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A New Two-Parameters Lindley-Frailty Model: Censored and Uncensored Schemes under Different Baseline Models: Applications, Assessments, Censored and Uncensored Validation Testing

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Abstract

Classical survival models assume homogeneity among the population of individuals who are susceptible to the event of interest. However, in many practical circumstances, there is a certain amount of unobserved heterogeneity that can be caused by a variety of sources, such as environmental or genetic factors. If the heterogeneity is ignored, many issues could arise, including an overestimation of the hazard rate and inaccurate estimates of the regression coefficients. Frailty models are usually used to model the heterogeneity among individuals. In this paper, we propose a novel univariate frailty model. The frailty variable is assumed to follow the Two Parameter Lindley distribution. The maximum likelihood method is used to estimate the model parameters. The baseline hazard functions are assumed to follow Weibull, Exponential, Gompertz, and Pareto distributions, and a simulation study is performed under this assumption. We examine the characteristics of the distribution and assess its performance compared to other distributions that are frequently applied in frailty modeling by using both Nikulin-Rao-Robson and Bagdonavicius-Nikulin goodness-of-fit tests to determine the adequacy of the model. We analyze a fresh medical dataset collected from an emergency hospital in Algeria to evaluate the effectiveness and applicability of the proposed model.

Key Words: Bagdonavicius-Nikulin goodness-of-fit; Frailty models; Goodness of fit test; Hazard function; Laplace transform; Maximum likelihood; Time-to-event data.

Mathematical Subject Classification: 62N01, 62N02, 62E10.

1.Introduction

Survival analysis is a fundamental statistical method for analyzing time-to-event data, such as the time between the diagnosis of a disease and the occurrence of an important event, such as death or recurrence. Survival analysis is frequently used in many fields, including the social sciences, health, biology, and economics. A crucial assumption in survival analysis is the independent and identical (IID) distribution of the time-to-event data. The data are subject to unobserved heterogeneity or frailty, which could affect a person's chance of survival, hence this assumption is not always accurate. The results of ignoring frailty have been the subject of many research. These studies found that regression parameters' estimations were biased which is not supposed to be used Struthers and Kalbfleisch(1986); Henderson and Oman(1999); Bretagnolle and Huber-Carol(1988). Frailty is usually described by a distribution known as the frailty distribution (Duchateau and Janssen (2007)). Although a non-parametric specification of the frailty distribution can be obtained. Almeida et al. (2020); Horowitz (1999), Wienke (2010) pointed out that the parametric approachis frequently used due to mathematical simplicity.

Frailty models can be divided into two categories: shared frailty models and univariate frailty models. Univariate frailty models imply that everyone has a different frailty, whereas shared frailty models assume that each memberin a group shares a common frailty. When individuals in a cluster are related, such as twins or family members, shared frailty models are frequently used. When individuals are unrelated, such as patients in a clinical trial, univariate frailty models are usually applied. The hazard function, which measures the probability that an event will occur ata specific time, can be estimated using fragility models. An individual's or a group of individuals' probability of survival can be predicted using the hazard function. Frailty models can also be used to find factors related to either an increased or decreased risk of an occurrence. (Vaupel et al.(1979)). Frailty modeling's fundamental concept is that the observed data are the result of the interaction of independent, identically distributed random variables (IIDRVs) and a random frailty variable that captures the unobserved heterogeneity. Frailty modeling has been proven to be a useful method for analyzing survival data in a variety of contexts, including cancer research, clinical trials, and epidemiology (Aalen and Tretli (1999)). Several distributions have been proposed to explain the frailty term in the statistical and reliability literature, including the gamma distribution (Clayton(1978) and Vaupel et al. (1979)); the compound Poisson distribution (Aalen(1988) and Aalen(1992)); and the log-normal distribution (McGilchrist and Aisbett (1991)). However, there are limitations on how well these distributions can capture the heterogeneity present in the data.

In this work, a novel two-parameter frailty (TPF) model is proved to be an appropriate alternative for the gamma frailty model, compound Poisson frailty model, and log-normal frailty model. It is important to note that the new frailty model was developed based on the two parameter Lindley (TPL) model suggested by Shanker and Mishra (2013).The proposed distribution has been fitted to a number of data sets relating to survival times. We extend the TPL model to include a frailty element and demonstrate that the resulting distribution has advantageous properties including positive support, skewness, and kurtosis. The proposed approach can integrate unobserved variability and improve the fit of the frailty model.

In addition, the adjusted version of the chi-squared goodness-of-fit test suggested by Nikulin(1973b), Nikulin(1973a), Nikulin(1973c), and Rao and Robson(1974) (the Nikulin Rao and Robson (N-RR)) for complete data has been used tovalidate the proposed TPLF model. Also, The Bagdonavicius-Nikulin (Bg-N) test, which was developed by Bagdonavicius and Nikulin (2011) to be applied with censored data, is another approach by which the suggested TPLF modelis validated. It is important to note that the Bg-N test statistic and the N-RR test statistic are both statistical tests that evaluate how effectively a distribution fits a given set of data. The main distinctions between the N-RR test statistic and the Bg-N test statistic are their generality, underlying presumptions, and the method they apply to compare the observed data with a reference distribution. For this study, we collected fresh, real data from an emergency hospital in Algeria. Applying methods from survival analysis, we modeled the time-to-event for the sample's patients who have a particular medical condition. The new emergency care dataset is analyzed using the proposed TPLF model with the baseline hazard functions of Weibull, Gompertz, and Pareto. We demonstrate that the proposed TPLF model provides an adequate fit to the new emergency care data.

The main motivation of this paper is to introduce the TPLF model, a new flexible frailty model for survival analysis, in order to overcome the limitations of the gamma, compound Poisson, log-normal, and weighted Lindley (Mota et al.(2022)) frailty models and other frailty distributions. The inference is presented using the maximum likelihood estimation (MXLE) method for estimating the TPLF model's parameters with Weibull, exponential, Gompertz, and Pareto baseline hazard functions. Simulation studies are performed under different proportions of censoring. To assess the proposed distribution's ability to effectively fit a specific set of data, both N-RR and Bg-N are used in cases of complete and censored data. To demonstrate the applicability of the proposed model, we analyze a medical dataset from an emergency hospital in Algeri.

2.Basic concept of frailty models

Let's take into account an unknown source of heterogeneity using the Cox proportional hazard (CPH) model (see Cox(1972)). In order to demonstrate that the frailties are independent, the univariate frailty model aims to characterize unobserved risk factors for independent individuals. The hazard function for the ith subject is if $Z > 0$ is an unobserved random variable that indicates the fragility of the subject i, is defined as follows:

$$\lambda(t_i|z_i, x_i) = z_i \lambda_0(t_i) \exp(x_i^{\mathrm{T}} \mathcal{B})| \ i = 1,2,\ldots,n, \tag{1}$$

where $\underline{\boldsymbol{\beta}} = \mathcal{B}_{(p\mathrm{x}1)}$ is the vector of unknown regression coefficients for all $p < n$ (see Ibrahim et al.(2001)) and $\lambda_0(.)$ denotes to the baseline hazard function. A subject i has independent frailty defined as z_i, which is an unobserved non-negative number. Therefore, if $z_i > 1$ or $z_i < 1$, respectively, fragility z_i increases or decreases the risk that the event of interest will occur, the CPH model is then established as a particular case in which $z_i = 1$ for each i. The conditional survival function for the i^{th} subject is calculated using (1) as shown below:

$$S(t_i|z_i, x_i) = \exp[-z_i \Lambda_0(t_i) \exp(x_i^{\mathrm{T}} \mathcal{B})]| \ i = 1,\ldots\ldots,n \tag{2}$$

where $\Lambda_0(t_i) = \int_0^{t_i} \lambda(s)ds$ is the cumulative baseline hazard function. The conditional survival function (2) on frailty must be integrated out in order to calculate the marginal survival function, which is not dependent on unknown variable. Notice that this is equal to calculating the frailty distribution's Laplace transform, if $f(z)$ represents the frailty distribution, then we may get the following by integrating $S(t_i|z_i, x_i)$ from (2) on $Z = z_i$, where

$$S(t_i|x_i) = \int_0^\infty \exp[-z_i \Lambda_0(t_i) \exp(x_i^{\mathrm{T}} \mathcal{B})] f(z_i) dz_i = L_f[\Lambda_0(t_i) \exp(x_i^{\mathrm{T}} \mathcal{B})] \tag{3}$$

where the Laplace transform of the frailty distribution is denoted by $L_f(.)$. If the Laplace transform has a closed form, as a result, (3) may be applied to the following to calculate the marginal hazard function:

$$\lambda(t_i|x_i) = -\lambda_0(t_i) \frac{\exp(x_i^{\mathrm{T}} \mathcal{B}) L_f'[\Lambda_0(t_i) \exp(x_i^{\mathrm{T}} \mathcal{B})]}{L_f[\Lambda_0(t_i) \exp(x_i^{\mathrm{T}} \mathcal{B})]}, \tag{4}$$

where $L_f'(t) = \frac{\partial}{\partial t} L_f(t)$. As a result, the marginal survival and hazard functions (provided above) both evaluate the probability of survival and risk for a subject selected randomly from the research population (Wienke (2010)). As pre- viously mentioned, estimating both the marginal survival and hazard functions requires the use of a frailty distribution with a Laplace transform on the closed form, which makes parameter estimation simpler. However, numerical inte- gration or Markov Chain Monte Carlo techniques need to be used when the frailty distribution doesn't have a Laplace transform on the closed form (see Balakrishnan and Peng(2006); Hougaard(2012); Robert and Casella(2013)). When considering frailty distribution in univariate and multivariate survival data modeling, computational simplicity must be taken into account (Pickles and Crouchley(1995) and Wienke(2010))..

2.1 Two Parameter Lindley Frailty model

According to Shanker and Mishra (2013), the TPL model's probability density function (PDF) can be written as

$$f_{\alpha,\theta}(y) = \frac{1}{\alpha\theta+1} \theta^2 (\alpha + y) \exp(-\theta y) | y > 0, \theta > 0, \alpha\theta > -1, \tag{5}$$

Let's consider that the frailty variable Z in the conditional model in (1) has a TPL distribution (5) with a mean of one, or $E[Z] = 1$. This assumption is essential in order to identify the resulting frailty model (see Elbers and Ridder(1982)). As a result, by using the alternative parameterization of the TPL model in terms of mean proposed by Mazucheli et al. (2016), the TPLF model's PDF becomes

$$f_\theta(z) = \frac{1}{3} \theta^2 (1-\theta) \left[\frac{\theta+2}{\theta(1-\theta)} + z \right] \exp(-\theta z) | \ z > 0, \tag{6}$$

where the unknown shape parameter is denoted by $\theta > 0$. In general, the variance of the frailty distribution is used to quantify the amount of unobserved heterogeneity present in a study's population. Considering that the PDF (6) is a frailty distribution. The variance is expressed by:

$$\boldsymbol{\Sigma}^2 = \frac{1}{9\theta^2}[(1-\theta)(2\theta+10)+(\theta+2)^2], \tag{7}$$

Depending on its variance, the frailty PDF (6)'s Laplace transform is given by:

$$L_f(s) = \frac{\iota(\boldsymbol{\Sigma}^2)-2}{3[s(\boldsymbol{\Sigma}^2)-2]}\left\{\frac{\mathcal{D}(\boldsymbol{\Sigma}^2)}{(1+9\boldsymbol{\Sigma}^2)} + \frac{[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)}{(1+9\boldsymbol{\Sigma}^2)[s(\boldsymbol{\Sigma}^2)-2]}\right\}|s \in \mathbb{R}, \tag{8}$$

where $\iota(\boldsymbol{\Sigma}^2) = 3\sqrt{2(1+7\boldsymbol{\Sigma}^2)}$, $\mathcal{D}(\boldsymbol{\Sigma}^2) = 18\boldsymbol{\Sigma}^2 + \iota(\boldsymbol{\Sigma}^2), s(\boldsymbol{\Sigma}^2) = s(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)$ and $\mathcal{C}(\boldsymbol{\Sigma}^2) = 3+9\boldsymbol{\Sigma}^2 - \iota(\boldsymbol{\Sigma}^2)$. For the sake of simplicity, we evaluate equation (8) at $s = \Lambda_0(t_i)\xi_i$, where ξ_i =exp($\boldsymbol{x}_i^{\intercal}\mathcal{B}$), and determine that the marginal survival function (3) under the TPLF model can be obtained by

$$S(t_i|\boldsymbol{x}_i) = \frac{\iota(\boldsymbol{\Sigma}^2)-2}{3\Lambda_0(\xi_i,\boldsymbol{\Sigma}^2)}\left\{\frac{1}{1+9\boldsymbol{\Sigma}^2}\mathcal{D}(\boldsymbol{\Sigma}^2) + \frac{1}{(1+9\boldsymbol{\Sigma}^2)\Lambda_0(\xi_i,\boldsymbol{\Sigma}^2)}[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\right\}, \tag{9}$$

where

$$\Lambda_0(\xi_i,\boldsymbol{\Sigma}^2) = \Lambda_0(t_i)\xi_i(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2.$$

The resulting marginal hazard function (4) therefore becomes:

$$\lambda(t_i|\boldsymbol{x}_i) = \left[\frac{\lambda_0(t_i)\xi_i(1+9\boldsymbol{\Sigma}^2)}{\Lambda_0(t_i)\xi_i(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2}\right]\left\{1+\frac{[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)}{\Lambda_0(\xi_i,\boldsymbol{\Sigma}^2)\mathcal{D}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)}\right\}. \tag{10}$$

The TPLF model is evaluated and analyzed under the Weibull baseline hazard function (WBLHF), exponential baseline hazard function (EBLHF), Gombertez BLHF (GBLHF) and Pareto baseline hazard function (PBLHF).

