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Censored and Uncensored Nikulin-Rao-Robson Distributional Validation: Characterizations, Classical and Bayesian estimation with Censored and Uncensored Applications

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Abstract

In our paper, we introduce a novel extension of the Lomax distribution, aiming to enhance its applicability in various contexts. We emphasize a pragmatic approach in deriving mathematical properties of the new distribution, prioritizing its practical implications. Three distinct methods for characterizing the distribution are thoroughly discussed to provide a comprehensive understanding. The parameters of this newly proposed distribution are estimated through a diverse set of classical methodologies as well as Bayes' method. Additionally, we develop the censored case maximum likelihood method to address scenarios where data may be incomplete. We meticulously compare the efficacy of likelihood estimation and Bayesian estimation using Pitman's proximity criterion, thereby offering insights into their relative performance. For Bayesian estimation, we employ three distinct loss functions: the generalized quadratic, the Linex, and the entropy functions, each offering unique perspectives on the estimation process. Through extensive simulation experiments, we meticulously evaluate the performance of all estimation methods under various conditions, providing valuable insights into their practical utility. Furthermore, we conduct a comparative analysis between the Bayesian technique and the censored maximum likelihood method using the BB algorithm, facilitating a nuanced understanding of their respective strengths and weaknesses. In addition to estimation methodologies, we delve into the construction of the Nikulin-Rao-Robson statistic for the new model under both uncensored and censored cases. Detailed simulation studies and the presentation of two real-world applications elucidate the practical significance of our proposed statistics in diverse scenarios. Overall, our paper not only introduces a novel extension of the Lomax distribution but also provides a comprehensive exploration of various estimation techniques and statistical measures, underpinning its practical relevance across different domains.

Key Words: Bayesian Estimation; BB algorithm; Censored Applications; Characterizations; Lomax Model; Nikulin-Rao-Robson; Pitman's Proximity.

Mathematical Subject Classification: 62E10, 60K10, 60N05

1 Introduction

Lomax (1954) looked into his continuous heavy-tail probability distribution to simulate the real-life data of business failure, actuarial science, queueing theory, and internet traffic modeling. The Lomax model is frequently referred to as the Pareto type-II (Pa-II) distribution. In his research of lifetime data on business failure, Lomax (1954) first postulated the heavy-tailed distribution known as the Lomax (Lx) distribution. It has several uses in actuarial science, economics, and business. The distribution, which is essentially a shifted Pareto distribution, is frequently employed in survival analysis. In applied statistics and allied domains like engineering, for instance, wealth inequality, income, medical, biological studies, and dependability, there are special efforts being made to broaden the Lomax distribution and its pertinent expansions. The Lomax model is used to simulate actual income and wealth statistics (see Harris (1968) and Asgharzadeh and Valiollahi (2011)), the type-II progressive censored competing risks data analysis (see Cramer and Schemiedt (2011), real data of the firm sizes (see Corbellini et al. (2007)).

In recent years, numerous authors have explored extensions of the Lomax (Lx) model. Shao (2004) focused on maximum likelihood estimation for the two-parameter Lomax (2PLx) model, while Shao et al. (2004) investigated statistical modeling for extreme events using the extended 2PLx model, particularly in flood frequency analysis. Soliman (2005) highlighted that the Lomax distribution encapsulates the curve shape characteristics of various other distributions. Silva et al. (2008) proposed a location-scale regression model based on the Lx model, and Paranaiba et al. (2011) introduced and analyzed the beta Lx (BLx) model.

Al-Saiari et al. (2014) delved into the Marshall-Olkin extended Lx (MOE Lx) model, while Gomes et al. (2015) explored two extended Lx models and their applications. Cordeiro et al. (2018) introduced the Lx G (Lx-G) family, which proved to be flexible, defining several important Lx models and their special cases. Altun et al. (2018a) proposed a novel Lx log-location regression model with diagnostic tools and real data applications. Altun et al. (2018b) studied the Zografos-Balakrishnan Lx (ZBLx) distribution and developed a regression model for prediction, showcasing its applicability with real data.

Nasir et al. (2018) presented a new Weibull Lx (WLx) distribution, and Korkmaz et al. (2018) investigated the odd Lindley Lx (OL Lx) model, along with Bayesian analysis and classical inference. Ibrahim (2019) derived and studied the compound Poisson-Rayleigh-Lx (PR Lx) distribution, elucidating its properties and applications. Yousof et al. (2019a) introduced the zero-truncated Poisson Topp-Leone-Lx model and provided characterizations, while Yousof et al. (2019b) proposed a new Lx lifetime model based on the Topp-Leone family, including regression models and applications.

Gad et al. (2019) explored the Lx-Lx (Lx-Lx) distribution, characterizing it and demonstrating its application alongside statistical properties. Elgohari and Yousof (2020) introduced a generalized version of the Lomax distribution, covering properties, copula, and real data applications. Chesneau and Yousof (2021) introduced a special generalized mixture class of probabilistic models, while Elsayed and Yousof (2021) extended the Lx model to derive the Poisson generalized Lx (PGLx) distribution, with four applications.

Aboraya et al. (2022) presented the Poisson exponentiated exponential Lomax distribution, detailing its statistical properties, applications, copulas, and various estimation methods including ordinary least squares, weighted least square, Cramér-von-Mises, Anderson-Darling, right-tail Anderson-Darling, maximum likelihood, and left-tail Anderson-Darling methods.

For further extensions of the Lomax model, real-life datasets, and additional applications, readers are referred to the works of Elbiely and Yousof (2018), Ali et al. (2019, 2021), Ibrahim and Yousof (2020), Elgohari and Yousof (2020a, b, c), Elgohari et al. (2021), Ansari et al. (2021), Hamed et al. (2022), and Yousof et al. (2023a, b, c).

To this end, we will be content with what has been reported as extensions of the Lomax distribution in the statistical literature, and we will refer the reader to those references for more details. But in what follows from this introduction, we will focus on how the new distribution was derived and originated, and on the most important motives that motivated us to present this work.

The cumulative distribution function (CDF) of the standard one-parameter Lomax (Lx) model is given by

$$\mathbf{G}_{\rho}(\mathcal{Y})|_{(\mathcal{Y}\geq 0)} = 1-(1+\mathcal{Y})^{-\rho}|\mathcal{Y}\geq 0, \quad (1)$$

the corresponding PDF is given by

$$\mathfrak{g}_{\rho}(\mathcal{Y})|_{(\mathcal{Y}\geq 0)} = \rho\,(1+\mathcal{Y})^{-\rho-1}|\mathcal{Y}>0, \quad (2)$$

where $\rho > 0$ is the shape parameter. For comprehensive insights into the Lomax (Lx) model and its mathematical properties, researchers can refer to several seminal works in the field. Burr's contributions in 1942, 1968, and 1973 provide foundational understanding, supplemented by the work of Burr and Cislak in 1968, Rodriguez in 1977, and Tadikamalla in 1980. These texts offer detailed discussions and analyses, serving as essential references for scholars and practitioners alike.

In extending the scope of the Lx model, Yousof et al. (2017a) introduced the Burr X-G (BX-G) family of distributions, which incorporates additional flexibility and versatility. By leveraging the CDF of the BX-G family and performing an inversion followed by substituting the CDF of the Lx model, a new model emerges. This novel model, coined the inverted Burr X Lomax (IBX-Lx) model, represents a fusion of the Lx model's characteristics with the innovative features of the BX-G family.

The CDF of the IBX-Lx model is expressed as a result of this transformation process, representing a synthesis of the underlying principles and properties of both the Lx model and the BX-G family. This formulation opens up new avenues for exploration and application, offering researchers a powerful tool for modeling and analyzing data in various domains. Through the integration of established methodologies and novel insights, the IBX-Lx model promises to enhance the analytical capabilities and predictive accuracy of researchers in fields ranging from statistics to engineering and beyond, where

$$F_{\mathcal{O},\rho}(\mathcal{Y}) = 1-(1-\exp\{-[(1+\mathcal{Y})^{\rho}-1]^{-2}\})^{\mathcal{O}}. \quad (3)$$

For $\mathcal{O}=1$ the IBX-Lx model reduces to the inverted Rayleigh Lx (IRLx) model. The PDF of the IBX-Lx is given by

$$f_{\mathcal{O},\rho}(\mathcal{Y}) = 2\mathcal{O}\rho(1+\mathcal{Y})^{-2\rho-1}\,\frac{\exp\{-[(1+\mathcal{Y})^{\rho}-1]^{-2}\}}{[1-(1+\mathcal{Y})^{-\rho}]^{3}}\,\underbrace{(1-\exp\{-[(1+\mathcal{Y})^{\rho}-1]^{-2}\})^{\mathcal{O}-1}}_{\mathcal{A}_{\mathcal{O},\rho}(\mathcal{Y})}. \quad (4)$$

The hazard rate function of the IBX-Lx distribution can be derived directly from

$$h_{\mathcal{O},\rho}(\mathcal{Y}) = f_{\mathcal{O},\rho}(\mathcal{Y})/[1-F_{\mathcal{O},\rho}(\mathcal{Y})].$$

We estimate the parameters of the new distribution using a variety of classical methods including the maximum likelihood method, the Cramér-von Mises method, the Anderson-Darling method and the right-tail Anderson-Darling method in addition to the Bayes' method. Moreover the maximum likelihood method in the censored case is derived in details and assessed via a comprehensive simulation. The Bayesian estimation is presented under different loss functions, the likelihood estimation and the Bayesian estimation are Compared under Pitman's closeness criterion. We use three loss functions namely the generalised quadratic, the Linex and the entropy to obtain the Bayesian estimators and many useful details are presented in its places below. All the estimation methods referred to have been evaluated through simulation studies under certain conditions and with certain controls, all of which are mentioned in their appropriate place in the paper.

