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A New Heavy-Tailed Exponential Distribution: Inference, Regression Model and Applications

Ahmed Z. Afify[1*], Rodrigo R. Pescim[2], Gauss M. Cordeiro[3], Hisham A. Mahran[4]

*Corresponding author

1. Department of Statistics, Mathematics and Insurance, Benha University, Benha 13511, Egypt, ahmed.afify@fcom.bu.edu.eg
2. Department of Statistics, Londrina State University, Londrina, PR, Brazil, rrpescim@uel.br
3. Departamento de Estatística, Universidade Federal de Pernambuco, Recife 50710-165, Brazil, gauss@de.ufpe.br
4. Department of Statistics, Mathematics and Insurance, Ain Shams University, Cairo 11566, Egypt, dr.hisham.abdeltawab@bus.asu.edu.eg

Abstract

A new weighted exponentiated-exponential distribution is defined to model financial data. It has heavy-tailed behavior that is suitable for data with the right tails. Some actuarial measures for the new model are determined, and simulations are reported. Its parameters are estimated using eight approaches. A new Log-weighted exponentiated-exponential regression model is constructed for right censored data. The importance of the new models is proved through applications to financial data.

Key Words: Actuarial measures; Censored data; Heavy-tailed distributions; Regression model.

Mathematical Subject Classification: 60E05, 62E15.

1. Introduction

The exponential distribution has been used for analyzing real-life data in several areas due to its simple analytical form and lack of memory property. The exponential distribution has only a constant hazard rate shape which makes it unable to fit real-life data with different hazard rate shapes including decreasing, increasing, unimodal or bathtub shaped failure rates, which are often encountered in reliability and engineering, among others.

Several authors have proposed new generalizations and modified forms of the exponential distribution to improve its applicability and flexibility. For example, the Marshall–Olkin exponential by Marshall and Olkin (1997), modified exponential by Rasekhi et al. (2017), Marshall–Olkin logistic-exponential by Mansour et al. (2019), Marshall–Olkin alpha power exponential by Nassar et al. (2019), extended odd Weibull exponential by Afify and Mohamed (2020), generalized odd log-logistic exponential by Afify et al. (2021), and power-modified Kies-exponential by Afify et al. (2022), among others.

Newly distributions are important in modelling real-life phenomenon which depend on extra parameters introduced in some way to a baseline model. These proposed distributions are being used to fit real-life data from some fields such as agriculture, engineering, industry, medical, among others. More recently, very few classes have been defined as functions of a baseline model without requiring extra parameters. In this context, we introduce a new class of distributions following the second approach by choosing the exponentiated exponential for baseline.

Heavy-tailed distributions are useful for modeling financial and lost insurance data, whereas the classical distributions are not very flexible to capture this behavior (Bhati and Ravi, 2018); see (Ahmad et al., 2020a), (Cooray and Ananda, 2005), (Punzo, 2019), and (Punzo et al., 2018). To overcome the deficiencies associated with classical models, it is important to construct new distributions which have right tail probabilities heavier than those of the exponential one.

Henceforth, the notation $\bar{G}(x) = 1-G(x)$ is used to denote the complementary cumulative distribution function (cdf). A distribution G possesses regularly varying behavior if the limit holds

$$\lim_{x\to\infty} = \frac{\bar{G}(tx)}{\bar{G}(x)} = t^{-a}, \qquad a \in [0,\infty],$$

where a is the index of regular variation. Among these distributions, the Weibull is the most popular to model financial and other related data sets. For more details, see (Ahmad et al., 2020c).

We propose a new flexible heavy-tailed *weighted exponentiated exponential* (WEx-Exponential) distribution by combining the exponentiated exponential model with the weighted T-X (WT-X) family.
The cdf of the exponentiated exponential (Ex-exponential) random variable X is

$$F(x) = \left(1 - \mathrm{e}^{-\gamma \mathrm{x}}\right)^{\alpha}, \qquad x > 0, \tag{1}$$

where $\alpha > 0$ is the shape and $\gamma > 0$ is the scale. The probability density function (pdf) of X becomes

$$f(x) = \alpha\gamma \mathrm{e}^{-\gamma \mathrm{x}} \left(1 - \mathrm{e}^{-\gamma \mathrm{x}}\right)^{\alpha-1}. \tag{2}$$

The cdf and pdf of the WT-X class(Ahmad et al., 2020b) follow from the T-X family (Alzaatreh et al., 2013) as

$$G(x) = 1 - \frac{1-F(x)}{\mathrm{e}^{\mathrm{F(x)}}}, \qquad x > 0, \tag{3}$$

and

$$g(x) = \frac{f(x)}{\mathrm{e}^{\mathrm{F(x)}}}\left[2 - F(x)\right], \tag{4}$$

respectively.

The Exp-exponential distribution has been investigated in many applications in dozens of papers over the past 20 years. Also, it was the first distributions to include an additional unknown parameter as the power of the parent cdf. This is the reason for choosing this distribution as the baseline. Further, we show in an application the superiority of the WEx-Exponential distribution in relation to its baseline. The cdf and pdf of the WEx-Exponential distribution have simple analytical forms and hence it can be used effectively in analyzing censored data.

Another important subject of actuarial sciences is the evaluation of the exposure to market risk in a portfolio of instruments. This arises from changes in underlying variables such as interest rates, prices of equity, and exchange rates. Then, our objective is to derive two important risk measures including value at risk and tail value at risk for the WEx-Exponential distribution. Additionally, we explore the parameter estimation of the WEx-Exponential distribution by eight estimation methods. The estimators are compared by using extensive computational simulations to develop a guideline for choosing the best method of estimation which provides better estimates for the WEx-Exponential parameters. We think that this procedure would be of a great interest to applied statisticians/actuaries/engineers.

Finally, we proposed a regression model based on the new WEx-Exponential distribution called log-WEx-Exponential regression model. The new regression model is fitted using the heart transplantation data showing its better fit as compared to other important regression models.

The paper is organized as follows. Section 2 introduces the WEx-Exponential model by inserting (1) in Equation (3).

Some actuarial measures and simulations for these measures are addressed in Section 3. Section 4 deals with eight frequentist estimation methods for the new distribution, and provides some simulation studies. Regression analysis is reported in Section 5. Real-life data application is analyzed in Section 6. Finally, some conclusions are offered in Section 7.

2. The WEx-Exponential Distribution

The cdf and pdf of the WEx-Exponential distribution are

$$G(x;\alpha,\gamma) = 1 - \frac{1-\left(1-\mathrm{e}^{-\gamma x}\right)^{\alpha}}{\mathrm{e}^{\left(1-\mathrm{e}^{-\gamma x}\right)^{\alpha}}}, \qquad x>0,\ \alpha,\ \gamma>0 \tag{5}$$

and

$$g(x;\alpha,\gamma) = \frac{\alpha\gamma\mathrm{e}^{-\gamma x}\left(1-\mathrm{e}^{-\gamma x}\right)^{\alpha-1}}{\mathrm{e}^{\left(1-\mathrm{e}^{-\gamma x}\right)^{\alpha}}}\left[2-\left(1-\mathrm{e}^{-\gamma x}\right)^{\alpha}\right], \tag{6}$$

respectively.

For $\gamma = 0.7$ (right panel) and $\gamma = 1$ (left panel) and varying α, some plots of (6) are reported in Figure 1. These plots indicate that the WEx-Exponential converges to a heavy-tailed distribution when α increases.

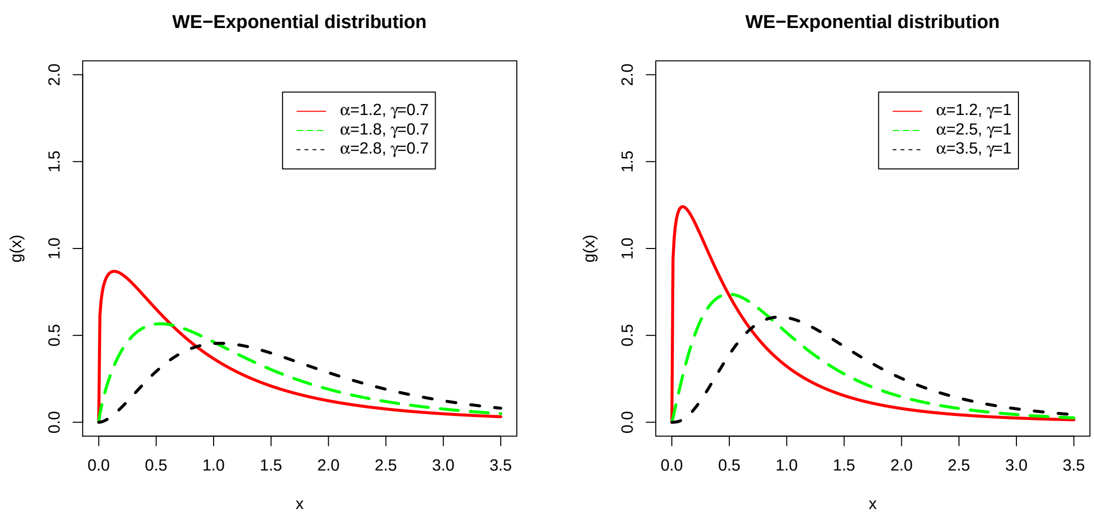

Figure 1: Plots of the WEx-Exponential density.

