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Multimodal Alpha Skew Normal Distribution: A New Distribution to Model Skewed Multimodal Observations

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Abstract

Multimodal alpha skew normal (MMASN) distribution is proposed for modelling skewed observations in the presence of multiple modality at arbitrary points. To this end the multimodal skew normal distribution of Chakraborty et al. (2015) is extended by integrating it with alpha skew normal distribution of Elal-Olivero (2010). Cumulative distribution function (cdf), moments, skewness and kurtosis of the proposed distribution are derived in compact form. The data modelling ability of the proposed distribution is checked by considering three multimodal data sets from literature in comparison to some nested and known distributions. Akaike Information Criterion (AIC) and the likelihood ratio (LR) test, both clearly favored proposed model over its nested models as expected.

Key Words: Skew Distribution, Alpha Skew Distribution, Multimodal Skew Normal Distribution, AIC, LR Test.

Mathematical Subject Classification: 60E05, 62H10, 62H12

1. Introduction

Azzalini (1985) introduced skew-normal (SN) distribution, as a natural extension of normal distribution by inducing an additional skewness parameter $\lambda \in R$ with probability density function (pdf) given by

$$f_Z(z;\lambda) = 2\,\phi(z)\,\Phi(\lambda z);\ -\infty < z < \infty \tag{1}$$

where $\varphi(.)$ and $\Phi(.)$ are the pdf and cdf of standard normal distribution respectively. A lot of extensions and generalizations of this distribution were proposed and studied (for details see Chakraborty and Hazarika, 2011, Ali et al. (2008), Arnold and Beaver (2000), Shah et al. (2022), Shah et al. (2023a), Shah et al. (2023b) and among others). We focus on two extensions namely the alpha skew normal (ASN) distribution of Elal-Olivero (2010) and the multimodal skew normal (*MMSN*) distribution of Chakraborty et al. (2015).

A random variable Z is said to follow alpha skew normal distribution of Elal-Olivero (2010) if its density function is given by

$$f(z;\alpha) = \left(\frac{(1-\alpha z)^2 + 1}{2+\alpha^2}\right)\varphi(z);\ z \in R. \tag{2}$$

This distribution denoted by $ASN(\alpha)$ can be bimodal unlike the skew normal distribution which is unimodal.

A random variable Z is said to be a multimodal skew normal distribution (Chakraborty et al., 2015), denoted by $MMSN(\lambda,\delta)$ if its density function is given by

$$f(z;\lambda,\delta)=\left(1+\frac{Sin(\lambda z)}{\delta}\right)\varphi(z),\ \delta\geq 1,\ z,\lambda\in R. \quad (3)$$

This distribution can be multimodal unlike the $ASN(\alpha)$ and thus can overcomes the limitations of both skew normal and alpha skew normal distributions with respect to multiple modality.

Intent we propose a new extension which includes both the ASN and MMSN as particular case and call it the multimodal alpha skew normal (MMASN) distribution. This new distribution is derived by integrating the ideas of ASN and MMSN to effectively overcomes the limitation of the periodic nature of MMSN's multiple modes as well as the inability of ASN to cater for more than two modes effectively. The prime motivation behind developing the MMASN is to do away with the periodic multimodality of MMSN to offer multiple modes at arbitrary locations. This distribution is shown to be very flexible to support unimodality, bimodality as well as multimodality and thus provides improved fit as compared to its sub-models and some other distributions while dealing with multimodal data.

To understand the relevance of the issue raised in the last paragraph let us look at a real life data. This data set provides the Oits IQ Scores for 52 Non-White males. It is easy to see from Figure 1 that this data is multimodal with 6 modes. Naturally the ASN which is a bimodal distribution fails to adjust the data well as can be seen if Figure 2. We then tried to fit the MMSN which is supposedly multimodal with an inherent periodic nature of modality. This distribution also fails to adjust the data correctly as apparent from Figure 3. This provided us a motivation for proposing a new distribution which can cater to multimodality at arbitrary location to fit such data sets. We shall discuss the three such multimodal data fitting examples in Section 5 by considering three data sets including the above to establish the benefit of the proposed distribution.

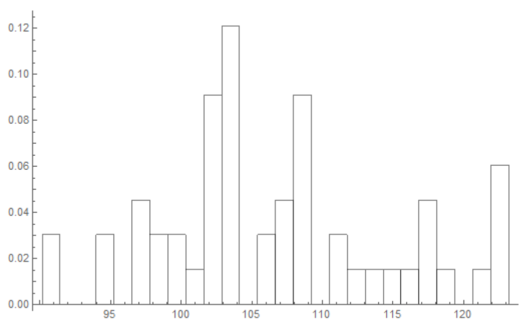


Figure 1. Histogram of the data

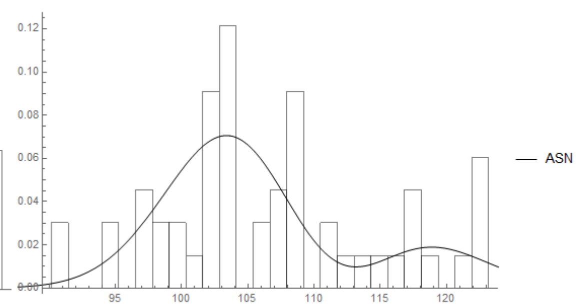


Figure 2. Histogram and fitted ASN of the data

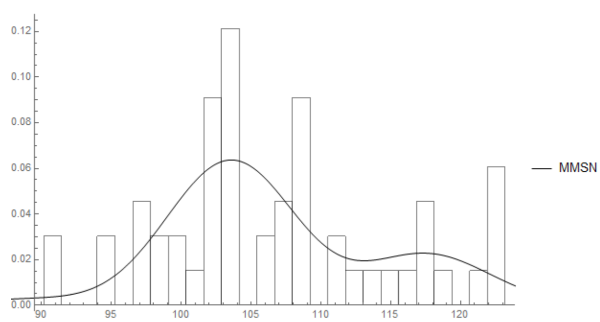


Figure 3. Histogram and fitted MMSN of the data

The article is summarized as follows: In the next Section, we define the proposed distribution and identify its basic properties like cdf, moments, skewness and kurtosis etc. The characterizations of the MMASN distribution via a simple relationship between two truncated moments are given in section 3. In section 4, the parameter estimation and a simulation study has been conducted. The real life comparative data modelling of the proposed distribution and LR test among the nested models are provided in Section 5. The article ends with concluding remarks in Section 6.

2. Multimodal Alpha Skew Normal Distribution

In this section we introduce a new form of multimodal skewed extension of skew normal distribution and investigate some of its basic properties.

If an r.v. Z has a density function

$$f(z;\alpha,\lambda,\delta)=\frac{1}{C(\alpha,\lambda,\delta)}\left(1+\frac{\text{Sin}(\lambda z)}{\delta}\right)[(1-\alpha z)^2+1]\phi(z)\,,\delta\geq 1\,,\, z,\alpha,\lambda\in R \tag{4}$$

where $C(\alpha,\lambda,\delta)=2+\alpha^2-2\alpha\lambda e^{-\lambda^2/2}/\delta$, then it is said to be *MMASN* distribution with parameters α, λ and δ, denoted by $MMASN(\alpha,\lambda,\delta)$. The normalizing constant $C(\alpha,\lambda,\delta)$ is easily obtained as shown in Appendix A. In particular, $\alpha=0$, results in the standard $MMSN(\lambda,\delta)$ distribution of Chakraborty et al. (2015) as given in (3), for $\lambda=0$, it reduces to the $ASN(\alpha)$ distribution of Elal-Olivero (2010) as given in (2) and for $\alpha=\lambda=0$, it reduces to the standard normal distribution given by $f(z)=\varphi(z)$. Also if $\delta\to\pm\infty$, then $MMASN(\alpha,\lambda,\delta)\to ASN(\alpha)$ and $-Z\sim MMASN(-\alpha,-\lambda,\delta)$.

The pdf of $MMASN(\alpha,\lambda,\delta)$ distribution for different choices of the parameters α, λ and δ are plotted in Figure 5.