The BB algorithm serves as a crucial tool for estimating processes involving censored samples, particularly in the context of comparing the effectiveness of the censored maximum likelihood method and the Bayesian method. Our paper explores the IBX-Lx distribution through three distinct characterizations: first, by utilizing two truncated moments; second, by examining the hazard function; and third, by analyzing the conditional expectation of a function of the random variable. We meticulously detail the construction of the Nikulin-Rao-Robson (NURR) statistic for the IBX-Lx model under both uncensored and censored cases. For the uncensored scenario, we present a comprehensive simulation study aimed at assessing the performance of the NURR statistics, followed by the presentation of two real-world data applications. The first application involves the analysis of heat exchanger tube crack data, while the second application focuses on the strengths of glass fibers. Furthermore, we provide an in-depth exploration of the construction of the NURR statistic for the IBX-Lx model under the censored case, accompanied by a detailed simulation study to evaluate the NURR statistics' efficacy. Additionally, we present two real data applications under the censored case: the first involves reliability data concerning capacitors, while the second pertains to medical data related to lung cancer. Through these analyses and applications, our paper offers valuable insights into the IBX-Lx distribution, its statistical properties, and its practical implications in diverse real-world scenarios, thereby contributing to the advancement of reliability analysis and risk assessment methodologies.

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In this paper, we embark on a novel exploration of the IBX-Lx distribution, diverging from the traditional methodologies commonly employed by researchers in this domain. Our approach breaks new ground by thoroughly investigating both the theoretical underpinnings and practical applications of this distribution. We achieve this through an intricate examination of four distinct applications and an extensive series of simulation experiments, each designed to shed light on different facets of the IBX-Lx distribution. This departure from conventional approaches has spurred us to adopt a diverse array of parameter estimation methods, blending classical techniques with the Bayesian framework to provide a more comprehensive understanding of the distribution's characteristics. Moreover, our methodological approach has led us to integrate three unique loss functions into our analysis, facilitating a rigorous comparison that is grounded in appropriate evaluation criteria. Within this analytical framework, we also place significant emphasis on statistical hypothesis testing and distributional verification, ensuring the robustness and validity of our findings. To illustrate the practical relevance of our study, we present four detailed applications, encompassing both complete datasets and scenarios involving censored data. Through these meticulously crafted analyses, we aim to offer a holistic perspective on the IBX-Lx distribution, elucidating its estimation methods and demonstrating its versatility across various real-world contexts. Ultimately, our endeavor seeks to contribute to a deeper understanding of this distribution and its potential applications in diverse fields.

2 Some properties

In this section, we introduce a valuable linear representation for the BXLx density function (4). This representation serves as a powerful tool for analyzing and understanding the behavior of the density function in various contexts. By expressing the density function in a linear form, we can uncover underlying patterns, relationships, and properties that may not be immediately apparent from its original formulation. This linear representation facilitates computational analysis, allows for efficient manipulation of the density function, and provides insights into its mathematical structure. Overall, the linear representation presented herein enhances our understanding of the BXLx density function and its applications. If $\left|\frac{\Upsilon_1}{\Upsilon_2}\right| < 1$ and $\Upsilon_3 > 0$ is a real non-integer, the power series holds

$$\left(1-\frac{1}{\Upsilon_2}\Upsilon_1\right)^{\Upsilon_3} = \sum_{\hbar=0}^{+\infty} \frac{\Gamma(1+\Upsilon_3)}{\hbar!\ \Gamma(1+\Upsilon_3-\hbar)} \left(\frac{1}{\Upsilon_2}\Upsilon_1\right)^{\hbar} (-1)^{\hbar}\,. \tag{5}$$

Applying (5) to and inserting the expansion of $\boldsymbol{\mathcal{A}}_{\mathcal{O},\rho}(\mathcal{Y})$ into (4), where

$$\boldsymbol{\mathcal{A}}_{\mathcal{O},\rho}(\mathcal{Y}) = (1-\exp\{-[(1+\mathcal{Y})^{\rho}-1]^{-2}\})^{\mathcal{O}-1},$$

we get

$$f_{\mathcal{O},\rho}(\mathcal{Y}) = 2\mathcal{O}\rho\ \frac{(1+\mathcal{Y})^{-2\rho-1}}{[1-(1+\mathcal{Y})^{-\rho}]^3} \sum_{\hbar=0}^{+\infty} \frac{(-1)^{\hbar}\ \Gamma(\mathcal{O})}{\hbar!\ \Gamma(\mathcal{O}-\hbar)} \underbrace{\exp[-(\hbar+1)[(1+\mathcal{Y})^{\rho}-1]^{-2}]}_{\boldsymbol{\mathcal{B}}_{\hbar,\rho}(\mathcal{Y})}. \tag{6}$$

Then, applying the power series to $\boldsymbol{\mathcal{B}}_{\hbar,\rho}(\mathcal{Y})$ and inserting the expansion of $\boldsymbol{\mathcal{B}}_{\hbar,\rho}(\mathcal{Y})$ into (6), the equation (6) can be summarized as

$$f_{\mathcal{O},\underline{\boldsymbol{\Phi}}}(\mathcal{Y}) = 2\mathcal{O}\rho\ (1+\mathcal{Y})^{-\rho-1} \sum_{\hbar,l=0}^{+\infty} \frac{(-1)^{\hbar+l}\ (\hbar+1)^{l}\Gamma(\mathcal{O})}{\hbar!\ l!\,\Gamma(\mathcal{O}-\hbar)}\ \frac{[1-(1+\mathcal{Y})]^{-2l-3}}{\underbrace{[(1+\mathcal{Y})]^{-2l-1}}_{\boldsymbol{\mathcal{B}}_{l,\rho}(\mathcal{Y})}}. \tag{7}$$

Applying (5) to $\boldsymbol{\mathcal{B}}_{l,\rho}(\mathcal{Y})$, Equation (7) This can be written as

$$f_{\mathcal{O},\rho}(\mathcal{Y}) = \sum_{l,\nu=0}^{+\infty} 2\mathcal{O}\, \frac{(-1)^{\hbar+l+\nu}\ \Gamma(\mathcal{O})\Gamma(2l+2)}{l!\,\nu!\,\Gamma(2l+2-\nu)} \sum_{\hbar=0}^{+\infty} \frac{(-1)^{\hbar}(\hbar+1)^{l}}{\hbar!\,\Gamma(\mathcal{O}-\hbar)}$$
$$\rho\ (1+\mathcal{Y})^{-\rho-1}[-(1+\mathcal{Y})^{-\rho}+1]^{[\nu-2(1+l)]-1} \tag{8}$$

$$f_{\mathcal{O},\rho}(\mathcal{Y}) = \sum_{\zeta=0}^{+\infty} \nabla_{\zeta}(\hbar,l,\nu)\mathfrak{g}_{\rho^*}(\mathcal{Y})|_{\rho^*=\rho(1+\zeta)} \tag{9}$$

where

$$\nabla_{\zeta}(\hbar,l,\nu)=\sum_{\hbar,l,\nu=0}^{+\infty}2\mathcal{O}\frac{(-1)^{\hbar+l+\nu+\zeta}\,(\hbar+1)^{l}\Gamma(\mathcal{O})\Gamma(2l+2)\,\Gamma\big(\nu-2(l+1)\big)}{\hbar!\;l!\,\nu!\,\zeta!\,\Gamma(\mathcal{O}-\hbar)\Gamma(2l+2-\nu)\Gamma(\nu-2(l+1)-\zeta)(1+\zeta)},$$

and

$$\mathfrak{g}_{\rho^*}(\mathcal{Y})=\rho^*\;(1+\mathcal{Y})^{-\rho^*-1}$$

is the PDF of the Lx model with parameters ρ^* and Equation (9) reveals that the density of Y can be expressed as a linear mixture of Lx densities. Indeed, by leveraging the properties of the exponential Lomax (exp-Lx) distribution, we can derive numerous mathematical properties for the new family of distributions. The exp-Lx distribution serves as a foundational component, offering a basis from which we can extend our understanding to encompass the broader family. One key advantage of building upon the exp-Lx distribution lies in its well-established properties and characteristics, which provide a solid framework for further analysis. By understanding the fundamental properties of the exp-Lx distribution, such as its probability density function, cumulative distribution function, moments, and moment generating function, we can derive analogous properties for the broader family. Additionally, the exp-Lx distribution allows us to explore various statistical measures and parameters, such as the mean, variance, skewness, and kurtosis, which serve as essential metrics for characterizing the distribution's behavior. Through careful analysis and manipulation of these properties, we can elucidate the intricacies of the new family and uncover its unique features and capabilities. Furthermore, by studying the relationships between the exp-Lx distribution and other related distributions, such as the Lomax distribution itself, we can gain further insights into the behavior and properties of the new family. This comparative analysis enables us to identify similarities, differences, and potential areas for further exploration and refinement. Overall, the exp-Lx distribution serves as a valuable starting point for investigating the mathematical properties of the new family of distributions. By leveraging its established framework and properties, we can expand our understanding and unlock the full potential of this innovative class of distributions. Similarly, the CDF of the new distribution can also be expressed as a mixture of exp-Lx CDFs given by

$$f_{\mathcal{O},\rho}(\mathcal{Y})=\sum_{\zeta=0}^{+\infty}\nabla_{\zeta}(\hbar,l,\nu)\mathbf{G}_{\rho^*}(\mathcal{Y})$$

where $\mathbf{G}_{\rho^*}(\mathcal{Y})$ is the CDF of the Lx model with parameter ρ^*. Let$W$ be a random variable having the Lx distribution with parameters ρ and . Then, the n^{th} ordinary and incomplete moments of W are, respectively, given by

$$\mu'_{n,W}|_{n<\rho}=\rho\,\mathcal{B}(\rho-n,1+n)$$

and

$$\Upsilon_{n,W}(t)|_{n<\rho}=\rho\,\mathcal{B}(t;\rho-n,1+n),$$

where

$$\mathcal{B}(a,b)=\int_0^{+\infty}\mathrm{w}^{a-1}\,(1+\mathrm{w})^{-(a+b)}d\mathrm{w}$$

and

$$\mathcal{B}(t;a,b)=\int_0^{t}\mathrm{w}^{a-1}\,(1+\mathrm{w})^{-(a+b)}d\mathrm{w}$$

are the beta and the incomplete beta functions of the second type, respectively. So, several structural properties of the IBX-Lx model can be obtained from (9) and those properties of the Lx distribution. The n^{th} ordinary moment of Y is given by

$$\mu'_{n,Y}=E(Y^n)=\sum_{\zeta=0}^{+\infty}\nabla_{\zeta}(\hbar,l,\nu)\;\int_0^{+\infty}\mathcal{Y}^n\,\mathfrak{g}_{\rho^*}(\mathcal{Y})d\mathcal{Y}.$$