The Weibull and exponentiated Weibull (Ex-Weibull) densities are

$$f(x) = \alpha\gamma x^{\alpha-1}\mathrm{e}^{-\gamma x^{\alpha}}, \qquad \mathrm{x}>0$$

and

$$f(x) = \alpha\gamma\theta x^{\alpha-1}\mathrm{e}^{-\gamma x^{\alpha}}\left(1-\mathrm{e}^{-\gamma x^{\alpha}}\right)^{\theta-1},$$

respectively, where $\alpha > 0$ and $\gamma > 0$.

Plots of these two densities for $\gamma = 1$ varying α are given in Figure 2. The plots in Figures 1 and 2 show that the WEx-Exponential density has heavier tails than the other two densities.

3. Simulations of the Risk Measures

The *value at risk* (VaR) measure can provide the risk of loss for investments in the financial market. The VaR of a random variable *X* is its qth quantile

$$VaR_q(X) = F_X^{-1}(t), \qquad 0<q<1,$$

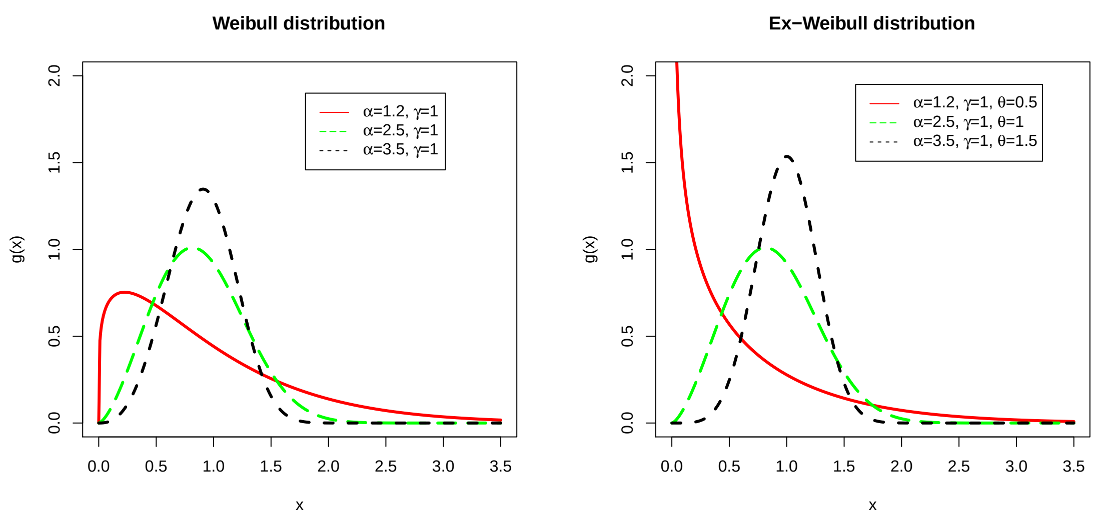

Figure 2: Plots of the Weibull and Ex-Weibull densities.

where t is the solution of $\log(1-t)-t=\log(1-q)$.

The *tail value at risk* (TVaR) describes the tail behavior beyond the threshold of VaR. The TVaR of X can be expressed from its pdf as

$$TVaR_q(X)=\frac{1}{1-q}\int_{VaR_q(X)}^{\infty} x\,g(x)dx.$$

Inserting (6) in the previous equation. the quantity TVaR follows after some algebra

$$TVaR_q(X)=\frac{\alpha\gamma}{1-q}\int_{VaR_q(X)}^{\infty}\frac{x\,\mathrm{e}^{-\gamma x}\left(1-\mathrm{e}^{-\gamma x}\right)^{\alpha-1}}{\mathrm{e}^{(1-\mathrm{e}^{-\gamma x})^{\alpha}}}\left[2-\left(1-\mathrm{e}^{-\gamma x}\right)^{\alpha}\right]dx,$$

and then

$$TVaR_q(X)=\frac{\alpha\gamma}{1-q}\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{(i+1)\alpha-1}{j}\int_{VaR_q(X)}^{\infty} x\mathrm{e}^{-\gamma(j+1)x}\left[2-\left(1-\mathrm{e}^{-\gamma x}\right)^{\alpha}\right]\mathrm{dx}.$$

Hence,

$$\begin{aligned}TVaR_q(X)=&\frac{2\alpha\gamma}{1-q}\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{(i+1)\alpha-1}{j}\Gamma(2,(j+1)\,\gamma\,VaR_q(X))\\&-\frac{\alpha\gamma}{1-q}\sum_{i,j,k=0}^{\infty}(-1)^{i+j+k}\binom{(i+1)\alpha-1}{j}\binom{\alpha}{k}\Gamma(2,(j+k+1)\,\gamma\,VaR_q(X)),\end{aligned}$$

where $\Gamma(p,w)=\int_w^{\infty}x^{p-1}\mathrm{e}^{-x}\mathrm{dx}$ is the upper incomplete gamma function.

Some simulations given below compare the VaR and TVaR quantities to check the heavy-tailed behavior of the WEx-Exponential and Weibull distributions. A model with the highest quantity values is called a heavy-tailed distribution. The simulation findings in Table 1 and displayed in Figure 3 indicate that the new distribution has higher measure values than the Weibull. In support to these results, the plots of the VaR and TVaR measures for the Weibull and WEx-Exponential models are reported in Figure 3.

Table 1: A simulation comparison of risk measures with *n*=150.

Model	scenario	Significance level	VaR	TVaR
Weibull	$\alpha = 1.2$ $\gamma = 1$	0.700	1.8748	4.3686
		0.750	2.2626	4.8301
		0.800	2.7608	5.4125
		0.850	3.4377	6.1900
		0.900	4.4506	7.3327
		0.950	6.3213	9.4002
		0.975	8.3431	11.5956
		0.999	19.2568	23.1389
WEx-Exponential	$\alpha = 1.2$ $\gamma = 1$	0.700	2.5415	5.9502
		0.750	3.0686	6.5812
		0.800	3.7448	7.3786
		0.850	4.6632	8.4457
		0.900	6.0398	10.0201
		0.950	8.5955	12.8896
		0.975	11.3842	15.9669
		0.999	26.8988	32.6021
Weibull	$\alpha = 0.9$ $\gamma = 1.2$	0.700	1.7019	4.6515
		0.750	2.1142	5.2019
		0.800	2.6600	5.9091
		0.850	3.4257	6.8725
		0.900	4.6150	8.3237
		0.950	6.9183	11.0397
		0.975	9.5293	14.0293
		0.999	25.0152	30.9963
WEx-Exponential	$\alpha = 0.9$ $\gamma = 1.2$	0.700	2.3655	6.8383
		0.750	2.9654	7.6754
		0.800	3.7651	8.7584
		0.850	4.8982	10.2459
		0.900	6.6802	12.5130
		0.950	10.2027	16.8352
		0.975	14.2986	21.7007
		0.999	40.2415	50.9370
Weibull	$\alpha = 0.5$ $\gamma = 0.8$	0.700	1.5871	6.4133
		0.750	2.1042	7.3296
		0.800	2.8361	8.5501
		0.850	3.9406	10.2846
		0.900	5.8050	13.0369
		0.950	9.8259	18.5757
		0.975	14.8990	25.1665
		0.999	52.2446	69.5608
WEx-Exponential	$\alpha = 0.5$ $\gamma = 0.8$	0.700	6.1138	20.1177
		0.750	7.8336	22.7547
		0.800	10.1794	26.2081
		0.850	13.5841	31.0224
		0.900	19.0909	38.4918
		0.950	30.3782	53.0891
		0.975	43.9853	69.9669
		0.999	136.7909	177.5591

4. Estimation and Simulations

Eight methods to estimate α and γ in the WE-Exponential model, and a simulation study to verity the accuracy of the estimators is addressed in this section.

Let $x_1, \cdots, x_m$ be observations from the WEx-Exponential density (6) and $x_{(1)}, \cdots, x_{(m)}$ be their order statistics.