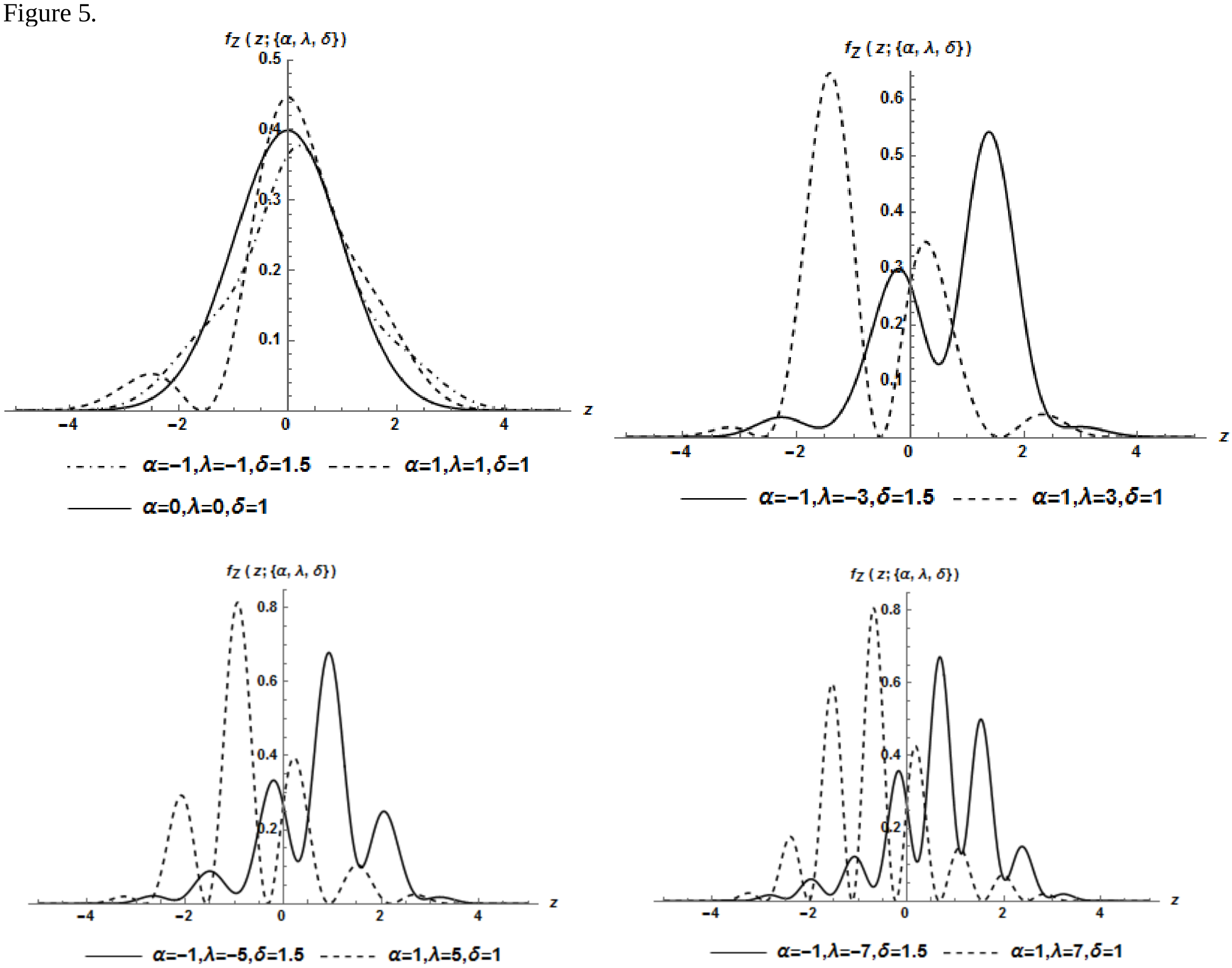


Figure 4. Plots of the pdf of $MMASN(\alpha,\lambda,\delta)$ distribution.

It is observed from Figure 4 that as λ increases the number of peaks also increase, as δ increases the curve tends to normal curve (but not in all cases) and the skewness is positive (negative) according to $\alpha>(<)0$. Also, the pdf approaches to the normal pdf as λ and α tend to zero. So the three parameters play an important role on determining the shape of the proposed distribution.

Theorem 1: *The cdf of* $MMASN(\alpha,\lambda,\delta)$ *distribution is given by*

$$F_Z(z)=[(2+\alpha^2)F_Y(z)+(2+\alpha^2)C(\alpha,b)(\varphi_X(\lambda)-\varphi_X(-\lambda))/2i\delta]/C(\alpha,\lambda,\delta) \quad (5)$$

where $C(\alpha,b)=\Phi(b)+\alpha(2-b\alpha)\phi(b)/(2+\alpha^2)$, $\varphi(.)$ *and* $\Phi(.)$ *respectively are the pdf and cdf of standard normal distribution,* $F_Y(z)$ *is the cdf of* $Y\sim ASN(\alpha)$ *and* $\phi_X(.)$ *is the characteristics function of* $X\sim Truncated\ ASN(\alpha)$ *distribution in the range* $(-\infty,b)$ *defined below.*
Proof: See Appendix B.

The pdf of $X\sim Truncated\ ASN(\alpha)$ distribution in the range $(-\infty,b)$ is given by

$$f(x;\alpha,b)=\frac{[(1-\alpha x)^2+1]\varphi(x)}{(2+\alpha^2)\ C(\alpha,b)};-\infty<x<b \quad (6)$$

where $C(\alpha,b)=\Phi(b)+\frac{\alpha(2-b\alpha)}{(2+\alpha^2)}\varphi(b)$, $\Phi(.)$ and $\varphi(.)$ are defined above. The characteristics function is given by

$$\varphi_X(t)=\frac{e^{-t^2/2}\ \phi(b)}{(2+\alpha^2)\ C(\alpha,b)}\left[e^{t(2ib+t)/2}\alpha(2-b\alpha-it\alpha)+e^{b^2/2}\sqrt{2\pi}(2-2it\alpha+\alpha^2-t^2\alpha^2)\Phi(b-it)\right].$$

Theorem 2: *The moment generating function (mgf) of* $MMASN(\alpha,\lambda,\delta)$ *distribution is given by*

$$M_Z(t)=[(2+\alpha^2)M_Y(t)+\frac{(2+\alpha^2)}{2i\delta}(M_Y(t+i\lambda)-M_Y(t-i\lambda))]/C(\alpha,\lambda,\delta) \quad (7)$$

where $M_Y(.)$ *is the mgf of* $Y\sim ASN(\alpha)$.
Proof: See Appendix C.

The n^{th} order moment of $MMASN(\alpha,\lambda,\delta)$ distribution is given by

$$E(Z^n)=[2E(X^n)-2\alpha E(X^{n+1})+\alpha^2E(X^{n+2})]/C(\alpha,\lambda,\delta) \quad (8)$$

where $E(X^n)$, $E(X^{n+1})$ and $E(X^{n+2})$ are the n^{th}, $(n+1)^{th}$ and $(n+2)^{th}$ moments of standard $X\sim MMSN(\alpha,\lambda)$ distribution (Chakraborty et al., 2015). By substituting n = 1, 2, 3, etc. in (8), one can easily get the moments. In particular the mean is given by

$$E[Z]=-(2\alpha\delta-\lambda e^{-\lambda^2/2}(\alpha^2\lambda^2-3\alpha^2-2))/\delta C(\alpha,\lambda,\delta).$$

2.1. Skewness and Kurtosis

The skewness or Pearson's β_1 coefficient of $MMASN(\alpha,\lambda,\delta)$ distribution is

$$\beta_1=(-2b_1^3+3b_1b_2\delta C(\alpha,\lambda,\delta)+b_3\delta^2(C(\alpha,\lambda,\delta))^2)^2/(-b_1^2+b_2\delta C(\alpha,\lambda,\delta))^3$$

where $b_1=2\alpha\delta-\lambda e^{-\lambda^2/2}(\alpha^2\lambda^2-3\alpha^2-2)$, $b_2=2\delta+3\alpha^2\delta+2\alpha\lambda e^{-\lambda^2/2}(-3+\lambda^2)$ and

$$b_3=-6\alpha\delta+e^{-\lambda^2/2}\left(3\lambda(2+5\alpha^2)-2\lambda^3(1+5\alpha^2)+\alpha^2\lambda^5\right).$$

The distribution is positively (negatively) skewed if $\lambda<(>)0$. The values of β_1 are plotted in Figure 5 against λ $(-5\le\lambda\le5)$ for different values of α and δ.

From Figure 5, it is observed that β_1 increases when α increase and remain constant for $|\lambda|>5$.

The kurtosis or Pearson's β_2 coefficient of $MMASN(\alpha,\lambda,\delta)$ distribution is

$$\beta_2=\frac{-3b_1^4+6b_1^2b_2\delta C(\alpha,\lambda,\delta)+4b_1b_3\delta^2(C(\alpha,\lambda,\delta))^2+b_4\delta^3(C(\alpha,\lambda,\delta))^3}{(b_1^2-b_2\delta C(\alpha,\lambda,\delta))^2}$$

where $b_4=6\delta+15\alpha^2\delta-2\alpha\lambda e^{-\lambda^2/2}(\lambda^4-10\lambda^2+15)$. The values of β_2 are plotted in Figure 6 against λ $(-5\le\lambda\le5)$ for different values of α and δ.

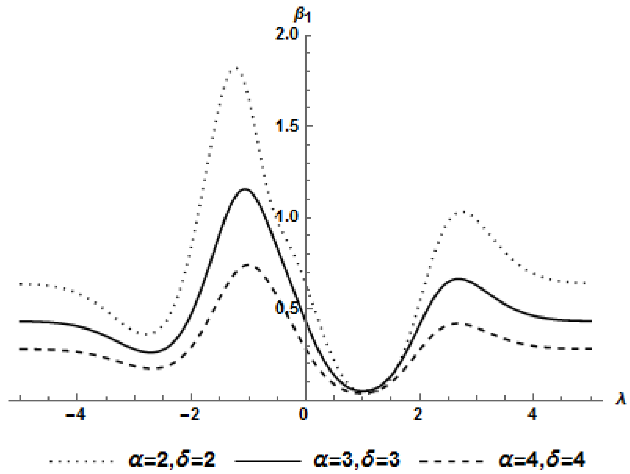


Figure 5. Plots of skewness of $MMASN(\alpha,\lambda,\delta)$ distribution.

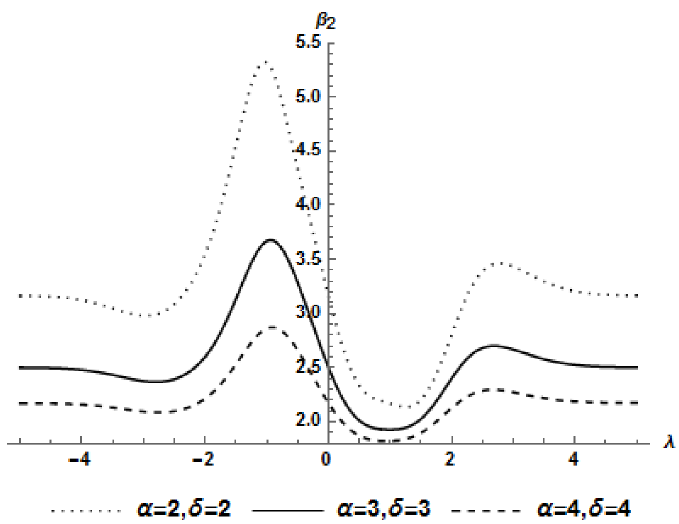


Figure 6. Plots of kurtosis of $MMASN(\alpha,\lambda,\delta)$ distribution.