For $n<\rho$, we obtain

$$\mu'_{n,Y}=E(Y^n)=\sum_{\zeta=0}^{+\infty}\nabla_{\zeta}(\hbar,l,\nu)\rho^*\,\mathcal{B}(\rho^*-n,1+n).\tag{10}$$

Setting $n=1$ in (10), we have the mean of Y. The moment generating function (MGF) of $\mathcal{Y}$, say $M_Y(t)=E[\exp(tY)]$, can be easily obtained either from (9) as

$$M_Y(t)=\sum_{\zeta=0}^{+\infty}\nabla_{\zeta}(\hbar,l,\nu)\,M_Y(t;\rho^*),$$

where $M_Y(t;\rho^*)$ is the MGF of the Lx distribution with parameter ρ^*. The n^{th} incomplete moment, say $\Upsilon_{\mathcal{S},Y}(t)$, of the IBX-Lx distribution is given by

$$\Upsilon_{\mathcal{S},Y}(t)=\int_0^{t}\mathcal{Y}^{\mathcal{S}}\,f(\mathcal{Y})d\mathcal{Y}.$$

We can write from equation (9),

$$\mathbf{\Upsilon}_{\mathcal{S},Y}(t) = \sum_{r=0}^{+\infty} b_r \int_0^t \mathcal{Y}^{\mathcal{S}}\, \mathfrak{g}_{\rho^*}(\mathcal{Y}) d\mathcal{Y},$$

and then we obtain (for $\mathcal{S} < \rho$)

$$\mathbf{\Upsilon}_{\mathcal{S},Y}(t) = \sum_{\zeta=0}^{+\infty} \nabla_\zeta(\hbar, l, \nu)\rho^* \boldsymbol{\mathcal{B}}(t; \rho^* - \mathcal{S}, 1 + \mathcal{S}).$$

The mean deviations, about the mean and about the median of Y, depend on $\mathbf{\Upsilon}_{1,Y}(t)$. The $\mathcal{S}^{\,th}$ moment of the reversed residual life, say

$$R_{\mathcal{S},\mathcal{Y}}(t) = E[(t - Y)^{\mathcal{S}} | Y \le t] \text{for } t > 0 \text{ and } \mathcal{S} = 1,2, \ldots$$

Then, $R_{\mathcal{S},Y}(t)$ is defined by

$$R_{\mathcal{S},Y}(t) = \frac{1}{F(t)} \int_0^t (t - \mathcal{Y})^{\mathcal{S}} dF(\mathcal{Y}).$$

The $\mathcal{S}^{\,th}$ moment of the reversed residual life of Y is

$$R_{\mathcal{S},Y}(t) = \frac{1}{F(t)} \sum_{\hbar=0}^{n} \sum_{\zeta=0}^{+\infty} \frac{(-1)^{\hbar} \mathcal{S}!}{\hbar!\,(\mathcal{S} - \hbar)!} \nabla_\zeta(\hbar, l, \nu)\rho^* \boldsymbol{\mathcal{B}}(t; \rho^* - \mathcal{S}, 1 + \mathcal{S}).$$

The amount of time it takes a person to obtain something, be it an item, money, or a service, is referred to as the waiting time. Technically speaking, mean waiting time refers to the average length of time needed to access a service in the system.

3 Characterization results

In this section, we explore three distinct approaches for defining the IBX-Lx distribution:

- **(i)** through the assessment of two truncated moments;
- **(ii)** by analyzing the hazard function; and
- **(iii)** by examining the conditional expectation of a function involving the random variable. Subsequent subsections will elaborate on each of these methods.

3.1 Characterizations based on two truncated moments

In this subsection, our attention is directed towards characterizing the IBX-Lx distribution by establishing a direct relationship between two truncated moments. We delve into the intricacies of this method to illuminate the distribution's properties and behavior. Central to this approach is the application of Glänzel's (1987) Theorem 3.1.1, a pivotal theorem that provides a framework for understanding the relationship between truncated moments. This theorem serves as a foundational tool, enabling us to derive meaningful insights into the IBX-Lx distribution's statistical characteristics. The utilization of Glänzel's theorem allows us to establish a clear and concise relationship between the truncated moments of the IBX-Lx distribution, thereby facilitating a deeper understanding of its central tendencies and variability. Through rigorous analysis and application of this theorem, we aim to elucidate the distribution's underlying structure and enhance our ability to interpret and utilize its statistical properties effectively. As we progress through this subsection, we will explore the nuances of Glänzel's theorem and its implications for characterizing the IBX-Lx distribution. By leveraging this powerful analytical tool, we endeavor to provide valuable insights into the distribution's behavior and foster a deeper appreciation for its role in statistical modeling and analysis.Clearly, the result holds when the H is not a closed interval.

" **Theorem 3.1.1.** Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ $(d = -\infty, e = +\infty \text{mightaswellbeallowed})$. Let $Y: \Omega \to H$ be a continuous random variable with the distribution function F and let g and $\hbar$ be two real functions defined on H such that

$$\mathbf{E}[g(Y) | Y \ge \mathcal{Y}] = \mathbf{E}[\hbar(Y) | Y \ge \mathcal{Y}] \varsigma(\mathcal{Y}), \quad \mathcal{Y} \in H,$$

is defined with some real function ς. Assume that $g, \hbar \in C^1(H)$, $\varsigma \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\xi h = g$ has no real solution in the interior of H. Then F is uniquely determined by the functions g, $\hbar$ and ς , particularly

$$F(\mathcal{Y}) = \int_a^{\mathcal{Y}} C \left| \frac{1}{\varsigma(\mathcal{Y})\hbar(\mathcal{Y}) - g(\mathcal{Y})} \varsigma'(\mathcal{Y}) \right| \exp\left(-\mathcal{S}(\mathcal{Y})\right) d\mathcal{Y},$$

where the function $\mathcal{S}$ is a solution of the differential equation $\mathcal{S}' = \frac{\varsigma' \hbar}{\varsigma\hbar - g}$ and C is the normalization constant, such that $\int_H dF = 1$. "

Proposition 3.1.1. Let the random variable $Y: \Omega \to (0, +\infty)$ be continuous, and assume that $\hbar(\mathcal{Y}) = \left[\boldsymbol{\mathcal{A}}_{\mathcal{O},\rho}(\mathcal{Y})\right]^{-1}$ and

$$g(\mathcal{Y}) = \hbar(\mathcal{Y})\exp\{-[(1+\mathcal{Y})^{\rho} - 1]^{-2}\}, \quad \mathcal{Y} > 0.$$

Then, the density of Y is given in (4) if and only if the function ς defined in Theorem 3.1.1 is

$$\varsigma(\mathcal{Y}) = \frac{1}{2}\left\{1 + \exp\{-[(1+\mathcal{Y})^{\rho} - 1]^{-2}\}\right\}, \quad \mathcal{Y} > 0.$$

Proof. If X has pdf (4), then

$$\left[1 - F_{\mathcal{O},\rho}(\mathcal{Y})\right] E[\hbar(Y) | Y \geq \mathcal{Y}] = \mathcal{O}\left\{1 - \exp\{-[(1+\mathcal{Y})^{\rho} - 1]^{-2}\}\right\}, \quad \mathcal{Y} > 0,$$

and

$$\left(1 - F_{\mathcal{O},\rho}(\mathcal{Y})\right) E[g(Y) | Y \geq \mathcal{Y}] = \frac{\mathcal{O}}{2}\{1 - \exp[-2((1+\mathcal{Y})^{\rho} - 1)^{-2}]\}, \quad \mathcal{Y} > 0,$$

and finally

$$\varsigma(\mathcal{Y})\hbar(\mathcal{Y}) - g(\mathcal{Y}) = \frac{1}{2}\hbar(\mathcal{Y})\left\{1 - \exp\{-[(1+\mathcal{Y})^{\rho} - 1]^{-2}\}\right\} > 0 \quad for \ \mathcal{Y} > 0.$$

Conversely, if ς has the above form, then

$$\mathcal{S}'(\mathcal{Y}) = \frac{\varsigma'(\mathcal{Y})\hbar(\mathcal{Y})}{\varsigma(\mathcal{Y})\hbar(\mathcal{Y}) - g(\mathcal{Y})} = \frac{\rho(1+\mathcal{Y})^{\rho-1}((1+\mathcal{Y})^{\rho} - 1)^{-3}\exp\{-[(1+\mathcal{Y})^{\rho} - 1]^{-2}\}}{1 - \exp\{-[(1+\mathcal{Y})^{\rho} - 1]^{-2}\}},$$

and hence

$$\mathcal{S}_{\mathcal{O},\rho}(\mathcal{Y}) = -\ln(1 - \exp\{-[(1+\mathcal{Y})^{\rho} - 1]^{-2}\}), \quad \mathcal{Y} > 0.$$

Now, according to Theorem 3.1.1, Y has density (4).

Corollary 3.1.1. If $Y: \Omega \to (0, +\infty)$ is a continuous random variable and $\hbar(\mathcal{Y})$ is as in Proposition 3.1.1, then, Y has pdf (2) if and only if there exist functions g and ς defined in Theorem 3.1.1 satisfying the following first order differential equation

$$\frac{\varsigma'(\mathcal{Y})\hbar(\mathcal{Y})}{\varsigma(\mathcal{Y})\hbar(\mathcal{Y}) - g(\mathcal{Y})} = \frac{\rho(1+\mathcal{Y})^{\rho-1}((1+\mathcal{Y})^{\rho} - 1)^{-3}\exp\{-[(1+\mathcal{Y})^{\rho} - 1]^{-2}\}}{1 - \exp\{-[(1+\mathcal{Y})^{\rho} - 1]^{-2}\}}.$$

Corollary 3.1.2. The general solution of the above differential equation is

$$\varsigma_{\mathcal{O},\rho}(\mathcal{Y}) = \left\{1 - \exp\{-[(1+\mathcal{Y})^{\rho} - 1]^{-2}\}\right\}^{-1} \begin{bmatrix} -\int \rho(1+\mathcal{Y})^{\rho-1}((1+\mathcal{Y})^{\rho} - 1)^{-3} \times \\ \exp\{-[(1+\mathcal{Y})^{\rho} - 1]^{-2}\}\left(\hbar(\mathcal{Y})\right)^{-1} g(\mathcal{Y}) + D \end{bmatrix},$$

where D is a constant. A set of functions satisfying this differential equation is presented in Proposition 3.1.1 with $D = 0$. Clearly, there are other triplets $(\hbar, g, \varsigma)$ satisfying the conditions of Theorem 3.1.1.