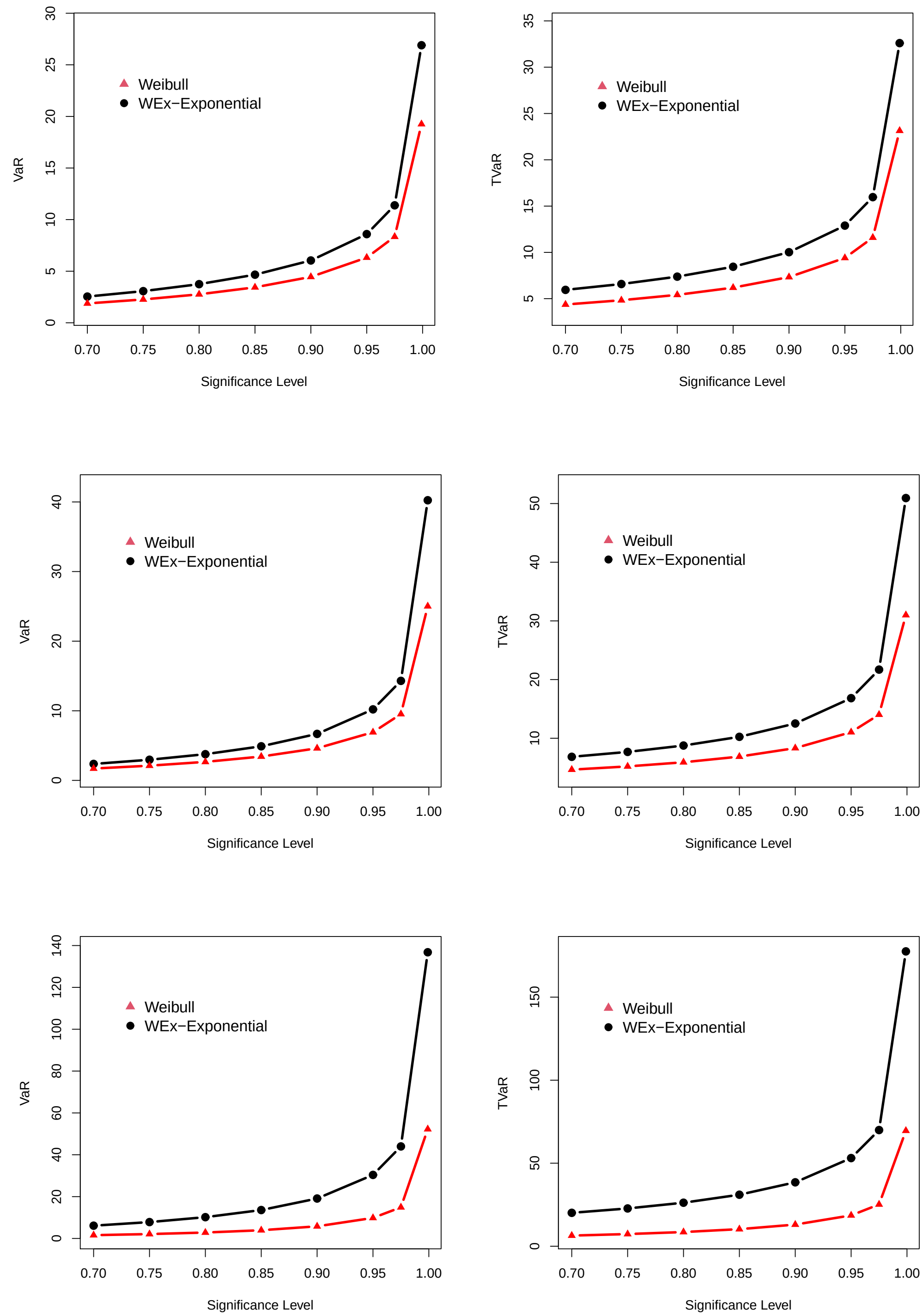

Figure 3: Plots of risk measures.

The log-likelihood function has the form

$$\ell(\alpha,\gamma) = m\log(\alpha\,\gamma) - \gamma\sum_{k=1}^{m}\log(x_k) + (\alpha-1)\sum_{k=1}^{m}\log\left(1-\mathrm{e}^{-\gamma \mathrm{x_k}}\right)$$
$$+\sum_{k=1}^{m}\log\left[2-\left(1-\mathrm{e}^{-\gamma \mathrm{x_k}}\right)^{\alpha}\right]-\sum_{k=1}^{m}\left(1-\mathrm{e}^{-\gamma \mathrm{x_k}}\right)^{\alpha}.$$

The maximum likelihood estimates (MLEs) are found by maximizing $\ell(\alpha,\gamma)$.
Cheng and Amin (1979) pioneered the maximum product of spacings (MPS) method. The MPS estimates (MPSEs) are determined by maximizing the function

$$M(\alpha,\gamma) = \frac{1}{m+1}\sum_{k=1}^{m+1}\log\left[\frac{1-\left(1-\mathrm{e}^{-\gamma \mathrm{x}_{(k-1)}}\right)^{\alpha}}{\mathrm{e}^{\left(1-\mathrm{e}^{-\gamma \mathrm{x}_{(k-1)}}\right)^{\alpha}}} - \frac{1-\left(1-\mathrm{e}^{-\gamma \mathrm{x}_{(k)}}\right)^{\alpha}}{\mathrm{e}^{\left(1-\mathrm{e}^{-\gamma \mathrm{x}_{(k)}}\right)^{\alpha}}}\right].$$

The percentile estimates (PCEs) follow by minimizing (Kao, 1958)

$$P(\alpha,\gamma) = \sum_{k=1}^{m}\left\{x_{(k)} + \frac{1}{\gamma}\log\left(1-\left[1-W_{-1}(\mathrm{e}\,(1-\mathrm{p_k}))\right]^{\frac{1}{\alpha}}\right)\right\}^2,$$

where $p_k = k/(n+1)$ is an unbiased estimate of $F(x_{(k)};\alpha,\gamma)$, and $W_{-1}(\cdot)$ denotes the negative branch of the Lambert function, i.e., the solution $w = W_0(z)$ of $w\,\exp(w) = z$ for $z \geq 0$.
The least squares estimates (LSEs) and weighted least squares estimates (WLSEs) (Swain et al., 1988) are found by minimizing

$$S(\alpha,\gamma) = \sum_{k=1}^{m}\delta_k\left\{\frac{1-\left(1-\mathrm{e}^{-\gamma \mathrm{x}_{(k)}}\right)^{\alpha}}{\mathrm{e}^{\left(1-\mathrm{e}^{-\gamma \mathrm{x}_{(k)}}\right)^{\alpha}}} - \frac{k}{m+1}\right\}^2,$$

where $\delta_k = 1$ for the LS method, and $\delta_k = [(m+2)\,(m+1)^2]/[k(m-k+1)]$ for the WLS method.
The Cramér-von Mises estimates (CVMEs) follow by minimizing

$$C(\alpha,\gamma) = \sum_{k=1}^{m}\left\{\frac{1-\left(1-\mathrm{e}^{-\gamma \mathrm{x}_{(k)}}\right)^{\alpha}}{\mathrm{e}^{\left(1-\mathrm{e}^{-\gamma \mathrm{x}_{(k)}}\right)^{\alpha}}} - \frac{2k-1}{2m}\right\}^2.$$

The Anderson-Darling estimates (ADEs) correspond to the minimum of the function

$$AD(\alpha,\gamma) = -m - \frac{1}{m}\sum_{k=1}^{m}(2k-1)[\log(F(x_{(k)}|\alpha,\gamma)) + \log(\bar{F}(x_{(m-k+1)}|\alpha,\gamma))].$$

In a similar manner, the right-tail Anderson-Darling estimates (RADEs) are found by minimizing

$$RAD(\alpha,\gamma) = \frac{m}{2} - 2\sum_{k=1}^{n}F\left(x_{(k)}\mid\alpha,\gamma\right) - \frac{1}{m}\sum_{k=1}^{n}(2k-1)\log\overline{F}\left(x_{(m-k+1)}\mid\alpha,\gamma\right).$$

The behavior of the estimates is explored using a detailed simulation study using the average mean square errors (MSEs), MSEs $= \frac{1}{N}\sum_{i=1}^{N}(\widehat{\boldsymbol{\phi}}-\boldsymbol{\phi})^2$, the average absolute biases (ABBs) ($|\text{Bias}(\widehat{\boldsymbol{\phi}})|$), $|\text{Bias}(\widehat{\boldsymbol{\phi}})| = \frac{1}{N}\sum_{i=1}^{N}|\widehat{\boldsymbol{\phi}}-\boldsymbol{\phi}|$, and the average of the mean relative errors (MREs), MREs $= \frac{1}{N}\sum_{i=1}^{N}|\widehat{\boldsymbol{\phi}}-\boldsymbol{\phi}|/\boldsymbol{\phi}$.

The chosen values for the parameters are: $\alpha = \{0.5, 1.5, 3\}$ and $\gamma = \{0.5, 0.75, 1.5, 4\}$. We generate $N = 5,000$ random samples using the R software for the sizes $n =$ 20, 50, 150, and 300 from the WEx-Exponential distribution using its quantile function (qf)

$$Q(p) = \frac{-1}{\gamma}\log\left(1-\left[1-W_0(\mathrm{e}\,(1-\mathrm{p}))\right]^{\frac{1}{\alpha}}\right).$$

For each sample and configuration, the parameters λ and α are estimated using the eight previous estimators: MLEs, MPSEs, LSEs, CVMEs, WLSEs, PCEs, ADEs, and RADEs, where the MSEs, ABBs, and MREs of the estimates are computed. The simulation findings are reported in Tables 2–5.