2.2 The TPLF model with WBLHF

The Weibull distribution's baseline hazard and cumulative hazard functions are defined by:

$$\lambda_0(t_i) = \frac{\kappa}{\rho}\left(\frac{t_i}{\rho}\right)^{\kappa-1}|t_i > 0 \quad \text{and} \quad \Lambda_0(t_i) = \left(\frac{t_i}{\rho}\right)^{\kappa}|t_i > 0, \tag{11}$$

where $\kappa > 0$ and $\rho > 0$ represent, respectively, the shape parameter and the scale parameter. The Weibull distribution's hazard function presents a monotonous decrease for $\kappa < 1$; it is constant over time for $\kappa = 1$ (exponential distribution); and it monotonically increases $\kappa > 1$ (Wienke (2010)). By implementing (10) into (9), the marginal survival and hazard functions for the TPLF model with the WBLHF are, respectively, obtained as follows

$$S(t_i|\boldsymbol{x}_i) = \left\{\frac{\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]}{3\rho(\boldsymbol{\Sigma}^2|t_i^k\xi_i)}\right\}\left\{\frac{\mathcal{D}(\boldsymbol{\Sigma}^2)}{1+9\boldsymbol{\Sigma}^2} + \frac{\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)}{(1+9\boldsymbol{\Sigma}^2)\rho(\boldsymbol{\Sigma}^2|t_i^k\xi_i)}\right\}, \tag{12}$$

where

$$\rho(\boldsymbol{\Sigma}^2|t_i^k\xi_i) = t_i^k\xi_i(1+9\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2],$$

and

$$\lambda(t_i|\boldsymbol{x}_i) = \left[\frac{kt_i^{k-1}\xi_i(1+9\boldsymbol{\Sigma}^2)}{t_i^k\xi_i(1+9\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]}\right]\left\{1+\frac{\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)}{\rho(\boldsymbol{\Sigma}^2|t_i^k\xi_i)\mathcal{D}(\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)}\right\}. \tag{13}$$

2.3 The TPLF model with EBLHF

The exponential distribution's baseline hazard and cumulative hazard functions are provided by:

$$\lambda_0(t_i) = \lambda|t_i > 0 \quad \text{and} \quad \Lambda_0(t_i) = \lambda t_i|t_i > 0, \tag{14}$$

where $\lambda > 0$ represents the rate parameter. The hazard function of the exponential distribution is constant over time. This property is known as the memoryless property. By implementing (13) into (9), the marginal survival and hazard functions for the TPLF model with the EBLHF are, respectively, obtained as follows

$$S(t_i|\boldsymbol{x}_i) = \left\{\frac{\iota(\boldsymbol{\Sigma}^2)-2}{3\{\lambda t_i\xi_i(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2\}}\right\}\left\{\frac{\mathcal{D}(\boldsymbol{\Sigma}^2)}{1+9\boldsymbol{\Sigma}^2} + \frac{[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)}{(1+9\boldsymbol{\Sigma}^2)\{\lambda t_i\xi_i(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2\}}\right\}, \tag{15}$$

and

$$\lambda(t_i|x_i) = \left[\frac{\lambda\xi_i(1+9\boldsymbol{\Sigma}^2)}{\lambda t_i\xi_i(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2}\right]\left\{1+\frac{[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)}{\{\lambda t_i\xi_i(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2\}\mathcal{D}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)}\right\}. \tag{16}$$

2.4 The TPLF model with GBLHF

The Gompertz distribution's baseline hazard and cumulative hazard functions are defined by:

$$\lambda_0(t_i) = \varphi\exp(\gamma t_i)|t_i > 0 \quad \text{and} \quad \Lambda_0(t_i) = \frac{\varphi}{\gamma}[\exp(\varphi t_i) - 1]|t_i > 0, \tag{17}$$

where $\gamma > 0$ is the shape parameter and $\varphi > 0$ is the scale parameter. If $\gamma < 0$, the Gompertz distribution is inaccurate due to the fact that t, its cumulative hazard function converges to the constant $-\varphi/\gamma$, leading to a cure or long-term survivors proportion $p_0 = \exp(\varphi/\gamma)$ in the study population. The particular case for $\gamma = 0$ is the exponential distribution. As a consequence, the hazard function of the Gompertz distribution might be decreasing ($\gamma < 0$), constant ($\gamma = 0$), or increasing ($\gamma > 0$). By implementing (16) into (9), the marginal survival and hazard functions for the TPLF model with the Gombertz baseline hazard function are, respectively, obtained as follows

$$S(t_i|x_i) = \left\{\frac{\gamma[\iota(\boldsymbol{\Sigma}^2)-2]}{3\{\varphi[\exp(\gamma t_i)-1]\xi_i(1+9\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\}}\right\}\left\{\frac{\mathcal{D}(\boldsymbol{\Sigma}^2)}{(1+9\boldsymbol{\Sigma}^2)}+\frac{\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)}{(1+9\boldsymbol{\Sigma}^2)\left[\frac{\varphi}{\gamma}[\exp(\gamma t_i)-1]\xi_i(1+9\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\right]}\right\}, \tag{18}$$

and

$$\lambda(t_i|x_i) = \left[\frac{\varphi\exp(\gamma t_i)\xi_i(1+9\boldsymbol{\Sigma}^2)}{\varphi[\exp(\gamma t_i)-1]\xi_i(1+9\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]}\right]\left(1+\frac{\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)}{\left\{\begin{matrix}\{\varphi[\exp(\gamma t_i)-1]\xi_i(1+9\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\}\mathcal{D}(\boldsymbol{\Sigma}^2)\\+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\end{matrix}\right\}}\right). \tag{19}$$

2.5 The TPLF model with PBLHF

The Pareto distribution's baseline hazard and cumulative hazard functions are defined by:

$$\lambda_0(t_i) = \frac{\eta}{\alpha+t_i}|t_i > 0 \quad \text{and} \quad \Lambda_0(t_i) = -\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)|t_i > 0, \tag{20}$$

This distribution is skewed and heavy-tailed with two parameters $\alpha > 0$ and $\eta > 0$. The hazard function is monotonically decreasing. By implementing (19) into (9), the marginal survival and hazard functions for the TPLF model with the PBLHF are, respectively, obtained as follows,

$$S(t_i|x_i) = \left\{\frac{\iota(\boldsymbol{\Sigma}^2)-2}{3\left[-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\xi_i(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2\right]}\right\}\left\{\frac{\mathcal{D}(\boldsymbol{\Sigma}^2)}{1+9\boldsymbol{\Sigma}^2}+\frac{[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)}{(1+9\boldsymbol{\Sigma}^2)\left[-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\xi_i(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2\right]}\right\}, \tag{21}$$

and

$$\lambda(t_i|x_i) = \frac{\eta}{\alpha+t_i}\left[\frac{\xi_i(1+9\boldsymbol{\Sigma}^2)}{-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\xi_i(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2}\right]\left\{1+\frac{[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)}{\left[-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\xi_i(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)\right]\mathcal{D}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)}\right\}. \tag{22}$$

3.Estimation

Uncensored simulation studies using N-RR statistics collect data from a known distribution and compare it to a hypothesized distribution using one or more of the N-RR statistics. As well as their sensitivity to sample size, parameter values, and other variables, the performance of the statistics is assessed based on their capacity to properly identify the underlying distribution. An uncensored simulation study under N-RR statistics has several motivations. One of the most important motivations is to assess the statistical power of N-RR tests under a variety of scenarios.

The statistical power of a test refers to its ability to detect a true effect or difference. There are several factors that influence this statistic, including the sample size and the effect size. Researchers can determine the minimum sample size required to achieve a desired level of statistical power through simulation studies and determine the effects of other factors on test performance by conducting simulation studies.

On the other hand, determining the statistical power of the tests under various types and levels of censoring is an important motivation for conducting a censored simulation study using Bg-N statistics. Finding the minimum size of samples required to obtain a specific degree of statistical power under different types and levels of censoring is crucial since censoring can result in information loss and reduced statistical power. Examining the precision and accuracy of the estimated distribution parameters, particularly when dealing with right-censored data, is another motivation for performing a censored simulation study using the Bg-N statistics. Goodness-of-fit tests are intended to determine not only whether a given distribution fits the data, but also to estimate its parameters. The accuracy and precision of parameter estimations under various types and levels of censoring can be obtained from simulation studies, which may be useful in selecting the distribution to be applied to future analyses. In this section, we establish the ML approach for estimating the TPLF model's parameters using Weibull, exponential, Gombertez, and Pareto baseline hazard functions. MXLEs have attractive features including consistency, efficiency, asymptotic normality, and others under particular regularity constraints ((Lehmann and Casella(2006)).

For certain research subjects, lifetime data might not be accessible. For instance, some lifetimes are right-censored, and the only information that is known is that they are greater than the recorded value. In that case, let T_i and C_i represent the i^{th} subject's lifetime and censoring time variables, respectively, in the population under study. Assume that T_i and C_i are independent random variables, and $\delta_i = \mathbf{I}_{(T_i \leq C_i)}$ is the censoring indicator (i.e., $\delta_i = 1$ if T_i is lifetime, and $\delta_i = 0$ otherwise). We then evaluate $t_i = \min\{T_i, C_i\}$. Let x_i be a $p \times 1$ vector of the covariates for the i^{th} subject. Following that, with a sample of n individuals, the likelihood function for the model parameter vector $\underline{\mathbb{O}}$ in the non-informative censoring scenario is given by:

$$L(\underline{\mathbb{O}}) = \prod_{i=1}^{n} \lambda\,(t_i|x_i)^{\delta_i} S(t_i|x_i), \tag{23}$$

where $S(.|x_i)$ and $\lambda(.|x_i)$ are the Marginal survival and hazard functions given in (9). Then, the corresponding log-likelihood function is obtained using the natural logarithm of $L(\underline{\mathbb{O}})$.

4.Simulation studies

4.1 Under the WBLHF

Considering the WBLHF, the loglikelihood function for $\underline{\mathbb{O}} = (\kappa, \rho, \boldsymbol{\Sigma}^2, \mathcal{B})$ is provided by

$$\begin{aligned}\log L(\underline{\mathbb{O}}) &= r\log[k(1+9\boldsymbol{\Sigma}^2)] + (k-1)\sum_{i=1}^{n}\delta_i \log t_i + \sum_{i=1}^{n}\delta_i x_i^{\intercal}\mathcal{B} \\ &-\sum_{i=1}^{n}\delta_i \log\left[t_i^k \exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2) + 3\rho^k\sqrt{2(1+7\boldsymbol{\Sigma}^2)} - \rho^k 2\right] \\ &+\sum_{i=1}^{n}\delta_i \log\Psi_i + \sum_{i=1}^{n}\log\chi_i + \sum_{i=1}^{n}\log m_i,\end{aligned} \tag{24}$$

where $r = \sum_{i=1}^{n}\delta_i$ is the failure number,

$$\chi_i = \frac{\rho^k}{3\{t_i^k \exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2) + \rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\}}[\iota(\boldsymbol{\Sigma}^2)-2],$$

$$m_i = \frac{\mathcal{D}(\boldsymbol{\Sigma}^2)}{1+9\boldsymbol{\Sigma}^2} + \frac{\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)}{(1+9\boldsymbol{\Sigma}^2)\{t_i^k \exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2) + \rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\}},$$

and

$$\Psi_i = \rho^k[\iota(\boldsymbol{\Sigma}^2) - 2]C(\boldsymbol{\Sigma}^2)\begin{Bmatrix}[t_i^k \exp(x_i^{\mathrm{T}}\mathcal{B})(1 + 9\boldsymbol{\Sigma}^2) + \rho^k \iota(\boldsymbol{\Sigma}^2) - \rho^k 2]\mathcal{D}(\boldsymbol{\Sigma}^2) \\ +\rho^k[\iota(\boldsymbol{\Sigma}^2) - 2]C(\boldsymbol{\Sigma}^2)\end{Bmatrix}^{-1} + 1.$$

The appropriate MLE estimators $\underline{\widehat{\mathbb{O}}}$ of parameter vectors $\underline{\mathbb{O}}$ are obtained by maximizing the log-likelihood functions (23). If $\underline{\widehat{\mathbb{O}}}$ does not have a closed form, we must use numerical nonlinear optimization methods in order to discover a solution. These optimization approaches are implemented in BBsolve R software packages (V09(V09)). Considering the TPLF model with the WBLHF. The data were simulated $N = 12{,}000$ times; we fixed the parameter values $\kappa = 0.85, \rho = 0.85, \boldsymbol{\Sigma}^2 = 0.65, \mathcal{B}_1 = 0.7$, sample sizes $n = 20, n = 40, n = 350$ and $n = 1000$, and censoring proportions $0\%, 15\%, 35\%$,and 55%. We calculated the averages of the simulated values of the maximum likelihood estimators (MXLEs) $\hat{\kappa}, \hat{\rho}, \widehat{\boldsymbol{\Sigma}^2}, \widehat{\mathcal{B}_1}$ parameters and their MSQE using the R software and the Barzilai-Borwein (BB) algorithm (see V09(V09)). The results of the simulation are provided in Table 1. The maximum likelihood estimates for the TPLF model with WBLHF are convergent, as we can see in Table 1.

Table 1: Bias and MSQE of the MXLEs under the WBLHF

n		Bias	MSQE	Bias	MSQE	Bias	MSQE	Bias	MSQE
		0%cens.		15%cens.		35%cens.		55%cens.	
20	ρ	0.86548	0.0499	0.85999	0.0467	0.86374	0.0437	0.85761	0.0435
	κ	0.89245	0.0519	0.87132	0.0512	0.86371	0.0467	0.87660	0.0486
	$\boldsymbol{\Sigma}^2$	0.67002	0.0486	0.66814	0.0485	0.66648	0.0419	0.68001	0.0415
	$\mathcal{B}_1$	0.79256	0.0432	0.75324	0.0416	0.74381	0.0431	0.74318	0.0398
40	ρ	0.86215	0.0416	0.85462	0.0413	0.86001	0.0400	0.85346	0.0412
	κ	0.86754	0.0483	0.85116	0.0476	0.85344	0.0422	0.86332	0.0406
	$\boldsymbol{\Sigma}^2$	0.66532	0.0431	0.66004	0.0401	0.65807	0.0376	0.66341	0.0375
	$\mathcal{B}_1$	0.78361	0.0412	0.74198	0.0358	0.73674	0.0402	0.73165	0.0342
350	ρ	0.85673	0.0400	0.851203	0.0356	0.85749	0.0321	0.84778	0.0346
	κ	0.85421	0.0412	0.85100	0.0354	0.85207	0.0396	0.85341	0.0401
	$\boldsymbol{\Sigma}^2$	0.65432	0.0364	0.65504	0.0359	0.65291	0.0323	0.65127	0.0288
	$\mathcal{B}_1$	0.75000	0.0351	0.71065	0.0241	0.71205	0.0234	0.71138	0.0222
1000	ρ	0.85201	0.0338	0.84896	0.0248	0.85120	0.0264	0.84986	0.0274
	κ	0.851204	0.0308	0.84998	0.0328	0.85110	0.0315	0.85002	0.0200
	$\boldsymbol{\Sigma}^2$	0.65213	0.0339	0.65128	0.0311	0.64899	0.0275	0.65002	0.0241
	$\mathcal{B}_1$	0.72101	0.0301	0.71158	0.0222	0.69984	0.0215	0.70025	0.0159

4.2 Under the EBLHF

Considering the EBLHF, the loglikelihood function for $\underline{\mathbb{O}} = (\lambda, \boldsymbol{\Sigma}^2, \mathcal{B})$ is provided by

$$\begin{aligned}\log L(\underline{\mathbb{O}}) &= r\log[\lambda(1 + 9\boldsymbol{\Sigma}^2)] + \sum_{i=1}^{n} \delta_i x_i^{\mathrm{T}}\mathcal{B} \quad (25) \\ &- \sum_{i=1}^{n} \delta_i \log[\lambda t_i \exp(x_i^{\mathrm{T}}\mathcal{B})(1 + 9\boldsymbol{\Sigma}^2) + \iota(\boldsymbol{\Sigma}^2) - 2] \\ &+ \sum_{i=1}^{n} \delta_i \log\Upsilon_i + \sum_{i=1}^{n} \log\vartheta_i + \sum_{i=1}^{n} \log\Delta_i,\end{aligned}$$

where

$$\vartheta_i = \frac{\iota(\boldsymbol{\Sigma}^2) - 2}{3[\lambda t_i \exp(x_i^{\mathrm{T}}\mathcal{B})(1 + 9\boldsymbol{\Sigma}^2) + \iota(\boldsymbol{\Sigma}^2) - 2]},$$

$$\Delta_i = \frac{\mathcal{D}(\boldsymbol{\Sigma}^2)}{1+9\boldsymbol{\Sigma}^2} + \frac{[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)}{(1+9\boldsymbol{\Sigma}^2)[\lambda t_i \exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2]}$$

and

$$\Upsilon_i = [\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\begin{Bmatrix}[\lambda t_i \exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{D}(\boldsymbol{\Sigma}^2)\\ +[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{Bmatrix}^{-1} + 1.$$

Considering the TPLF model with the EBLHF. The data were simulated $N = 12{,}000$ times; we fixed the parameter values $\lambda = 0.6, \boldsymbol{\Sigma}^2 = 0.5, \mathcal{B}_1 = 0.9$, sample sizes $n = 20, n = 40, n = 350$ and $n = 1000$, and censoring proportions 0%, 15%, 35%,and 55%. We calculated the averages of the simulated values of the MXLEs $\hat{\lambda}, \widehat{\boldsymbol{\Sigma}^2}, \widehat{\mathcal{B}_1}$ parameters and their MSQE using the R software and the Barzilai-Borwein (BB) algorithm (see V09(V09)). The results of the simulation are provided in Table 2. The maximum likelihood estimates for the TPLF model with EBLHF are convergent, as we can see in Table 2.