3.2 Characterization based on hazard function

The hazard function, $\hbar_F$, of a twice differentiable distribution function, F with density f, satisfies the first following trivial first differential equation

$$\frac{1}{f(\mathcal{Y})}f'(\mathcal{Y}) = \frac{1}{\hbar_F(\mathcal{Y})}\hbar'_F(\mathcal{Y}) - \hbar_F(\mathcal{Y}).$$

This is the only characterisation based on hazard function for many univariate continuous distributions, as we have stated in previous publications. The proposal given below offers a complex description of the IBXLx distribution.

Proposition 3.2.1. Suppose $Y: \Omega \to (0, +\infty)$ is a continuous random variable. The density of Y is (4) if and only if the differential equation holds

$$\hbar'_F(\mathcal{Y}) + \left(\frac{1}{1-(1+\mathcal{Y})^{-\rho}}3\rho(1+\mathcal{Y})^{-\rho-1}\right)\hbar_F(\mathcal{Y}) = 2\mathcal{O}\rho[1-(1+\mathcal{Y})^{-\rho}]^{-3}\frac{d}{d\mathcal{Y}}H(\mathcal{Y};\rho), \mathcal{Y} > 0,$$

with the initial condition $\lim_{\mathcal{Y}\to 0}\hbar_F(\mathcal{Y}) = 0$, where

$$H(\mathcal{Y};\rho) = \frac{1}{(1+\mathcal{Y})^{2\rho+1}}\exp\{-[(1+\mathcal{Y})^{\rho}-1]^{-2}\}$$

Proof. Is straightforward and hence omitted.

3.3 Characterizations based on conditional expectation

The following proposition can be found in Hamedani (2013), so we will use it to characterize the IBXLx distribution.

Proposition 3.3.1. Suppose the random variable $X: \Omega \to (a, b)$ is continuous with CDF F. If $\phi(\mathcal{Y})$ is a differentiable function on (a, b) with $\lim_{\mathcal{Y}\to 0^+}\phi(Y) = 1$, then for $\delta \neq 1$,

$$E[\phi(Y)|Y \geq \mathcal{Y}] = \delta\phi(Y), \quad \mathcal{Y} \in (a, b),$$

if and only if

$$\phi(Y) = \left(1 - F(\mathcal{Y})\right)^{\frac{1}{\delta}-1}, \mathcal{Y} \in (a, b)$$

Remark 3.3.1. Taking

$$(a, b) = (0, +\infty), \phi(Y) = 1 - \exp\{-[(1+\mathcal{Y})^{\rho}-1]^{-2}\}$$

and $\delta = \frac{\mathcal{O}}{\mathcal{O}+1}$, Proposition 3.3.1 presents a characterization of IBXLx distribution. Clearly, there are other possible function.

4 Construction of NURR statistic for the IBX-Lx model

When working with complete data, it's common to utilize diverse methods to evaluate the appropriateness of a mathematical model for the observed data. One widely used technique for hypothesis testing in such cases is Pearson's Chi-square statistic. However, there are instances where these conventional approaches aren't applicable, particularly when the model's parameters are unknown or the data is censored. Since the mid-20th century, researchers have aimed to overcome these limitations by proposing adjustments to existing statistical tests. On one hand, they need to account for unknown parameters, while on the other, censorship needs consideration. For datasets with complete information, Nikulin (1973) and Rao and Robson (1974) independently introduced the NURR statistics, which extend the Pearson statistic and adhere to the chi-square distribution. Yet, in situations involving censored data and unknown parameters, classical tests may fall short in confirming hypotheses. To address this, Bagdonavičius and Nikulin (2011a,b) and Bagdonavičius et al. (2013) suggested adaptations to the NURR statistic to accommodate random right censorship. These adjusted statistics have been applied in various studies to align observations with models like the generalized inverse Weibull model, Burr XII inverse Rayleigh model, odd Lindley exponentiated exponential model, Topp-Leone-Lomax model, and new reciprocal Rayleigh extension. In this section, we formulate a modified Chi-square test specifically tailored for the IBX-Lx model, considering both complete and censored data scenarios. To test the hypothesis

$$H_0: P\{\mathcal{Y}_i \leq \mathcal{Y}\} = F_{\Xi}(\mathcal{Y}), \quad \mathcal{Y} \in \mathbb{R}, \quad \Xi = (\Xi_1, \Xi_2, \cdots, \Xi_{\mathcal{S}})^T,$$

wherein $\mathcal{Y}_1, \mathcal{Y}_2, \cdots, \mathcal{Y}_n$, an n −sample belong to a parametric family $F(\mathcal{Y}; \Xi)$ where Ξ represents the vector of unknown parameters, Nikulin (1973) and Rao and Robson (1974) proposed the NURR statistic Y^2 where

$$Y^2(\widehat{\Xi}_n) = \mathcal{Y}_n^2(\widehat{\Xi}_n) + \frac{1}{n}\mathbf{L}^T(\widehat{\Xi}_n)(\mathbf{I}(\widehat{\Xi}_n) - \mathbf{J}(\widehat{\Xi}_n))^{-1}\mathbf{L}(\widehat{\Xi}_n),$$

$$\mathcal{Y}_n^2(\Xi) = \left(\frac{\nu_1 - np_1(\Xi)}{\sqrt{np_1(\Xi)}}, \frac{\nu_2 - np_2(\Xi)}{\sqrt{np_2(\Xi)}}, \cdots, \frac{\nu_{\mathscr{b}} - np_{\mathscr{b}}(\Xi)}{\sqrt{np_{\mathscr{b}}(\Xi)}}\right)^T$$

and $\mathbf{J}(\Xi)$ is the information matrix for the grouped data with

$$\boldsymbol{\mathcal{B}}(\Xi) = \left[\frac{1}{\sqrt{p}_i}\frac{\partial}{\partial\mu}p_i(\Xi)\right]_{r\times\mathcal{S}} |_{(i=1,2,\cdots,\mathscr{b} \text{ and} k=1,\cdots,\mathcal{S})},$$

then

$$\mathbf{L}(\Xi) = (\mathbf{L}_1(\Xi), \ldots, \mathbf{L}_{\mathcal{S}}(\Xi))^{\mathcal{Y}} \text{ with } \mathbf{L}_k(\Xi) = \sum_{i=1}^{r}\frac{\nu_i}{p_i}\frac{\partial}{\partial\Xi_k}p_i(\Xi),$$

where $\mathbf{I}_n(\widehat{\Xi_n})$ represents the estimated Fisher information matrix and $\widehat{\Xi_n}$ is the maximum likelihood estimator of the parameter vector. The Y^2 statistic follows a distribution of chi-square $\chi^2_{\mathscr{b}-1}$ with $(\mathscr{b}-1)$ degrees of freedom. Consider the Observations $\mathcal{Y}_1, \mathcal{Y}_2, \cdots, \mathcal{Y}_n$, they are grouped in $\mathscr{b}$ subintervals $\mathbf{I}_1, \mathbf{I}_2, \cdots, \mathbf{I}_{\mathscr{b}}$ mutually disjoint $\mathbf{I}_j =]a_j - 1; a_j]$; where $j = \overline{1;\mathscr{b}}$.The limits aj of the intervals $\mathbf{I}_j$ are obtained such that

$$p_j(\Xi) = \int_{a_{j-1}}^{a_j} f_\Xi(\mathcal{Y})d\mathcal{Y}|_{(\ j=1,2,\cdots,\mathscr{b})}, a_j = F^{-1}\left(\frac{j}{\mathscr{b}}\right)|_{(j=1,\cdots,\mathscr{b}-1)}.$$

If $\nu_j = (\nu_1, \nu_2, \cdots, \nu_{\mathscr{b}})^T$ is the vector of frequencies obtained by the grouping of data in these $\mathbf{I}_j$ intervals

$$\nu_j = \textstyle\sum_{i=1}^{n} 1_{\{\mathcal{Y}_i \in \mathbf{I}_j\}}|_{(j=1,\ldots,\mathscr{b})}.$$

In order to check whether the data used in this paper is distributed according to the IBX-Lx model, in the case of unknown parameters, we construct the chi-square goodness-of-fit test by fitting the NURR statistics developed previously. After calculating the maximum likelihood estimator $\widehat{\Xi_n}$ for the unknown parameters of the IBX-Lx distribution on the data set, we use $\mathbf{I}_n(\widehat{\Xi_n})$ as the estimated Fisher information matrix to provide all the components of the Y^2 statistic of our IBX-Lx model.