It is noted from the figures in these tables that the property of consistency holds for all estimation methods, i.e., the MSEs and MREs decrease when n increases for all cases implying that the eight estimators are consistent for the

WEx-Exponential parameters. Further, all estimation methods give small biases and MSEs for all cases, which means that these estimates are quite reliable. On the other hand, the proposed estimators behaved as asymptotically unbiased estimators since the biases are very close to 0 when n increases. Additionally, their performance in terms of MSEs, ABBs, and MREs, from the best to the worst, is MLE, MPSEs, WLSEs, ADEs, CVMEs, LSEs, RADEs, and PCEs for all scenarios.

5. The Log-WEx-Exponential Regression Model

If X has density (6), the random variable $Y = \log(X)$ defines the *log-WEx-Exponential* (LWEx-Exponential) distribution. By setting $\gamma = \mathrm{e}^{-\mu}$, the pdf of Y (for $y \in \mathbb{R}$) has the form

$$f(y) = \frac{\alpha \, \exp(y-\mu) \, \{1-\exp[-\exp(y-\mu)]\}^{\alpha-1}}{\exp[\{1-\exp[-\exp(y-\mu)]\}^{\alpha}]} \, \{2-\{1-\exp[-\exp(y-\mu)]\}^{\alpha}\}, \tag{7}$$

where $\mu \in \mathbb{R}$ and $\alpha > 0$. We can write $Y \sim$ LWEx-Exponential(α, μ). The survival function (sf) of Y becomes

$$S(y) = \frac{1-\{1-\exp[-\exp(y-\mu)]\}^{\alpha}}{\exp[\{1-\exp[-\exp(y-\mu)]\}^{\alpha}]}.$$

The random variable $Z = (Y - \mu)$ has pdf (for $z \in \mathbb{R}$)

$$f(z) = \frac{\alpha \, \exp(z) \, \{1-\exp[-\exp(z)]\}^{\alpha-1}}{\exp[\{1-\exp[-\exp(z)]\}^{\alpha}]} \, \{2-\{1-\exp[-\exp(z)]\}^{\alpha}\}. \tag{8}$$

Let y_i be the response variable having density (7) with unknown parameters $\mu_i \in \mathbb{R}$ and $\alpha > 0$, and $\mathbf{v}_i^\top = (v_{i1}, \cdots, v_{ip})$ be the explanatory variable vector. We consider the regression model

$$y_i = \boldsymbol{v}_i^\top \boldsymbol{\beta} + z_i, \; i = 1, \ldots, n, \tag{9}$$

where $\mu_i = \mathbf{v}_i^\top \boldsymbol{\beta}$. The vector $\mu = (\mu_1, \ldots, \mu_n)^\top$ is defined as $\mu = \mathbf{V}\boldsymbol{\beta}$, where $\mathbf{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)^\top$ is a known model matrix.

Consider that the observed (t_i) and censored (c_i) times are independent. The observations $(y_1, \mathbf{v}_1), \cdots, (y_n, \mathbf{v}_n)$ are assumed independent, where $y_i = \min\{\log(t_i), \log(c_i)\}$.

The log-likelihood function for $\eta = (\boldsymbol{\beta}^\top, \alpha)^\top$ has the form

$$\begin{aligned} l(\eta) \;\; = \;\; & r \, \log(\alpha) + \sum_{i \in F} y_i - \mathbf{v}_i^\top \boldsymbol{\beta} + (\alpha - 1) \sum_{i \in F} \log\left\{1 - \exp\left[-\exp\left(y_i - \mathbf{v}_i^\top \boldsymbol{\beta}\right)\right]\right\} \\ & + \sum_{i \in F} \log\left[2 - \left\{1 - \exp\left[-\exp\left(y_i - \mathbf{v}_i^\top \boldsymbol{\beta}\right)\right]\right\}^{\alpha}\right] - \sum_{i \in F} \left\{1 - \exp\left[-\exp\left(y_i - \mathbf{v}_i^\top \boldsymbol{\beta}\right)\right]\right\}^{\alpha} \\ & + \sum_{i \in C} \log\left[1 - \left\{1 - \exp\left[-\exp\left(y_i - \mathbf{v}_i^\top \boldsymbol{\beta}\right)\right]\right\}^{\alpha}\right] \\ & - \sum_{i \in C} \left\{1 - \exp\left[-\exp\left(y_i - \mathbf{v}_i^\top \boldsymbol{\beta}\right)\right]\right\}^{\alpha}, \end{aligned} \tag{10}$$

where F and C denote the sets of the observed and censoring times, respectively, and r and c to their cardinalities.

5.1. Residuals Analysis

The martingale residuals (Fleming and Harrington, 1994) to investigate the true functional form of a particular covariate of the LWEx-Exponential regression models are

$$r_{M_i} = \begin{cases} 1 - u_i^{\hat{\alpha}} + \log\left(1 - u_i^{\hat{\alpha}}\right) & \text{if } i \in F \\ - u_i^{\hat{\alpha}} + \log\left(1 - u_i^{\hat{\alpha}}\right) & \text{if } i \in C, \end{cases}$$

where $u_i = 1 - \exp\left[-\exp\left(z_i\right)\right]$, and $z_i = y_i - \mathbf{v}_i^T \hat{\boldsymbol{\beta}}$.
These residuals take values in $[-\infty, 1]$ for uncensored observations and $[-\infty, 0]$ for censored observations. They are not symmetrically distributed around zero and then should be transformed to be closer to the normal distribution (Collett, 2003). The modified deviance residuals (r_{D_i}) can be used for solving this problem. For the LWEx-Exponential regression models, they can be expressed as

$$r_{D_i} = \begin{cases} \operatorname{sign}(r_{M_i}) \left\{-2\left[1 - u_i^{\hat{\alpha}} + \log\left(1 - u_i^{\hat{\alpha}}\right)\right] + \log\left[u_i^{\hat{\alpha}} - \log\left(1 - u_i^{\hat{\alpha}}\right)\right]\right\}^{1/2} & \text{if } i \in F \\ \operatorname{sign}(r_{M_i}) \left\{-2\left[1 - u_i^{\hat{\alpha}} + \log\left(1 - u_i^{\hat{\alpha}}\right)\right]\right\}^{1/2} & \text{if } i \in C. \end{cases}$$

5.2. Heart Transplantation Data

Consider 103 patients, 69 of them received heart transplants, where the number of deaths is 75 (Kalbfleisch and Prentice, 1980). The response variable t_i is the time (in days) between the transplant and death. The explanatory variables are (for $i = 1, \ldots, 103$): x_{i1} = age (at acceptance), x_{i2} = previous surgery (1 = yes; 0 = no), and x_{i3} = transplant (1 = yes; 0 = no).
Consider the regression model

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + z_i,$$

where the random variable $y_i = \log(t_i)$ has density (7).

The maximization of (10) using the R software gives the MLEs of the parameters. The Akaike information criterion (AIC) and Bayesian information criterion (BIC) compare the proposed model with the Log-Weibull, the LEx-Exponential, the log-odd-logistic Weibull (LOLLW) (da Cruz et al., 2016), the log Topp-Leone Fréchet (LTLFr) (Yousof et al., 2018) and Cox proportional hazards (Cox, 1972) regression models. Table 6 reports the MLEs, their standard errors (SEs) in parentheses and p-values in $[\cdot]$, and the criteria values for the fitted models. The explanatory variables x_1, x_2 and x_3 are significant at 5% for the fitted LWEx-Exponential regression model.
Figure 4 reports the plot of the modified deviance residuals (r_{D_i}) versus the index, which reveals that the LWEx-Exponential regression model fits these data well.

6. Newspaper Advertising Data

The histogram of the newspaper advertising data in Figure 5 shows that they are right-skewed and heavy-tailed. These features can be effectively managed by the WEx-Exponential distribution (see Figure 1).

The MLEs and SEs (in parentheses) of the fitted models are reported in Table 7. The AIC, BIC, Hannan-Quinn (HQIC), Corrected Akaike (CAIC), Anderson-Darling (AD), Cramér-von Mises (CM) and Kolmogorov-Smirnov (KS) (and its p-vlaue) reported in Table 8 indicate that the proposed distribution is the best model for the current data. It gives KS $= 0.026$ against KS $= 0.043$ for the second-best distribution. Further, the new model yields a p-value 0.973, which is relatively close to one (maximum value), against the second-best model with p-value $= 0.939$.
The estimated cdf and pdf from the fitted WEx-Exponential distribution are

$$G(x; 1.977, 0.025) = 1 - \frac{1 - \left(1 - \mathrm{e}^{-0.025\mathrm{x}}\right)^{1.977}}{\mathrm{e}^{\left(1 - \mathrm{e}^{-0.025\mathrm{x}}\right)^{1.977}}}, \quad x \geq 0,$$

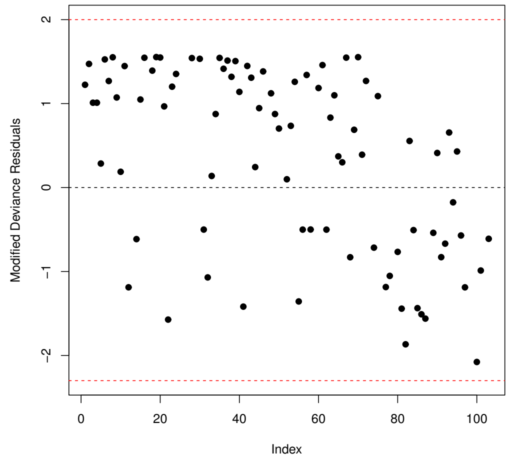

Figure 4: Plot of r_{D_i} versus Index.