From Figure 6, it is observed that β_2 increases when α increase and remain constant for $|\lambda| > 5$.

3. Characterization of MMASN distribution

This section is devoted to the characterizations of the MMASN distribution via a simple relationship between two truncated moments. The characterization applies a theorem of Glänzel (1987), Theorem 3 given below. Clearly, the result holds as well when the H is not a closed interval. This characterization is stable of weak convergence. (Glänzel (1990)).

Theorem 3: Let, (Ω, F, P) be given probability space and let $H = [d, e]$ be an interval for some $d < e$ ($d = -\infty$, $e = \infty$ might as well be allowed). Let, $Z : \Omega \to H$ be a continuous random variable with the distribution function F and let k and h be two real functions defined on H such that

$$E\left[k(Z)\,\middle|\,Z \geq z\right] = E\left[h(Z)\,\middle|\,Z \geq z\right]\eta(z), \quad z \in H,$$

is defined with some real function η. Assume that $k,\, h \in C^1(H),\ \eta \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta h = k$ has no real solution in the interior of *H*. Then *F* is uniquely determined by the functions *k*, *h* and η, particularly,

$$F(z) = \int_a^z C\left|\frac{\eta'(u)}{\eta(u)h(u) - k(u)}\right| \exp(-(s(u)))du,$$

where, the function s is a solution of the differential equation $s' = \dfrac{\eta' h}{\eta h - k}$ and C is the normalizing constant, such that $\int_H dF = 1.$

Proposition 1: Let, the random variable $Z : \Omega \to R$ be continuous, and let $h(z) = \left\{\left(1 + \dfrac{Sin(\lambda z)}{\delta}\right)\left[(1 - \alpha x)^2 + 1\right]\right\}^{-1}$ and $k(z) = h(z)\Phi(z)$ for $z \in R$. Then, Z has density (4) if and only if the function η defined in Theorem A.1 is

$$\eta(z) = \frac{1}{2}\{1 + \Phi(z)\}, \qquad z \in R.$$

Proof: If *Z* has pdf (4), then

$$(1 - F(z))\, E[h(Z) \,|\, X \geq z] = \frac{1}{C(\alpha,\lambda,\delta)}\{1 - \Phi(z)\}, \qquad z \in R,$$

and

$$(1 - F(z))\, E[k(Z) \,|\, X \geq z] = \frac{1}{2C(\alpha,\lambda,\delta)}\{1 - \Phi^2(z)\}, \qquad z \in R,$$

and finally,

$$\eta(z)h(z) - k(z) = \frac{1}{2}h(z)\{1 - \Phi(z)\} > 0 \qquad \text{for} \qquad z \in R.$$

Conversely, if η has the above form, then

$$s'(z) = \frac{\eta'(z)h(z)}{\eta(z)h(z) - k(z)} = \frac{\varphi(z)}{1 - \Phi(z)},$$

and hence

$$s(z) = -\log\{1 - \Phi(z)\}, \qquad z \in R.$$

In view of theorem 3, *Z* has pdf (4).

Corollary 1: If $Z : \Omega \to R$ is a continuous random variable and $h(z)$ is as in Proposition 1. Then, Z has pdf (4) if and only if there exist functions k and η defined in theorem 3 satisfying the following first order differential equation

$$\frac{\eta'(z)h(z)}{\eta(z)h(z) - k(z)} = \frac{\varphi(z)}{1 - \Phi(z)}.$$

Corollary 2: The general solution of the above differential equation is

$$\eta(z) = \{1 - \Phi(z)\}^{-1}\left\{- \int \varphi(z)(h(z))^{-1}k(z) + D\right\},$$

where, D is a constant. A set of functions satisfying this differential equation is presented in Proposition 1 with $D = \frac{1}{2}$. Clearly, there are other triplets (h,k,η) satisfying the conditions of Theorem 3.

4. Maximum Likelihood Estimation and Simulation

A location and scale extension of $MMASN(\alpha,\lambda,\delta)$ distribution is introduced as follows. If $Z \sim MMASN(\alpha,\lambda,\delta)$ then $Y = \mu + \sigma Z$ is said to be the location (μ) and scale (σ) extension of Z and has the density function given by

$$f(y;\alpha,\lambda,\delta,\mu,\sigma)=\left(\frac{1}{\delta C(\alpha,\lambda,\delta)}\right)\left[\delta+Sin\left(\lambda\left(\frac{y-\mu}{\sigma}\right)\right)\right]\left[\left(1-\alpha\left(\frac{y-\mu}{\sigma}\right)\right)^2+1\right]\varphi\left(\frac{y-\mu}{\sigma}\right) \tag{9}$$

where $(\mu,\alpha,\lambda)\in R^3$ and $(\sigma,\delta)\in(0,\infty)\times[1,\infty)$ and denoted by $Y\sim MMASN(\alpha,\lambda,\delta,\mu,\sigma)$.

Let $y_1, y_2,..., y_n$ be a random sample from the distribution of the random variable $Y\sim MMASN(\mu,\sigma,\alpha,\lambda,\delta)$ so that the log-likelihood function for $\theta=(\mu,\sigma,\alpha,\lambda,\delta)$ is given by

$$\begin{aligned}\log L(\theta)=&\sum_{i=1}^{n}\log\left[\left\{1-\alpha\left(\frac{y_i-\mu}{\sigma}\right)\right\}^2+1\right]-n\log C(\alpha,\lambda,\delta)-n\log(\sigma)-\frac{n}{2}\log(2\pi)\\&-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{y_i-\mu}{\sigma}\right)^2-n\log(\delta)+\sum_{i=1}^{n}\log\left[\delta+Sin\left\{\lambda\left(\frac{y_i-\mu}{\sigma}\right)\right\}\right]\end{aligned} \tag{10}$$

The corresponding normal equations and information matrix are provided in Appendix D.

4.1 Simulation

In order to study the efficiency of the Maximum Likelihood estimates of the parameters of $MMASN(\alpha,\lambda,\delta,\mu,\sigma)$ distribution a simulation study was conducted. To generate the random number Z from $MMASN(\alpha,\lambda,\delta,\mu,\sigma)$ distribution for different choices of the values of the parameters $\mu,\sigma,\alpha,\lambda$ and δ one can adopt the acceptance sampling method with the following steps:

I: Rewrite the pdf of $MMASN(\alpha,\lambda,\delta,\mu,\sigma)$ distribution as

$$f(z;\alpha,\lambda,\delta)=\frac{(2+\alpha^2)}{C(\alpha,\lambda,\delta,\mu,\sigma)}\left(1+\frac{Sin(\lambda(z-\mu)/\sigma)}{\delta}\right)f_1(z;\alpha,\mu,\sigma)$$

where $f_1(z;\alpha,\mu,\sigma)$ is the pdf of $ASN(\alpha,\mu,\sigma)$ distribution of Elal-Olivero (2010).

II: Generate random number $U=u$ from Uniform $(0,1)$ distribution.

III: For $U=u$, generate a random number H from $ASN(\alpha,\mu,\sigma)$ distribution.

IV: Set $Z=H$ if $U<\frac{1}{\Delta}\frac{f(H)}{f_1(H)}$, otherwise, step back to I and continue the process.

Here, $\Delta=\frac{(2+\alpha^2)}{C(\alpha,\lambda,\delta,\mu,\sigma)}\left(1+\frac{1}{\delta}\right)$, and $f(.)$ and $f_1(.)$ are the pdfs of $MMASN(\alpha,\lambda,\delta,\mu,\sigma)$ and $ASN(\alpha,\mu,\sigma)$ distributions respectively.

Applying this method a simulation study was conducted for sample sizes $n=100$, 300 and 500 with different combinations of the true values of the parameters α, δ and λ for fixed values $\mu=0$ and $\sigma=1$. The number of replications *r* = 1500. For each sample the MLEs were computed using GenSA package in R and then Bias and MSE were computed.

From the simulation results presented in Table 1, 2, and 3 (in Appendix E), it is observed that the estimated values of the average bias and MSE gradually decrease as the sample size increases as expected.

5. Data Modelling Applications

We provide three applications of the new proposed distribution using real data for illustrative purposes to show the flexibility and usefulness of the new proposed distribution.

The first data set is concerning the Oits IQ Scores for 52 Non-White males hired by a large insurance company in 1971, given in Roberts (1988).

Data set-I: 91, 102, 100, 117, 122, 115, 97, 109, 108, 104, 108, 118, 103, 123, 123, 103, 106, 102, 118, 100, 103, 107, 108, 107, 97, 95, 119, 102, 108, 103, 102, 112, 99, 116, 114, 102, 111, 104, 122, 103, 111, 101, 91, 99, 121, 97, 109, 106, 102, 104, 107, 95.

The second data set is related to N latitude degrees in 69 samples from world lakes, which appear in Column 5 of the Diversity data set in website http://users.stat.umn.edu/sandy/courses/8061/ datasets/lakes.lsp.

Data set-II: 47.5, 44, 62, 42, 52, 39.1, 33.8, 43.2, 39, 45.1, 47.6, 42.9, 43.1, 46, 42.4, 28, 68.6, 43.1, 46, 71.3, 74.7, 46, 33.8, 49.7, 41.4, 49.3, 46, 40.1, 43.9, 49.3, 49.3, 44, 41.3, 42.3, 42.4, 41.4, 46.2, 50.3, 43, 42.4, 38.8, 40.6, 46.2, 40, 39, 43.6, 41.4, 41.6, 39, 42.2, 42.5, 42.5, 71.3, 44.1, 32.8, 38.7, 71.3, 71.3, 38.6, 39, 43, 45.3, 37.2, 32.8, 38.6, 38.6, 43, 52.8, 37.1.