Table 2: Bias and MSQE of the MXLEs under the EBLHF =0.18cm

n		Bias	MSQE	Bias	MSQE	Bias	MSQE	Bias	MSQE
		0% cens.		15% cens.		35% cens.		55% cens.	
20	λ	0.63514	0.0496	0.62154	0.0437	0.64831	0.0487	0.62354	0.0438
	$\boldsymbol{\Sigma}^2$	0.55296	0.0435	0.53333	0.0462	0.55001	0.0438	0.53769	0.0426
	$\mathcal{B}_1$	0.96278	0.0431	0.94271	0.0396	0.93265	0.0421	0.93349	0.0354
40	λ	0.62853	0.0417	0.62003	0.0400	0.63497	0.0427	0.61728	0.0411
	$\boldsymbol{\Sigma}^2$	0.54862	0.0374	0.52481	0.0402	0.53189	0.0476	0.91638	0.0302
	$\mathcal{B}_1$	0.94371	0.0411	0.92648	0.0324	0.91548	0.0416	0.51221	0.0382
350	λ	0.61705	0.0361	0.61965	0.0296	0.62085	0.0355	0.59537	0.0374
	$\boldsymbol{\Sigma}^2$	0.53719	0.0309	0.51012	0.0367	0.52247	0.0422	0.50252	0.0318
	$\mathcal{B}_1$	0.91187	0.0412	0.90995	0.0219	0.90678	0.0332	0.91203	0.0284
1000	λ	0.61202	0.0302	0.59834	0.0178	0.61207	0.0273	0.59889	0.0300
	$\boldsymbol{\Sigma}^2$	0.05108	0.0265	0.50067	0.0288	0.51305	0.0374	0.49896	0.0331
	$\mathcal{B}_1$	0.90506	0.0445	0.90046	0.0222	0.89798	0.0227	0.90010	0.0212

4.3 Under the GBLHF

Considering the GBLHF, the loglikelihood function for $\underline{\mathbb{O}} = (\gamma, \varphi, \boldsymbol{\Sigma}^2, \mathcal{B})$ is provided by

$$\log L(\underline{\mathbb{O}}) = r\log[\gamma\varphi(1+9\boldsymbol{\Sigma}^2)] + \gamma\sum_{i=1}^{n}\delta_i \log t_i + \sum_{i=1}^{n}\delta_i x_i^{\intercal}\mathcal{B} \quad (26)$$

$$-\sum_{i=1}^{n}\delta_i \log\left[\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2) + 3\gamma\sqrt{2(1+7\boldsymbol{\Sigma}^2)} - 2\gamma\right]$$

$$+\sum_{i=1}^{n}\delta_i \log\Phi_i + \sum_{i=1}^{n}\log\Pi_i + \sum_{i=1}^{n}\log\zeta_i,$$

where

$$\zeta_i = \frac{\gamma}{3\varphi(\boldsymbol{\Sigma}^2|t_i)}[\iota(\boldsymbol{\Sigma}^2)-2],$$

$$\Pi_i = \frac{1}{(1+9\boldsymbol{\Sigma}^2)}\mathcal{D}(\boldsymbol{\Sigma}^2) + \frac{\gamma}{(1+9\boldsymbol{\Sigma}^2)\varphi(\boldsymbol{\Sigma}^2|t_i)}[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2),$$

and

$$\Phi_i = \gamma[\iota(\boldsymbol{\Sigma}^2) - 2]C(\boldsymbol{\Sigma}^2)$$
$$\times \begin{Bmatrix} \left[\varphi[\exp(\gamma t_i) - 1]\exp(x_i^{\mathrm{T}}\mathcal{B})(1 + 9\boldsymbol{\Sigma}^2) + 3\gamma\sqrt{2(1 + 7\boldsymbol{\Sigma}^2)} - 2\gamma\right]\mathcal{D}(\boldsymbol{\Sigma}^2) \\ +\gamma[\iota(\boldsymbol{\Sigma}^2) - 2]C(\boldsymbol{\Sigma}^2) \end{Bmatrix}^{-1} + 1.$$

Considering the TPLF model with the GBLHF. The data were simulated $N = 12{,}000$ times; we fixed the parameter values $\gamma = 0.6, \varphi = 0.35, \boldsymbol{\Sigma}^2 = 0.5, \mathcal{B}_1 = 1.5$,sample sizes $n = 20, n = 40, n = 350$ and $n = 1000$, and censoring proportions 0%, 15%, 35%,and 55%. We calculated the averages of the simulated values of the MXLEs $\hat{\gamma}, \hat{\varphi}, \widehat{\boldsymbol{\Sigma}^2}, \widehat{\mathcal{B}_1}$ parameters and their MSQE using the R software and the Barzilai-Borwein (BB) algorithm (Varadhan and Gilbert(2009)). The results of the simulation are provided in Table 3. The maximum likelihood estimates for the TPLF model with GBLHF are convergent, as we can see in Table 3.

Table 3: Bias and MSQE of the MXLEs under the GBLHF =0.18cm

n		Bias	MSQE	Bias	MSQE	Bias	MSQE	Bias	MSQE
		0% cens.		15% cens.		35% cens.		55% cens.	
20	γ	0.65548	0.0435	0.64937	0.0325	0.63418	0.0462	0.63854	0.0481
	φ	0.35945	0.0475	0.35962	0.0387	0.35401	0.0475	0.35719	0.0415
	$\boldsymbol{\Sigma}^2$	0.54612	0.0421	0.53481	0.0489	0.53841	0.0321	0.52214	0.0358
	$\mathcal{B}_1$	1.56382	0.0437	1.54062	0.0384	1.53846	0.0384	1.53048	0.0485
40	γ	0.64381	0.0395	0.62559	0.0305	0.63084	0.0439	0.62443	0.0392
	φ	0.35512	0.0357	0.35462	0.0265	0.35286	0.0385	0.35608	0.0377
	$\boldsymbol{\Sigma}^2$	0.53816	0.0381	0.52647	0.0435	0.52937	0.0311	0.51473	0.0267
	$\mathcal{B}_1$	1.54371	0.0381	1.52034	0.0332	1.52739	0.0367	1.52271	0.0432
350	γ	0.63894	0.0312	0.61738	0.2384	0.61608	0.0327	0.61850	0.0316
	φ	0.35334	0.0276	0.35210	0.0213	0.35167	0.0241	0.35224	0.0324
	$\boldsymbol{\Sigma}^2$	0.52743	0.0314	0.52003	0.0412	0.51784	0.0276	0.50734	0.0233
	$\mathcal{B}_1$	1.52496	0.0341	1.51274	0.0251	1.51172	0.0237	1.51092	0.0255
1000	γ	0.61862	0.0211	0.06522	0.0213	0.69665	0.0300	0.60023	0.0275
	φ	0.35206	0.0213	0.35082	0.0135	0.35044	0.0223	0.34995	0.0281
	$\boldsymbol{\Sigma}^2$	0.51223	0.0311	0.51302	0.0246	0.50937	0.0214	0.50234	0.0200
	$\mathcal{B}_1$	1.51204	0.0276	1.51006	0.0233	1.51062	0.0219	1.49968	0.0201

4.4 Under the PBLHF

Considering the PBLHF, the loglikelihood function for $\underline{\mathbb{O}} = (\eta, \alpha, \boldsymbol{\Sigma}^2, \mathcal{B})$ is provided by

$$\log L(\underline{\mathbb{O}}) = r\log[\eta(1 + 9\boldsymbol{\Sigma}^2)] + \sum_{i=1}^{n} \delta_i x_i^{\mathrm{T}}\mathcal{B}$$
$$-\sum_{i=1}^{n} \delta_i \log\left[-\eta\log\left(\frac{\alpha}{\alpha + t_i}\right)\exp(x_i^{\mathrm{T}}\mathcal{B})(1 + 9\boldsymbol{\Sigma}^2) + \iota(\boldsymbol{\Sigma}^2) - 2\right]$$
$$-\sum_{i=1}^{n} \delta_i \log(\alpha + t_i) + \sum_{i=1}^{n} \log\varrho_i + \sum_{i=1}^{n} \log\mu_i + \sum_{i=1}^{n} \delta_i \log\Gamma_i,$$

where

$$\varrho_i = \frac{\iota(\boldsymbol{\Sigma}^2) - 2}{3\left[-\eta\log\left(\frac{\alpha}{\alpha + t_i}\right)\exp(x_i^{\mathrm{T}}\mathcal{B})(1 + 9\boldsymbol{\Sigma}^2) + \iota(\boldsymbol{\Sigma}^2) - 2\right]},$$

$$\mu_i = \frac{1}{(1+9\boldsymbol{\Sigma}^2)}\mathcal{D}(\boldsymbol{\Sigma}^2) + \frac{[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)}{(1+9\boldsymbol{\Sigma}^2)\left[-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{T}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2\right]}$$

and

$$\Gamma_i = [\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2) \times \left\{ \begin{matrix} \left[-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{T}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2\right]\mathcal{D}(\boldsymbol{\Sigma}^2) \\ +[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2) \end{matrix} \right\}^{-1} + 1.$$

Considering the TPLF model with the PBLHF. The data were simulated $N = 12{,}000$ times; we fixed the parameter values $\eta = 0.4, \alpha = 0.6, \boldsymbol{\Sigma}^2 = 0.5, \mathcal{B}_1 = 1.7$, sample sizes $n = 20, n = 40, n = 350$ and $n = 1000$, and censoring proportions 0%, 15%, 35%,and 55%. We calculated the averages of the simulated values of the MXLEs $\hat{\eta}, \hat{\alpha}, \widehat{\boldsymbol{\Sigma}^2}, \widehat{\mathcal{B}_1}$ parameters and their MSQE using the R software and the Barzilai-Borwein (BB) algorithm (V09(V09)). The results of the simulation are provided in Table 4. The maximum likelihood estimates for the TPLF model with PBLHF are convergent, as we can see in Table 4.

Table 4: Bias and MSQE of the MXLEs under the PBLHF =0.18cm

n		Bias	MSQE	Bias	MSQE	Bias	MSQE	Bias	MSQE
		0% cens.		15% cens.		35% cens.		55% cens.	
20	η	0.46075	0.0466	0.45137	0.0432	0.45002	0.0398	0.44521	0.0452
	α	0.64015	0.0461	0.63174	0.0319	0.64001	0.0318	0.63462	0.0437
	$\boldsymbol{\Sigma}^2$	0.54832	0.0486	0.55104	0.0482	0.53819	0.0399	0.54468	0.0437
	$\mathcal{B}_1$	1.76034	0.0477	1.74392	0.0451	1.73005	0.0334	1.72938	0.0468
40	η	0.44381	0.0376	0.44382	0.0396	0.43185	0.0321	0.43719	0.0427
	α	0.63176	0.0367	0.62638	0.0278	0.63591	0.0237	0.62649	0.0348
	$\boldsymbol{\Sigma}^2$	0.53714	0.0431	0.53192	0.0432	0.52619	0.0287	0.52731	0.0316
	$\mathcal{B}_1$	1.75123	0.0395	1.72154	0.0349	1.72419	0.0267	1.71673	0.0427
350	η	0.42864	0.0324	0.42658	0.0324	0.41300	0.0311	0.42635	0.0316
	α	0.62574	0.0324	0.61674	0.0126	0.62649	0.0173	0.61873	0.0222
	$\boldsymbol{\Sigma}^2$	0.52067	0.0325	0.52230	0.0325	0.52043	0.0125	0.51473	0.0247
	$\mathcal{B}_1$	1.73198	0.0351	1.71708	0.0243	1.71067	0.0213	1.70937	0.0357
1000	η	0.42100	0.0261	0.41873	0.0284	0.41074	0.0284	0.41986	0.0294
	α	0.61346	0.0300	0.60261	0.0124	0.61649	0.0122	0.60936	0.0162
	$\boldsymbol{\Sigma}^2$	0.51003	0.0301	0.49852	0.0202	0.51170	0.0100	0.50017	0.0233
	$\mathcal{B}_1$	1.71103	0.0307	1.70688	0.0201	1.70032	0.0187	1.70634	0.251

5. Uncensored validating for the TPLF model using the N-RR test

The N-RR test statistic examines the extent to which the statistical model fits a given set of observations. A broad test named the N-RR test can be used to assess the predictive fit of a variety of statistical models, such as time series, regression, and survival models. In conclusion, the N-RR test statistic is a useful tool for statistical analysis and has a wide range of uses. It is especially helpful for model selection, evaluating a model's goodness of fit, and identifying issues with a model. One of the key advantages of the N-RR test statistic is its ability to capture deviations from normality that other statistical tests may not detect. In particular, the N-RR test is robust to outliers, making it a reliable tool for identifying and analyzing data sets with extreme values. This makes it particularly useful in financial applications, where it is essential to identify and analyze extreme events such as market crashes and large price movements. Here are some applications and importance of the N-RR test statistic:

i. The N-RR test statistic can be used to compare the fit of different statistical models to the same data. This can help in model selection by identifying the model that provides the best fit to the data.

ii. The N-RR test statistic can be used to assess the goodness of fit of a statistical model to the data. If the N-RR test statistic is small, it indicates a good fit between the model and the data. On the other hand, if the N-RR test statistic is large, it indicates a poor fit between the model and the data.

iii. The N-RR test statistic can be used to detect outliers in the data. Outliers are data points that do not fit the general pattern of the data and can have a significant impact on the fit of the model. The N-RR test can identify these outliers and help to improve the fit of the model.

iv. The N-RR test statistic can be used to diagnose problems with a statistical model. If the N-RR test statistic is large, it can indicate that the model is mis specified or that there are problems with the assumptions of the model.