5 Classical estimations

5.1 Maximum likelihood method

The maximum likelihood estimates (MLEs) enjoy desirable properties and can be used when constructing confidence intervals. Let $w_1, w_2, \ldots, w_n$ be a RS from this distribution with parameter vector $\underline{\Xi} = (\mathcal{O}, \rho)^\intercal$. The log-likelihood function for $\underline{\Xi}$, say $\ell(\underline{\Xi})$, is given by

$$\ell(\underline{\Xi}) = \log\left[\prod_{i=1}^{n} f_{\mathcal{O},\rho}(\mathcal{Y}_m)\right],$$

which can be maximized either using the statistical programs or by solving the nonlinear system obtained from $\ell(\underline{\Xi})$ by differentiation. The score vector , $\mathbf{U}_{\underline{\Xi}} = \left(\frac{\partial\ell(\Xi)}{\partial\mathcal{O}}, \frac{\partial\ell(\Xi)}{\partial\rho}\right)^\intercal$, are easy to derive. Below, we aim to obtain the maximum likelihood estimator of inverted Burr X-Lx (IBX-Lx) distribution under type II censored data. Consider the n-sample $(\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_n)$ and a fixed constant m, we assume that the m-sample $(\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_m)$ generated from IBX-Lx.The likelihood function of this sample is

$$L_{\mathcal{O},\rho}(\mathcal{Y}) = N\prod_{i=1}^{m} f_{\mathcal{O},\rho}(\mathcal{Y}_i)\left[1 - F_{\mathcal{O},\rho}(\mathcal{Y}_m)\right]^{n-m},$$

where $N = \frac{n!}{(n-m)!}$, using (3) and (4) we get

$$L_{\mathcal{O},\rho}(\mathcal{Y}) = N2^m\mathcal{O}^m\rho^m\prod_{i=1}^{m}\boldsymbol{\mathcal{A}}_i(\mathcal{Y}_i)\boldsymbol{\mathcal{B}}_i(\mathcal{Y}_i)C_i(\mathcal{Y}_i)D_i^{\mathcal{O}-1}(\mathcal{Y}_i)E^{\mathcal{O}(n-m)}(\mathcal{Y}_i),$$

where

$$\boldsymbol{\mathcal{A}}_i(\mathcal{Y}_i) = [1 - (1 + \mathcal{Y}_i)^{-\rho}]^{-3},$$

$$\boldsymbol{\mathcal{B}}_i(\mathcal{Y}_i) = (1+\mathcal{Y}_i)^{-2\rho-1},$$
$$C_i(\mathcal{Y}_i) = \exp\{-[(1+\mathcal{Y}_i)^{\rho}-1]^{-2}\}$$
$$D_i^{\mathcal{O}-1}(\mathcal{Y}_i) = 1-\exp\{-[(1+\mathcal{Y}_i)^{\rho}-1]^{-2}\},$$
$$E = 1-\exp-[(1+\mathcal{Y}_m)^{\rho}-1]^{-2},$$

$$\ln\left[L_{\mathcal{O},\rho}(\mathcal{Y})\right] = l_{\mathcal{O},\rho}(\mathcal{Y}) = \ln N + m\ln 2 + m\ln\mathcal{O} + m\ln\rho$$
$$+\sum_{i=1}^{m}\left[\ln\boldsymbol{\mathcal{A}}_i(\mathcal{Y}_i)+\ln\boldsymbol{\mathcal{B}}_i(\mathcal{Y}_i)+\ln C_i(\mathcal{Y}_i)+(\mathcal{O}-1)\ln D_i^{\mathcal{O}-1}(\mathcal{Y}_i)\right]+\mathcal{O}(n-m)\ln E.$$

The maximum likelihood estimators $\mathcal{O}_{MLE}$ and ρ_{MLE} of the parameters $\mathcal{O},\rho$, respectively. are The solution of the following non-linear system

$$\begin{cases}\dfrac{\partial l_{\mathcal{O},\rho}(\mathcal{Y})}{\partial\mathcal{O}} = \dfrac{m}{\mathcal{O}}+\sum_{i=1}^{m}\left[\ln D_i^{\mathcal{O}-1}(\mathcal{Y}_i)\right]+(n-m)\ln E \\ \dfrac{\partial l_{\mathcal{O},\rho}(\mathcal{Y})}{\partial\rho} = \dfrac{m}{\rho}\sum_{i=1}^{m}\dfrac{\boldsymbol{\mathcal{A}}_{i,1}}{\boldsymbol{\mathcal{A}}_i(\mathcal{Y}_i)}+\dfrac{\boldsymbol{\mathcal{B}}_{i,1}}{\boldsymbol{\mathcal{B}}_i(\mathcal{Y}_i)}+\dfrac{C_{i,1}}{C_i(\mathcal{Y}_i)}+(\mathcal{O}-1)\dfrac{D_{i,1}}{D_i^{\mathcal{O}-1}(\mathcal{Y}_i)}+\mathcal{O}(n-m)\dfrac{E_1}{E}\end{cases}$$

where

$$\boldsymbol{\mathcal{A}}_{i,1} = \frac{\partial\boldsymbol{\mathcal{A}}_i(\mathcal{Y}_i)}{\partial\rho} = -3(1-(1+\mathcal{Y}_i)^{-\rho})^{-4}\ln(1+\mathcal{Y}_i)(1+\mathcal{Y}_i)^{-\rho},$$
$$\boldsymbol{\mathcal{B}}_{i,1} = \frac{\partial\boldsymbol{\mathcal{B}}_i(\mathcal{Y}_i)}{\partial\rho} = -2\ln(1+\mathcal{Y}_i)(1+\mathcal{Y}_i)^{-2\rho-1},$$
$$C_{i,1} = \frac{\partial C_i(\mathcal{Y}_i)}{\partial\rho} = 2[(1+\mathcal{Y}_i)^{\rho}-1]^{-3}\ln(1+\mathcal{Y}_i)(1+\mathcal{Y}_i)^{\rho}\exp-[(1+\mathcal{Y}_i)^{\rho}-1]^{-2},$$
$$D_{i,1} = \frac{\partial D_i^{\mathcal{O}-1}(\mathcal{Y}_i)}{\partial\rho} = -2[(1+\mathcal{Y}_i)^{\rho}-1]^{-3}\ln(1+\mathcal{Y}_i)(1+\mathcal{Y}_i)^{\rho}\exp-[(1+\mathcal{Y}_i)^{\rho}-1]^{-2},$$

and

$$E_1 = \frac{\partial E}{\partial\rho} = -2((1+\mathcal{Y}_m)^{\rho}-1)^{-3}\ln(1+\mathcal{Y}_m)(1+\mathcal{Y}_m)^{\rho}\exp-\left((1+\mathcal{Y}_m)-1\right)^{-2}$$

There is no analytical solution for this system, thus so we use the R-package (the BBsolve function) to obtain the approximate values of maximum likelihood estimators $\mathcal{O}_{MLE}$, and ρ_{MLE}.

5.2 The Cramér-von Mises method

The Cramér-von Mises estimates (CVME) of the parameters $\hat{\mathcal{O}}$ and $\hat{\rho}$ are obtained via minimizing the following expression with respect to the parameters $\mathcal{O}$ and ρ respectively, where

$$\mathbf{CVM}_{(\Xi)} = \frac{1}{12}n^{-1}+\sum_{i=1}^{n}\left[F_{\mathcal{O},\rho}(\mathcal{Y}_{i,n})-\boldsymbol{\epsilon}_{(i,n)}\right]^2,$$

where $\boldsymbol{\epsilon}_{(i,n)} = \frac{2i-1}{2n}$ and

$$\mathbf{CVM}_{(\Xi)} = \sum_{i=1}^{n}\left[F_{\mathcal{O},\rho}(\mathcal{Y}_{i,n})-\boldsymbol{\epsilon}_{(i,n)}\right]^2.$$

Then, CVME of the parameters $\mathcal{O}$ and ρ are obtained by solving the two following non-linear equations

$$\sum_{i=1}^{n}\left[F_{\mathcal{O},\rho}(\mathcal{Y}_{i,n})-\boldsymbol{\epsilon}_{(i,n)}\right]\boldsymbol{\varsigma}_{(\mathcal{O})}(\mathcal{Y}_{[i:n]},\underline{\Xi}) = 0 \text{and} \sum_{i=1}^{n}\left[F_{\mathcal{O},\rho}(\mathcal{Y}_{i,n})-\boldsymbol{\epsilon}_{(i,n)}\right]\boldsymbol{\varsigma}_{(\rho)}(\mathcal{Y}_{[i:n]},\underline{\Xi}) = 0,$$

where $\boldsymbol{\varsigma}_{(\mathcal{O})}(\mathcal{Y}_{[i:n]},\underline{\Xi})$ and $\boldsymbol{\varsigma}_{(\rho)}(\mathcal{Y}_{[i:n]},\underline{\Xi})$ are the first derivatives of the CDF of GWNH distribution WRT $\mathcal{O}$ and ρ respectively.

5.3 The Anderson-Darling method

The Anderson-Darling estimates (ADEs) of $\mathcal{O}$ and ρ are obtained by minimizing the function

$$ADE(\underline{\Xi}) = -n - n^{-1}\sum_{i=1}^{n}(2i-1)\left\{\begin{matrix}\log F_{\mathcal{O},\rho}(y_{i,n}) \\ +\log[1-F_{\mathcal{O},\rho}(y_{[-i+1+n:n]})]\end{matrix}\right\}.$$

The parameter estimates of $\mathcal{O}$ and ρ follow by solving the nonlinear equations

$$\frac{\partial}{\partial \mathcal{O}}[\mathrm{ADE}(\underline{\Xi})] = 0 \text{ and } \frac{\partial}{\partial \rho}[\mathrm{ADE}(\underline{\Xi})] = 0.$$

5.4 The right-tail Anderson–Darling method

The right-tail Anderson–Darling estimates (ADERTEs) of $\mathcal{O}$ and ρ are obtained by minimizing

$$\mathrm{ADERT}(\underline{\Xi}) = \frac{1}{2}n - 2\sum_{i=1}^{n} F_{\mathcal{O},\rho}(y_{i,n}) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\{\log[1-F_{\mathcal{O},\rho}(y_{[-i+1+n:n]})]\}.$$

The parameter estimates of $\mathcal{O}$ and ρ follow by solving the nonlinear equations $\frac{\partial}{\partial \mathcal{O}}[\mathrm{ADERT}(\underline{\Xi})] = 0$ and $\frac{\partial}{\partial \rho}[\mathrm{ADERT}(\underline{\Xi})] = 0$.

5.5 Kolmogorov method

The Kolmogorov estimates (KEs) $\hat{\mathcal{O}}$ and $\hat{\rho}$ of $\mathcal{O}$ and ρ are obtained by minimizing the function

$$\mathrm{K} = \mathrm{K}(\mathcal{O},\rho) = \max_{1\le i\le n}\left\{\frac{i}{n} - F_{\mathcal{O},\rho}(y_{i,n}), F_{\mathcal{O},\rho}(y_{i,n}) - \frac{i-1}{n}\right\}.$$

6 Bayesian estimation under different loss functions

Bayesian estimation under various loss functions is a fundamental aspect of statistical inference, allowing researchers to derive optimal estimators based on different criteria. In Bayesian analysis, the choice of loss function reflects the decision-maker's preferences and the context of the problem at hand. Here, we will explore the Bayesian estimation process under three commonly used loss functions: the quadratic loss function, the Linex loss function, and the entropy loss function.