The last data set gives the observed 72 survival times data (in days) of infected guinea pigs. It's infected with virulent tubercle bacilli originally observed and reported by (Bjerkedal, 1960).

Data set-III: 0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

The summary statistics of the data sets considered above are given in Table 4.

Table 4. Summary Statistics of the Data sets.

Min.	Median	Mean	Max.	SD	Skewness	Kurtosis
Data set-I						
91.0	105.0	106.7	123.0	8.31	0.37	2.43
Data set-II						
28	43	45.17	74.70	9.62	1.66	5.59
Data set-III						
0.10	1.59	1.79	5.55	1.01	1.29	5.05

For these data sets, we compare the fits of the $MMASN(\mu,\sigma,\alpha,\lambda,\delta)$ distribution to many well-known distributions but reported only the distributions which are nested within it namely the normal distribution, the skew-normal $SN(\mu,\sigma,\lambda)$ distribution of Azzalini (1985), the alpha-skew-normal distribution $ASN(\alpha,\mu,\sigma)$ of Elal-Olivero (2010) and the multimodal skew normal distribution $MMSN(\mu,\sigma,\lambda,\delta)$ of Chakraborty et al. (2015). It may be worth noting that other related distributions we have fitted namely the alpha-skew-logistic distribution of Hazarika and Chakraborty (2014), the alpha-skew-laplace distribution of Harandi and Alamatsaz (2013), the alpha-beta-skew-normal distribution and beta-skew-normal distribution of Shafiei et al. (2016), the generalized alpha skew normal distribution of Sharafi et al. (2017), the Balakrishnan alpha skew normal distribution of Hazarika et al. (2020), the Log-Balakrishnan alpha skew normal distribution of Shah et al. (2020a), the Balakrishnan alpha skew logistic distribution of Shah et al. (2020b), the Balakrishnan alpha skew Laplace distribution of Shah et al. (2020c) have not been reported here as all those distributions too have inferior performance than the proposed one.

Using GenSA package in R, the MLE of the parameters are obtained by using global numerical optimization routine. In order to compare the models, we consider the model selection criteria viz., the AIC.

Table 5 shows the MLE's, log-likelihood and AIC of the above mentioned distributions. Graphical representation of the results taking only the top three competitors of the proposed model is given in Figure 7.

It is found from Table 5 that the proposed $MMASN(\mu,\sigma,\alpha,\lambda,\delta)$ distribution provides best fit to all the three data sets in terms of AIC value. The plots of observed and expected densities presented in Figure 7 clearly confirm our findings. It is important to note that the proposed distribution could capture the multiple modes in all the three examples in much better way than the others.

5.1. Likelihood Ratio Test

Since $N(\mu,\sigma^2)$, $ASN(\mu,\sigma,\alpha)$, $MMSN(\mu,\sigma,\lambda,\delta)$ and $MMASN(\mu,\sigma,\alpha,\lambda,\delta)$ distributions are nested models, the LR test is used to discriminate between them with the following procedure:

i) To discriminate $MMASN(\mu,\sigma,\alpha,\lambda,\delta)$ and $N(\mu,\sigma^2)$, we have to test the null hypothesis $H_0:\alpha=\lambda=0$ vs $H_1:\alpha\neq 0,\ \lambda\neq 0$ and the test statistic is $-2\log(LR)=-2\left[\log L(\tilde{\mu},\tilde{\sigma}^2,\alpha=\lambda=0\mid y)-\log L(\hat{\mu},\hat{\sigma},\hat{\alpha},\hat{\lambda},\hat{\delta}\mid y)\right]\sim\chi_r^{\,2}$, where $\tilde{\mu},\tilde{\sigma}^2$ and $\hat{\mu},\hat{\sigma},\hat{\alpha},\hat{\lambda},\hat{\delta}$ are the MLEs of $N(\mu,\sigma^2)$ and $MMASN(\mu,\sigma,\alpha,\lambda,\delta)$ distributions respectively and $r=3$ (difference between the number of parameters). Similarly,

ii) to discriminate $MMASN(\mu,\sigma,\alpha,\lambda,\delta)$ and $ASN(\mu,\sigma,\alpha)$, we have to test the null hypothesis $H_0:\lambda=0$ vs $H_1:\lambda\neq 0$ the test statistic is $-2\log(LR)=-2\left[\log L(\tilde{\mu},\tilde{\sigma},\tilde{\alpha},\ \lambda=0\mid y)-\log L(\hat{\mu},\hat{\sigma},\hat{\alpha},\hat{\lambda},\hat{\delta}\mid y)\right]\sim\chi_2^{\,2}$, where, $\tilde{\mu},\tilde{\sigma},\tilde{\alpha}$ and $\hat{\mu},\hat{\sigma},\hat{\alpha},\hat{\lambda},\hat{\delta}$ are the MLEs of $ASN(\mu,\sigma,\alpha)$ and $MMASN(\mu,\sigma,\alpha,\lambda,\delta)$ distributions respectively.

iii) Again, to discriminate $MMASN(\mu,\sigma,\alpha,\lambda,\delta)$ and $MMSN(\mu,\sigma,\lambda,\delta)$, we have to test the null hypothesis $H_0:\alpha=0$ vs $H_1:\alpha\neq 0$, the test statistic is $-2\log(LR)=-2\left[\log L(\tilde{\mu},\tilde{\sigma},\tilde{\delta},\tilde{\lambda},\alpha=0\,|\,y)-\log L(\hat{\mu},\hat{\sigma},\hat{\alpha},\hat{\lambda},\hat{\delta}\,|\,y)\right]\sim\chi_1^2$ where, $\tilde{\mu},\tilde{\sigma},\tilde{\delta},\tilde{\lambda}$ and $\hat{\mu},\hat{\sigma},\hat{\alpha},\hat{\lambda},\hat{\delta}$ are the MLEs of $MMSN(\mu,\sigma,\lambda,\delta)$ and $MMASN(\mu,\sigma,\alpha,\lambda,\delta)$ distributions respectively.

The result of the LR test is shown in Table 6.

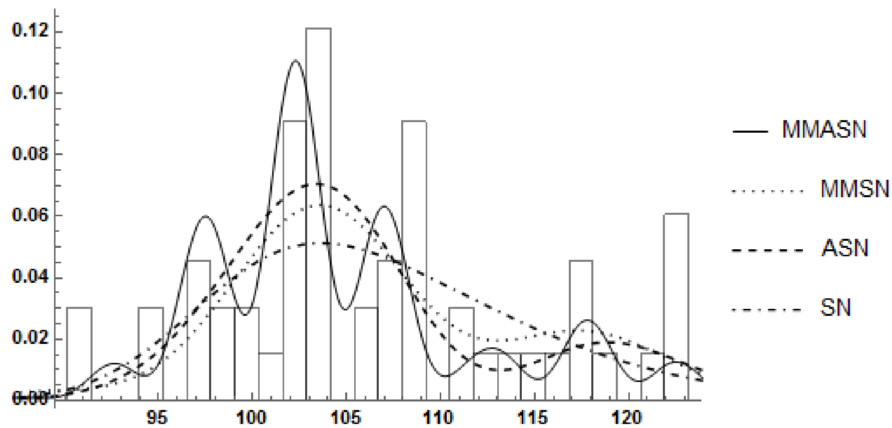
Table 5. MLE's, log-likelihood, and AIC of Data Set I, II, III.

Distributions	μ	σ	δ	λ	α	$\log L$	AIC
Data Set I							
$N(\mu,\sigma^2)$	106.654	8.229	--	--	--	-183.387	370.774
$SN(\mu,\sigma,\lambda)$	97.455	12.343	--	2.546	--	-182.140	370.28
$MMSN(\mu,\sigma,\lambda,\delta)$	107.042	8.239	2.016	-2.564	--	-180.011	368.022
$ASN(\mu,\sigma,\alpha)$	109.907	5.779	--	--	2.025	-180.278	366.556
$MMASN(\mu,\sigma,\alpha,\lambda,\delta)$	108.756	5.670	1.880	-6.859	1.814	**-177.461**	**364.922**
Data Set II							
$N(\mu,\sigma^2)$	45.165	9.549	--	--	--	-253.599	511.198
$SN(\mu,\sigma,\lambda)$	35.344	13.70	--	3.687	--	-243.036	492.072
$ASN(\mu,\sigma,\alpha)$	52.147	7.714	--	--	2.042	-235.370	476.739
$MMSN(\mu,\sigma,\lambda,\delta)$	48.94	10.63	1.130	-1.876	--	-228.014	464.028
$MMASN(\mu,\sigma,\alpha,\lambda,\delta)$	52.432	7.739	1.153	-1.272	3.168	**-226.406**	**462.811**
Data Set III							
$N(\mu,\sigma^2)$	1.791	1.004	--	--	--	-108.206	220.411
$MMSN(\mu,\sigma,\lambda,\delta)$	2.134	1.062	1.287	-1.669	--	-99.345	206.689
$ASN(\mu,\sigma,\alpha)$	2.429	0.843	--	--	1.408	-100.243	206.486
$SN(\mu,\sigma,\lambda)$	0.624	1.540	--	4.833	--	-98.8569	203.714
$MMASN(\mu,\sigma,\alpha,\lambda,\delta)$	2.522	0.866	2.292	-4.893	1.499	**-96.0753**	**202.151**

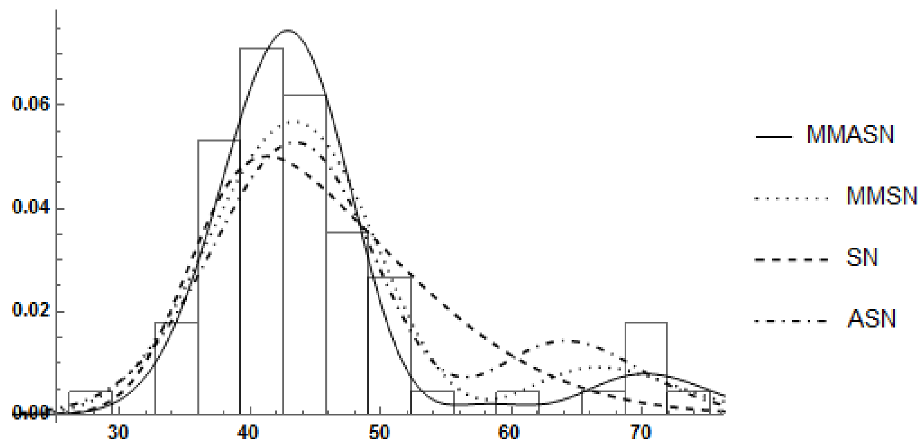
Table 6. The values of LR test statistic for different hypothesis.

