\lambda}\right)^{c}\right\}+a\sum_{i=1}^{n}\Delta_1^{a-1}(x_i)\left\{\left(\frac{x_i}{\lambda}\right)^{c}\ln\left(\frac{x_i}{\lambda}\right)e^{-(x/\lambda)^{c}}\right\}\\&\times\left[\frac{b\alpha(\beta-1)\Delta_2^{\alpha b-1}(x_i)}{\Delta_3(x_i)}-\frac{(\alpha\beta-1)}{\Delta_2(x_i)}\right]+(a-1)\sum_{i=1}^{n}\frac{1}{\Delta_1(x_i)}\left\{\left(\frac{x_i}{\lambda}\right)^{c}\ln\left(\frac{x_i}{\lambda}\right)e^{-(x/\lambda)^{c}}\right\},\end{aligned} \tag{50}$$

and

$$\begin{aligned}\frac{\partial}{\partial \lambda}\ell(\mathbf{x})=&-\frac{nc}{\lambda}+\left(\frac{c}{\lambda}\right)\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{c}-\frac{ac}{\lambda}\sum_{i=1}^{n}\Delta_1^{a-1}(x_i)\left\{\left(\frac{x_i}{\lambda}\right)^{c}e^{-(x/\lambda)^{c}}\right\}\left[\frac{b\alpha(\beta-1)\Delta_2^{\alpha b-1}(x_i)}{\Delta_3(x_i)}-\frac{(\alpha\beta-1)}{\Delta_2(x_i)}\right]\\&-\frac{c(a-1)}{\lambda}\sum_{i=1}^{n}\frac{1}{\Delta_1(x_i)}\left\{\left(\frac{x_i}{\lambda}\right)^{c}e^{-(x/\lambda)^{c}}\right\}\end{aligned} \tag{51}$$

The maximum likelihood estimates are obtained by equating the above derivatives to zero and numerically solving the resulting equations.

5.1.2 Using the Dual Generalized Order Statistics

In this section, we have discussed the maximum likelihood estimation of the parameters of the *NIKW* distribution by using the *dgos*. For this, suppose that $X_{1:n,m,k}\le X_{2:n,m,k}\le\cdots\le X_{n:n,m,k}$ be a random sample of n *dgos* from the *NIKW* distribution. The log-likelihood function in the basis of *dgos* is
The log of likelihood function, (24), is

$$\begin{aligned}\ell_{dgos}(\mathbf{x})=&\ln k+n\ln a+n\ln b+n\ln\alpha+n\ln\beta+\sum\nolimits_{j=1}^{n-1}\ln\gamma_j+\ln(c/\lambda)+(c-1)\ln(x_n/\lambda)-(x_n/\lambda)^{c}\\&+(a-1)\Delta_1(x_n)+(\alpha b-1)\ln\Delta_2(x_n)+(\beta k-1)\ln\Delta_3(x_n)+\sum\nolimits_{i=1}^{n-1}\ln\left\{\frac{c}{\lambda}\left(\frac{x_i}{\lambda}\right)^{c-1}e^{-(x_i/\lambda)^{c}}\right\}\\&+(a-1)\sum\nolimits_{i=1}^{n-1}\ln\Delta_1(x_i)+(\alpha b-1)\sum\nolimits_{i=1}^{n-1}\ln\Delta_2(x_i)+\left[\beta(m+1)-1\right]\sum\nolimits_{i=1}^{n-1}\ln\Delta_3(x_i).\end{aligned} \tag{52}$$

The maximum likelihood estimates of unknown parameters, using *dgos*, are obtained by differentiating the above log-likelihood function with respect to unknown parameters, equating the resulting derivatives to zero and then solving the equations. Now, the derivatives of (52) with respect to the unknown parameters are

$$\frac{\partial}{\partial a}\ell_{dgos}(\mathbf{x})=\frac{n}{a}+\ln\Delta_1(x_n)\left[1+\Delta_1^a(x_n)\left\{\frac{(\alpha b-1)}{\Delta_2(x_n)}+\frac{\alpha b(\beta k-1)\Delta_2^{\alpha b-1}(x_n)}{\Delta_3(x_n)}\right\}\right]+\sum_{i=1}^{n-1}\ln\Delta_1(x_i)$$
$$-\sum_{i=1}^{n-1}\ln\Delta_1(x_i)\left[\Delta_1^a(x_i)\left\{\frac{(\alpha b-1)}{\Delta_2(x_i)}-\frac{\alpha b\{\beta(m+1)-1\}\Delta_2^{\alpha b-1}(x_i)}{\Delta_3(x_i)}\right\}\right], \tag{53}$$