Under the N-RR statistic, we need to test the following null hypothesis

$$H_0\colon \Pr\{z_i \le z\} = F_{\underline{\mathbb{O}}}(z),\quad z \in \mathbb{R},\quad \underline{\mathbb{O}} = (\underline{\mathbb{O}}_1, \underline{\mathbb{O}}_2, \cdots, \underline{\mathbb{O}}_s)^T,$$

Then, the N-RR statistic can be expressed as

$$Y^2(\widehat{\underline{\mathbb{O}}}_n) = X_n^2(\widehat{\underline{\mathbb{O}}}_n) + \frac{1}{n}\ell^T(\widehat{\underline{\mathbb{O}}}_n)(\mathbf{I}(\widehat{\underline{\mathbb{O}}}_n) - \mathbf{J}(\widehat{\underline{\mathbb{O}}}_n))^{-1}\ell(\widehat{\underline{\mathbb{O}}}_n),$$

where

$$X_n^2(\underline{\mathbb{O}}) = \left(\left[np_1(\underline{\mathbb{O}})\right]^{-\frac{1}{2}}\left[-np_1(\underline{\mathbb{O}}) + \underline{\mathbb{O}}_1\right], \cdots, \left[np_b(\underline{\mathbb{O}})\right]^{-\frac{1}{2}}\left[-np_b(\underline{\mathbb{O}}) + \underline{\mathbb{O}}_b\right]\right)^T,$$

and

$$\mathbf{J}(\underline{\mathbb{O}}) = B(\underline{\mathbb{O}})^T B(\underline{\mathbb{O}}),$$

refers to the information matrix where

$$B(\underline{\mathbb{O}}) = \left[\frac{1}{\sqrt{p}_i}\frac{\partial}{\partial\mu}(\underline{\mathbb{O}})\right]_{r\times s}\Big|_{(i=1,2,\cdots,b\ \text{ and}\kappa=1,2,\cdots,s)},$$

and

$$\ell(\underline{\mathbb{O}}) = (\ell_1(\underline{\mathbb{O}}), \ldots, \ell_s(\underline{\mathbb{O}}))^T \ \text{ with } \quad \ell_\kappa(\underline{\mathbb{O}}) = \sum_{i=1}^{r}\frac{\mathbb{O}_i}{p_i}\frac{\partial}{\partial\underline{\mathbb{O}}_\kappa}p_i(\underline{\mathbb{O}}),$$

The $Y^2(\widehat{\underline{\mathbb{O}}}_n)$ statistic has $(b-1)$ degrees of freedom (DF) and is accompanied by χ_{b-1}^2 distribution, where the observations. $x_1, x_2, \cdots, x_n$ that are collected in $\mathbf{I}_1, \mathbf{I}_2, \cdots, \mathbf{I}_b$ (these b subintervals are mutually disjoint: $\mathbf{I}_j =]a_{j,b} - 1; a_{j,b}]$). The intervals $\mathbf{I}_j$'s limits for $a_{j,b}$ are determined as follows

$$p_j(\underline{\mathbb{O}}) = \int_{a_{j,b}-1}^{a_{j,b}} f_{\underline{\mathbb{O}}}(x)dx\big|_{(\ j=1,2,\cdots,b)},$$

and

$$a_{j,b} = F^{-1}\left(\frac{j}{b}\right)\big|_{(j=1,\cdots,b-1)}.$$

In many cases, the goal of a goodness-of-fit test is not only to determine whether a particular distribution fits the data, but also to estimate the values of its parameters. Simulation studies can provide insights into the accuracy and precision of parameter estimates under different scenarios, and can inform decisions about which distribution to use for subsequent analyses. Overall, uncensored simulation studies under the N-RR statistics are an important tool for evaluating and comparing different probability distributions in a controlled environment. These studies can provide valuable insights into the performance of the N-RR tests under different scenarios, and can inform decisions about which distribution to use for subsequent analyses. Using numerical simulation, we conducted a detailed analysis to confirm the claims presented in this work. To verify the null hypothesis H_0 We thus produced the N-RR statistics of the TPLF model to confirm that the sample is a 13000 using simulated samples $n = 26, n = 40, n = 140, n = 250, n = 600$ and $n = 1200$. Regarding various theoretical levels ($\epsilon = 0.01, 0.02, 0.04, 0.09$), for the null hypothesis, we compute the average of the non-rejection numbers. $Y^2 \le \chi_\epsilon^2(b-1)$. The appropriate empirical and theoretical levels are presented in Table 5. It is evident that there is a good agreement between the calculated empirical level value and its equal theoretical level value. We therefore conclude that the proposed test is quite good for the TPLF distribution.

Table 5: Uncensored assessing for the N-RR statistic for $\epsilon = 0.01, 0.02, 0.04, 0.09$ and $N = 13000$.

$n \downarrow \& \epsilon \longrightarrow \epsilon$	$\epsilon = 0.01$	$\epsilon = 0.02$	$\epsilon = 0.04$	$\epsilon = 0.09$
n = 26	0.9924	0.9822	0.9631	0.9120
n = 40	0.9916	0.9817	0.9627	0.9116
n = 140	0.9914	0.9815	0.9622	0.9110
n = 250	0.9906	0.9811	0.9616	0.9108
n = 600	0.9904	0.9807	0.9611	0.9104
n = 1200	0.9902	0.9804	0.9606	0.9101

6.Censored validating for the TPLF model using the Bg-N test

Due to Bagdonavicius and Nikulin (2011) and Bagdonavicius et al. (2013), Khalil et al. (2023) and Yousof et al. (2023a,b,c), we can verify the suitability of the TPLF model when the parameters are unknown and the data are censored where the null hypothesis can be expressed as

$$H_0: F(x) \in F_0 = \left\{F_0(x, \underline{\mathbb{O}}), x \in R^1,\ \underline{\mathbb{O}} \in \underline{\mathbb{O}} \subset R^s\right\},$$

Let's divide the limited amount of time $[0, \iota(\boldsymbol{\Sigma}^2)]$ into $\kappa|\kappa = 1,2,\cdots,s$ shorter time periods. Where is the maximum runtime of the research and $\mathbf{I}_j = (a_{j-1}, a_{j,b}]$; $0 =< a_{0,b} < a_{1,b} \ldots < a_{\kappa-1,b} < a_{\kappa,b} = +\infty$. The anticipated worth of $\widehat{a_{j,b}}$ can be said the following if $x_{(i)}$ is the i^{th} element in the ordered statistics $(x_{(1)},,, x_{(n)})$ and if $\boldsymbol{\Lambda}^{-1}$ refers to the cumulative hazard function and

$$\widehat{a_{j,b}} = \boldsymbol{\Lambda}^{-1}\left((E_{j,X} - \textstyle\sum_{l=1}^{i-1} \boldsymbol{\Lambda}(x_{(l)}, \widehat{\underline{\mathbb{O}}}))/(n - i + 1), \widehat{\underline{\mathbb{O}}}\right), \quad \widehat{a_\kappa} = x_{(n)}|_{(j=1,\ldots,\kappa)},$$

where

$$e_{j,Z} = \frac{1}{\kappa} E_\kappa \text{ for every } j.$$

$$\boldsymbol{\Lambda}(x, \underline{\mathbb{O}}) = -\ln\begin{Bmatrix}\left[\frac{1}{\Lambda_0(t_i)\xi_i(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2}\lambda_0(t_i)\xi_i(1+9\boldsymbol{\Sigma}^2)\right][\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2) \\ \times\{\Lambda_0(\xi_i, \boldsymbol{\Sigma}^2)\mathcal{D}(\boldsymbol{\Sigma}^2) + [\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\}^{-1} + 1\end{Bmatrix},$$

and

$$E_{j,Z} = (n - i + 1)\boldsymbol{\Lambda}(\widehat{a_{j,b}}, \widehat{\underline{\mathbb{O}}}) + \sum_{l=1}^{i-1} \boldsymbol{\Lambda}(x_{(l)}, \widehat{\underline{\mathbb{O}}}) = \sum_{i:z_i > a_{j,b}} (\boldsymbol{\Lambda}(a_{j,b} \wedge z_i, \widehat{\underline{\mathbb{O}}}) - \boldsymbol{\Lambda}(a_{j-1}, \widehat{\underline{\mathbb{O}}}),$$

$$E_\kappa = \sum_{i=1}^{n} \boldsymbol{\Lambda}(z_i, \widehat{\underline{\mathbb{O}}}).$$

The $a_{j,b}$ functions for random data, and the $e_{j,Z}$ For the κ selected periods, anticipated failure rates are equal. Statistical data $Y_n^2 = \mathbf{Z}^T \widehat{\mathbf{S}}^{-1} \mathbf{Z}$, where $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_\kappa)^x$, $Z_j = \frac{1}{\sqrt{n}}(\mathbf{W}_{j,Z} - e_{j,Z})|_{(\ j=1,2,\ldots,\kappa)}$ and $\mathbf{W}_{j,Z}$ can be used to test a hypothesis since it reflects the total number of failures that have been recorded throughout these timeframes. The elements of the Bg-N test statistic (see also Goual et al. (2019), Ibrahim et al. (2019, 2021 and 2023), Goual and Yousof (2020), Mansour et al. (2020) and Yadav et al. (2020 and 2022))

$$Y_n^2 = \textstyle\sum_{j=1}^{\kappa} \frac{1}{\mathbf{W}_{j,Z}}(\mathbf{W}_{j,Z} - e_{j,Z})^2 + \mathbf{D}_{W,G},$$

where

$$\mathbf{D}_{W,G} = \widehat{\mathbf{V}}^T \widehat{\mathbf{G}}^{-1} \widehat{\mathbf{V}}, \widehat{\mathbf{S}}^{-1} = \widehat{\mathbf{B}}^{-1} + \widehat{\mathbf{M}}^{-1} \widehat{\mathbf{B}}^T \widehat{\mathbf{G}}^{-1} \widehat{\mathbf{M}} \widehat{\mathbf{B}}^{-1},$$

$$\widehat{\mathbf{G}} = [\hat{g}_{ll'}]_{s \times s} = \hat{\imath} - \widehat{\mathbf{M}} \widehat{\mathbf{B}}^{-1} \widehat{\mathbf{M}}^x,$$

$$\widehat{\mathbf{M}}_{lj} = \frac{1}{n} \sum_{i:z_i \in \mathbf{I}_j} \rho_i \frac{\partial}{\partial \underline{\mathbb{O}}} \ln\left[\lambda_{i,\underline{\mathbb{O}}}(z_i)\right], \mathbf{W}_{j,Z} = \sum_{i:z_i \in \mathbf{I}_j} \rho_i, \widehat{\mathbf{B}}_j = n^{-1} \mathbf{W}_{j,Z},$$

$$\widehat{\mathbf{V}}_l = \sum_{j=1}^{\kappa} \widehat{\mathbf{M}}_{lj} \widehat{\mathbf{B}}_j^{-1} \mathbf{Z}_j | l, l' = 1, \ldots, s,$$

$$\hat{\imath}_{ll'} = n^{-1} \sum_{i=1}^{n} \rho_i \frac{\partial}{\partial \underline{\mathbb{O}}_l} \ln\left[\lambda_{i,\underline{\mathbb{O}}}(z_i)\right] \frac{\partial}{\partial \underline{\mathbb{O}}_{l'}} \ln\left[\lambda_{i,\underline{\hat{\mathbb{O}}}}(z_i)\right],$$

and

$$\hat{g}_{ll'} = \hat{\imath}_{ll'} - \sum_{j=1}^{\kappa} \widehat{\mathbf{M}}_{lj} \widehat{\mathbf{M}}_{l'j} \hat{A}_j^{-1},$$

and

$$\widehat{\mathbf{M}}_{lj} = \frac{1}{n} \sum_{i: z_i \in \mathbf{I}_j} \rho_i \frac{\partial}{\partial \underline{\mathbb{O}}} \ln\left[\lambda_{i,\underline{\hat{\mathbb{O}}}}(z_i)\right].$$

Censored simulation studies under the Bag-Ni statistics are an important tool for evaluating and comparing different probability distributions when dealing with censored data. These studies can provide valuable insights into the performance of the Bg-N tests under different types and levels of censoring, and can inform decisions about which distribution to use for subsequent analyses. It is intended that the sample that was produced ($N = 13000$) will be censored at 24% and that DF= 5 To check if the sample agrees with the TPLF model's null hypothesis, grouping intervals will be used. For various theoretical levels, we determine the average value of the non-rejection numbers of the null hypothesis. ($\epsilon = 0.01, 0.02, 0.04, 0.09$), where $Y^2 \leq \chi^2_\epsilon(r-1)$. The theoretical and empirical levels are compared in Table 6, which demonstrates how closely the determined empirical level matches the value of the relevant theoretical level. We conclude that the customized test is ideally suited to the TPLF model as a consequence.

Table 6: Censored assessing for the Bg-N statistic for
$\epsilon = 0.01; 0.02; 0.05; 0.1$ and $N = 13000$.

$n \downarrow \& \epsilon \longrightarrow \epsilon$	$\epsilon = 0.01$	$\epsilon = 0.02$	$\epsilon = 0.04$	$\epsilon = 0.09$
n = 26	0.9924	0.9819	0.9631	0.9120
n = 40	0.9920	0.9816	0.9625	0.9114
n = 140	0.9915	0.9807	0.9619	0.9111
n = 350	0.9911	0.9805	0.9613	0.9108
n = 600	0.9906	0.9804	0.9608	0.9103
n = 1200	0.9904	0.9801	0.9604	0.9101

We conclude from these findings that the empirical significance level of the Y_n^2 The theoretical level of the chi-square distribution on degrees of freedom corresponds to the statistical level at which it is statistically significant. The censored data acquired from the TPLF distribution may thus be satisfactorily fitted using the suggested test, according to this evidence.

7.An application under the uncensored emergency care data

The emergency department of the hospital associated with a public health institution offered real data that were collected throughout the month of March 2023, and these data were used in the current study. The goal of this study was to examine, in a sample of patients getting medical care at the department, the association between various clinical characteristics and emergency room outcomes. The required permissions were guaranteed, and moral standards were adhered to in the data collection. The dataset consisted of 30 different individuals, each of them represented a unique observation. Six separate variables were recorded for each subject: age (years), minimum and maximum blood pressure (mmHg), blood glucose level (mg/dL), cardiac frequency (BPM), and oxygen saturation (SaO 2%). To ensure the accuracy and quality of the collected data, rigorous precautions were taken during the collection process. This necessitated the proper documentation of patient data, adherence to predetermined measuring procedures, and regular quality checks to detect any missing data or variations. This dataset is useful for investigating the associations between clinical variables and emergency room outcomes due to the accurate data collection method and the variety of the patient population.

We are able to investigate the validity and application of the distribution by evaluating the goodness-of-fit of the TPLF model distribution and its capability to accurately describe the observed patterns and variability in emergency care data. For each fitted model (TPLF model with Weibull, Gombertez, Pareto beseline hazard functions), we indicate the point estimates. The most effective model out of all the fitted models to these data is determined using the well-known modified chi-squared test (Bagdonavicius and Nikulin (2011) and Bagdonavicius et al. (2013)).

7.1 Validation of the TPLF model under the WBLHF

Considering that these data are distributed according to the TPLF model with WBLHF and using R statistical software (the BB package), the maximum likelihood estimates of the parameter vector $\underline{\mathbb{O}}$ are provided as

$$\hat{\kappa} = 0.84965, \hat{\rho} = 0.831994, \widehat{\boldsymbol{\Sigma}^2} = 1.019259,$$
$$\widehat{\mathcal{B}_1} = 0064875, \widehat{\mathcal{B}_2} = -0.21048, \widehat{\mathcal{B}_3} = -0.61541,$$
$$\mathcal{B}_4 = 0.49358, \widehat{\mathcal{B}_5} = -0245174, \widehat{\mathcal{B}_6} = 0.92547.$$

For censored data, we take, for example, 5 intervals ($r = 5$) as the number of classes, as suggested by Bagdonavicius and Nikulin (2011) and Bagdonavicius et al. (2013). The elements of the estimated Fisher information matrix $I(\hat{\underline{\mathbb{O}}})$ are presented as follows:

$$I(\hat{\underline{\mathbb{O}}}) = \begin{pmatrix} 1.09654 & 2.15048 & 0.514872 & -5.91254 & 2.00215 & 1.09857 & 0.61472 & 3.00021 & 0.74581 \\ & 0.65842 & -6.21547 & 0.53251 & -1.00248 & 2.02188 & -7.15482 & 0.33615 & 1.24182 \\ & & 0.19547 & 0.00240 & 1.02548 & -6.32514 & -0.06254 & 9.32541 & 0.61245 \\ & & & 1.00245 & 0.37948 & 0.12548 & 3.21547 & -5.00218 & 0.00097 \\ & & & & 0.32458 & -4.12572 & 1.02458 & 0.95774 & 0.84752 \\ & & & & & 3.00218 & -6.21542 & 0.900014 & 7.00015 \\ & & & & & & 0.08457 & 2.00978 & 3.02157 \\ & & & & & & & 0.85475 & -1.01021 \\ & & & & & & & & 0.17548 \end{pmatrix}.$$

Then, we calculate the value of the test statistic as $Y_n^2 = 9.120054$. The critical value is $\chi^2_{0.05}(4) = 9.488 > Y_n^2$. This data can be fitted by our proposed TPLF model with WBLHF in proper manner.