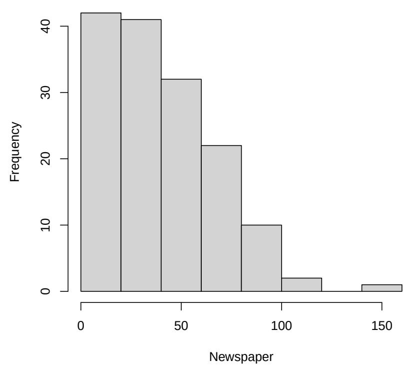

Figure 5: Histogram of the newspaper advertising data.

and

$$g(x;1.977,0.025)=\frac{0.049\mathrm{e}^{-0.025\mathrm{x}}\left(1-\mathrm{e}^{-0.025\mathrm{x}}\right)^{0.977}}{e^{(1-\mathrm{e}^{-0.025\mathrm{x}})^{1.977}}}\left[2-\left(1-\mathrm{e}^{-0.025\mathrm{x}}\right)^{1.977}\right],\quad x>0,$$

respectively.

Then, the plots of the estimated pdf, cdf, and probability-probability (PP) from the fitted new distribution and Kaplan–Meier plots are displayed in Figures 6 and 7. These plots reveal that the WEx-Exponential curves are closer to the corresponding empirical pdf, cdf, Kaplan–Meier survival, and PP plots. These findings show that the new distribution fits the current data well.

7. Concluding Remarks

The heavy-tailed distributions have great applicability in actuarial and financial sciences. A new weighted exponentiated-exponential (WEx-Exponential) distribution was proposed, which has a heavier tail than the tails of the Weibull and exponentiated-Weibull models. A comprehensive simulation study is conducted to determine the risk measures. The parameters of the new distribution were estimated through eight different methods and some Monte Carlo simulations were conducted. It was defined the Log-WEx-Exponential regression model with a residual analysis to check the model assumptions. We presented two real applications to heart transplantation data and newspaper advertising data to show empirically the utility of the proposed models.

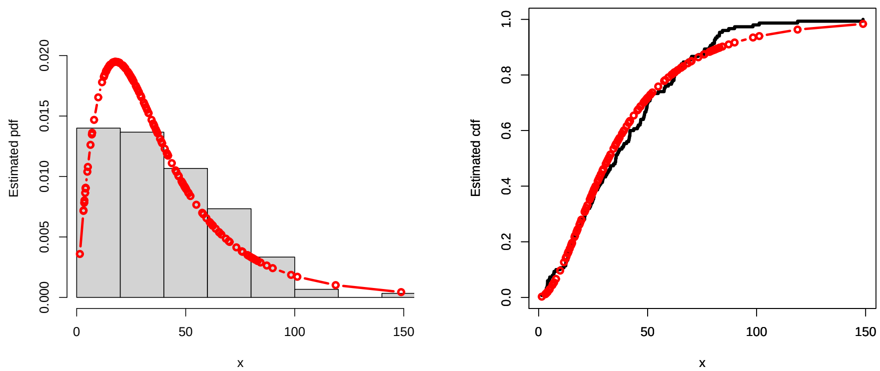

Figure 6: Estimated pdf and cdf of the WEx-Exponential distribution and empirical cdf.

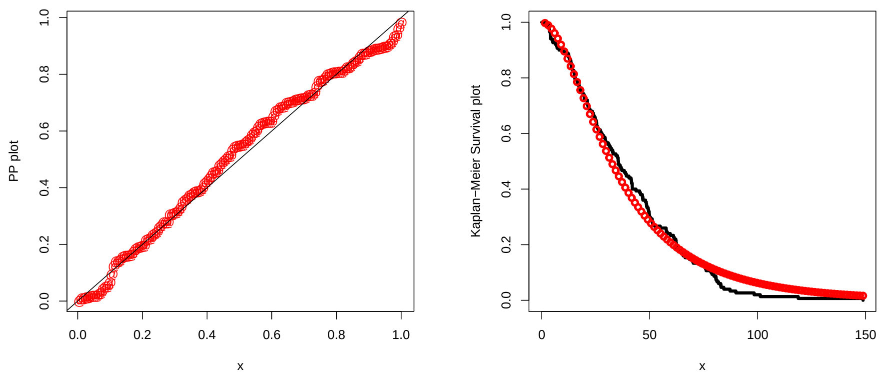

Figure 7: PP and Kaplan-Meier survival plots of the WEx-Exponential distribution.

References

1. Afify, A. Z., Gemeay, A. M., Alfaer, N. M., Cordeiro, G. M., and Hafez, E. H. (2022). Power-modified Kies-exponential distribution: properties, classical and Bayesian inference with an application to engineering data. *Entropy*, 24:883.
2. Afify, A. Z. and Mohamed, O. A. (2020). A new three-parameter exponential distribution with variable shapes for the hazard rate: Estimation and applications. *Mathematics*, 8:135.
3. Afify, A. Z., Suzuki, A. K., Zhang, C., and Nassar, M. (2021). On three-parameter exponential distribution: properties, Bayesian and non-Bayesian estimation based on complete and censored samples. *Communications in Statistics-Simulation and Computation*, 50:3799–3819.
4. Ahmad, Z., Mahmoudi, E., and Dey, S. (2020a). A new family of heavy tailed distributions with an application to the heavy tailed insurance loss data. *Communications in Statistics-Simulation and Computation*, 49:1–24.
5. Ahmad, Z., Mahmoudi, E., Dey, S., and Khosa, S. K. (2020b). Modeling vehicle insurance loss data using a new member of TX family of distributions. *Journal of Statistical Theory and Applications*, 19:133–147.
6. Ahmad, Z., Mahmoudi, E., and Hamedani, G. G. (2020c). A class of claim distributions: Properties, characterizations and applications to insurance claim data. *Communications in Statistics-Theory and Methods*, 49:1–26.
7. Alzaatreh, A., Lee, C., and Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, 71:63–79.
8. Bhati, D. and Ravi, S. (2018). On generalized log-Moyal distribution: a new heavy tailed size distribution. *Insurance: Mathematics and Economics*, 79:247–259.
9. Cheng, R. and Amin, N. (1979). Maximum product of spacings estimation with application to the lognormal distribution. *Mathematical Report 79-1; University of Wales: Cardiff, UK*.
10. Collett, D. (2003). *Modelling Survival Data in Medical Research*. Chapman and Hall/CRC, London.

11. Cooray, K. and Ananda, M. M. (2005). Modeling actuarial data with a composite lognormal-Pareto model. *Scandinavian Actuarial Journal*, 2005:321–334.
12. Cox, D. R. (1972). Regression models and life tables (with discussion). *Journal of the Royal Statistical Society Series B (Statistical Methodology)*, 34:187–220.
13. da Cruz, J. N., Ortega, E. M. M., and Cordeiro, G. M. (2016). The log-odd log-logistic Weibull regression model: modelling, estimation, influence diagnostics and residual analysis. *Journal of Statistical Computation and Simulation*, 86:1516–1538.
14. Fleming, T. R. and Harrington, D. P. (1994). *Counting Process and Survival Analysis*. Wiley, New York.
15. Kalbfleisch, J. D. and Prentice, R. L. (1980). *The Statistical Analysis of Failure Time Data*. John Wiley, New York.
16. Kao, J. (1958). Computer methods for estimating Weibull parameters in reliability studies. *IRE Reliab. Qual. Control*, 13:15–22.
17. Mansour, M. M., Abd Elrazik, E. M., Afify, A. Z., Ahsanullah, M., and Altun, E. (2019). The transmuted transmuted-G family: Properties and applications. *J. Nonlinear Sci. Appl.*, 12:217–229.
18. Marshall, A. W. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, 84:641–652.
19. Nassar, M., Kumar, D., Dey, S., Cordeiro, G. M., and Afify, A. Z. (2019). The Marshall-Olkin alpha power family of distributions with applications. *J. Comput. Appl. Math.*, 351:41–53.
20. Punzo, A. (2019). A new look at the inverse gaussian distribution with applications to insurance and economic data. *Journal of Applied Statistics*, 46:1260–1287.
21. Punzo, A., Bagnato, L., and Maruotti, A. (2018). Compound unimodal distributions for insurance losses. *Insurance: Mathematics and Economics*, 81:95–107.
22. Rasekhi, M., Alizadeh, M., Altun, E., Hamedani, G. G., Afify, A. Z., and Ahmad, M. (2017). The modified exponential distribution with applications. *Pak. J. Stat.*, 33:383–398.
23. Swain, J. J., Venkatraman, S., and Wilson, J. R. (1988). Least-squares estimation of distribution functions in johnson's translation system. *J. Stat. Comput. Simul.*, 29:271–297.
24. Yousof, H. M., Jahanshahi, S. M. A., Ramires, T. G., Aryal, G. R., and Hamedani, G. G. (2018). Anew distribution for extreme values: regression model, characterizations and applications. *Journal of Data Science*, 16:677–706.