Hypothesis	LR test statistic			Degrees of Freedom	Critical values
	Data set-I	Data set-II	Data set-III		
$H_0:\alpha=0$ vs $H_1:\alpha\neq 0$	5.100	3.216	6.539	1	3.841
$H_0:\lambda=0$ vs $H_1:\lambda\neq 0$	5.634	17.928	8.335	2	5.991
$H_0:\alpha=\lambda=0$ vs $H_1:\alpha\neq 0,\ \lambda\neq 0$	11.852	54.386	24.261	3	7.815

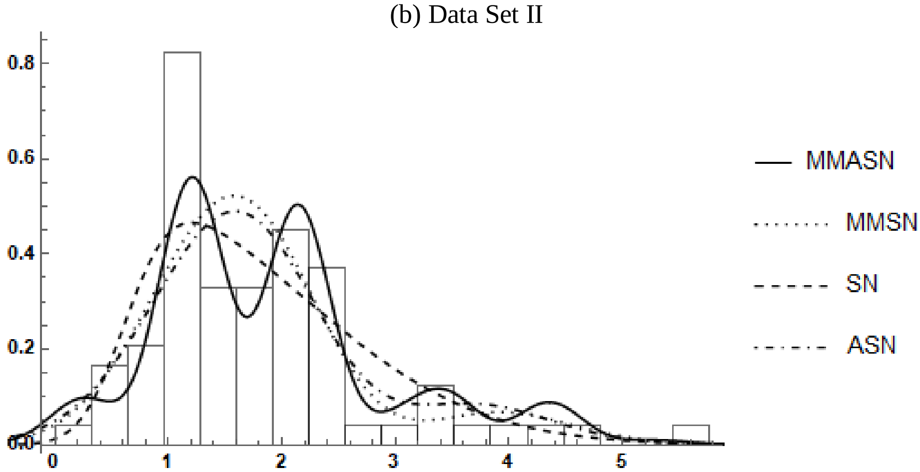
From Table 6, we observe that, in seven out of the nine test cases, the value of LR test statistic exceeds the corresponding critical value at 5% level of significance. Thus, there is evidence in support of the alternative hypothesis. Thus, we may conclude that the sampled data comes from $MMASN(\mu,\sigma,\alpha,\lambda,\delta)$ distribution and not from other distribution considered in seven out of nine tests.



(a) Data Set I



(b) Data Set II



(c) Data set-III

Figure 7. Plots of observed and expected densities of Data Set I, II, III.

6. Conclusion

In this article, a new family of skew distribution is introduced which can cater to unimodal, bimodal as well as multimodal data modelling. Some of its distributional properties are investigated. To study the behaviour of MLE's a simulation study has been conducted. The numerical results show that the $MMASN(\alpha,\lambda,\delta,\mu,\sigma)$ distribution provides better fit compared to the other known distributions applied here. The methodology applied in this article can be applied to extend Logistic and the Laplace distributions which will be considered in follow up works. Further, bivariate generalizations and logarithmic transformed distributions can also be considered in future.

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Appendices

A: The normalizing constant $C(\alpha,\lambda,\delta)$ is easily obtained as follows:

$$C(\alpha,\lambda,\delta)=\int_{-\infty}^{\infty}\left(1+\frac{Sin(\lambda z)}{\delta}\right)[(1-\alpha z)^2+1]\varphi(z)\,dz$$

$$=\int_{-\infty}^{\infty}[(1-\alpha z)^2+1]\varphi(z)\,dz+\int_{-\infty}^{\infty}\left(\frac{Sin(\lambda z)}{\delta}\right)[(1-\alpha z)^2+1]\varphi(z)\,dz$$

$$=(2+\alpha^2)+\int_{-\infty}^{\infty}\left(\frac{\exp(i\lambda z)-\exp(-i\lambda z)}{2i\delta}\right)[(1-\alpha z)^2+1]\varphi(z)\,dz$$

$$=(2+\alpha^2)+\int_{-\infty}^{\infty}\left(\frac{\exp(i\lambda z)}{2i\delta}\right)[(1-\alpha z)^2+1]\varphi(z)\,dz-\int_{-\infty}^{\infty}\left(\frac{\exp(-i\lambda z)}{2i\delta}\right)[(1-\alpha z)^2+1]\varphi(z)\,dz$$

$$=(2+\alpha^2)+\frac{1}{2i\delta}\left[(2+\alpha^2)\{\phi_Y(\lambda)-\phi_Y(-\lambda)\}\right],$$

where, $\phi_Y(.)$ is the characteristic function of $Y \sim ASN(\alpha)$

$$=(2+\alpha^2)+\frac{1}{2i\delta}\left[e^{-\lambda^2/2}(2+\alpha^2-2i\alpha\lambda-\alpha^2\lambda^2)-e^{-\lambda^2/2}(2+\alpha^2+2i\alpha\lambda-\alpha^2\lambda^2)\right]$$

$$=2+\alpha^2-\frac{2\alpha\lambda e^{-\lambda^2/2}}{\delta}$$

B: Proof of cdf:

$$F_Z(z)=\int_{-\infty}^{z}\frac{1}{C(\alpha,\lambda,\delta)}\left(1+\frac{Sin(\lambda z)}{\delta}\right)[(1-\alpha z)^2+1]\varphi(z)\,dz$$

$$=\frac{1}{C(\alpha,\lambda,\delta)}\left[\int_{-\infty}^{z}[(1-\alpha z)^2+1]\varphi(z)\,dz+\int_{-\infty}^{z}\left(\frac{Sin(\lambda z)}{\delta}\right)[(1-\alpha z)^2+1]\varphi(z)\,dz\right]$$

$$=\frac{1}{C(\alpha,\lambda,\delta)}\left[(2+\alpha^2)F_Y(z)+\int_{-\infty}^{z}\left(\frac{\exp(i\lambda z)-\exp(-i\lambda z)}{2i\delta}\right)[(1-\alpha z)^2+1]\varphi(z)\,dz\right];Y\sim ASN(\alpha)$$

$$=\frac{1}{C(\alpha,\lambda,\delta)}\left[(2+\alpha^2)F_Y(z)+\frac{1}{2i\delta}\left[\int_{-\infty}^{z}\exp(i\lambda z)[(1-\alpha z)^2+1]\varphi(z)\,dz\right.\right.$$

$$\left.\left.-\int_{-\infty}^{z}\exp(-i\lambda z)[(1-\alpha z)^2+1]\varphi(z)\,dz\right]\right]$$

$$=\frac{1}{C(\alpha,\lambda,\delta)}\left[(2+\alpha^2)F_Y(z)+\frac{1}{2i\delta}\left[(2+\alpha^2)C(\alpha,b)\{\phi_X(\lambda)-\phi_X(-\lambda)\}\right]\right]$$

C: Proof of mgf:

$$M_Z(t)=\int_{-\infty}^{\infty} e^{tz}\frac{1}{C(\alpha,\lambda,\delta)}\left(1+\frac{Sin(\lambda z)}{\delta}\right)[(1-\alpha z)^2+1]\varphi(z)\,dz$$

$$=\frac{1}{C(\alpha,\lambda,\delta)}\left[\int_{-\infty}^{\infty} e^{tz}[(1-\alpha z)^2+1]\varphi(z)\,dz+\int_{-\infty}^{\infty} e^{tz}\left(\frac{Sin(\lambda z)}{\delta}\right)[(1-\alpha z)^2+1]\varphi(z)\,dz\right]$$

$$=\frac{1}{C(\alpha,\lambda,\delta)}\left[(2+\alpha^2)M_Y(t)+\int_{-\infty}^{\infty} e^{tz}\left(\frac{\exp(i\lambda z)-\exp(-i\lambda z)}{2i\delta}\right)[(1-\alpha z)^2+1]\varphi(z)\,dz\right];Y\sim ASN(\alpha)$$

$$=\frac{1}{C(\alpha,\lambda,\delta)}\left[(2+\alpha^2)M_Y(t)+\frac{1}{2i\delta}\left[\int_{-\infty}^{z}\exp[(t+i\lambda)z][(1-\alpha z)^2+1]\varphi(z)\,dz\right.\right.$$

$$\left.\left.-\int_{-\infty}^{z}\exp[(t-i\lambda)z][(1-\alpha z)^2+1]\varphi(z)\,dz\right]\right]$$

$$=\frac{1}{C(\alpha,\lambda,\delta)}\left[(2+\alpha^2)M_Y(t)+\frac{1}{2i\delta}\left[(2+\alpha^2)\{M_Y(t+i\lambda)-M_Y(t-i\lambda)\}\right]\right]$$

where $Y\sim ASN(\alpha)$, $M_Y(t)$ is the mgf of $ASN(\alpha)$ distribution.