$$\frac{\partial}{\partial b}\ell_{dgos}(\mathbf{x})=\frac{n}{b}+\alpha\ln\Delta_2(x_n)\left\{1+\frac{(\beta k-1)\Delta_2^{\alpha b}(x_n)}{\Delta_3(x_n)}\right\}+\alpha\sum_{i=1}^{n-1}\ln\Delta_2^{\alpha b}(x_i)$$
$$+\alpha b\left[\beta(m+1)-1\right]\sum_{i=1}^{n-1}\frac{\Delta_2^{\alpha b}(x_i)\ln\Delta_2(x_i)}{\Delta_3(x_i)}, \tag{54}$$

$$\frac{\partial}{\partial \alpha}\ell_{dgos}(\mathbf{x})=\frac{n}{\alpha}+b\ln\Delta_2(x_n)\left\{1-\frac{(\beta k-1)\Delta_2^{\alpha b}(x_n)}{\Delta_3(x_n)}\right\}-b\sum_{i=1}^{n-1}\ln\Delta_2(x_i)\left[1-\{\beta(m+1)-1\}\frac{\Delta_2^{\alpha b}(x_i)}{\Delta_3(x_i)}\right], \tag{55}$$

$$\frac{\partial}{\partial \beta}\ell_{dgos}(\mathbf{x})=\frac{n}{\beta}+k\ln\Delta_3(x_n)+(m+1)\sum_{i=1}^{n-1}\ln\Delta_3(x_i), \tag{56}$$

$$\frac{\partial}{\partial c}\ell(\mathbf{x})=\frac{1}{c}+\ln\left(\frac{x_n}{\lambda}\right)\left\{1-\left(\frac{x_n}{\lambda}\right)^c\right\}+\left\{\left(\frac{x_n}{\lambda}\right)^c\ln\left(\frac{x_n}{\lambda}\right)e^{-(x_n/\lambda)^c}\right\}\left\{\frac{(a-1)}{\Delta_1(x_n)}-\frac{a(\alpha b-1)\Delta_1^{a-1}(x_n)}{\Delta_2(x_n)}\right.$$
$$\left.+\frac{ab\alpha(\beta k-1)\Delta_2^{\alpha b-1}(x_n)\Delta_1^{a-1}(x_n)}{\Delta_3(x_n)}\right\}+\frac{(n-1)}{c}\sum_{i=1}^{n-1}\left[\ln\left(\frac{x_i}{\lambda}\right)\left\{1-\left(\frac{x_n}{\lambda}\right)^c\right\}\right]$$
$$+\sum_{i=1}^{n-1}\left\{\left(\frac{x_i}{\lambda}\right)^c\ln\left(\frac{x_i}{\lambda}\right)e^{-(x_i/\lambda)^c}\right\}\left[\frac{(a-1)}{\Delta_1(x_i)}-\frac{a(\alpha b-1)\Delta_1^{a-1}(x_i)}{\Delta_2(x_i)}\right.$$
$$\left.+\frac{ab\alpha\{\beta(m+1)-1\}\Delta_2^{\alpha b-1}(x_i)\Delta_1^{a-1}(x_i)}{\Delta_3(x_i)}\right] \tag{57}$$

and

$$\frac{\partial}{\partial \lambda}\ell_{dgos}(\mathbf{x})=-\frac{nc}{\lambda}+\left(\frac{c}{\lambda}\right)\left(\frac{x_n}{\lambda}\right)^c-\frac{c}{\lambda}\left\{\left(\frac{x_n}{\lambda}\right)^c e^{-(x_n/\lambda)^c}\right\}\left[\frac{a-1}{\Delta_1(x_n)}-\frac{a(\alpha b-1)\Delta_1^{a-1}(x_n)}{\Delta_2(x_n)}\right.$$
$$\left.-\frac{a\alpha b(\beta k-1)\Delta_1^{a-1}(x_n)\Delta_2^{\alpha b-1}(x_n)}{\Delta_3(x_n)}\right]+\left(\frac{c}{\lambda}\right)\sum_{i=1}^{n-1}\left(\frac{x_i}{\lambda}\right)^c+\sum_{i=1}^{n-1}\left\{\left(\frac{x_i}{\lambda}\right)^c e^{-(x_i/\lambda)^c}\right\}$$
$$\left[\frac{a-1}{\Delta_1(x_n)}-\frac{a(\alpha b-1)\Delta_1^{a-1}(x_i)}{\Delta_2(x_i)}-\frac{a\alpha b(\beta k-1)\Delta_1^{a-1}(x_i)\Delta_2^{\alpha b-1}(x_i)}{\Delta_3(x_i)}\right]. \tag{58}$$