Table 2: Simulation results for $\alpha = 0.5$ and $\gamma = 0.5, 0.75$.

n	Stat.	Par.	MLEs	MPSEs	LSEs	CVMEs	WLSEs	PCEs	ADEs	RADEs
$\alpha = 0.5, \gamma = 0.5$										
20	MSEs	α	0.00620	0.00703	0.00743	0.00799	0.00716	0.03248	0.00580	0.00835
		γ	0.02941	0.03344	0.03774	0.04383	0.03211	0.06383	0.03165	0.03230
	ABBs	α	0.07872	0.08382	0.08620	0.08939	0.08464	0.18023	0.07614	0.09137
		γ	0.17150	0.18287	0.19428	0.20936	0.17918	0.25266	0.17789	0.17971
	MREs	α	0.15744	0.16764	0.17239	0.17877	0.16929	0.36046	0.15228	0.18274
		γ	0.34301	0.36575	0.38856	0.41873	0.35836	0.50531	0.35579	0.35943
50	MSEs	α	0.00252	0.00268	0.00306	0.00324	0.00267	0.01535	0.00244	0.00325
		γ	0.01114	0.01362	0.01592	0.01579	0.01318	0.02604	0.01296	0.01231
	ABBs	α	0.05018	0.05177	0.05535	0.05695	0.05171	0.12389	0.04942	0.05703
		γ	0.10552	0.11670	0.12617	0.12567	0.11482	0.16136	0.11386	0.11097
	MREs	α	0.10036	0.10354	0.11070	0.11390	0.10342	0.24777	0.09884	0.11406
		γ	0.21105	0.23340	0.25234	0.25133	0.22965	0.32272	0.22771	0.22194
150	MSEs	α	0.00076	0.00080	0.00093	0.00108	0.00088	0.00825	0.00082	0.00100
		γ	0.00358	0.00379	0.00562	0.00601	0.00460	0.01055	0.00446	0.00436
	ABBs	α	0.02753	0.02823	0.03058	0.03282	0.02959	0.09081	0.02860	0.03161
		γ	0.05983	0.06158	0.07498	0.07753	0.06781	0.10270	0.06677	0.06602
	MREs	α	0.05506	0.05646	0.06115	0.06564	0.05917	0.18162	0.05720	0.06322
		γ	0.11966	0.12316	0.14996	0.15506	0.13562	0.20540	0.13354	0.13204
300	MSEs	α	0.00042	0.00036	0.00052	0.00050	0.00040	0.00448	0.00043	0.00056
		γ	0.00188	0.00168	0.00270	0.00283	0.00211	0.00488	0.00206	0.00216
	ABBs	α	0.02040	0.01889	0.02273	0.02246	0.01990	0.06695	0.02064	0.02365
		γ	0.04336	0.04104	0.05196	0.05324	0.04592	0.06985	0.04539	0.04645
	MREs	α	0.04079	0.03779	0.04546	0.04491	0.03979	0.13390	0.04127	0.04730
		γ	0.08672	0.08208	0.10393	0.10649	0.09185	0.13970	0.09078	0.09290
$\alpha = 0.5$ and $\gamma = 0.75$										
20	MSEs	α	0.00606	0.00664	0.00778	0.00795	0.00716	0.03202	0.00603	0.00744
		γ	0.06161	0.07494	0.08776	0.08871	0.07697	0.12492	0.06880	0.06745
	ABBs	α	0.07782	0.08148	0.08818	0.08915	0.08463	0.17895	0.07763	0.08627
		γ	0.24822	0.27375	0.29624	0.29784	0.27743	0.35344	0.26230	0.25972
	MREs	α	0.15563	0.16297	0.17636	0.17830	0.16926	0.35790	0.15527	0.17253
		γ	0.33095	0.36500	0.39499	0.39712	0.36990	0.47125	0.34973	0.34629
50	MSEs	α	0.00223	0.00263	0.00288	0.00313	0.00251	0.01864	0.00242	0.00294
		γ	0.02421	0.02863	0.03407	0.03572	0.02856	0.06379	0.02785	0.02715
	ABBs	α	0.04720	0.05124	0.05367	0.05597	0.05013	0.13652	0.04915	0.05419
		γ	0.15560	0.16921	0.18457	0.18901	0.16899	0.25257	0.16688	0.16477
	MREs	α	0.09439	0.10248	0.10734	0.11193	0.10026	0.27303	0.09830	0.10838
		γ	0.20746	0.22561	0.24610	0.25201	0.22531	0.33676	0.22251	0.21969
150	MSEs	α	0.00073	0.00085	0.00100	0.00099	0.00087	0.00784	0.00073	0.00099
		γ	0.00808	0.00883	0.01175	0.01252	0.00959	0.02152	0.00912	0.00873
	ABBs	α	0.02701	0.02919	0.03169	0.03145	0.02952	0.08856	0.02711	0.03148
		γ	0.08990	0.09398	0.10838	0.11191	0.09791	0.14669	0.09551	0.09344
	MREs	α	0.05401	0.05837	0.06338	0.06290	0.05904	0.17713	0.05422	0.06295
		γ	0.11987	0.12531	0.14451	0.14922	0.13055	0.19558	0.12734	0.12459
300	MSEs	α	0.00036	0.00042	0.00055	0.00053	0.00040	0.00407	0.00039	0.00055
		γ	0.00375	0.00477	0.00718	0.00658	0.00477	0.01122	0.00473	0.00474
	ABBs	α	0.01884	0.02061	0.02345	0.02297	0.02008	0.06383	0.01978	0.02340
		γ	0.06124	0.06909	0.08473	0.08112	0.06907	0.10592	0.06878	0.06887
	MREs	α	0.03769	0.04123	0.04689	0.04593	0.04015	0.12766	0.03955	0.04679
		γ	0.08165	0.09212	0.11297	0.10816	0.09210	0.14122	0.09171	0.09183

Table 3: Simulation results for $\alpha = 0.5$ and $\gamma = 1.5, 4$.