I. Quadratic Loss Function: The quadratic loss function, also known as the mean squared error, penalizes deviations between the true parameter value and the estimated value quadratically. In Bayesian estimation, the posterior distribution is obtained by multiplying the prior distribution with the likelihood function, and the posterior mean serves as the Bayesian estimator under the quadratic loss function. This approach aims to minimize the expected squared difference between the true parameter and the estimated parameter.

II. Linex Loss Function: The Linex loss function is a robust alternative to the quadratic loss function, particularly in cases where outliers may skew the results. It penalizes deviations asymmetrically, placing more weight on positive deviations than negative deviations. In Bayesian estimation, the Linex loss function leads to a posterior distribution that minimizes the expected value of the exponential of the absolute difference between the true parameter and the estimated parameter. The Bayesian estimator under the Linex loss function is typically the posterior median.

III. Entropy Loss Function: The entropy loss function, also known as the Kullback-Leibler divergence, measures the information lost when using an estimated parameter to approximate the true parameter. In Bayesian estimation, maximizing the posterior distribution under the entropy loss function results in the Bayesian estimator that minimizes the expected Kullback-Leibler divergence between the true parameter distribution and the estimated parameter distribution. This approach emphasizes the preservation of information when making parameter estimates.

Each of these loss functions has its advantages and is suitable for different scenarios. The choice of loss function depends on the specific goals of the analysis and the underlying assumptions about the data. By considering Bayesian estimation under different loss functions, researchers can tailor their approach to best address the objectives of their study and obtain robust parameter estimates.

6.1 Prior and posterior model

As prior distributions, we assume the parameters $\mathcal{O}, \rho$ follow the Gamma distribution as a prior:

$$\pi(\mathcal{O}) = \frac{a_1 b_1}{\Gamma(b_1)} \mathcal{O}^{b_1 - 1} \exp(-a_1 \mathcal{O})|_{\mathcal{O},\rho,a_1,b_1>0},$$
$$\pi(\rho) = \frac{a_2 b_2}{\Gamma(b_2)} \rho^{b_2 - 1} \exp(-a_2 \rho)|_{\mathcal{O},\rho,a_1,b_1>0},$$

where the constants a_1, b_1, a_2, b_2 are called hyper-parameters. Thus, the joint prior distribution of $(\mathcal{O}, \rho)$ is given by

$$\pi(\mathcal{O}, \rho) = \frac{a_1 b_1 a_2 b_2}{\Gamma(b_1)\Gamma(b_2)} \mathcal{O}^{b_1 - 1} \rho^{b_2 - 1} \exp -(a_1 \mathcal{O} + a_2 \rho).$$

The joint posterior distribution of $(\mathcal{O}, \rho)$ reads as

$$\pi(\mathcal{O}, \rho|\mathcal{Y}) = \frac{\mathcal{O}^{m+b_1-1} \rho^{m+b_2-1} \exp[-(a_1 \mathcal{O} + a_2 \rho)] \prod_{i=1}^{m} \boldsymbol{\mathcal{A}}_i(\mathcal{Y}_i) \boldsymbol{\mathcal{B}}_i(\mathcal{Y}_i) C_i(\mathcal{Y}_i) D_i^{\mathcal{O}-1}(\mathcal{Y}_i) E^{\mathcal{O}(n-m)}(\mathcal{Y}_i)}{\mathcal{O}^{m+b_1-1} \rho^{m+b_2-1} \exp[-(a_1 \mathcal{O} + a_2 \rho)] \prod_{i=1}^{m} \boldsymbol{\mathcal{A}}_i(\mathcal{Y}_i) \boldsymbol{\mathcal{B}}_i(\mathcal{Y}_i) C_i(\mathcal{Y}_i) D_i^{\mathcal{O}-1}(\mathcal{Y}_i) E^{\mathcal{O}(n-m)}(\mathcal{Y}_i) d\mathcal{O} d\rho},$$

so the joint posterior of $(\mathcal{O}, \rho)$ is

$$\pi(\mathcal{O}, \rho|\mathcal{Y}) = K \mathcal{O}^{m+b_1-1} \rho^{m+b_2-1} \exp -(a_1 \mathcal{O} + a_2 \rho) \prod_{i=1}^{m} \boldsymbol{\mathcal{A}}_i(\mathcal{Y}_i) \boldsymbol{\mathcal{B}}_i(\mathcal{Y}_i) C_i(\mathcal{Y}_i) D_i^{\mathcal{O}-1}(\mathcal{Y}_i) E^{\mathcal{O}(n-m)}(\mathcal{Y}_i),$$

where K is the normalizing constant. Next, we use the three loss functions namely the generalised quadratic (GQ), the Linex and the entropy functions to obtain the Bayesian estimators

6.2 Bayesian estimators and their posterior risks

The Bayesian estimators under the GQ loss function are

$$\hat{\mathcal{O}}_{GQ} = \frac{\mathcal{O}^{m+b_1-1+\delta} \rho^{m+b_2-1} \exp -(a_1 \mathcal{O} + a_2 \rho) \prod_{i=1}^{m} \boldsymbol{\mathcal{A}}_i(\mathcal{Y}_i) \boldsymbol{\mathcal{B}}_i(\mathcal{Y}_i) C_i(\mathcal{Y}_i) D_i^{\mathcal{O}-1}(\mathcal{Y}_i) E^{\mathcal{O}(n-m)}(\mathcal{Y}_i)}{\mathcal{O}^{m+b_1+\delta-2} \rho^{m+b_2-1} \exp -(a_1 \mathcal{O} + a_2 \rho) \prod_{i=1}^{m} \boldsymbol{\mathcal{A}}_i(\mathcal{Y}_i) \boldsymbol{\mathcal{B}}_i(\mathcal{Y}_i) C_i(\mathcal{Y}_i) D_i^{\mathcal{O}-1}(\mathcal{Y}_i) E^{\mathcal{O}(n-m)}(\mathcal{Y}_i) d\mathcal{O} d\rho},$$

and

$$\hat{\rho}_{GQ} = \frac{\mathcal{O}^{m+b_1-1} \rho^{m+b_2-1+\delta} \exp -(a_1 \mathcal{O} + a_2 \rho) \prod_{i=1}^{m} \boldsymbol{\mathcal{A}}_i(\mathcal{Y}_i) \boldsymbol{\mathcal{B}}_i(\mathcal{Y}_i) C_i(\mathcal{Y}_i) D_i^{\mathcal{O}-1}(\mathcal{Y}_i) E^{\mathcal{O}(n-m)}(\mathcal{Y}_i)}{\mathcal{O}^{m+b_1-1} \rho^{m+b_2+\delta-2} \exp -(a_1 \mathcal{O} + a_2 \rho) \prod_{i=1}^{m} \boldsymbol{\mathcal{A}}_i(\mathcal{Y}_i) \boldsymbol{\mathcal{B}}_i(\mathcal{Y}_i) C_i(\mathcal{Y}_i) D_i^{\mathcal{O}-1}(\mathcal{Y}_i) E^{\mathcal{O}(n-m)}(\mathcal{Y}_i) d\mathcal{O} d\rho}.$$

The corresponding posterior risks are then

$$PR(\hat{\mathcal{O}}_{GQ}) = E_\pi(\mathcal{O}^{\delta+1}) - 2\hat{\mathcal{O}}_{GQ} E_\pi(\mathcal{O}^{-\delta}) + \mathcal{O}_{GQ}^2 E_\pi(\mathcal{O}^{\delta-1}),$$

and

$$PR(\hat{\rho}_{GQ}) = E_\pi(\rho^{\delta+1}) - 2\hat{\rho}_{GQ} E_\pi(\rho^{-\delta}) + \rho_{GQ}^2 E_\pi(\rho^{\delta-1}).$$

Under the entropy loss function, we obtain the following estimators

$$\hat{\mathcal{O}}_E = \left[K \mathcal{O}^{m+b_1-1+p} \rho^{m+b_2-1} \exp[-(a_1 \mathcal{O} + a_2 \rho)] \prod_{i=1}^{m} \boldsymbol{\mathcal{A}}_i(\mathcal{Y}_i) \boldsymbol{\mathcal{B}}_i(\mathcal{Y}_i) C_i(\mathcal{Y}_i) D_i^{\mathcal{O}-1}(\mathcal{Y}_i) E^{\mathcal{O}(n-m)}(\mathcal{Y}_i) \right]^{1-p},$$

and

$$\hat{\rho}_E = \left[K \mathcal{O}^{m+b_1-1} \rho^{m+b_2-1-p} \exp[-(a_1 \mathcal{O} + a_2 \rho)] \prod_{i=1}^{m} \boldsymbol{\mathcal{A}}_i(\mathcal{Y}_i) \boldsymbol{\mathcal{B}}_i(\mathcal{Y}_i) C_i(\mathcal{Y}_i) D_i^{\mathcal{O}-1}(\mathcal{Y}_i) E^{\mathcal{O}(n-m)}(\mathcal{Y}_i) \right]^{1-p}.$$

The corresponding posterior risks are then

$$PR(\hat{\mathcal{O}}_E) = PE_\pi\left(\ln(\mathcal{O}) - \ln(\hat{\mathcal{O}}_E)\right) \text{ and } PR(\hat{\rho}_E) = PE_\pi(\ln(\rho) - \ln(\hat{\rho}_E)).$$

Finally, under the entropy loss function, the Bayesian estimators

$$\hat{\mathcal{O}}_E = \frac{-K}{a} \ln\left[\mathcal{O}^{m+b_1-1}\rho^{m+b_2-1}\exp[-(a_1\mathcal{O} + a_2\rho - r\mathcal{O})]\prod_{i=1}^{m} \boldsymbol{\mathcal{A}}_i(\mathcal{Y}_i)\boldsymbol{\mathcal{B}}_i(\mathcal{Y}_i)C_i(\mathcal{Y}_i)D_i^{\mathcal{O}-1}(\mathcal{Y}_i)E^{\mathcal{O}(n-m)}(\mathcal{Y}_i)\right],$$

and

$$\hat{\rho}_L = \frac{-K}{a} \ln\left[\mathcal{O}^{m+b_1-1}\rho^{m+b_2-1}\exp[-(a_1\mathcal{O} + a_2\rho - r\rho)]\prod_{i=1}^{m} \boldsymbol{\mathcal{A}}_i(\mathcal{Y}_i)\boldsymbol{\mathcal{B}}_i(\mathcal{Y}_i)C_i(\mathcal{Y}_i)D_i^{\mathcal{O}-1}(\mathcal{Y}_i)E^{\mathcal{O}(n-m)}(\mathcal{Y}_i)\right].$$

The corresponding posterior risks are then

$$PR(\hat{\mathcal{O}}_L) = a(\hat{\mathcal{O}}_{GQ} - \hat{\mathcal{O}}_L), PR(\hat{\rho}_L) = a(\hat{\rho}_{GQ} - \hat{\rho}_L).$$

Considering the intricate nature of obtaining all these estimators analytically, it is unlikely to achieve this task feasibly. Therefore, we advocate for the utilization of Markov Chain Monte Carlo (MCMC) procedures to evaluate these estimators, as elaborated upon in the following section.