The maximum likelihood estimates can be obtained by equating the above derivatives to zero and solving the resulting equations.

5.2 The Least Square Estimation

In this part, we have discussed the least square estimation of the parameters of the *NIKW* distribution. For this suppose that $X_{(1)}\le X_{(2)}\le\cdots\le X_{(n)}$ be the order statistics of a random sample of size n from distribution that has cumulative distribution function $F(x)$. The mean is $E(X)=i/(n+1)$. The least square function is

$$SS(\theta\,|\,\mathbf{x})=\sum_{i=1}^{n}\left[F\left(x_{(i)};\theta\right)-\frac{i}{n+1}\right]^2. \tag{59}$$

Using (9) in (31), the least square function for the *NIKW* distribution is

$$S=SS(a,b,\alpha,\beta\,|\,\mathbf{x})=\sum_{i=1}^{n}\left[\Delta_3^{\beta}\left(x_{(i)}\right)-\frac{i}{n+1}\right]^2. \tag{60}$$

The derivative of (32) with respect to the unknown parameters are

$$\frac{\partial}{\partial a}S = 2b\alpha\beta\sum_{i=1}^{n}\left[\Delta_1^{a}\left(x_{(i)}\right)\Delta_2\left(x_{(i)}\right)\ln\Delta_1\left(x_{(i)}\right)\Delta_3^{\beta-1}\left(x_{(i)}\right)\left(\Delta_3^{\beta}\left(x_{(i)}\right)-\frac{i}{n+1}\right)\right], \tag{61}$$

$$\frac{\partial}{\partial b}S = -2\alpha\beta\sum_{i=1}^{n}\left[\Delta_2^{\alpha b}\left(x_{(i)}\right)\Delta_3^{\beta-1}\left(x_{(i)}\right)\ln\Delta_2\left(x_i\right)\left\{\Delta_3^{\beta}\left(x_{(i)}\right)-\frac{i}{n+1}\right\}\right] \tag{62}$$

$$\frac{\partial}{\partial \alpha}S = -2b\beta\sum_{i=1}^{n}\left[\Delta_2^{\alpha b}\left(x_{(i)}\right)\Delta_3^{\beta-1}\left(x_{(i)}\right)\ln\Delta_2\left(x_{(i)}\right)\left\{\Delta_3^{\beta}\left(x_{(i)}\right)-\frac{i}{n+1}\right\}\right] \tag{63}$$

$$\frac{\partial}{\partial \beta}S = 2\sum_{i=1}^{n}\left[\Delta_3^{\beta}\left(x_{(i)}\right)\ln\Delta_3\left(x_{(i)}\right)\left\{\Delta_3^{\beta}\left(x_{(i)}\right)-\frac{i}{n+1}\right\}\right] \tag{64}$$

$$\frac{\partial}{\partial c}S = 2ab\alpha\beta\sum_{i=1}^{n}\left[\Delta_1^{a-1}\left(x_{(i)}\right)\Delta_2^{\alpha b-1}\left(x_{(i)}\right)\Delta_3^{\beta-1}\left(x_{(i)}\right)\left\{\Delta_3^{\beta}\left(x_{(i)}\right)-\frac{i}{n+1}\right\}\right] \times\left(x_{(i)}/\lambda\right)^{c}\ln\left(x_{(i)}/\lambda\right)\exp\left[-\left(x_{(i)}/\lambda\right)^{c}\right], \tag{65}$$

and

$$\frac{\partial}{\partial c}S = -2\frac{ab\alpha\beta}{\lambda}\sum_{i=1}^{n}\left[\Delta_1^{a-1}\left(x_{(i)}\right)\Delta_2^{\alpha b-1}\left(x_{(i)}\right)\Delta_3^{\beta-1}\left(x_{(i)}\right)\left\{\Delta_3^{\beta}\left(x_{(i)}\right)-\frac{i}{n+1}\right\}\right] \times\left(x_{(i)}/\lambda\right)^{c}\exp\left[-\left(x_{(i)}/\lambda\right)^{c}\right], \tag{66}$$

The least square estimates of the unknown parameters are obtained by equating the above derivatives to zero and then solving the resulting equations.