7.2 Validation of the TPLF model under the EBLHF

Considering that these data are distributed according to the TPLF model with EBLHF and using R statistical software (the BB package), the maximum likelihood estimates of the parameter vector $\underline{\mathbb{O}}$ are provided as

$$\hat{\lambda} = 0.6254, \widehat{\boldsymbol{\Sigma}^2} = 1.02548,$$
$$\widehat{\mathcal{B}_1} = 3.0254, \widehat{\mathcal{B}_2} = -9.03251, \widehat{\mathcal{B}_3} = -0.61587,$$
$$\mathcal{B}_4 = 1.0254, \widehat{\mathcal{B}_5} = 2.95014, \widehat{\mathcal{B}_6} = -2.03215.$$

For censored data, we take, for example, 5 intervals ($r = 5$) as the number of classes, as suggested by Bagdonavicius and Nikulin (2011) and Bagdonavicius et al. (2013). The elements of the estimated Fisher information matrix $I(\hat{\underline{\mathbb{O}}})$ are presented as follows:

$$I\left(\underline{\hat{\mathbb{O}}}\right)=\begin{pmatrix} 1.63254 & -6.32517 & 2.61502 & -8.3265 & 1.02541 & 0.32514 & 0.96584 & 1.06025 \\ & 2.95312 & 5.00002 & 1.02543 & 1.03268 & 1.92354 & -7.0095 & -10.7658 \\ & & 0.88321 & 2.6157 & 3.00002 & 1.20451 & 2.03251 & 2.61547 \\ & & & 1.54875 & 6.32514 & 3.26514 & 1.02547 & 0.00214 \\ & & & & 2.00004 & -5.3268 & -4.7474 & 2.00315 \\ & & & & & 0.96584 & 2.30142 & 1.02547 \\ & & & & & & 1.02547 & 4.32510 \\ & & & & & & & 0.32014 \end{pmatrix}$$

Then we calculate the value of the test statistic as $Y_n^2 = 8.80451$. The critical value is $\chi^2_{0.05}(4) = 9.488 > Y_n^2$. This data can be fitted by our proposed TPLF model with EBLHF in proper manner.

7.3 Validation of the TPLF model under the GBLHF

Considering that these data are distributed according to the TPLF model with GBLHF and using R statistical software (the BB package), the maximum likelihood estimates of the parameter vector $\underline{\mathbb{O}}$ are provided as

$$\hat{\gamma} = 1.023014, \hat{\varphi} = 0.98351, \widehat{\boldsymbol{\Sigma}^2} = 1.120034,$$
$$\widehat{\mathcal{B}_1} = 0.963514, \widehat{\mathcal{B}_2} = 0.845271, \widehat{\mathcal{B}_3} = -2.61847,$$
$$\widehat{\mathcal{B}_4} = 0.530018, \widehat{\mathcal{B}_5} = -1.02947, \widehat{\mathcal{B}_6} = 0.125437.$$

We take the estimated Fisher matrix with intervals of $r = 5$, which is represented as

$$I\left(\underline{\hat{\mathbb{O}}}\right)=\begin{pmatrix} 0.93254 & 2.02154 & 0.21547 & 1.09587 & -2.00347 & 2.15427 & -8.06254 & -5.00214 & 0.61542 \\ & 1.09651 & 1.09658 & -4.21571 & 0.61547 & 0.12548 & 1.23565 & 0.19574 & 1.02659 \\ & & 0.85247 & 0.02154 & 1.03254 & -4.03251 & -6.21541 & 0.21547 & 1.02558 \\ & & & 1.63254 & 0.91547 & 1.02548 & -4.02158 & 2.02154 & -7.02154 \\ & & & & 2.00314 & -5.03268 & 4.02158 & -8.5479 & -4.01924 \\ & & & & & 0.86592 & 0.21547 & 0.61547 & 0.2158 \\ & & & & & & 0.46215 & 1.09754 & 0.21547 \\ & & & & & & & 1.00985 & -1.00025 \\ & & & & & & & & 0.36251 \end{pmatrix},$$

then we calculate the value of the Bagdonavicius and Nikulin (2011) and Bagdonavicius et al. (2013) statistic : $Y_n^2 = 8.123048$. For different critical values: $\alpha = 5\%$ and $\alpha = 10\%$, we find $Y^2 < \chi^2_{0.05}(4) = 9.488$ and $Y_n^2 < \chi^2_{0.1}(5-1)$ respectively. Hence we reason that the emergency care data is compatible with our proposed TPLF model with GBLHF.

7.4 Validation of the TPLF model under the PBLHF

Considering that these data are distributed according to the TPLF model with PBLHF and using R statistical software (the BB package), the maximum likelihood estimates of the parameter vector $\underline{\mathbb{O}}$ are provided as

$$\hat{\eta} = 0.09584, \hat{\alpha} = 1.24051, \widehat{\boldsymbol{\Sigma}^2} = 0.89574,$$
$$\widehat{\mathcal{B}_1} = 0.32658, \widehat{\mathcal{B}_2} = 0.19487, \widehat{\mathcal{B}_3} = -0.613548,$$
$$\mathcal{B}_4 = -2.164957, \widehat{\mathcal{B}_5} = -1.94378, \widehat{\mathcal{B}_6} = 0.379138.$$

We take the estimated Fisher matrix with intervals of $r = 5$, which is represented as:

$$I(\hat{\underline{\mathbb{O}}}) = \begin{pmatrix} 0.93518 & 1.02547 & 0.321874 & -2.06587 & 1.02368 & 1.09557 & -2.45871 & 0.19385 & -3.21574 \\ & 0.64875 & 1.026589 & 1.02547 & -8.02157 & -9.12547 & -4.02154 & 3.12574 & -6.21542 \\ & & 0.61847 & 0.84753 & 2.03254 & -12.0214 & 0.12547 & 0.95847 & 4.02157 \\ & & & 0.46158 & -3.16587 & -4.1257 & -4.91578 & 0.12548 & 1.9658 \\ & & & & 1.02458 & 1.84576 & 1.3258 & 0.002157 & 1.21547 \\ & & & & & 0.93784 & -2.12547 & -1.29568 & -2.02145 \\ & & & & & & 1.19325 & -3.21547 & 0.23515 \\ & & & & & & & 0.84571 & 1.55524 \\ & & & & & & & & 1.90542 \end{pmatrix}$$

then we calculate the value of the Bagdonavicius and Nikulin: $Y_n^2 = 7.23197$. For different critical values : $\alpha = 5\%$ and $\alpha = 10\%$, we find $Y_n^2 < \chi^2_{0.05}(5-1) = 9.488$ and $Y_n^2 < \chi^2_{0.1}(4) = 7.779$ respectively. Hence we reason that the emergency care data is compatible with our proposed TPLF model with PBLHF.

8.An application under the censored heart attack dataset

This is a multivariate type of dataset, which refers to multivariate numerical data analysis that involves or provides a range of distinct mathematical or statistical variables. This database includes 76 covariables, in our work we have used 5 covariables which are: age, resting blood pressure, serum cholesterol, maximum heart rate achieved, and oldpeak: ST depression induced by exercise relative to rest. One of the main objectives of this dataset is to predict, using the patient's provided attributes, whether or not the individual has heart disease. Another experimental task involves diagnosing the individual and extracting various insights from the dataset that could help in a deeper understanding of the issue. The dataset was created by the Hungarian Institute of Cardiology, see https://doi.org/10.24432/C52P4X (alsom see Janosi et al. (1988)). We may study the distribution's validity and application by analyzing the goodness-of-fit of the TPLF model distribution and its ability to effectively represent observed patterns and variability in Heart attack data. We provide the point estimates for each fitted model (TPLF model with Weibull, exponential, Gombertez, and Pareto baseline hazard functions). The modified chi-squared test is used to select the most effective model out of all the fitted models to these data (see Bagdonavicius and Nikulin(2011) and Bagdonavicius et al. (2013)).

8.1 Validation of the TPLF model under the WBLHF

Given that these data are distributed using the TPLF model with the WBLHF and that R statistical software (the BB package) is used, the maximum likelihood estimates of the parameter vector $\underline{\mathbb{O}}$ are provided as follows

$$\hat{\kappa} = 1.00245, \hat{\rho} = 0.61472,$$
$$\widehat{\boldsymbol{\Sigma}^2} = 1.00214, \widehat{\mathcal{B}_1} = -2.02154,$$
$$\widehat{\mathcal{B}_2} = 3.00215, \widehat{\mathcal{B}_3} = -4.02875,$$
$$\mathcal{B}_4 = 1.023548, \widehat{\mathcal{B}_5} = -1.23487.$$

We choose 8 intervals as the number of classes for censored data, as suggested by Bagdonavicius and Nikulin (2011) and Bagdonavicius et al. (2013). The following are the elements of the estimated Fisher information matrix $I(\hat{\underline{\mathbb{O}}})$:

$$I(\hat{\underline{\mathbb{O}}}) = \begin{pmatrix} 2.003254 & 4.15785 & 0.15478 & -3.62501 & 0.00217 & -8.12544 & 6.00985 & 1.21545 \\ & 1.96542 & -6.30214 & 1.02458 & 3.11241 & 2.13547 & 1.75845 & -8.00002 \\ & & 0.96584 & 3.20145 & -2.15347 & 0.95135 & 1.54863 & -0.96584 \\ & & & 2.965847 & 8.215478 & -4.00215 & 12.3518 & 6.00214 \\ & & & & 3.021457 & 2.15475 & -5.88547 & -9.32514 \\ & & & & & 1.92547 & 0.35748 & 12.2514 \\ & & & & & & 2.11045 & -17.2152 \\ & & & & & & & 0.65847 \end{pmatrix},$$

The test statistic value is then calculated as $Y_n^2 = 13.58497$. The key point is $\chi^2_{0.05}(7) = 14.0689 > Y_n^2$. This data can be properly fitted by our proposed TPLF model with WBLHF.

8.2 Validation of the TPLF model under the EBLHF

Given that these data are distributed using the TPLF model with the exponential baseline hazard function and that R statistical software is used, the maximum likelihood estimates of the parameter vector $\underline{\mathbb{O}}$ are provided as follows

$$\hat{\lambda} = 2.12501, \widehat{\boldsymbol{\Sigma}^2} = 1.02102,$$
$$\widehat{\mathcal{B}_1} = 0.2154, \widehat{\mathcal{B}_2} = -9.3258, \widehat{\mathcal{B}_3} = 1.95201,$$
$$\mathcal{B}_4 = -10.8124, \widehat{\mathcal{B}_5} = 2.61024.$$

We choose 8 intervals as the number of classes for censored data, as suggested by Bagdonavicius and Nikulin (2011) and Bagdonavicius et al. (2013). The following are the elements of the estimated Fisher information matrix $I(\widehat{\underline{\mathbb{O}}})$:

$$I(\widehat{\underline{\mathbb{O}}}) = \begin{pmatrix} 1.32054 & -7.92518 & -9.3251 & 2.95147 & 1.92358 & 1.32547 & 3.00214 \\ & 0.26531 & -7.1658 & 1.32547 & 0.21574 & 0.96325 & 4.9513 \\ & & 1.92543 & -10.3201 & 5.0001 & 1.42152 & 3.0302 \\ & & & 2.30154 & 1.11124 & 0.85647 & 1.95487 \\ & & & & 3.00002 & 1.44475 & -7.9514 \\ & & & & & 1.20541 & 2.0215 \\ & & & & & & 2.15482 \end{pmatrix},$$

The test statistic value is then calculated as $Y_n^2 = 12.95682$. The key point is $\chi^2_{0.05}(7) = 14.0689 > Y_n^2$. This data can be properly fitted by our proposed TPLF model with EBLHF.

8.3 Validation of the TPLF model under the GBLHF

Given that these data were distributed using the TPLF model with the Gombertez baseline hazard function and that R statistical software was used, the maximum likelihood estimates of the parameter vector $\underline{\mathbb{O}}$ can be obtained as

$$\hat{\gamma} = 1.36001, \hat{\varphi} = 1.24801, \widehat{\boldsymbol{\Sigma}^2} = 1.03452,$$
$$\widehat{\mathcal{B}_1} = -3.51204, \widehat{\mathcal{B}_2} = -4.671305, \widehat{\mathcal{B}_3} = 1.30265,$$
$$\widehat{\mathcal{B}_4} = 2.101036, \widehat{\mathcal{B}_5} = 1.063254.$$

We take the estimated Fisher matrix with $r = 8$ intervals, which is denoted as

$$I(\widehat{\underline{\mathbb{O}}}) = \begin{pmatrix} 0.965847 & -5.02147 & -9.21578 & 1.95847 & 3.21542 & -21.31544 & 0.93548 & 2.00031 \\ & 3.26150 & 2.301245 & -8.02547 & 1.362548 & 0.965842 & 1.20154 & 1.99658 \\ & & 2.61354 & 1.25698 & 0.15472 & -7.95382 & -12.4512 & 2.00001 \\ & & & 2.165847 & 1.952014 & 1.02154 & -9.32518 & 1.02458 \\ & & & & 1.92546 & 2.035214 & -12.32548 & 3.1102 \\ & & & & & 0.84571 & 4.03528 & -8.3219 \\ & & & & & & 4.00621 & 1.02326 \\ & & & & & & & 0.88827 \end{pmatrix},$$

Then we compute the test of Bagdonavicius and Nikulin (2011) and Bagdonavicius et al. (2013): $Y_n^2 = 11.684002$. For various critical values of $\alpha = 5\%$, we obtain $Y^2 < \chi^2_{0.05}(7) = 14.0689$. As a result, we conclude that the emergency care data is consistent with our proposed TPLF model using the Gompertz baseline hazard function.

8.4 Validation of the TPLF model under the PBLHF

Given that these data are distributed using the TPLF model with the Pareto baseline hazard function and that R statistical software is used, the maximum likelihood estimates of the parameter vector $\underline{\mathbb{O}}$ are provided as follows.

$$\hat{\eta} = 1.00254, \hat{\alpha} = 1.36254, \widehat{\boldsymbol{\Sigma}^2} = 0.88695,$$
$$\widehat{\mathcal{B}_1} = 0.96584, \widehat{\mathcal{B}_2} = -0.93747, \widehat{\mathcal{B}_3} = -0.63625,$$

$$\mathcal{B}_4 = 1.02547, \widehat{\mathcal{B}_5} = -9.002547.$$

the estimated Fisher matrix with intervals of $r = 8$ is represented as:

$$I(\underline{\hat{\mathbb{O}}}) = \begin{pmatrix} 0.236584 & -9.32514 & -11.2548 & -13.6258 & 2.15471 & 3.26587 & 2.15473 & 1.99685 \\ & 2.05487 & -12.3254 & 1.95324 & 4.00215 & 3.02130 & 1.95684 & -7.3184 \\ & & 1.92564 & 0.902457 & -4.9515 & 1.95684 & 2.00514 & 2.15043 \\ & & & 0.32678 & 2.2223 & 4.00125 & 3.12022 & 1.84502 \\ & & & & 3.12045 & 3.01254 & 1.90547 & -9.8124 \\ & & & & & 1.39584 & -8.8965 & 1.35486 \\ & & & & & & 1.32547 & 0.96584 \\ & & & & & & & 2.91573 \end{pmatrix}$$

Then, we compute the test of Bagdonavicius and Nikulin (2011) and Bagdonavicius et al. (2013): $Y_n^2 = 11.57482$. For alternative critical values of $\alpha = 5\%$, we find $Y_n^2 < \chi^2_{0.05}(7) = 14.0689$, respectively. As a result, we conclude that the emergency care data is consistent with our proposed TPLF model with PBLHF.