D: Normal equations and information matrix:

$$\frac{\partial\log L(\theta)}{\partial\mu}=-\left(\frac{\lambda}{\sigma}\right)\sum_{i=1}^{n}C_i+\left(\frac{1}{\sigma}\right)\sum_{i=1}^{n}V_i+\left(\frac{2\alpha}{\sigma}\right)\sum_{i=1}^{n}\frac{(1-\alpha V_i)}{(1+(1-\alpha V_i)^2)}$$

$$\frac{\partial\log L(\theta)}{\partial\sigma}=-\frac{n}{\sigma}-\left(\frac{\lambda}{\sigma}\right)\sum_{i=1}^{n}V_i\,C_i+\left(\frac{1}{\sigma}\right)\sum_{i=1}^{n}V_i^{\,2}+\left(\frac{2}{\sigma}\right)\sum_{i=1}^{n}\frac{(1-\alpha V_i)\,\alpha V_i}{(1+(1-\alpha V_i)^2)}$$

$$\frac{\partial\log L(\theta)}{\partial\alpha}=-\frac{n(2\alpha-2\lambda e^{-\lambda^2/2}/\delta)}{C(\alpha,\lambda,\delta)}-\sum_{i=1}^{n}\frac{2(1-\alpha V_i)V_i}{(1+(1-\alpha V_i)^2)}$$

$$\frac{\partial\log L(\theta)}{\partial\lambda}=-\frac{2n\alpha e^{-\lambda^2/2}(\lambda^2-1)}{\delta\,C(\alpha,\lambda,\delta)}+\sum_{i=1}^{n}C_iV_i \text{ and}$$

$$\frac{\partial\log L(\theta)}{\partial\delta}=-\frac{n}{\delta}-\frac{2n\alpha\,\lambda e^{-\lambda^2/2}}{\delta^2C(\alpha,\lambda,\delta)}+\sum_{i=1}^{n}\frac{1}{(\delta+\sin(\lambda V_i))}$$

where $V_i=(y_i-\mu)/\sigma$, $S_i=\sin(\lambda V_i)/(\delta+\sin(\lambda V_i))$ and $C_i=\cos(\lambda V_i)/(\delta+\sin(\lambda V_i))$. Solving them simultaneously one may get the estimates of the parameters but solving them is not mathematically tractable. Hence the maximum likelihood estimates of $\theta=(\mu,\sigma,\alpha,\lambda,\delta)$ are obtained by numerically maximizing $\log L(\theta)$ with respect to $\theta=(\mu,\sigma,\alpha,\lambda,\delta)$. The generalized simulated annealing algorithm implemented in R software package is used in numerical optimization. The variance-covariance matrix of the estimators can be obtained by inverting the Fisher Information Matrix (**I**) given by

$$\mathbf{I}=\left[E\left(-\frac{\partial^2\log L(\theta)}{\partial\theta_i\partial\theta_j}\right)\right],\quad i,j=1,2,3,4,5\quad \text{where } (\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)=(\mu,\sigma,\alpha,\lambda,\delta)$$

and

$$\frac{\partial^2\log L(\theta)}{\partial\mu^2}=-\left(\frac{n}{\sigma^2}\right)-\left(\frac{\lambda^2}{\sigma^2}\right)\sum_{i=1}^{n}(C_i^{\,2}+S_i)-\left(\frac{2\alpha^2}{\sigma^2}\right)\sum_{i=1}^{n}\left(\frac{2(1-\alpha V_i)^2}{(1+(1-\alpha V_i)^2)^2}-\frac{1}{(1+(1-\alpha V_i)^2)}\right)$$

$$\frac{\partial^2 \log L(\theta)}{\partial \sigma^2} = \frac{n}{\sigma^2} - \frac{3}{\sigma^2}\sum_{i=1}^{n} V_i^{\,2} + \left(\frac{\lambda}{\sigma^2}\right)\sum_{i=1}^{n}(2C_iV_i - \lambda C_i^{\,2}V_i^{\,2} - \lambda S_iV_i)$$
$$+\left(\frac{2\alpha}{\sigma^2}\right)\sum_{i=1}^{n}\left(-\frac{2\alpha V_i^{\,2}(1-\alpha V_i)^2}{(1+(1-\alpha V_i)^2)^2} + \frac{\alpha V_i^{\,2}}{(1+(1-\alpha V_i)^2)} - \frac{2V_i(1-\alpha V_i)}{(1+(1-\alpha V_i)^2)}\right)$$

$$\frac{\partial^2 \log L(\theta)}{\partial \alpha^2} = -\frac{n(-(2\alpha - \frac{2\lambda e^{-\lambda^2/2}}{\delta})^2 + 2)}{C(\alpha,\lambda,\delta)} + \sum_{i=1}^{n}\left(-\frac{4(1-\alpha V_i)^2 V_i^2}{(1+(1-\alpha V_i)^2)^2} + \frac{2V_i^2}{(1+(1-\alpha V_i)^2)}\right)$$

$$\frac{\partial^2 \log L(\theta)}{\partial \lambda^2} = -n\left(\frac{[2\alpha e^{-\lambda^2/2}(1-\lambda^2)]^2}{\delta^2 C(\alpha,\lambda,\delta)^2} + \frac{2\alpha\lambda e^{-\lambda^2/2}(3-\lambda^2)}{\delta C(\alpha,\lambda,\delta)}\right) + \sum_{i=1}^{n}\left(-C_i^2V_i^2 - S_iV_i^2\right)$$

$$\frac{\partial^2 \log L(\theta)}{\partial \delta^2} = \frac{n}{\delta^2} + n\left(\frac{4\alpha^2\lambda^2 e^{-\lambda^2}}{\delta^4 C(\alpha,\lambda,\delta)^2} + \frac{4\alpha\lambda e^{-\lambda^2/2}}{\delta^3 C(\alpha,\lambda,\delta)}\right) - \sum_{i=1}^{n}\frac{1}{(\delta + \sin(\lambda V_i))^2}$$

$$\frac{\partial^2 \log L(\theta)}{\partial \mu \partial \sigma} = -\frac{2}{\sigma^2}\sum_{i=1}^{n} V_i + \left(\frac{\lambda}{\sigma^2}\right)\sum_{i=1}^{n}(C_i - \lambda C_i^{\,2}V_i - \lambda S_iV_i) +$$
$$\left(\frac{2\alpha}{\sigma^2}\right)\sum_{i=1}^{n}\left(-\frac{2\alpha V_i(1-\alpha V_i)^2}{(1+(1-\alpha V_i)^2)^2} + \frac{\alpha V_i}{(1+(1-\alpha V_i)^2)} - \frac{(1-\alpha V_i)}{(1+(1-\alpha V_i)^2)}\right)$$

$$\frac{\partial^2 \log L(\theta)}{\partial \mu \partial \alpha} = \frac{2}{\sigma}\sum_{i=1}^{n}\left(\frac{2\alpha(1-\alpha V_i)^2 V_i}{(1+(1-\alpha V_i)^2)^2} - \frac{\alpha V_i}{(1+(1-\alpha V_i)^2)} + \frac{(1-\alpha V_i)}{(1+(1-\alpha V_i)^2)}\right)$$

$$\frac{\partial^2 \log L(\theta)}{\partial \mu \partial \lambda} = \frac{1}{\sigma}\sum_{i=1}^{n}\left(-C_i + \lambda C_i^{\,2}V_i + \lambda S_iV_i\right), \quad \frac{\partial^2 \log L(\theta)}{\partial \mu \partial \delta} = \frac{\lambda}{\sigma}\sum_{i=1}^{n}\frac{C_i^{\,2}}{\cos(\lambda V_i)}$$

$$\frac{\partial^2 \log L(\theta)}{\partial \sigma \partial \alpha} = \frac{2}{\sigma}\sum_{i=1}^{n}\left(\frac{2\alpha(1-\alpha V_i)^2 V_i^{\,2}}{(1+(1-\alpha V_i)^2)^2} - \frac{\alpha V_i^{\,2}}{(1+(1-\alpha V_i)^2)} + \frac{(1-\alpha V_i)V_i}{(1+(1-\alpha V_i)^2)}\right)$$

$$\frac{\partial^2 \log L(\theta)}{\partial \sigma \partial \lambda} = \frac{1}{\sigma}\sum_{i=1}^{n}(-C_iV_i + \lambda C_i^{\,2}V_i^{\,2} + \lambda S_iV_i^{\,2}), \quad \frac{\partial^2 \log L(\theta)}{\partial \sigma \partial \delta} = \frac{1}{\sigma}\sum_{i=1}^{n}\lambda C_iV_i$$

$$\frac{\partial^2 \log L(\theta)}{\partial \alpha \partial \lambda} = \frac{2n\delta e^{\frac{\lambda^2}{2}}(\alpha^2-2)(\lambda^2-1)}{(\delta e^{\frac{\lambda^2}{2}}(\alpha^2+2) - 2\alpha\lambda)^2}, \quad \frac{\partial^2 \log L(\theta)}{\partial \alpha \partial \delta} = \frac{2n\lambda e^{\frac{\lambda^2}{2}}(\alpha^2-2)}{(\delta e^{\frac{\lambda^2}{2}}(\alpha^2+2) - 2\alpha\lambda)^2} \text{ and}$$

$$\frac{\partial^2 \log L(\theta)}{\partial \lambda \partial \delta} = \frac{2n\alpha e^{\frac{\lambda^2}{2}}(\alpha^2+2)(\lambda^2-1)}{(\delta e^{\frac{\lambda^2}{2}}(\alpha^2+2) - 2\alpha\lambda)^2} - \sum_{i=1}^{n}\frac{C_iV_i}{\cos(\lambda V_i)}.$$

While it is not easy to get the closed-form expression for the elements of **I**, the estimate of the elements of **I** can be well approximated by substituting the parameters by their corresponding MLEs.