6. Simulation

In this section, the simulation study is given to see the performance of maximum likelihood estimation. The simulation study is conducted by draw random samples of sizes 50, 100, 150, 200, and 250 from the *NIKW* distribution by using specific values of the parameters.

The maximum likelihood estimated of the parameters for different sample sizes are computed. The procedure is repeated for 10000 times for 3 combinations of fixed parameters which are written at the top of each table. The average value of the maximum likelihood estimates of the parameters and the mean square errors (*MSE*'s) are computed which are given in Table 2 below.

Table 2: Average Estimates and Mean Square Error for *NIKW* Distribution

n		***a = 0.5***	**α = 1.5**	***b = 1.5***	**β = 0.5**	***c = 0.5***	**λ = 1**
50	Estimate	0.467	1.578	1.575	0.635	0.795	1.114
	MSE	0.203	0.493	0.479	0.583	0.731	0.705
100	Estimate	0.477	1.551	1.546	0.556	0.631	1.136
	MSE	0.162	0.372	0.377	0.319	0.344	0.627
150	Estimate	0.479	1.512	1.511	0.537	0.582	1.052
	MSE	0.139	0.278	0.281	0.231	0.231	0.475
200	Estimate	0.488	1.506	1.512	0.523	0.555	1.036
	MSE	0.123	0.236	0.245	0.161	0.174	0.407
250	Estimate	0.495	1.516	1.516	0.512	0.542	1.048
	MSE	0.105	0.227	0.237	0.14	0.149	0.395
n		***a = 1.5***	**α = 2.5**	***b = 0.5***	**β = 1.5**	***c = 0.5***	**λ = 1**
50	Estimate	1.656	2.535	0.554	1.941	0.61	0.992
	MSE	0.722	0.647	0.37	1.637	0.596	0.832
100	Estimate	1.579	2.551	0.562	1.696	0.532	1.11
	MSE	0.485	0.636	0.241	0.857	0.212	0.639
150	Estimate	1.566	2.505	0.511	1.609	0.524	0.941
	MSE	0.422	0.588	0.226	0.596	0.144	0.495

200	Estimate MSE	1.557 0.385	2.519 0.537	0.504 0.198	1.58 0.473	0.517 0.124	0.941 0.417
250	Estimate MSE	1.531 0.35	2.576 0.476	0.508 0.176	1.586 0.395	0.51 0.109	1.001 0.387
n		***a = 2.5***	**α = 0.5**	***b = 1.5***	**β = 0.5**	***c = 0.5***	**λ = 1**
50	Estimate MSE	2.694 1.295	0.441 0.353	1.248 0.604	0.505 0.309	0.696 0.381	1.12 1.126
100	Estimate MSE	2.587 0.85	0.495 0.299	1.313 0.622	0.5 0.176	0.604 0.216	1.122 0.946
150	Estimate MSE	2.554 0.672	0.502 0.272	1.345 0.58	0.501 0.141	0.57 0.168	1.047 0.781
200	Estimate MSE	2.562 0.617	0.495 0.262	1.405 0.559	0.496 0.12	0.556 0.144	1.042 0.732
250	Estimate MSE	2.562 0.551	0.459 0.218	1.474 0.563	0.488 0.103	0.552 0.132	1.018 0.674

The results show that the estimated values are very close to the fixed parameters which are used in this simulation. The consistency of the estimates can be seen with a decrease in *MSE*'s of the estimates as the sample size increases.

7. Data Analysis

In the following, two real data sets have been used to see the applicability of the *NIKW* distribution. The following probability distributions are also fitted on the data: Exponential (EXP), Rayleigh (RAY), Weibull (WEI), Exponentiated Exponential (EEXP), Exponentiated Rayleigh (ERAY), Exponentiated Weibull (EWEI), and the IK-W distributions. To compare these fitted models the following goodness-of-fit criteria are considered: the Kolmogorov–Smirnov (KS) test with its P-value, the negative log-likelihood ($-\ell$), the Akaike information criterion (AIC) (Akaike, 1998), and the Bayesian information criterion (BIC) (Schwarz, 1978).