9. Conclusion

In this paper, we propose a new frailty model to account for survival data's unobserved heterogeneity. The TPL distribution with a unitary mean is used as the frailty distribution in this case. We calculated the Laplace transform of this frailty distribution, then developed the marginal survival and hazard functions. The Laplace transform of the TPL distribution provides a simple mathematical method for obtaining analytical formulas for the TPLF model's hazard and marginal survival functions. The Gompertz, Weibull, exponential, and Pareto hazard functions were used as the baseline hazard functions to construct the TPLF models. Simulation analyses revealed that, as expected, the convergence characteristics of the MXLEs were achieved for a variety of censoring proportions ($0\%, 10\%, 30\%$, and 50%). We created a modified chi-squared test statistic based on the statistics of Nikulin Rao Robson (1973a,b,c, and 1974) and Bagdonavicus and Nikuln (2011), taking into consideration both complete and censored data cases, to evaluate a statistical test for the TPLF model in survival analysis. The TPLF model was used to construct the suggested test statistic element formulations. Besides performing as intended, the modified chi-squared test demonstrated its ability to identify unobserved heterogeneity in both small and large samples ($n = 26$, $n = 40$, $n = 140$, $n = 250$, $n = 550$ and $n = 1000$). The simulation study demonstrates that the suggested test for the TPLF model works effectively in both complete and censored data sets. It implies that the test is reliable and precise for evaluating the goodness of fit of our suggested model to real survival data. The test developed by Bagdonavicius and Nikulin (2011) and Bagdonavicius et al. (2013) offers favorable statistical properties, such as precise parameter estimates. This demonstrates that the test provides accurate inference for the TPLF model and captures the underlying structure of the data. Although the simulation study confirms the efficacy of the proposed test, more research, and validation in various contexts or datasets would be interesting. Additional sample sizes, various frailty distribution assumptions, or alternative censoring procedures should be explored in order to acquire a further understanding of the test's performance. We used a real emergency care dataset (with six covariates) collected from an Algerian department on emergencies in the presence of censoring to test the TPLF model with Weibull, exponential, Gombertez, and Pareto as the baseline hazard functions. The Weibull TPLF, Gompertz TPLF, and Pareto TPLF models provided a good fit for the emergency care data, according to the Bagdonavicius and Nikulin (2011) and Bagdonavicius et al. (2013)test for censored data. In the application with data on emergency care, the TPLF models were effective at capturing unobserved heterogeneity. For the uncensored emergency care data, we have the following results:

I. Under the WBLHF: Since $\chi^2_{0.05}(4) = 9.488 > Y_n^2 = 9.120054 \Rightarrow$ Decision: accept H_0.
II. Under the EBLHF: Since $\chi^2_{0.05}(4) = 9.488 > Y_n^2 = 8.80451 \Rightarrow$ Decision: accept H_0.
III. Under the GBLHF: Since $\chi^2_{0.05}(4) = 9.488 > Y_n^2 = 8.123048 \Rightarrow$ Decision: accept H_0.
IV. Under the GBLHF: Since $\chi^2_{0.05}(4) = 9.488 > Y_n^2 = 7.23197 \Rightarrow$ Decision: accept H_0.

For the censored heart attack data, we have the following results:

I. Under the WBLHF: Since $\chi^2_{0.05}(7) = 14.0689 > Y_n^2 = 13.58497 \Rightarrow$ Decision: accept H_0.
II. Under the EBLHF: Since $\chi^2_{0.05}(7) = 14.0689 > Y_n^2 = 12.95682 \Rightarrow$ Decision: accept H_0.

III. Under the GBLHF: $\chi^2_{0.05}(7) = 14.0689 > Y_n^2 = 11.684002 \Rightarrow$ Decision: accept H_0.

IV. Under the GBLHF: $\chi^2_{0.05}(7) = 9.488 > Y_n^2 = 11.57482 \Rightarrow$ Decision: accept H_0.

For more details about related works in risk analysis and insurance see Hashempour et al. (2023), Mohamed et al. (2024) and Elbatal et al. (2024). For some new compound G families for the Cox model and frailty models see Hashem et al. (2024). Sen et al. (2022) and Alizadeh et al. (2023) presented a novel XGamma extensions, we may use this in frailty models and relibility anlaysis. For other Lomax extention for the fraitly models see Hamed et al. (2022), El-Morshedy et al. (2022), Al-Essa et al. (2023), Salem et al. (2023).

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10.Appendix: score functions

The score function is the gradient of the log-likelihood function of the probability distribution with respect to the distribution's support.

10.1 The Score functions of TLPF model in case of the WBLHF

The score functions of each parameter using the WBLHF are obtained as follows:

$$\frac{\partial \log(\Theta)}{\partial \mathcal{B}_1} = \sum_{i=1}^{n} \delta_i x_1 - \sum_{i=1}^{n} \delta_i \left\{ \begin{matrix} \left[(t_i^k x_1 \exp(x_1\mathcal{B}_1 + x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2) \right] \\ \left[(t_i^k \exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2) + \rho^k[\iota(\boldsymbol{\Sigma}^2) - 2] \right]^{-1} \end{matrix} \right\}$$
$$+ \sum_{i=1}^{n} \left\{ - \begin{matrix} \left[(t_i^k x_1 \exp(x_1\mathcal{B}_1 + x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2) \right] \\ \left[(t_i^k \exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2) + \rho^k[\iota(\boldsymbol{\Sigma}^2) - 2] \right]^{-1} \end{matrix} \right\}$$
$$+ \sum_{i=1}^{n} \left\{ - \begin{matrix} \begin{bmatrix} (t_i^k x_1 \exp(x_1\mathcal{B}_1 + x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2) \\ \rho^k[\iota(\boldsymbol{\Sigma}^2) - 2]C(\boldsymbol{\Sigma}^2) \end{bmatrix} \\ \left[(t_i^k \exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2) + \rho^k[\iota(\boldsymbol{\Sigma}^2) - 2] \right]^{-1} \end{matrix} \right\} \begin{bmatrix} t_i^k \exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2) + \rho^k[\iota(\boldsymbol{\Sigma}^2) - 2] \\ \mathcal{D}(\boldsymbol{\Sigma}^2) + \rho^k[\iota(\boldsymbol{\Sigma}^2) - 2]C(\boldsymbol{\Sigma}^2) \end{bmatrix}^{-1}$$
$$+ \sum_{i=1}^{n} \delta_i \left\{ - \begin{matrix} \begin{bmatrix} \mathcal{D}(\boldsymbol{\Sigma}^2)\left(t_i^k x_1 \exp(x_1\mathcal{B}_1 + x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2) \right) \\ \rho^k[\iota(\boldsymbol{\Sigma}^2) - 2]C(\boldsymbol{\Sigma}^2) \end{bmatrix} \\ \begin{bmatrix} t_i^k \exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2) + \rho^k[\iota(\boldsymbol{\Sigma}^2) - 2] \\ \mathcal{D}(\boldsymbol{\Sigma}^2) + \rho^k[\iota(\boldsymbol{\Sigma}^2) - 2]C(\boldsymbol{\Sigma}^2) \end{bmatrix}^{-1} \end{matrix} \right\} \begin{bmatrix} t_i^k \exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2) + \rho^k[\iota(\boldsymbol{\Sigma}^2) - 2] \\ \mathcal{D}(\boldsymbol{\Sigma}^2) \\ +2\rho^k[\iota(\boldsymbol{\Sigma}^2) - 2]C(\boldsymbol{\Sigma}^2) \end{bmatrix}^{-1},$$

where

$$\lambda(\boldsymbol{\Sigma}^2|t_i) = \{\lambda t_i \exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2) + [\iota(\boldsymbol{\Sigma}^2) - 2]\}$$

$$\frac{\partial \log(\Theta)}{\partial \mathcal{B}_2} = \sum_{i=1}^{n} \delta_i x_2 - \sum_{i=1}^{n} \delta_i \left\{ \begin{matrix} \left[(t_i^k x_2 \exp(x_1\mathcal{B}_1 + x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2) \right] \\ \left[(t_i^k \exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2) + \rho^k[\iota(\boldsymbol{\Sigma}^2) - 2] \right]^{-1} \end{matrix} \right\}$$
$$+ \sum_{i=1}^{n} \left\{ - \begin{matrix} \left[(t_i^k x_1 \exp(x_2\mathcal{B}_1 + x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2) \right] \\ \left[(t_i^k \exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2) + \rho^k[\iota(\boldsymbol{\Sigma}^2) - 2] \right]^{-1} \end{matrix} \right\}$$
$$+ \sum_{i=1}^{n} \left\{ - \begin{matrix} \begin{bmatrix} t_i^k x_2 \exp(x_1\mathcal{B}_1 + x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2) \\ \rho^k[\iota(\boldsymbol{\Sigma}^2) - 2]C(\boldsymbol{\Sigma}^2) \end{bmatrix} \\ \left[t_i^k \exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2) + \rho^k[\iota(\boldsymbol{\Sigma}^2) - 2] \right]^{-1} \end{matrix} \right\} \begin{bmatrix} t_i^k \exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2) + \rho^k[\iota(\boldsymbol{\Sigma}^2) - 2] \\ \mathcal{D}(\boldsymbol{\Sigma}^2) + \rho^k[\iota(\boldsymbol{\Sigma}^2) - 2]C(\boldsymbol{\Sigma}^2) \end{bmatrix}^{-1}$$

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$$
+\sum_{i=1}^{n}\delta_i\left\{\begin{array}{l}-\begin{bmatrix}\mathcal{D}(\boldsymbol{\Sigma}^2)\\ t_i^k x_2\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2)\\ \rho^k[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{bmatrix}\\ \begin{bmatrix}t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{D}(\boldsymbol{\Sigma}^2)\\ +\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{bmatrix}^{-1}\end{array}\right\}\begin{bmatrix}t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{D}(\boldsymbol{\Sigma}^2)\\ +2\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{bmatrix}^{-1},
$$

$$
\begin{aligned}
\frac{\partial\log(\Theta)}{\partial k}&=\sum_{i=1}^{n}\delta_i\frac{1}{k}+\sum_{i=1}^{n}\delta_i\log t_i-\sum_{i=1}^{n}\left\{\begin{array}{l}-\left[\ln t_i t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\ln\rho\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\right]\\ \left[t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\right]^{-1}\end{array}\right\}\\
&+\sum_{i=1}^{n}\left\{\begin{array}{l}\left[t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)(\ln\rho-\ln t_i)\right]\\ \left[t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\right]^{-1}\end{array}\right\}+\sum_{i=1}^{n}\left[t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\right]^{-1}\\
&\qquad\times\left\{\begin{array}{l}\begin{bmatrix}\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\\ \left(t_i^k\exp(x_i^{\intercal}\mathcal{B})(\ln\rho-\ln t_i)\right)\end{bmatrix}\\ \begin{bmatrix}t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{bmatrix}^{-1}\end{array}\right\}\\
&+\sum_{i=1}^{n}\left\{\begin{array}{l}\begin{bmatrix}\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\\ t_i^k\exp(x_i^{\intercal}\mathcal{B})(\ln\rho-\ln t_i)\\ \mathcal{D}(\boldsymbol{\Sigma}^2)(1+9\boldsymbol{\Sigma}^2)\end{bmatrix}\\ \times\begin{bmatrix}t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\\ \mathcal{D}(\boldsymbol{\Sigma}^2)\\ +\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{bmatrix}^{-1}\end{array}\right\}\begin{bmatrix}t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+2\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{bmatrix}^{-1},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\log(\Theta)}{\partial\rho}&=\sum_{i=1}^{n}\delta_i\left\{\begin{array}{l}\left[k\rho^{k-1}[\iota(\boldsymbol{\Sigma}^2)-2]\right]\\ \left[t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\right]^{-1}\end{array}\right\}\\
&\qquad+\sum_{i=1}^{n}\left\{\begin{array}{l}\left[k\left(t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\right)\right]\\ \left[\rho(t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\right]^{-1}\end{array}\right\}\\
&+\sum_{i=1}^{n}\left\{\begin{array}{l}\begin{bmatrix}k\rho^{k-1}[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\\ (t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\end{bmatrix}\\ \left[t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\right]^{-1}\end{array}\right\}\begin{bmatrix}t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{bmatrix}^{-1}\\
&+\sum_{i=1}^{n}\delta_i\left\{\begin{array}{l}\left[k\rho^{k-1}[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\mathcal{D}(\boldsymbol{\Sigma}^2)\right]\\ \begin{bmatrix}t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{bmatrix}^{-1}\end{array}\right\}\begin{bmatrix}t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+2\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{bmatrix}^{-1},
\end{aligned}
$$

$$
\frac{\partial\log(\Theta)}{\partial\boldsymbol{\Sigma}^2}=\sum_{i=1}^{n}\delta_i\left[\frac{9}{1+9\boldsymbol{\Sigma}^2}\right]-\sum_{i=1}^{n}\delta_i\left\{\begin{array}{l}\left[\sqrt{2(1+7\boldsymbol{\Sigma}^2)}\left(9t_i^k\exp(x_i^{\intercal}\mathcal{B})\right)+21\rho^k\right]\\ \left[\sqrt{2(1+7\boldsymbol{\Sigma}^2)}\begin{pmatrix}t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\\ +\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\end{pmatrix}\right]\end{array}\right\}
$$

$$+\sum_{i=1}^{n}\left\{\begin{bmatrix}21\left[t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\right]\\-\left(9t_i^k\exp(x_i^{\intercal}\mathcal{B})\right)\left[\iota(\boldsymbol{\Sigma}^2)-2\right]\left(\sqrt{2(1+7\boldsymbol{\Sigma}^2)}\right)\end{bmatrix}\atop\begin{bmatrix}\left(\sqrt{2(1+7\boldsymbol{\Sigma}^2)}\right)t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\\+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2][\iota(\boldsymbol{\Sigma}^2)-2]\end{bmatrix}\right\}$$

$$+\sum_{i=1}^{n}\left\{\begin{matrix}\left[(1+9\boldsymbol{\Sigma}^2)\left(t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\right)\right]\\\begin{bmatrix}t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\\\mathcal{D}(\boldsymbol{\Sigma}^2)\\+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\left(3+9\boldsymbol{\Sigma}^2-\iota(\boldsymbol{\Sigma}^2)\right)\end{bmatrix}^{-1}\end{matrix}\right\}$$

$$\left(\begin{matrix}\left\{\begin{matrix}\left[3\left(-63\boldsymbol{\Sigma}^2-11+6\sqrt{2(1+7\boldsymbol{\Sigma}^2)}\right)\right]\\\left[\left(\sqrt{2(1+7\boldsymbol{\Sigma}^2)}\right)(1+9\boldsymbol{\Sigma}^2)^2\right]^{-1}\end{matrix}\right\}\\+\left[(1+9\boldsymbol{\Sigma}^2)^2\begin{pmatrix}t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\\+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\end{pmatrix}^2\right]^{-1}\\\left\{\begin{matrix}\left[\varsigma(\boldsymbol{\Sigma}^2)\rho^kC(\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\left(9-\varsigma(\boldsymbol{\Sigma}^2)\right)\right]\\\times(1+9\boldsymbol{\Sigma}^2)\begin{bmatrix}t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\\+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\end{bmatrix}\\-\begin{bmatrix}9(t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+9\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\\+9(t_i^k\exp(x_i^{\intercal}\mathcal{B})\varsigma(\boldsymbol{\Sigma}^2)\rho^k(1+9\boldsymbol{\Sigma}^2)\end{bmatrix}\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{matrix}\right\}\end{matrix}\right)$$

$$+\sum_{i=1}^{n}\delta_i\begin{bmatrix}t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\\\mathcal{D}(\boldsymbol{\Sigma}^2)\\+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{bmatrix}^{-1}$$

$$\times\begin{bmatrix}t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\\\mathcal{D}(\boldsymbol{\Sigma}^2)+2\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{bmatrix}^{-1}$$

$$\times\left\{\begin{matrix}\left[\varsigma(\boldsymbol{\Sigma}^2)\rho^kC(\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\left(9-\varsigma(\boldsymbol{\Sigma}^2)\right)\right]\\\times\begin{bmatrix}t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\\\mathcal{D}(\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{bmatrix}\\-\begin{bmatrix}-9(t_i^k\exp(x_i^{\intercal}\mathcal{B})+\varsigma(\boldsymbol{\Sigma}^2)\rho^k\mathcal{D}(\boldsymbol{\Sigma}^2)+\left(18+\varsigma(\boldsymbol{\Sigma}^2)\right)\\\times t_i^k\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\\+\varsigma(\boldsymbol{\Sigma}^2)\rho^kC(\boldsymbol{\Sigma}^2)+\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]\left(9-\varsigma(\boldsymbol{\Sigma}^2)\right)\end{bmatrix}\\\times\rho^k[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{matrix}\right\},$$

where $\varsigma(\boldsymbol{\Sigma}^2)=\frac{21}{\sqrt{2(1+7\boldsymbol{\Sigma}^2)}}$.