n	Stat.	Par.	MLEs	MPSEs	LSEs	CVMEs	WLSEs	PCEs	ADEs	RADEs
						$\alpha = 0.5, \gamma = 1.5$				
20	MSEs	α	0.00675	0.00743	0.00779	0.00887	0.00669	0.03105	0.00628	0.00857
		γ	0.23202	0.28677	0.35671	0.36350	0.30029	0.45351	0.25579	0.29229
	ABBs	α	0.08216	0.08620	0.08825	0.09420	0.08177	0.17620	0.07927	0.09257
		γ	0.48169	0.53551	0.59725	0.60291	0.54798	0.67343	0.50575	0.54063
	MREs	α	0.16432	0.17241	0.17650	0.18840	0.16353	0.35239	0.15854	0.18515
		γ	0.32112	0.35700	0.39817	0.40194	0.36532	0.44896	0.33717	0.36042
50	MSEs	α	0.00249	0.00250	0.00325	0.00309	0.00252	0.01654	0.00261	0.00325
		γ	0.09288	0.12020	0.15705	0.15534	0.11238	0.24459	0.11800	0.10935
	ABBs	α	0.04992	0.04995	0.05698	0.05563	0.05016	0.12861	0.05109	0.05697
		γ	0.30476	0.34670	0.39630	0.39413	0.33523	0.49457	0.34351	0.33069
	MREs	α	0.09985	0.09991	0.11396	0.11125	0.10031	0.25722	0.10217	0.11394
		γ	0.20317	0.23113	0.26420	0.26275	0.22349	0.32971	0.22901	0.22046
150	MSEs	α	0.00074	0.00078	0.00109	0.00112	0.00085	0.00843	0.00079	0.00102
		γ	0.03136	0.03756	0.05086	0.05108	0.03615	0.09084	0.03725	0.03784
	ABBs	α	0.02727	0.02787	0.03309	0.03340	0.02911	0.09182	0.02816	0.03196
		γ	0.17710	0.19381	0.22553	0.22602	0.19014	0.30140	0.19301	0.19452
	MREs	α	0.05454	0.05574	0.06618	0.06680	0.05822	0.18363	0.05631	0.06393
		γ	0.11807	0.12921	0.15035	0.15068	0.12676	0.20094	0.12867	0.12968
300	MSEs	α	0.00039	0.00042	0.00049	0.00050	0.00042	0.00457	0.00043	0.00047
		γ	0.01573	0.01642	0.02452	0.02248	0.02064	0.04572	0.02045	0.01679
	ABBs	α	0.01987	0.02045	0.02210	0.02232	0.02057	0.06759	0.02070	0.02178
		γ	0.12543	0.12815	0.15660	0.14994	0.14365	0.21383	0.14301	0.12958
	MREs	α	0.03973	0.04090	0.04420	0.04463	0.04113	0.13518	0.04139	0.04357
		γ	0.08362	0.08544	0.10440	0.09996	0.09577	0.14256	0.09534	0.08639
						$\alpha = 0.5, \gamma = 4$				
20	MSEs	α	0.00612	0.00710	0.00788	0.00834	0.00662	0.03213	0.00650	0.00810
		γ	1.90409	2.10270	2.38104	2.97101	2.16293	3.31341	1.80495	1.82101
	ABBs	α	0.07824	0.08429	0.08877	0.09130	0.08135	0.17926	0.08063	0.08997
		γ	1.37989	1.45007	1.54306	1.72366	1.47069	1.82028	1.34348	1.34945
	MREs	α	0.15648	0.16858	0.17755	0.18261	0.16270	0.35852	0.16127	0.17995
		γ	0.34497	0.36252	0.38577	0.43092	0.36767	0.45507	0.33587	0.33736
50	MSEs	α	0.00237	0.00272	0.00313	0.00293	0.00272	0.01776	0.00249	0.00295
		γ	0.68468	0.82296	1.03794	1.06375	0.81651	1.72583	0.82939	0.77294
	ABBs	α	0.04872	0.05215	0.05595	0.05417	0.05215	0.13327	0.04986	0.05431
		γ	0.82746	0.90717	1.01879	1.03138	0.90361	1.31371	0.91071	0.87917
	MREs	α	0.09743	0.10429	0.11189	0.10833	0.10430	0.26654	0.09971	0.10863
		γ	0.20686	0.22679	0.25470	0.25785	0.22590	0.32843	0.22768	0.21979
150	MSEs	α	0.00069	0.00082	0.00102	0.00108	0.00080	0.00722	0.00083	0.00101
		γ	0.23210	0.22136	0.33129	0.34647	0.28045	0.59959	0.27358	0.25609
	ABBs	α	0.02630	0.02858	0.03200	0.03288	0.02830	0.08499	0.02887	.03183
		γ	0.48176	0.47048	0.57558	0.58861	0.52958	0.77433	0.52305	0.50606
	MREs	α	0.05259	0.05715	0.06400	0.06576	0.05659	0.16998	0.05774	0.06366
		γ	0.12044	0.11762	0.14390	0.14715	0.13239	0.19358	0.13076	0.12651
300	MSEs	α	0.00036	0.00042	0.00054	0.00051	0.00042	0.00452	0.00041	0.00050
		γ	0.10752	0.12371	0.19136	0.17592	0.13260	0.31601	0.13624	0.14278
	ABBs	α	0.01888	0.02042	0.02324	0.02254	0.02058	0.06720	0.02027	0.02233
		γ	0.32790	0.35173	0.43744	0.41943	0.36414	0.56215	0.36910	0.37786
	MREs	α	0.03777	0.04084	0.04648	0.04509	0.04117	0.13440	0.04053	0.04466
		γ	0.08197	0.08793	0.10936	0.10486	0.09103	0.14054	0.09228	0.09447

Table 4: Simulation results for $\alpha = 1.5, 3$ and $\gamma = 0.75$.

n	Stat.	Par.	MLEs	MPSEs	LSEs	CVMEs	WLSEs	PCEs	ADEs	RADEs
					$\alpha = 1.5, \gamma = 0.75$					
20	MSEs	α	0.09111	0.10044	0.12196	0.11745	0.09997	0.21706	0.08620	0.12269
		γ	0.02923	.03653	.03748	0.03809	0.03325	0 .05667	.02912	0.03450
	ABBs	α	0.30185	0.31693	0.34923	0.34271	0.31618	0.46590	0.29360	0.35027
		γ	0.17098	0.19112	0.19361	0.19517	0.18234	0.23805	.17064	0.18574
	MREs	α	0.20123	0.21129	0.23282	0.22847	0.21078	0.31060	0.19574	0.23351
		γ	0.22797	0.25482	0.25814	0.26023	0.24313	0.31741	0.22752	0.24765
50	MSEs	α	0.03129	0.03637	0.04868	0.04779	0.03908	0.10754	0.03556	0.04830
		γ	0.01088	0.01357	0.01624	0.01499	0.01390	0.02453	0.01095	0.01335
	ABBs	α	0.17689	0.19071	0.22064	0.21861	0.19769	0.32793	0.18858	0.21977
		γ	0.10431	0.11650	0.12743	0.12243	0.11788	0.15663	0.10466	0.11552
	MREs	α	0.11793	0.12714	0.14710	0.14574	0.13179	0.21862	0.12572	0.14651
		γ	0.13908	0.15533	0.16991	0.16324	0.15718	0.20884	0.13955	0.15403
150	MSEs	α	0.01081	0.01154	0.01453	0.01500	0.01188	0.04941	0.01108	0.01566
		γ	0.00358	0.00380	0.00445	0.00482	0.00415	0.00889	0.00383	0.00437
	ABBs	α	0.10395	0.10743	0.12055	0.12248	0.10899	0.22229	0.10527	0.12515
		γ	0.05981	0.06165	0.06671	0.06945	0.06445	0.09430	0.06190	0.06611
	MREs	α	0.06930	0.07162	0.08037	0.08166	0.07266	0.14819	0.07018	0.08343
		γ	0.07975	0.08220	0.08894	0.09260	0.08593	0.12573	0.08254	0.08815
300	MSEs	α	0.00502	0.00548	0.00735	0.00771	0.00580	0.02561	0.00613	0.00805
		γ	0.00182	0.00188	0.00233	0.00262	0.00191	0.00487	0.00213	0.00222
	ABBs	α	0.07085	0.07405	0.08571	0.08781	0.07614	0.16003	0.07827	0.08970
		γ	0.04267	0.04339	0.04827	0.05115	0.04367	0.06976	0.04613	0.04709
	MREs	α	0.04724	0.04937	0.05714	0.05854	0.05076	0.10669	0.05218	0.05980
		γ	0.05689	0.05785	0.06435	0.06820	0.05823	0.09302	0.06150	0.06279
					$\alpha = 3, \gamma = 0.75$					
20	MSEs	α	0.42040	0.57572	0.66410	0.68775	0.56203	0.91309	0.47120	0.78675
		γ	0.01948	0.02762	0.02837	0.02654	0.02455	0.03546	0.01977	0.02618
	ABBs	α	0.64838	0.75876	0.81493	0.82931	0.74969	0.95556	0.68644	0.88699
		γ	0.13957	0.16619	0.16843	0.16292	0.15668	0.18832	0.14059	0.16181
	MREs	α	0.21613	0.25292	0.27164	0.27644	0.24990	0.31852	0.22881	0.29566
		γ	0.18609	0.22158	0.22457	0.21722	0.20890	0.25109	0.18745	0.21575
50	MSEs	α	0.20123	0.19383	0.27111	0.25936	0.23298	0.47842	0.20439	0.26670
		γ	0.00791	0.00900	0.01080	0.01140	0.00961	0.01666	0.00898	0.00983
	ABBs	α	0.44859	0.44026	0.52068	0.50927	0.48268	0.69168	0.45210	0.51643
		γ	0.08893	0.09486	0.10391	0.10678	0.09803	0.12909	0.09476	0.09913
	MREs	α	0.14953	0.14675	0.17356	0.16976	0.16089	0.23056	0.15070	0.17214
		γ	0.11858	0.12648	0.13855	0.14237	0.13071	0.17212	0.12635	0.13217
150	MSEs	α	0.05420	0.06578	0.08783	0.08853	0.07548	0.18760	0.06248	0.08960
		γ	0.00242	0.00257	0.00343	0.00341	0.00320	0.00597	0.00276	0.00328
	ABBs	α	0.23281	0.25647	0.29636	0.29753	0.27474	0.43312	0.24996	0.29934
		γ	0.04921	0.05068	0.05853	0.05842	0.05658	0.07725	0.05250	0.05724
	MREs	α	0.07760	0.08549	0.09879	0.09918	0.09158	0.14437	0.08332	0.09978
		γ	0.06561	0.06757	0.07804	0.07789	0.07544	0.10299	0.07000	0.07633
300	MSEs	α	0.02736	0.03165	0.04461	0.04059	0.03412	0.10216	0.03587	0.04551
		γ	0.00123	0.00128	0.00175	0.00174	0.00138	0.00281	0.00143	0.00158
	ABBs	α	0.16540	0.17790	0.21121	0.20147	0.18471	0.31962	0.18938	0.21334
		γ	0.03510	0.03580	0.04183	0.04173	0.03710	0.05302	0.03775	0.03974
	MREs	α	0.05513	0.05930	0.07040	0.06716	0.06157	0.10654	0.06313	0.07111
		γ	0.04680	0.04773	0.05577	0.05564	0.04946	0.07070	0.05033	0.05299