7.1 Monthly Meters on Unemployment Insurance Data

The first data set contains 58 observations of monthly metrics on unemployment insurance in millions of dollars from July 2008 to April 2013 as reported by the Department of Labor, Licensing and Regulation. The data set has been recently used by Afify et al. (2022). The outcomes of the data analysis are reported in Table 3.

Table 3: Parameter Estimates and Goodness of Fit Criterion for Different Distributions

Distribution	EXP	RAY	WEI	EEXP	ERAY	EWEI	IK-W
a	-	-	-	6.503	1.747	0.628	2.707
α	-	-	-	-	-	-	0.165
b	-	-	-	-	-	-	1.836
β	-	-	-	-	-	-	0.1556
c	-	-	2.948	-	-	3.918	5.664
λ	46.934	50.082	52.527	18.970	42.552	59.370	56.693
$-\ell$	281.228	253.920	248.224	254.674	249.801	247.790	247.647
KS	0.30099	0.19465	0.10594	0.16721	0.13693	0.09277	0.08894
p-value	0.00005	0.02468	0.53311	0.07808	0.22685	0.70031	0.74858
AIC	564.455	509.839	500.447	513.348	503.602	501.579	507.294
BIC	566.516	511.899	504.568	517.469	507.722	507.761	519.657

Clearly, the table above indicates that the *NIKW* distribution outperformed all models in terms of the *KS* test and the log-likelihood; nevertheless, the Weibull distribution provided a better fit for the data in terms of information-based criteria since it has two parameters.

7.2 Monthly Actual Taxes Revenues

The second data set contains 59 observations of the monthly actual taxes revenue of Egypt in 1000 million Egyptian pounds from January 2006 to November 2010 considered by Nassar and Nada (2011). The data analysis results are summarized in Table 4.

Table 4: Parameter Estimates and Goodness of Fit Criterion for Different Distributions

Distribution	EXP	RAY	WEI	EEXP	ERAY	EWEI	IK-W
a	-	-	-	5.526	1.031	57.154	11.953
α	-	-	-	-	-	-	2.715
b	-	-	-	-	-	-	0.375
β	-	-	-	-	-	-	20.122
c	-	-	1.840	-	-	0.517	0.390
λ	13.488	15.673	15.306	5.599	15.512	0.642	0.125
$-\ell$	212.507	197.711	197.291	191.224	197.696	188.714	188.362
KS	0.30345	0.17206	0.14316	0.11489	0.17635	0.07352	0.06452
p-value	0.00004	0.06080	0.17798	0.41740	0.05097	0.90732	0.96670
AIC	427.014	397.422	398.581	386.447	399.393	383.428	388.725
BIC	429.091	399.499	402.736	390.602	403.548	389.661	401.190

Again, the above table shows that the *NIKW* distribution outperformed all models in terms of the *KS* test and the log-likelihood; however, it did not outperform the exponentiated exponential distribution in terms of information-based criteria since the former probability distribution has six parameters while the latter probability distribution has two parameters.

8. Conclusion

In this paper, we have developed a new family of inverse Kumaraswamy distribution and have investigated some of its statistical properties including hazard rate function, moment, moment generating function, and Shannon entropy. we have studied the estimation of parameters of the new inverse Kumaraswamy family of distribution in various ways, such as maximum likelihood estimation, maximum likelihood estimation by dual generalized order statistics, and least square estimation. we have introduced inverse Kumaraswamy-Weibull distribution and have explored some of its statistical properties including hazard rate function, moment, moment generating function, and Shannon entropy. we have studied the estimation of parameters of the inverse Kumaraswamy-Weibull distribution in a variety of ways, such as maximum likelihood estimation, maximum likelihood estimation by dual generalized order statistics, and least square estimation. we have also conducted a simulation study to see the performance of the estimation procedure and have shown that the estimated parameters are very close to the fixed parameters which are used in this simulation. Finally, two real data sets were analyzed and the analyses showed that the *NIKW* distribution outperformed its well-known sub-models in terms of the *KS* test and the log-likelihood. There are many research directions that need to be addressed for the model. An important research direction that is currently being considered is estimating the model parameters using Bayesian theory and comparing the resulting estimators to their *MLE* counterparts.

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