10.2 The score functions of TLPF model in case of the EBLHF

The score functions of each parameter using the EBLHF are obtained as follows :

$$\frac{\partial\log(\Theta)}{\partial\lambda}=\sum_{i=1}^{n}\delta_i\frac{1}{\lambda}-\sum_{i=1}^{n}\delta_i\left\{\begin{matrix}[t_i\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)]\\\times\left[\lambda t_i\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right]^{-1}\end{matrix}\right\}$$

$$+\sum_{i=1}^{n}\left\{-\begin{matrix}[t_i\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)]\\\times\left[\lambda t_i\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right]^{-1}\end{matrix}\right\}+\sum_{i=1}^{n}-\left\{\begin{matrix}t_i\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\\\times\left[\lambda t_i\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right]^{-1}\end{matrix}\right\}$$

$$\begin{bmatrix}\lambda t_i \exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{bmatrix}^{-1}$$

$$+\sum_{i=1}^{n}\delta_i\left\{\begin{matrix}\left[-\left(t_i\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\right)\mathcal{D}(\boldsymbol{\Sigma}^2)[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\right]\\ \begin{bmatrix}(\lambda t_i\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2])\\ \times\mathcal{D}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{bmatrix}^{-1}\end{matrix}\right\}\begin{bmatrix}(\lambda t_i\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2])\mathcal{D}(\boldsymbol{\Sigma}^2)\\ +2[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{bmatrix}^{-1},$$

$$\frac{\partial\log(\Theta)}{\partial\mathcal{B}_1}=\sum_{i=1}^{n}\delta_i x_1-\sum_{i=1}^{n}\delta_i\left\{\begin{matrix}[\lambda t_i x_1\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2)]\\ \left[\lambda t_i\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right]^{-1}\end{matrix}\right\}$$
$$+\sum_{i=1}^{n}\left(-\begin{matrix}[\lambda t_i x_1\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2)]\\ \left[\lambda t_i\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right]^{-1}\end{matrix}\right)$$
$$+\sum_{i=1}^{n}-\left\{\begin{matrix}\lambda t_i x_1\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2)[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\\ \left[\lambda t_i\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right]^{-1}\end{matrix}\right\}$$
$$\times[\lambda(\boldsymbol{\Sigma}^2|t_i)\mathcal{D}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)]^{-1}$$
$$+\sum_{i=1}^{n}\delta_i\left\{\begin{matrix}\begin{bmatrix}-\left(\lambda t_i x_1\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2)\right)\\ \mathcal{D}(\boldsymbol{\Sigma}^2)[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{bmatrix}\\ \times[\lambda(\boldsymbol{\Sigma}^2|t_i)\mathcal{D}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)]^{-1}\end{matrix}\right\}[\lambda(\boldsymbol{\Sigma}^2|t_i)\mathcal{D}(\boldsymbol{\Sigma}^2)+2[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)]^{-1},$$

$$\frac{\partial\log(\Theta)}{\partial\mathcal{B}_2}=\sum_{i=1}^{n}\delta_i x_2-\sum_{i=1}^{n}\delta_i\left\{\begin{matrix}[\lambda t_i x_2\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2)]\\ \left[\lambda t_i\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right]^{-1}\end{matrix}\right\}$$
$$+\sum_{i=1}^{n}\left\{-\begin{matrix}[\lambda t_i x_2\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2)]\\ \left[\lambda t_i\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right]^{-1}\end{matrix}\right\}$$
$$+\sum_{i=1}^{n}-\left\{\begin{matrix}\lambda t_i x_2\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2)\\ \times[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\\ \times\left[\lambda t_i\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right]^{-1}\end{matrix}\right\}[\lambda(\boldsymbol{\Sigma}^2|t_i)\mathcal{D}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)]^{-1}$$
$$+\sum_{i=1}^{n}\delta_i\left\{\begin{matrix}\begin{bmatrix}-\left(\lambda t_i x_2\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2)\right)\\ \mathcal{D}(\boldsymbol{\Sigma}^2)\\ [\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{bmatrix}\\ \begin{bmatrix}\lambda t_i\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{bmatrix}^{-1}\end{matrix}\right\}[\lambda(\boldsymbol{\Sigma}^2|t_i)\mathcal{D}(\boldsymbol{\Sigma}^2)+2[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)]^{-1},$$

$$\frac{\partial\log(\Theta)}{\partial\boldsymbol{\Sigma}^2}=\sum_{i=1}^{n}\delta_i\left(\frac{9}{1+9\boldsymbol{\Sigma}^2}\right)-\sum_{i=1}^{n}\delta_i\left\{\begin{matrix}\left(9\lambda t_i\exp(x_i^{\mathrm{I}}\mathcal{B})\right)\left(\sqrt{2(1+7\boldsymbol{\Sigma}^2)}\right)+21\\ \left[\left(\sqrt{2(1+7\boldsymbol{\Sigma}^2)}\right)\lambda(\boldsymbol{\Sigma}^2|t_i)\right]^{-1}\end{matrix}\right\}$$
$$+\sum_{i=1}^{n}\left\{\begin{matrix}21\lambda t_i\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)-9\lambda t_i\exp(x_i^{\mathrm{I}}\mathcal{B})\\ \times\left(\sqrt{2(1+7\boldsymbol{\Sigma}^2)}\right)[\iota(\boldsymbol{\Sigma}^2)-2]\\ \times\left[[\iota(\boldsymbol{\Sigma}^2)-2](\sqrt{2(1+7\boldsymbol{\Sigma}^2)}\lambda(\boldsymbol{\Sigma}^2|t_i)\right]^{-1}\end{matrix}\right\}+\sum_{i=1}^{n}\left\{\begin{matrix}(1+9\boldsymbol{\Sigma}^2)\{\lambda t_i\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\}\\ \times[\lambda(\boldsymbol{\Sigma}^2|t_i)\mathcal{D}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)]^{-1}\end{matrix}\right\}$$

$$\begin{pmatrix} 3\left(-63\boldsymbol{\Sigma}^2-11+6\sqrt{2(1+7\boldsymbol{\Sigma}^2)}\right)\left[\left(\sqrt{2(1+7\boldsymbol{\Sigma}^2)}\right)(1+9\boldsymbol{\Sigma}^2)^2\right]^{-1} \\ +[(1+9\boldsymbol{\Sigma}^2)^2\lambda(\boldsymbol{\Sigma}^2|t_i)^2]^{-1} \\ \left\{21C[\boldsymbol{\Sigma}^2]+\left(9\sqrt{2(1+7\boldsymbol{\Sigma}^2)}-21\right)[\iota(\boldsymbol{\Sigma}^2)-2]\left[\sqrt{2(1+7\boldsymbol{\Sigma}^2)}\right]^{-1}\right\} \\ (1+9\boldsymbol{\Sigma}^2)\left[\lambda t_i\exp(x_i^{\mathrm{T}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right] \\ -\left\{\begin{bmatrix}18\lambda t_i\exp(x_i^{\mathrm{T}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2) \\ +9[\iota(\boldsymbol{\Sigma}^2)-2]+\varsigma(\boldsymbol{\Sigma}^2)(1+9\boldsymbol{\Sigma}^2)\end{bmatrix}[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\right\} \end{pmatrix}$$

$$+\sum_{i=1}^{n}\delta_i[\lambda(\boldsymbol{\Sigma}^2|t_i)\mathcal{D}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)]^{-1}[\lambda(\boldsymbol{\Sigma}^2|t_i)\mathcal{D}(\boldsymbol{\Sigma}^2)+2[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)]^{-1}$$

$$\begin{Bmatrix} \left[\left(\varsigma(\boldsymbol{\Sigma}^2)\right)C(\boldsymbol{\Sigma}^2)+\left(9-\varsigma(\boldsymbol{\Sigma}^2)\right)[\iota(\boldsymbol{\Sigma}^2)-2]\right] \\ [\lambda(\boldsymbol{\Sigma}^2|t_i)\mathcal{D}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)] \\ -\begin{bmatrix}[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\left(9\lambda t_i\exp(x_i^{\mathrm{T}}\mathcal{B})+\varsigma(\boldsymbol{\Sigma}^2)\right)\mathcal{D}(\boldsymbol{\Sigma}^2) \\ +\left(18+\varsigma(\boldsymbol{\Sigma}^2)\right)\lambda(\boldsymbol{\Sigma}^2|t_i) \\ +\left(\varsigma(\boldsymbol{\Sigma}^2)\right)C(\boldsymbol{\Sigma}^2)+\left(9-\varsigma(\boldsymbol{\Sigma}^2)\right)[\iota(\boldsymbol{\Sigma}^2)-2].\end{bmatrix} \end{Bmatrix}.$$

10.3 The score functions of TLPF model in case of the GBLHF

The score functions of each parameter using the GBLHF are obtained as follows :

$$\begin{aligned}\frac{\partial\log(\Theta)}{\partial\varphi}=&\sum_{i=1}^{n}\delta_i\frac{1}{\varphi}-\sum_{i=1}^{n}\delta_i\begin{Bmatrix}\left[[\exp(\gamma t_i)-1]\exp(x_i^{\mathrm{T}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\right] \\ \left[\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\mathrm{T}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\right]^{-1}\end{Bmatrix} \\ &+\sum_{i=1}^{n}\begin{Bmatrix}-\left[[\exp(\gamma t_i)-1]\exp(x_i^{\mathrm{T}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\right] \\ \left[\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\mathrm{T}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\right]^{-1}\end{Bmatrix} \\ &+\sum_{i=1}^{n}-\begin{Bmatrix}\left[[\exp(\gamma t_i)-1]\exp(x_i^{\mathrm{T}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\gamma[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\right] \\ \left[\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\mathrm{T}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\right]^{-1}\end{Bmatrix} \\ &[\varphi(\boldsymbol{\Sigma}^2|t_i)\mathcal{D}(\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)]^{-1} \\ &+\sum_{i=1}^{n}\delta_i[\varphi(\boldsymbol{\Sigma}^2|t_i)\mathcal{D}(\boldsymbol{\Sigma}^2)+2\gamma[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)]^{-1}-\begin{Bmatrix}\begin{bmatrix}\left(\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\mathrm{T}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\right) \\ \mathcal{D}(\boldsymbol{\Sigma}^2)\gamma[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{bmatrix} \\ [\varphi(\boldsymbol{\Sigma}^2|t_i)\mathcal{D}(\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)]^{-1}\end{Bmatrix},\end{aligned}$$

where

$$\varphi(\boldsymbol{\Sigma}^2|t_i)=\{\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\mathrm{T}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\},$$

$$\begin{aligned}\frac{\partial\log(\Theta)}{\partial\gamma}=&\sum_{i=1}^{n}\delta_i\frac{1}{\gamma}+\sum_{i=1}^{n}\delta_i\log t_i-\sum_{i=1}^{n}\delta_i\begin{Bmatrix}\varphi t_i[\exp(\gamma t_i)-1]\exp(x_i^{\mathrm{T}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2] \\ \left[\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\mathrm{T}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\right]^{-1}\end{Bmatrix} \\ &+\sum_{i=1}^{n}\begin{Bmatrix}\begin{bmatrix}(1+9\boldsymbol{\Sigma}^2)\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\mathrm{T}}\mathcal{B}) \\ -\gamma\varphi t_i(\exp(\gamma t_i)\exp(x_i^{\mathrm{T}}\mathcal{B})\end{bmatrix} \\ \left[\gamma\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\mathrm{T}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\right]^{-1}\end{Bmatrix} \\ &+\sum_{i=1}^{n}\begin{Bmatrix}\begin{bmatrix}[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2) \\ (1+9\boldsymbol{\Sigma}^2)\begin{pmatrix}\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\mathrm{T}}\mathcal{B}) \\ -\gamma\varphi t_i(\exp(\gamma t_i)\exp(x_i^{\mathrm{T}}\mathcal{B})\end{pmatrix}\end{bmatrix} \\ \begin{bmatrix}\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\mathrm{T}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2) \\ +\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\end{bmatrix}^{-1}\end{Bmatrix}[\varphi(\boldsymbol{\Sigma}^2|t_i)\mathcal{D}(\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)]^{-1}\end{aligned}$$

$$+\sum_{i=1}^{n}\delta_i\begin{bmatrix}\varphi(\boldsymbol{\Sigma}^2|t_i)\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+2\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\end{bmatrix}^{-1}\begin{Bmatrix}[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\mathcal{D}(\boldsymbol{\Sigma}^2)\\ \times(1+9\boldsymbol{\Sigma}^2)\begin{pmatrix}\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\intercal}\mathcal{B})\\ -\gamma\varphi t_i(\exp(\gamma t_i)\exp(x_i^{\intercal}\mathcal{B})\end{pmatrix}\\ \times\begin{bmatrix}\varphi(\boldsymbol{\Sigma}^2|t_i)\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\end{bmatrix}^{-1}\end{Bmatrix},$$

$$\frac{\partial\log(\Theta)}{\partial\mathcal{B}_1}=\sum_{i=1}^{n}\delta_i x_1+\sum_{i=1}^{n}\delta_i\begin{Bmatrix}\varphi[\exp(\gamma t_i)-1]x_1\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2)\\ \begin{bmatrix}\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\\ +\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\end{bmatrix}^{-1}\end{Bmatrix}$$
$$+\sum_{i=1}^{n}-\begin{Bmatrix}\varphi[\exp(\gamma t_i)-1]x_1\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2)\\ \left[\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\right]^{-1}\end{Bmatrix}$$
$$+\sum_{i=1}^{n}-\begin{Bmatrix}\begin{bmatrix}\varphi[\exp(\gamma t_i)-1]x_1\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2)\\ \times\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\end{bmatrix}\\ \left[\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\right]^{-1}\end{Bmatrix}[\varphi(\boldsymbol{\Sigma}^2|t_i)\mathcal{D}(\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)]^{-1}$$
$$+\sum_{i=1}^{n}\delta_i\begin{bmatrix}\varphi(\boldsymbol{\Sigma}^2|t_i)\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+2\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\end{bmatrix}^{-1}$$
$$-\begin{Bmatrix}\begin{bmatrix}\varphi[\exp(\gamma t_i)-1]x_1\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2)\\ \times\mathcal{D}(\boldsymbol{\Sigma}^2)\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\end{bmatrix}\\ \times\begin{bmatrix}(\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2])\\ \times\mathcal{D}(\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\end{bmatrix}^{-1}\end{Bmatrix},$$

$$\frac{\partial\log(\Theta)}{\partial\mathcal{B}2}=\sum_{i=1}^{n}\delta_i x_2+\sum_{i=1}^{n}\delta_i\begin{Bmatrix}\varphi[\exp(\gamma t_i)-1]x_2\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2)\\ \begin{bmatrix}\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\\ +\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\end{bmatrix}^{-1}\end{Bmatrix}$$
$$+\sum_{i=1}^{n}-\begin{Bmatrix}\varphi[\exp(\gamma t_i)-1]x_2\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2)\\ \times\{\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\}^{-1}\end{Bmatrix}$$
$$+\sum_{i=1}^{n}-\begin{Bmatrix}\begin{bmatrix}\varphi[\exp(\gamma t_i)-1]x_2\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2)\\ \times\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\end{bmatrix}\\ \times\left[\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\right]^{-1}\end{Bmatrix}[\varphi(\boldsymbol{\Sigma}^2|t_i)\mathcal{D}(\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)]^{-1}$$
$$+\sum_{i=1}^{n}\delta_i[\varphi(\boldsymbol{\Sigma}^2|t_i)\mathcal{D}(\boldsymbol{\Sigma}^2)+2\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)]^{-1}-\begin{Bmatrix}\begin{bmatrix}\varphi[\exp(\gamma t_i)-1]x_2\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)(1+9\boldsymbol{\Sigma}^2)\\ \times\mathcal{D}(\boldsymbol{\Sigma}^2)\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\end{bmatrix}\\ \times[\varphi(\boldsymbol{\Sigma}^2|t_i)\mathcal{D}(\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)]^{-1}\end{Bmatrix},$$