E. Results of Simulation:

Table 1: Results of Simulation

			$\mu=0$		$\sigma=1$		$\delta=1$		λ		α	
λ	α	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
-2	-2	100	-0.0300	0.0114	0.0434	0.0510	0.0444	0.0249	-0.0652	0.0473	0.0562	0.0467
		300	0.0109	0.0130	-0.0233	0.0124	0.0390	0.0231	0.0450	0.0310	-0.0420	0.0191
		500	-0.0090	0.0096	0.0112	0.0189	0.0164	0.0169	0.0196	0.0089	0.0198	0.0099
	-1	100	0.0414	0.0321	-0.0300	0.0258	0.0590	0.0231	-0.0421	0.0257	0.0765	0.0900
		300	-0.0199	0.0129	-0.0107	0.0092	-0.0247	0.0144	0.0129	0.0102	0.0635	0.0427
		500	0.0131	0.0111	0.0098	0.0081	0.0107	0.0104	-0.0190	0.0120	0.0380	0.0221
	0	100	0.0531	0.0502	0.0259	0.0130	-0.0645	0.0871	0.0865	0.0656	-0.0520	0.0600
		300	0.0196	0.0163	0.0113	0.0093	-0.0611	0.0431	-0.0564	0.0544	0.0361	0.0368
		500	0.0158	0.0123	-0.0091	0.0106	0.0483	0.0534	0.0321	0.0190	0.0189	0.0244
	1	100	0.0351	0.0467	-0.0941	0.1101	-0.0520	0.0484	-0.1009	0.0963	0.1110	0.1010
		300	-0.0251	0.0267	0.0803	0.0901	-0.0297	0.0479	0.0531	0.0943	-0.1123	0.0905
		500	-0.0209	0.0111	-0.0631	0.0437	-0.0098	0.0124	0.0399	0.0279	-0.0195	0.0356
	2	100	-0.0611	0.0431	0.0432	0.0396	0.0419	0.0299	0.0698	0.0950	-0.1160	0.0910
		300	0.0241	0.0374	0.0198	0.0116	-0.0361	0.0198	-0.0450	0.0364	-0.0907	0.0605
		500	-0.0091	0.0197	-0.0099	0.0119	-0.0113	0.0149	0.0063	0.0111	-0.0610	0.0564
1	-2	100	0.0967	0.1001	0.0713	0.0514	-0.0755	0.0698	0.0964	0.1003	0.0631	0.0550
		300	-0.0771	0.0799	-0.0299	0.0316	0.0821	0.0746	0.0448	0.0555	-0.0513	0.0413
		500	-0.0350	0.0333	0.0109	0.0103	-0.0308	0.0111	-0.0232	0.0180	0.0099	0.0096
	-1	100	0.2010	0.0877	-0.0394	0.0462	-0.0222	0.0420	-0.0229	0.0147	0.0989	0.1200
		300	0.0955	0.0936	-0.0233	0.0200	0.0314	0.0478	0.0108	0.0123	-0.0665	0.0532
		500	0.0681	0.0154	-0.0090	0.0113	-0.0074	0.0221	-0.0085	0.0091	-0.0511	0.0365
	0	100	-0.0190	0.0192	-0.0193	0.0244	0.0330	0.0361	0.0228	0.0293	-0.0243	0.0360
		300	-0.0100	0.0097	0.0105	0.0136	0.0204	0.0222	0.0121	0.0164	0.0191	0.0213
		500	-0.0099	0.0097	0.0090	0.0106	-0.0130	0.0125	-0.0099	0.0100	0.0093	0.0107
	1	100	-0.0698	0.0412	0.1200	0.0910	-0.0999	0.0961	0.0311	0.0318	-0.0479	0.0455
		300	0.0345	0.0256	0.0672	0.0346	-0.0645	0.0369	-0.0153	0.0105	-0.0298	0.0181
		500	0.0135	0.0192	-0.0180	0.0195	0.0260	0.0398	-0.0108	0.0111	0.0150	0.0118
	2	100	-0.0655	0.0538	-0.1003	0.0843	-0.0506	0.0640	-0.0666	0.0964	-0.1956	0.1057
		300	0.0292	0.0100	0.0977	0.0506	-0.0361	0.0471	0.0320	0.0411	0.1059	0.1000
		500	0.0110	0.0133	-0.0296	0.0381	0.0098	0.0191	-0.0183	0.0139	-0.0610	0.0439
2	-2	100	-0.1004	0.0735	-0.0630	0.0341	-0.0444	0.0410	0.0391	0.0393	-0.0989	0.0907
		300	-0.0654	0.0535	-0.0306	0.0261	-0.0326	0.0265	-0.0198	0.0165	0.0598	0.0623
		500	0.0357	0.0199	-0.0221	0.0190	-0.0174	0.0196	0.0180	0.0111	-0.0232	0.0150
	-1	100	-0.0440	0.0231	0.2130	0.1036	-0.0331	0.0500	-0.0666	0.0626	0.0635	0.0477
		300	-0.0191	0.0121	0.1022	0.0956	-0.0269	0.0247	0.0510	0.0356	-0.0206	0.0230
		500	0.0098	0.0140	0.0954	0.0531	-0.0121	0.0094	-0.0209	0.0149	-0.0149	0.0148
	0	100	0.0198	0.0138	-0.0199	0.0103	0.0486	0.0302	-0.0591	0.0789	-0.0639	0.0480
		300	-0.0115	0.0089	0.0167	0.0101	0.0231	0.0292	-0.0209	0.0420	0.0444	0.0397
		500	0.0083	0.0090	0.0093	0.0099	-0.0079	0.0103	-0.0196	0.0123	0.0163	0.0099
	1	100	0.0826	0.0366	-0.0523	0.0414	0.0696	0.0426	0.0964	0.1002	-0.0635	0.0361
		300	0.0420	0.0356	0.0249	0.0279	0.0545	0.0170	0.0650	0.0365	0.0299	0.0332
		500	-0.0187	0.0132	-0.0163	0.0182	-0.0183	0.0165	-0.0400	0.0183	0.0167	0.0209
	2	100	0.1240	0.0930	-0.0962	0.1023	-0.0360	0.0214	-0.1802	0.0653	0.2422	0.1922
		300	-0.0950	0.0598	-0.1329	0.0895	-0.0204	0.0333	0.1063	-0.0752	0.1560	0.0901
		500	0.0390	0.0470	0.0356	0.0423	-0.0153	0.0210	0.0933	0.0659	0.0990	0.0224