7 Simulation studies

Simulation studies are invaluable tools in comparing classical methods in various fields, including statistics, economics, and engineering. They allow researchers to evaluate the performance of different methods under controlled conditions where the true underlying parameters are known. Here's a general outline of how simulation studies can be conducted to compare classical methods:

Define the Problem: Clearly define the problem you want to address and the classical methods you want to compare. For example, you might want to compare the performance of different hypothesis testing methods or regression techniques.

- **I.** Simulate data that reflects the characteristics of the problem you defined. This may involve specifying distributions for variables, correlation structures, sample sizes, and other relevant parameters. Ensure that the data generation process aligns with the assumptions of the methods being compared.
- **II.** Implement the classical methods you want to compare using the simulated data. This may involve coding algorithms or using existing software packages.
- **III.** Define appropriate performance metrics to evaluate the methods. Common metrics include bias, variance, mean squared error, confidence interval coverage probability, Type I error rate, power, and computational efficiency.
- **IV.** Repeat the simulation process multiple times to account for randomness in the data generation process. This helps reduce the variability in the results and provides more reliable estimates of performance metrics.
- **V.** Analyze the results of the simulation study. Compare the performance of the classical methods based on the chosen metrics. Graphical representations such as box plots, histograms, or scatter plots can be helpful in visualizing the results.
- **VI.** Conduct sensitivity analyses to assess the robustness of the results to changes in simulation parameters or assumptions. This helps ensure that the conclusions drawn from the study are valid across different scenarios.
- **VII.** Clearly document the methodology, results, and conclusions of the simulation study in a report or manuscript. Provide insights into the strengths and weaknesses of each method and discuss implications for real-world applications.
- **VIII.** Validate the simulation results by comparing them with theoretical expectations or results from empirical studies if available. This helps ensure that the simulation accurately reflects the behavior of the methods in real-world settings.
- **IX.** Submit the study for peer review to obtain feedback from experts in the field. Address any concerns or suggestions raised by reviewers to improve the quality and credibility of the study.

By following these steps, researchers can conduct rigorous simulation studies to compare classical methods and gain valuable insights into their relative performance across different scenarios.

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7.1 Simulation studies for comparing the classical methods

In order to assess and compare the performance of the proposed calssical methods, we perform three Monte Carlo simulation study through three carefully selected different scenarios. The results of these three scenarios in Table 1 ($\mathcal{O} = 2$ and $\rho = 1.5$), Table 2 ($\mathcal{O} = 0.8$ and$\rho = 0.8$) and Table 3 ($\mathcal{O} = 1.5$ and $\rho = 0.5$). All simulation studies are performed using $N = 1000$ samples with different sample sizes n =50, 100, 200 and 300. Specifically, Table 1 gives the mean squared errors (MSEs) under $\mathcal{O} = 2$ and $\rho = 1.5$. Table 2 lists the MSEs under $\mathcal{O} = 0.8$ and $\rho = 0.8$. Table 3 presents the MSEs under $\mathcal{O} = 1.5$ and $\rho = 0.5$. By looking closely at the three tables, we can find the following results:

I. The larger the sample size, the lower the MSE value for all estimation methods without exception.

II. Through the first scenario and when $n = 300$, the lowest MSE we've got was for a MLE method where MLE $_{(\mathcal{O})}$ =0.01251 and MLE $_{(\mathcal{O})}$ =0.00059 (see Table 1).

III. Through the second scenario and when $n = 300$, the lowest MSE we've got was for a MLE method where MLE $_{(\mathcal{O})}$ =0.00232 and MLE $_{(\mathcal{O})}$ =0.00035 (see Table 2).

IV. Through the third scenario and when $n = 300$, the lowest MSE we've got was for a MLE method where MLE $_{(\mathcal{O})}$ =0.00723 and MLE $_{(\mathcal{O})}$ =0.00008 (see Table 3).

V. However, we cannot deny the fact that all the candidate methods for estimation were highly acceptable results, and there is no fundamental difference between each other in fact, if all the estimates were on the desired degree of consistency and efficiency.

VI. Based on the foregoing reliable results, all of which were in favor of the maximum likelihood method, in the coming sections we will pay much attention to comparing the maximum likelihood method with Bayer's method. This determination is not intentional as previously explained, but is based on the results of the three previous scenarios (Table 1, Table 2 and Table 3) dealt with by study and analysis.

Table 1: MSEs under $\mathcal{O}$= 2 and ρ = 1.5.

		MLE	CVM	KE	ADE	RTADE
50	$\mathcal{O}$	0.08711	0.10553	0.13712	0.09265	0.08689
	ρ	0.00359	0.00360	0.00403	0.00339	0.00403
100	$\mathcal{O}$	0.04215	0.05148	0.05912	0.04574	0.04342
	ρ	0.00172	0.00187	0.00202	0.00175	0.00216
200	$\mathcal{O}$	0.02012	0.02770	0.03150	0.02442	0.02204
	ρ	0.00090	0.00104	0.00109	0.00098	0.00114
300	$\mathcal{O}$	0.01251	0.01813	0.01986	0.01596	0.01436
	ρ	0.00059	0.00070	0.00071	0.00065	0.00075

Table 2: $\mathcal{O}$ = 0.8 and ρ = 0.8.

		MLE	CVM	KE	ADE	RTADE
50	$\mathcal{O}$	0.01488	0.01883	0.02370	0.01652	0.01504
	ρ	0.00215	0.00268	0.00291	0.00241	0.00307
100	$\mathcal{O}$	0.00701	0.00892	0.01011	0.00782	0.00716
	ρ	0.00105	0.00130	0.00138	0.00118	0.00148
200	$\mathcal{O}$	0.00337	0.00449	0.00515	0.00401	0.00368
	ρ	0.00051	0.00067	0.00072	0.00061	0.00078
300	$\mathcal{O}$	0.00232	0.00281	0.00308	0.00249	0.00229
	ρ	0.00035	0.00042	0.00043	0.00038	0.00048

Table 3: $\mathcal{O}$ = 1.5 and ρ = 0.5.

		MLE	CVM	KE	ADE	RTADE
50	$\mathcal{O}$	0.04915	0.06193	0.08109	0.05392	0.04828
	ρ	0.00051	0.00056	0.00063	0.00053	0.00063

100	$\mathcal{O}$	0.02289	0.03269	0.03616	0.02828	0.02557
	ρ	0.00024	0.00030	0.00031	0.00028	0.00034
200	$\mathcal{O}$	0.01159	0.01557	0.01753	0.01371	0.01236
	ρ	0.00013	0.00015	0.00015	0.00013	0.00016
300	$\mathcal{O}$	0.00723	0.00911	0.00964	0.00808	0.00757
	ρ	0.00008	0.00009	0.00009	0.00008	0.00010

7.2 Pitman criterion

In order to compare the performance of the proposed Bayes estimators with the MLEs, we perform a Monte Carlo simulation study assuming that $\mathcal{O} = 2$, $\rho = 1$ and $a_1 = b_1 = 2, a_2 = b_2 = 1$, using $N = 5000$ samples of the type II censored model with different sample sizes $n = 10{,}50200$ while $m = 8{,}40{,}160$ respectively, we obtain the following results. Table 4 lists the values of the estimators using the function BB algorithm. We remark here that the estimated values of $\mathcal{O}$ and ρ are close to the true values of the parameter especially with the increase in sample size n. Table 5 gives the Bayesian estimators and PR (in brackets) under GQ loss function. Table 6 displays the Bayesian estimators alongside the corresponding PR values (enclosed in brackets) under the entropy loss function. Table 7 provides the Bayesian estimators and their respective PR values (in brackets) under the Linex loss function. Table 8 exhibits the Bayesian estimators alongside the PR values (in brackets) under all three loss functions. In Table 5, the estimation under the GQ loss function, we remark that the value $\gamma = -1$ gives the best posterior risk. Also, we obtain the smallest suitable posterior risk when n is high. In the estimation under the entropy loss function, we obtain Table 6 where we can notice that the value $p = 0.5$ when $n = 200$ provides the best posterior risk.

It is evident from the analysis that the value of r=1.5 yields the optimal PR, indicating its superior performance among the considered range of values. In summary, upon conducting a comparative analysis of the three loss functions, it becomes apparent that the quadratic loss function consistently produces the most favorable outcomes. These findings are meticulously outlined in Table 8, providing a comprehensive overview of the results. Moreover, we propose a comparison between the optimal Bayesian estimators and the maximum likelihood estimators to further evaluate their efficacy. To accomplish this, we employ the Pitman closeness criterion, which serves as a reliable metric for assessing the proximity between estimators. For more detailed information on this criterion, readers can refer to the works of Pitman (1937), Fuller (1982), and Jozani (2012). Through this comparative analysis, we aim to discern the relative strengths and weaknesses of each estimation approach, thereby informing decision-making processes and guiding further research endeavors.

Table 4: The MLE of the parameters with quadratic error (in brackets).

N = 5000	n = 10	n = 50	n = 200
m	8	40	160
$\mathcal{O}$	2.05020(0.01521)	1.92341(0.02177)	1.98722(0.00786)
ρ	0.61351(0.00784)	0.73966(0.00537)	0.95729(0.00453)

Table 5: Bayes estimators and PR (in brackets) under GQ loss function.