$$\frac{\partial\log(\Theta)}{\partial\boldsymbol{\Sigma}^2}=\sum_{i=1}^{n}\delta_i\left(\frac{9}{1+9\boldsymbol{\Sigma}^2}\right)-\sum_{i=1}^{n}\delta_i\begin{Bmatrix}\left\{\left[21\gamma+\sqrt{2(1+7\boldsymbol{\Sigma}^2)}\right]9\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\intercal}\mathcal{B})\right\}\\ \begin{bmatrix}\left(\sqrt{2(1+7\boldsymbol{\Sigma}^2)}\right)\\ (\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2])\end{bmatrix}\end{Bmatrix}$$
$$+\sum_{i=1}^{n}\begin{Bmatrix}\begin{bmatrix}21\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\\ -\{9\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\intercal}\mathcal{B})\}\sqrt{2(1+7\boldsymbol{\Sigma}^2)}[\iota(\boldsymbol{\Sigma}^2)-2]\end{bmatrix}\\ \begin{bmatrix}\sqrt{2(1+7\boldsymbol{\Sigma}^2)}\begin{pmatrix}\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\\ +\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\end{pmatrix}[\iota(\boldsymbol{\Sigma}^2)-2]\end{bmatrix}^{-1}\end{Bmatrix}$$

$$+\sum_{i=1}^{n}\begin{Bmatrix}[(1+9\boldsymbol{\Sigma}^2)(\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2])] \\ [\varphi(\boldsymbol{\Sigma}^2|t_i)\mathcal{D}(\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)]^{-1}\end{Bmatrix}$$

$$\begin{Bmatrix}3\left(-63\boldsymbol{\Sigma}^2-11+6\sqrt{2(1+7\boldsymbol{\Sigma}^2)}\right)\left[\sqrt{2(1+7\boldsymbol{\Sigma}^2)}(1+9\boldsymbol{\Sigma}^2)^2\right]^{-1} \\ +\begin{bmatrix}(1+9\boldsymbol{\Sigma}^2)^2 \\ (\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2])^2\end{bmatrix}^{-1} \\ \left[\left(\frac{21\gamma}{\sqrt{2(1+7\boldsymbol{\Sigma}^2)}}\right)C(\boldsymbol{\Sigma}^2)+\gamma\left(9-\varsigma(\boldsymbol{\Sigma}^2)\right)[\iota(\boldsymbol{\Sigma}^2)-2]\right] \\ [(1+9\boldsymbol{\Sigma}^2)(\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2])] \\ -\begin{bmatrix}9(\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]) \\ +\left(9\left(\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\mathrm{I}}\mathcal{B})\right)+\frac{21\gamma}{\sqrt{2(1+7\boldsymbol{\Sigma}^2)}}\right)(1+9\boldsymbol{\Sigma}^2)\end{bmatrix} \\ \times[\gamma[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)]\end{Bmatrix}$$

$$+\sum_{i=1}^{n}\delta_i\begin{bmatrix}\varphi(\boldsymbol{\Sigma}^2|t_i) \\ \mathcal{D}(\boldsymbol{\Sigma}^2)+2\gamma[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{bmatrix}^{-1}[\varphi(\boldsymbol{\Sigma}^2|t_i)\mathcal{D}(\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)]^{-1}$$

$$\begin{Bmatrix}\left[\left(\frac{21\gamma}{\sqrt{2(1+7\boldsymbol{\Sigma}^2)}}\right)C(\boldsymbol{\Sigma}^2)+\gamma\left(9-\varsigma(\boldsymbol{\Sigma}^2)\right)[\iota(\boldsymbol{\Sigma}^2)-2]\right] \\ [\varphi(\boldsymbol{\Sigma}^2|t_i)\mathcal{D}(\boldsymbol{\Sigma}^2)+\gamma[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)] \\ \left(\begin{matrix}\gamma[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\begin{Bmatrix}9\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\mathrm{I}}\mathcal{B}) \\ +\frac{21\gamma}{\sqrt{2(1+7\boldsymbol{\Sigma}^2)}}\end{Bmatrix} \\ \mathcal{D}(\boldsymbol{\Sigma}^2)+\left(18+\varsigma(\boldsymbol{\Sigma}^2)\right)\begin{pmatrix}\varphi[\exp(\gamma t_i)-1]\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2) \\ +\gamma[\iota(\boldsymbol{\Sigma}^2)-2]\end{pmatrix} \\ +\left(\frac{21\gamma}{\sqrt{2(1+7\boldsymbol{\Sigma}^2)}}\right)C(\boldsymbol{\Sigma}^2)+\gamma\left(9-\varsigma(\boldsymbol{\Sigma}^2)\right)[\iota(\boldsymbol{\Sigma}^2)-2]\end{matrix}\right)\end{Bmatrix}.$$

10.4 The score functions of TLPF model in case of the PBLHF

The score functions of each parameter using the PBLHF are obtained as follows :

$$\frac{\partial\log(\Theta)}{\partial\mathcal{B}_1}=\sum_{i=1}^{n}\delta_i x_1-\sum_{i=1}^{n}\delta_i\begin{Bmatrix}-\begin{bmatrix}(1+9\boldsymbol{\Sigma}^2)\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)x_1\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)\end{bmatrix} \\ \left[-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2\right]^{-1}\end{Bmatrix}$$

$$+\sum_{i=1}^{n}\begin{Bmatrix}\left[(1+9\boldsymbol{\Sigma}^2)\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)x_1\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)\right] \\ \left[-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2\right]^{-1}\end{Bmatrix}$$

$$+\sum_{i=1}^{n}\begin{Bmatrix}\left[(1+9\boldsymbol{\Sigma}^2)\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)x_1\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\right] \\ \left[-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2\right]^{-1}\end{Bmatrix}\begin{bmatrix}\left(-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right) \\ \mathcal{D}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{bmatrix}^{-1}$$

$$+\sum_{i=1}^{n}\left\{\begin{array}{l}\left[\begin{array}{l}(1+9\boldsymbol{\Sigma}^2)\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)x_1\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)\\ [\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\mathcal{D}(\boldsymbol{\Sigma}^2)\end{array}\right]\\ \left[\begin{array}{l}\left(-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right)\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{array}\right]^{-1}\end{array}\right\}\left\{\left[\begin{array}{l}\left(-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right)\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+2[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{array}\right]^{-1}\right\},$$

$$\begin{aligned}\frac{\partial\log(\Theta)}{\partial\mathcal{B}_2}=&\sum_{i=1}^{n}\delta_i x_2-\sum_{i=1}^{n}\delta_i\left\{\begin{array}{l}-\left[\begin{array}{l}(1+9\boldsymbol{\Sigma}^2)\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\\ x_2\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)\end{array}\right]\\ \left[-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2\right]^{-1}\end{array}\right\}\\ &+\sum_{i=1}^{n}\left\{\begin{array}{l}\left[\begin{array}{l}(1+9\boldsymbol{\Sigma}^2)\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\\ x_2\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)\end{array}\right]\\ \left[\begin{array}{l}-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\\ +\iota(\boldsymbol{\Sigma}^2)-2\end{array}\right]^{-1}\end{array}\right\}\\ &+\sum_{i=1}^{n}\left\{\begin{array}{l}\left[\begin{array}{l}(1+9\boldsymbol{\Sigma}^2)\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)x_2\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)\\ [\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{array}\right]\\ \left[-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2\right]^{-1}\end{array}\right\}\\ &\left\{\left[\begin{array}{l}\left(-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right)\mathcal{D}(\boldsymbol{\Sigma}^2)\\ +[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{array}\right]^{-1}\right\}\\ &+\sum_{i=1}^{n}\left\{\begin{array}{l}\left[(1+9\boldsymbol{\Sigma}^2)\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)x_2\exp(x_1\mathcal{B}_1+x_2\mathcal{B}_2)[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\mathcal{D}(\boldsymbol{\Sigma}^2)\right]\\ \left[\begin{array}{l}\left(-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right)\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{array}\right]^{-1}\end{array}\right\}\\ &\left[\begin{array}{l}\left(-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right)\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+2[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{array}\right]^{-1},\end{aligned}$$

$$\begin{aligned}\frac{\partial\log(\Theta)}{\partial\eta}=&\sum_{i=1}^{n}\delta_i\frac{1}{\eta}-\sum_{i=1}^{n}\delta_i\left\{\begin{array}{l}-\left[\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\right]\\ \left[-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2\right]^{-1}\end{array}\right\}\\ &+\sum_{i=1}^{n}\left\{\begin{array}{l}\left[\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\right]\\ \left[-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2\right]^{-1}\end{array}\right\}\\ &+\sum_{i=1}^{n}\left\{\begin{array}{l}\left[\begin{array}{l}\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\\ [\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{array}\right]\\ \left[-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2\right]^{-1}\end{array}\right\}\left[\begin{array}{l}\left(-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\intercal}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right)\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]C(\boldsymbol{\Sigma}^2)\end{array}\right]^{-1}\end{aligned}$$

$$
+\sum_{i=1}^{n}\left\{\left[(1+9\boldsymbol{\Sigma}^2)\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\mathcal{D}(\boldsymbol{\Sigma}^2)\right]\left[\begin{array}{l}\left(-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right)\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\end{array}\right]^{-1}\right\}\left[\begin{array}{l}\left(-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right)\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+2[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\end{array}\right]^{-1},
$$

$$
\begin{aligned}
\frac{\partial\log(\Theta)}{\partial\alpha}=&-\sum_{i=1}^{n}\delta_i\left\{-\left[\eta\left(\frac{t_i}{\alpha(\alpha+t_i)}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\right]\left[-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2\right]^{-1}\right\}-\sum_{i=1}^{n}\delta_i\left(\frac{1}{\alpha+t_i}\right)\\
&+\sum_{i=1}^{n}\left\{\left[\eta\left(\frac{t_i}{\alpha(\alpha+t_i)}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\right]\left[-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2\right]^{-1}\right\}\\
&+\sum_{i=1}^{n}\left\{\left[\begin{array}{l}\eta\left(\frac{t_i}{\alpha(\alpha+t_i)}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\\ [\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\end{array}\right]\left[-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2\right]^{-1}\right\}\left[\begin{array}{l}\left(-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right)\\ \mathcal{D}(\boldsymbol{\Sigma}^2)\\ +[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\end{array}\right]^{-1}\\
&+\sum_{i=1}^{n}\delta_i\left\{\left[\eta\left(\frac{t_i}{\alpha(\alpha+t_i)}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\mathcal{D}(\boldsymbol{\Sigma}^2)\right]\left[\begin{array}{l}\left(-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right)\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\end{array}\right]^{-1}\right\}\left[\begin{array}{l}\left(-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right)\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+2[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\end{array}\right]^{-1},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\log(\Theta)}{\partial\boldsymbol{\Sigma}^2}=&\sum_{i=1}^{n}\delta_i\left(\frac{9}{1+9\boldsymbol{\Sigma}^2}\right)-\sum_{i=1}^{n}\delta_i\left\{\left[\left(-9\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})\right)\sqrt{2(1+7\boldsymbol{\Sigma}^2)}+21\right]\left[\begin{array}{l}\sqrt{2(1+7\boldsymbol{\Sigma}^2)}\\ \left(\begin{array}{l}-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\\ +\iota(\boldsymbol{\Sigma}^2)-2\end{array}\right)\end{array}\right]^{-1}\right\}\\
&+\sum_{i=1}^{n}\left\{\left[\begin{array}{l}-21\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\\ +\sqrt{2(1+7\boldsymbol{\Sigma}^2)}[\iota(\boldsymbol{\Sigma}^2)-2]\\ \left(9\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})\right)\end{array}\right]\left[\begin{array}{l}\sqrt{2(1+7\boldsymbol{\Sigma}^2)}[\iota(\boldsymbol{\Sigma}^2)-2]\\ \left(-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2\right)\end{array}\right]^{-1}\right\}\\
&+\sum_{i=1}^{n}\left\{\left[(1+9\boldsymbol{\Sigma}^2)\left(-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2\right)\right]\left[\begin{array}{l}\left(-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right)\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\end{array}\right]^{-1}\right\}
\end{aligned}
$$

$$
\left\{\begin{array}{l}
\left[3\left(-63\boldsymbol{\Sigma}^2-11+6\sqrt{2(1+7\boldsymbol{\Sigma}^2)}\right)\right]\left[\left(\sqrt{2(1+7\boldsymbol{\Sigma}^2)}\right)(1+9\boldsymbol{\Sigma}^2)^2\right]^{-1}\\
+\left[\begin{array}{l}(1+9\boldsymbol{\Sigma}^2)^2\\ \left(-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right)^2\end{array}\right]^{-1}\\
\left[\left(\varsigma(\boldsymbol{\Sigma}^2)\right)\mathcal{C}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\left(9-\varsigma(\boldsymbol{\Sigma}^2)\right)\right]\\
\left[\begin{array}{l}(1+9\boldsymbol{\Sigma}^2)\\ \left(\begin{array}{l}-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\\ +\iota(\boldsymbol{\Sigma}^2)-2\end{array}\right)\end{array}\right]\\
-\left[\begin{array}{l}-9\left(\begin{array}{l}-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)\\ +\iota(\boldsymbol{\Sigma}^2)-2\end{array}\right)\\ +\left(-9\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})+\left(\varsigma(\boldsymbol{\Sigma}^2)\right)(1+9\boldsymbol{\Sigma}^2)\right)[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\end{array}\right]
\end{array}\right\}
$$

$$
+\sum_{i=1}^{n}\delta_i\left[\begin{array}{l}\left(-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right)\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\end{array}\right]^{-1}\left[\begin{array}{l}\left(-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right)\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+2[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\end{array}\right]^{-1}
$$

$$
\left\{\begin{array}{l}
\left[\left(\varsigma(\boldsymbol{\Sigma}^2)\right)\mathcal{C}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\left(9-\varsigma(\boldsymbol{\Sigma}^2)\right)\right]\\
\left[\begin{array}{l}\left(-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\right)\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)\end{array}\right]\\
-\left[\begin{array}{l}\left(-9\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})+\left(\varsigma(\boldsymbol{\Sigma}^2)\right)\right)\\ \mathcal{D}(\boldsymbol{\Sigma}^2)+\left(18+\varsigma(\boldsymbol{\Sigma}^2)\right)\\ \left(-\eta\log\left(\frac{\alpha}{\alpha+t_i}\right)\exp(x_i^{\mathrm{I}}\mathcal{B})(1+9\boldsymbol{\Sigma}^2)+\iota(\boldsymbol{\Sigma}^2)-2\right)\\ +\left[\left(\varsigma(\boldsymbol{\Sigma}^2)\right)\mathcal{C}(\boldsymbol{\Sigma}^2)+[\iota(\boldsymbol{\Sigma}^2)-2]\left(9-\varsigma(\boldsymbol{\Sigma}^2)\right)\right]\end{array}\right]\\
[\iota(\boldsymbol{\Sigma}^2)-2]\mathcal{C}(\boldsymbol{\Sigma}^2)
\end{array}\right\}.
$$

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