Table 5: Simulation results for $\alpha = 3$ and $\gamma = 1.5, 4$.

n	Stat.	Par.	MLEs	MPSEs	LSEs	CVMEs	WLSEs	PCEs	ADEs	RADEs
					$\alpha = 3, \gamma = 1.5$					
	MSEs	α	0.43418	0.57944	0.72564	0.64364	0.58058	0.88660	0.46215	0.68917
		γ	0.07186	0.09427	0.10738	0.11319	0.09416	0.14004	0.07305	0.09983
	ABBs	α	0.65893	0.76121	0.85185	0.80227	0.76196	0.94159	0.67982	0.83017
20		γ	0.26808	0.30703	0.32769	0.33644	0.30685	0.37422	0.27028	0.31595
	MREs	α	0.21964	0.25374	0.28395	0.26742	0.25399	0.31386	0.22661	0.27672
		γ	0.17872	0.20468	0.21846	0.22429	0.20457	0.24948	0.18018	0.21064
	MSEs	α	0.18544	0.21048	0.26225	0.25081	0.21757	0.49198	0.22160	0.26262
		γ	0.02841	0.03703	0.03874	0.04038	0.03378	0.06608	0.03252	0.03984
	ABBs	α	0.43063	0.45878	0.51210	0.50081	0.46644	0.70142	0.47074	0.51246
50		γ	0.16854	0.19243	0.19683	0.20094	0.18378	0.25705	0.18032	0.19960
	MREs	α	0.14354	0.15293	0.17070	0.16694	0.15548	0.23381	0.15691	0.17082
		γ	0.11236	0.12828	0.13122	0.13396	0.12252	0.17137	0.12021	0.13307
	MSEs	α	0.05251	0.06562	0.08767	0.08856	0.06704	0.17739	0.06694	0.10446
		γ	0.00974	0.01132	0.01384	0.01414	0.01106	0.02108	0.01013	0.01491
	ABBs	α	0.22916	0.25616	0.29610	0.29759	0.25893	0.42117	0.25873	0.32320
150		γ	0.09871	0.10642	0.11762	0.11892	0.10517	0.14518	0.10064	0.12211
	MREs	α	0.07639	0.08539	0.09870	0.09920	0.08631	0.14039	0.08624	0.10773
		γ	0.06581	0.07094	0.07842	0.07928	0.07012	0.09678	0.06709	0.08141
	MSEs	α	0.02948	0.03138	0.04280	0.04511	0.03205	0.09070	0.03524	0.04985
		γ	0.00503	0.00512	0.00706	0.00723	0.00517	0.01116	0.00552	0.00711
	ABBs	α	0.17171	0.17714	0.20688	0.21239	0.17901	0.30116	0.18773	0.22326
300		γ	0.07092	0.07155	0.08402	0.08504	0.07192	0.10566	0.07430	0.08430
	MREs	α	0.05724	0.05905	0.06896	0.07080	0.05967	0.10039	0.06258	0.07442
		γ	0.04728	0.04770	0.05602	0.05669	0.04795	0.07044	0.04953	0.05620
					$\alpha = 3, \gamma = 4$					
	MSEs	α	0.46585	0.56819	0.65343	0.65525	0.55194	0.92188	0.50561	0.74543
		γ	0.56604	0.67274	0.79679	0.75505	0.67794	1.07295	0.58962	0.80637
	ABBs	0.68253	0.75379	0.80835	0.80947	0.74292	0.96015	0.71106	0.86338	
20		γ	0.75236	0.82020	0.89263	0.86894	0.82337	1.03583	0.76787	0.89798
	MREs	α	0.22751	0.25126	0.26945	0.26982	0.24764	0.32005	0.23702	0.28779
		γ	0.18809	0.20505	0.22316	0.21723	0.20584	0.25896	0.19197	0.22450
	MSEs	α	0.18164	0.22547	0.26084	0.26744	0.22464	0.49603	0.19531	0.27981
		γ	0.21036	0.26100	0.28759	0.30018	0.24001	0.47869	0.24653	0.28694
	ABBs	α	0.42620	0.47483	0.51072	0.51714	0.47396	0.70429	0.44194	0.52897
50		γ	0.45865	0.51088	0.53627	0.54789	0.48991	0.69187	0.49652	0.53567
	MREs	α	0.14207	0.15828	0.17024	0.17238	0.15799	0.23476	0.14731	0.17632
		γ	0.11466	0.12772	0.13407	0.13697	0.12248	0.17297	0.12413	0.13392
	MSEs	α	0.06095	0.06763	0.07806	0.09146	0.06737	0.19450	0.07130	0.08910
		γ	0.07444	0.07593	0.08491	0.10471	0.07717	0.16830	0.07798	0.09058
	ABBs	α	0.24687	0.26006	0.27939	0.30242	0.25956	0.44103	0.26701	0.29850
150		γ	0.27285	0.27555	0.29139	0.32359	0.27779	0.41024	0.27925	0.30096
	MREs	α	0.08229	0.08669	0.09313	0.10081	0.08652	0.14701	0.08900	0.09950
		γ	0.06821	0.06889	0.07285	0.08090	0.06945	0.10256	0.06981	0.07524
	MSEs	α	0.02644	0.02704	0.04402	0.04498	0.03355	0.10352	0.03162	0.04771
		γ	0.03103	0.03452	0.04935	0.04707	0.03683	0.08363	0.03908	0.04922
	ABBs	α	0.16260	0.16445	0.20982	0.21208	0.18317	0.32174	0.17783	0.21843
300		γ	0.17616	0.18578	0.22215	0.21696	0.19191	0.28919	0.19769	0.22185
	MREs	α	0.05420	0.05482	0.06994	0.07069	0.06106	0.10725	0.05928	0.07281
		γ	0.04404	0.04645	0.05554	0.05424	0.04798	0.07230	0.04942	0.05546

Table 6: Fitted regression models to the heart transplantation data.

Model	β_0	β_1	β_2	β_3	α		AIC	BIC
LWEx-Exponential	**12.8003**	**-0.2069**	**0.5776**	**3.6810**	**6.0919**		**230.7**	**243.9**
	(0.4285)	**(0.0099)**	**(0.2552)**	**(0.1569)**	**(0.5086)**			
	[< 0.0001]	[< 0.0001]	[< 0.0001]	[0.0258]				
LEx-Exponential	9.1347	-0.0855	0.2803	1.9132	3.4607		627.0	640.2
	(0.4018)	(0.0097)	(0.2779)	(0.1860)	(0.3498)			
	[< 0.0001]	[< 0.0001]	[0.3158]	[< 0.0001]				
	β_0	β_1	β_2	β_3	θ	σ		
LOLLW	8.7448	-0.0769	1.4055	2.5919	4.6283	6.2032	347.6	363.4
	(1.7603)	(0.0198)	(0.6470)	(0.5746)	(3.5306)	(4.6852)		
	[< 0.001]	[< 0.001]	[0.015]	[< 0.001]				
Log-Weibull	7.9742	-0.0924	1.2143	2.5375		1.4658	353.4	366.5
	(0.9339)	(0.0206)	(0.6470)	(0.3733)		(0.1314)		
	[< 0.0001]	[< 0.0001]	[0.063]	[< 0.0001]				
	β_0	β_1	β_2	β_3	α_1	σ_1		
LTLFr	7.3238	0.1061	3.0613	2.9030	0.2428	3.4497	390.7	406.5
	(3.6369)	(0.1974)	(4.9455)	(0.8957)	(0.1679)	(3.8376)		
	[< 0.0001]	[< 0.0001]	[0.042]	[< 0.0001]				
	β_1	β_2	β_3					
Cox model	0.0589	-0.7421	-1.6528				555.2	563.1
	(0.0149)	(0.4422)	(0.2761)					
	[< 0.0001]	[0.0932]	[< 0.0001]					

Table 7: Findings from the fitted models.

Distribution	$\hat{\alpha}$	$\hat{\gamma}$
WEx-Exponential	1.977 (0.2074)	0.025 (0.0026)
Weibull	1.505 (0.0416)	0.003 (0.0005)
Ex-Exponential	1.916 (0.2272)	0.036 (0.0032)

Table 8: Adequacy measures for the fitted models.

Distribution	AIC	BIC	CAIC	HQIC	CM	AD	KS	p-value
WEx-Exponential	1367.956	1373.977	1368.037	1370.402	0.018	0.232	0.026	0.973
Weibull	1376.039	1382.060	1376.120	1378.485	0.042	0.371	0.043	0.939
Ex-Exponential	1381.587	1387.608	1381.668	1384.033	0.111	0.864	0.061	0.617