Table 2: Results of Simulation

			$\mu=0$		$\sigma=1$		$\delta=2$		λ		α	
λ	α	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
-2	-2	100	0.0670	0.0544	0.0565	0.0318	0.0598	0.0447	0.0630	0.0980	0.0653	0.0995
		300	0.0369	0.0220	0.0223	0.0108	-0.0233	0.0211	-0.0499	0.0400	-0.0366	0.0400
		500	0.0210	0.0106	-0.0200	0.0138	0.0194	0.0099	0.0197	0.0067	-0.0132	0.0107
	-1	100	0.0523	0.0431	0.0210	0.0100	-0.0356	0.0476	-0.0111	0.0216	-0.2260	0.1902
		300	-0.0279	0.0202	-0.0100	0.0125	-0.0207	0.0223	0.0123	0.0162	0.1310	0.0920
		500	0.0131	0.0164	-0.0093	0.0076	0.0197	0.0184	-0.0099	0.0100	0.0380	0.0440
	0	100	0.0966	0.0597	-0.0254	0.0231	0.0933	0.1006	-0.0765	0.0600	-0.0430	0.0333
		300	-0.0460	0.0563	0.0253	0.0107	-0.0654	0.0871	0.0542	0.0334	-0.0298	0.0320
		500	0.0090	0.0124	0.0088	0.0176	0.0500	0.0666	-0.0201	0.0211	0.0109	0.0190
	1	100	-0.0444	0.0601	-0.0920	0.0954	0.0304	0.0417	-0.0354	0.0463	0.1160	0.0988
		300	-0.0320	0.0291	-0.1003	0.1046	-0.0197	0.0198	-0.0230	0.0149	0.0978	0.0947
		500	-0.0193	0.0120	-0.0185	0.0240	0.0098	0.0155	-0.0161	0.0189	-0.0354	0.0463
	2	100	0.0665	0.0498	-0.0445	0.0403	-0.0364	0.0299	-0.0636	0.0541	0.3005	0.1502
		300	-0.0659	0.0651	0.0225	0.0190	0.0222	0.0240	0.0426	0.0382	-0.0950	0.0755
		500	0.0191	0.0142	0.0172	0.0097	0.0190	0.0100	-0.0097	0.0210	-0.0810	0.0657
1	-2	100	0.1007	0.0789	-0.0813	0.0536	-0.0410	0.0506	0.0355	0.0554	-0.0631	0.0536
		300	-0.0765	0.0567	-0.0543	0.0345	-0.0322	0.0220	-0.0215	0.0249	-0.0513	0.0409
		500	-0.0250	0.0201	-0.0219	0.0191	0.0174	0.043	-0.0170	0.0187	0.0199	0.0100
	-1	100	-0.0978	0.0876	0.0475	0.0431	-0.0279	0.0320	-0.0457	0.0384	-0.0998	0.1001
		300	0.0901	0.0900	0.0354	0.0218	0.0180	0.0144	-0.0222	0.0325	-0.0657	0.0532
		500	0.0382	0.0350	-0.0254	0.0119	0.0099	0.0105	0.0210	0.0090	-0.0503	0.0398
	0	100	-0.0230	0.0198	0.0236	0.0219	-0.0113	0.0251	-0.0222	0.0200	-0.0365	0.0269
		300	0.0224	0.0187	-0.0125	0.0126	-0.0100	0.0329	0.0123	0.0162	-0.0250	0.0220
		500	-0.0159	0.0098	0.0090	0.0091	-0.0100	0.0120	-0.0098	0.0109	0.0118	0.0198
	1	100	-0.0636	0.0555	-0.0542	0.0480	0.0755	0.0844	-0.0432	0.0550	-0.0465	0.0515
		300	0.0400	0.0390	-0.0422	0.0396	-0.0665	0.0498	0.0265	0.0230	0.0391	0.0147
		500	-0.0102	0.0142	0.0187	0.0195	0.0224	0.0298	-0.0118	0.0106	-0.0129	0.0097
	2	100	0.0991	0.0851	-0.0654	0.0753	0.0563	0.0301	-0.0987	0.0990	-0.0909	0.0936
		300	-0.0762	0.0653	0.0324	0.0420	-0.0211	0.0351	-0.0221	0.0430	0.1059	0.0897
		500	-0.0330	0.0256	-0.0211	0.0191	0.0185	0.0222	0.0192	0.0154	0.0430	0.0487
2	-2	100	-0.0996	0.0777	-0.0456	0.0511	0.0455	0.0310	0.0245	0.0491	-0.0987	0.0756
		300	0.0594	0.0565	0.0327	0.0335	-0.0315	0.0209	-0.0300	0.0256	0.0532	0.0600
		500	0.0327	0.0209	-0.0291	0.0165	-0.0198	0.0188	0.0121	0.0111	-0.0210	0.0130s
	-1	100	0.0409	0.0598	-0.0901	0.1006	0.0230	0.0312	0.0756	0.0576	-0.0901	0.0542
		300	-0.0400	0.0364	-0.0673	0.0485	-0.0197	0.0099	0.0431	0.0427	-0.0369	0.0294
		500	0.0095	0.0100	-0.0219	0.0193	0.0101	0.0111	-0.0221	0.0122	-0.0201	0.0231
	0	100	-0.0698	0.0360	-0.0236	0.0198	-0.0221	0.0241	0.0352	0.0250	-0.0989	0.1013
		300	-0.0162	0.0109	0.0195	0.0181	-0.0154	0.0100	-0.0187	0.0196	0.0622	0.0652
		500	0.0085	0.0091	-0.0097	0.0109	-0.0090	0.0079	-0.0100	0.0094	0.0390	0.0199
	1	100	0.0846	0.0536	-0.0462	0.0455	-0.0792	0.0815	0.1118	0.1004	0.0693	0.0565
		300	0.0620	0.0342	-0.0324	0.0256	-0.0661	0.0674	-0.0652	0.0576	0.0320	0.0656
		500	-0.0387	0.0252	-0.0211	0.0193	0.0212	0.0165	-0.0134	0.0112	0.0198	0.0120
	2	100	0.1365	0.0931	0.2161	0.1094	-0.0450	0.0440	-0.0903	0.0991	0.0993	0.1006
		300	-0.1002	0.0599	-0.0943	0.0460	0.0413	0.0540	0.1201	0.0898	-0.0653	0.0561
		500	0.0972	0.0465	-0.0623	0.0165	-0.0103	0.0190	-0.0610	0.0754	0.0210	0.0136

Table 3: Results of Simulation

			$\mu=0$		$\sigma=1$		$\delta=3$		λ		α	
λ	α	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
-2	-2	100	0.0444	0.0323	-0.0652	0.0459	0.0756	0.0890	-0.0635	0.0679	-0.0982	0.0694
		300	-0.0301	0.0222	-0.0223	0.0210	-0.0612	0.0520	0.0555	0.0265	-0.0469	0.0110
		500	-0.0180	0.0101	-0.0135	0.0199	0.0126	0.0199	0.0210	0.0159	-0.0227	0.0164
	-1	100	0.0655	0.0329	0.0333	0.0140	-0.0509	0.0325	0.1020	0.0930	0.0942	0.0697
		300	-0.0422	0.0339	-0.0254	0.0190	-0.0402	0.0301	-0.0987	0.0501	-0.1004	0.0982
		500	0.0156	0.0193	-0.0180	0.0176	-0.0107	0.0087	-0.0326	0.0190	0.0262	0.0234
	0	100	0.0523	0.0509	0.0653	0.0550	0.0997	0.1026	0.1240	0.0615	-0.0714	0.0831
		300	0.0493	0.0303	-0.0457	0.0336	-0.0697	0.0601	-0.0660	0.0556	-0.0543	0.0364
		500	0.0208	0.0223	0.0109	0.0200	-0.0101	0.0320	0.0212	0.0098	-0.0169	0.0190
	1	100	-0.0651	0.0460	-0.2201	0.1031	0.0424	0.0467	-0.0465	0.0366	-0.1103	0.1000
		300	0.0237	0.0261	0.1069	0.0699	-0.0347	0.0283	-0.0356	0.0214	0.0998	0.0923
		500	-0.0102	0.0113	-0.0099	0.0401	0.0190	0.0180	-0.0120	0.0099	0.0655	0.0413
	2	100	-0.0636	0.0439	0.0660	0.0747	0.0444	0.0540	0.0813	0.0518	0.1111	0.0988
		300	-0.0225	0.0370	0.0352	0.0251	-0.0365	0.0351	-0.0244	0.0311	0.0942	0.0570
		500	-0.0101	0.0123	-0.0190	0.0195	0.0200	0.0301	-0.0099	0.0103	-0.0366	0.0279
1	-2	100	-0.0922	0.1020	0.0852	0.0655	-0.0387	0.0564	-0.0998	0.0999	0.0666	0.0598
		300	-0.0710	0.0723	0.0654	0.0231	0.0213	0.0346	0.0546	0.0634	-0.0565	0.0465
		500	0.0120	0.0100	-0.0215	0.0244	-0.0099	0.0111	0.0117	0.0215	-0.0199	0.0093
	-1	100	0.0998	0.0823	-0.0398	0.0400	-0.0350	0.0406	-0.0321	0.0406	-0.1111	0.1097
		300	-0.0653	0.0562	-0.0221	0.0298	0.0201	0.0322	0.0220	0.0231	0.0984	0.0656
		500	0.0602	0.0330	-0.0187	0.0102	0.0119	0.0205	-0.0183	0.0108	-0.0364	0.0421
	0	100	-0.0450	0.0392	0.0165	0.0209	0.0200	0.0291	-0.0223	0.0202	0.0365	0.0545
		300	-0.0322	0.0247	0.0194	0.0117	-0.0145	0.0198	-0.0191	0.0168	-0.0216	0.0241
		500	-0.0109	0.0097	0.0055	0.0106	0.0045	0.0100	-0.0019	0.0114	-0.0144	0.0211
	1	100	0.0663	0.0702	0.0564	0.0710	0.0988	0.1007	-0.0322	0.0298	0.0638	0.0510
		300	0.0495	0.0306	0.0324	0.0376	-0.0651	0.0415	0.0215	0.0288	-0.0536	0.0540
		500	-0.0137	0.0194	0.0210	0.0100	-0.0220	0.0138	-0.0100	0.0119	-0.0296	0.0221
	2	100	-0.0685	0.0708	-0.0777	0.0654	0.0322	0.0456	0.1000	0.0983	0.0985	0.0960
		300	0.0287	0.0365	0.0568	0.0325	0.0256	0.0325	-0.0921	0.0498	-0.0850	0.0762
		500	-0.0100	0.0141	-0.0245	0.0110	-0.0100	0.0140	0.0329	0.0220	0.0200	0.0322
2	-2	100	0.0955	0.0798	0.0470	0.0377	0.0446	0.0315	0.0952	0.0544	0.0655	0.0715
		300	-0.0666	0.0665	-0.0356	0.0227	-0.0361	0.0298	-0.0742	0.0310	-0.0502	0.0698
		500	0.0241	0.0107	-0.0210	0.0183	0.0255	0.0169	0.0335	0.0210	-0.0197	0.0436
	-1	100	-0.0608	0.0498	0.0991	0.1074	-0.0191	0.0190	-0.0566	0.0429	-0.0555	0.0321
		300	0.0277	0.0301	-0.0733	0.0469	0.0189	0.0097	0.0211	0.0129	-0.0406	0.0291
		500	0.0136	0.0147	-0.0146	0.0132	-0.0086	0.0087	0.0104	0.0190	0.0143	0.0150
	0	100	-0.0362	0.0258	0.0155	0.0274	0.0196	0.0291	0.0501	0.0496	0.0405	0.0351
		300	-0.0242	0.0179	0.0165	0.0204	0.0180	0.0252	0.0398	0.0496	-0.0325	0.0292
		500	-0.0161	0.0098	-0.0084	0.0146	0.0059	0.0103	-0.0200	0.0118	0.0169	0.0190
	1	100	-0.0898	0.0486	0.0666	0.0854	0.0854	0.0555	0.0963	0.0742	-0.0654	0.0794
		300	0.0660	0.0446	0.0534	0.0457	0.0527	0.0431	0.0653	0.0300	0.0123	0.0321
		500	-0.0237	0.0264	-0.0294	0.0133	-0.0183	0.0109	-0.0293	0.0161	-0.0102	0.0109
	2	100	-0.1365	0.0993	-0.1238	0.0891	-0.0635	0.0564	0.1004	0.0879	0.1111	0.0987
		300	-0.0995	0.0964	-0.0956	0.0465	0.0456	0.0343	-0.0968	0.0800	0.1198	0.0903
		500	-0.0897	0.0512	0.0107	0.0307	-0.0097	0.0240	0.0124	0.0134	0.0470	0.0198