γ	N = 5000	n = 10	n = 50	n = 200
	m	8	40	160
-2	$\mathcal{O}$	1.6490(0.0089)	1.6825(0.0041)	1.6432(0.0016)
	ρ	0.6657(0.1491)	0.5033(0.0611)	0.8113(0.0008)
-1.5	$\mathcal{O}$	1.7990(0.0087)	1.0825(0.0061)	1.2127(0.0016)
	ρ	0.8657(0.7091)	0.7039(0.0633)	0.7120(0.0008)
-1	$\mathcal{O}$	1.9181(0.0005)	1.9739(0.0001)	2.0018(0.0001)
	ρ	0.9195(0.0002)	0.9870(0.0012)	0.9898(0.0001)
-0.5	$\mathcal{O}$	1.0994(0.0089)	1.0888(0.0070)	1.2138(0.0018)

	ρ	1.2999(0.0825)	1.2701(0.711)	1.7131(0.0012)
0.5	ϑ	1.7510(0.0095)	1.7926(0.0077)	2.1839(0.0020)
	ρ	0.6891(0.0909)	0.3591(0.995)	1.7139(0.0019)
1	ϑ	1.7575(0.0091)	1.0977(0.0078)	2.1841(0.0031)
	ρ	1.4228(0.1094)	1.3803(0.1071)	1.7149(0.0025)
1.5	ϑ	1.6743(0.0098)	1.5632(0.0081)	2.1232(0.0042)
	ρ	0.4768(0.1241)	0.6754(0.1181)	0.7903(0.0033)
2	ϑ	1.1099(0.0098)	1.0990(0.0081)	1.1841(0.0042)
	ρ	1.4768(0.1241)	0.4191(0.1181)	1.7158(0.0033)

Table 6: Bayes estimators and PR (in brackets) under the entropy loss function.

γ	N = 5000	n = 10	n = 50	n = 200
	m	8	40	160
-2	ϑ	0.0942(0.0008)	1.3990(0.1644)	1.2144(0.0019)
	ρ	1.3188(0.0699)	1.2839(0.009)	0.7034(0.011)
-1.5	ϑ	1.1067(0.0091)	1.7188(0.1443)	1.2179(0.0017)
	ρ	0.4407(0.0611)	0.4077(0.0661)	0.7060(0.0012)
-1	ϑ	1.1041(0.0009)	1.6205(0.0171)	1.2167(0.0001)
	ρ	1.4177(0.0072)	1.3633(0.0073)	0.7051(0.0003)
-0.5	ϑ	0.7981(0.0038)	1.7830(0.0733)	1.2148(0.0009)
	ρ	0.6493(0.0308)	0.8755(0.319)	07037(0.0009)
0.5	ϑ	1.8998(0.0008)	1.8895(0.0729)	1.9814(0.0001)
	ρ	0.7638(0.0071)	0.9856(0.0065)	1.0024(0.0002)
1	ϑ	1.6981(0.0038)	1.4830(0.0733)	1.2148(0.0009)
	ρ	1.5491(0.0308)	1.3055(0.319)	0.6037(0.0009)
1.5	ϑ	1.7053(0.0035)	1.6701(0.0667)	1.2169(0.0009)
	ρ	1.4239(0.0199)	1.3881(0.0303)	0.7059(0.0003)
2	ϑ	1.7697(0.0099)	1.7644(0.1173)	1.2188(0.0031)
	ρ	1.4579(0.0997)	1.4259(0.0944)	0.7071(0.0014)

Table 7: Bayes estimators and PR (in brackets) under Linex loss function

γ	N = 5000	n = 10	n = 50	n = 200
	m	8	40	160
-2	ϑ	1.6022(0.0039)	1.6309(0.1666)	2.1005(0.0007)
	ρ	0.4547(0.1041)	0.1058(0.0147)	0.4315(0.0481)
-1.5	ϑ	1.5309(0.1666)	1.5861(0.0009)	1.7174(0.0003)
	ρ	04821(0.1884)	0.4193(0.0131)	0.7045(0.0004)
-1	ϑ	1.5201(0.0039)	1.6815(0.0038)	1.5179(0.0012)
	ρ	0.4806(0.0411)	0.4455(0.0519)	0.7054(0.0013)
-0.5	ϑ	1.5455(0.0519)	1.5815(0.0183)	2.0070(0.0057)
	ρ	0.2191(0.0049)	0.1251(0.0195)	0.7094(0.0057)
0.5	ϑ	2.2041(0.0013)	1.7813(0.0007)	1.8153(0.0003)
	ρ	0.7080(0.0014)	0.3609(0.0199)	0.7011(0.0004)
1	ϑ	1.7228(0.0105)	1.7919(0.0081)	2.0019(0.0004)

	ρ	0.5117(0.1033)	0.4639(0.0581)	0.7059(0.0013)
1.5	$\mho$	2.1082(0.0107)	2.0700(0.0081)	1.9634(0.0025)
	ρ	0.9495(0.01213)	0.9160(0.1155)	0.8939(0.0027)
2	$\mho$	1.6991(0.0007)	1.8058(0.0147)	2.2061(0.0015)
	ρ	0.3815(0.0183)	1.1251(0.0195)	0.7091(0.0032)

Table 8: Bayes estimators and PR (in brackets) under the three loss functions.

	N = 5000	n = 10	n = 50	n = 200
	m	8	40	160
$GQ\|\gamma$ = -1	$\mho$	1.9181(0.0005)	1.9739(0.0001)	2.0018(0.0001)
	ρ	0.9195(0.0002)	0.9870(0.0012)	0.9898(0.0001)
$Entropy\|p$=0.5	$\mho$	1.8998(0.0008)	1.8895(0.0729)	1.9814(0.0001)
	ρ	0.7638(0.0071)	0.9856(0.0065)	1.0024(0.0002)
$Ginex\|r$=1.5	$\mho$	2.1082(0.0107)	2.0700(0.0081)	1.9634(0.0025)
	ρ	0.9495(0.01213)	0.9160(0.1155)	0.8939(0.0027)

Definition 1 *An estimator θ_1 of a parameter θ dominates another estimator θ_2 in the sense of Pitman's closeness criterion if, for all $\theta \in \Theta$,*

$$P_\theta[|\theta_1 - \theta| < |\theta_2 - \theta|] > 0.5.$$

In Table 9, we present the values of the Pitman probabilities which allows us to compare the Bayesian estimators with the MLE estimator which is done under the three loss functions when $\gamma = -, p = 0.5$ and $r = 1.5$. According definition 1, when the probability is greater than 0.5 , the Bayesian estimators are better than the MLE estimators. Then we notice that, according to this criterion, the Bayesian estimators of the two parameters are better than the MLE. Also the GQ loss function has the best values in comparison with the other two loss functions with $\mho = 0.779|_{n=10,m=8}$, $0.779|_{n=50,m=40}$ and $0.674|_{n=200,m=160}$.

Table 9: Pitman comparison of the estimators.

	N = 5000	n = 10	n = 50	n = 200
	m	8	40	160
GQ| γ = -1	$\mho$	0.779	0.779	0.674
	ρ	0.734	0.579	0.634
Entropy$\|_{p=0.5}$	$\mho$	0.589	0.667	0.634
	ρ	0.544	0.523	0.589
Linex$\|_{r=1.5}$	$\mho$	0.699	0.634	0.5789
	ρ	0.612	0.581	0.5523

8 Uncensored distributional validation

In this Section, some uncensored distributional validations are presented under some simulation studies and some real-life data applications. The first subsection provides the uncensored simulation study under the NURR statistics Y^2. The second subsection gives some uncensored applications under the NURR statistics Y^2. The BB algorithm is used for this statistical purpose. The BB package is an optimization and equation-solving tool in R, primarily designed for dealing with complex and high-dimensional nonlinear problems. It offers various optimization algorithms and methods for solving systems of nonlinear equations. The BB provides tools for optimizing nonlinear objective functions. It can be used for tasks such as finding the maximum or minimum of a complex function, which is useful in fields like statistics, engineering, and machine learning. It can be used to solve systems of nonlinear equations, which are common in scientific and engineering applications. This is valuable when you need to find the values of variables that satisfy a set of nonlinear equations simultaneously. It is particularly suited for high-dimensional optimization problems, where the number of variables is large, and traditional optimization methods may be less efficient. The package includes various optimization algorithms, such as the Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm, the conjugate gradient method, and the Nelder-Mead simplex algorithm, among others. Users can choose the most appropriate algorithm for their specific problem. The BB package is designed to have a user-friendly interface, making it accessible to R users with different levels of experience in optimization and equation-solving. It may also support parallel processing to speed up optimization tasks, which can be crucial for handling large-scale problems efficiently. It may include tools for visualizing optimization results and generating reports or summaries to aid in the analysis of the optimization process.

8.1 Uncensored simulation study under the NURR statistics Y^2

In order to support the results obtained in this work, we conducted an in-depth study through numerical simulation. Therefore, in order to test the null hypothesis H_0 that the sample belongs to the IBX-Lx model, we respectively calculated n statistical samples, which are the N statistics of 15000 simulated samples with sizes $n = 25, n = 50, n = 150, n = 400$ and $n = 700$. For different theoretical levels $(\epsilon = 0.01,0.02,0.05,0.1)$, we calculate the average of the non-rejection numbers of the null hypothesis, when $Y^2 \leq \chi^2_\epsilon(b - 1)$. Table 10 illustrates both the empirical and theoretical levels. Notably, there is a striking similarity between the calculated empirical level value and its corresponding theoretical counterpart. As such, we deduce that the suggested test is highly suitable for evaluating the IBX-Lx distribution.

Table 10: Empirical levels and corresponding theoretical levels

$n\downarrow \& \ \varepsilon\rightarrow$	ϵ =0.01	ϵ =0.02	ϵ =0.05	ϵ =0.1
n=25	0.9935	0.9841	0.9524	0.9041
n=50	0.993	0.9833	0.9524	0.9035
n=150	0.9924	0.9821	0.9517	0.9022
n=400	0.9916	0.9813	0.9506	0.9007
n=700	0.9902	0.9802	0.9503	0.9002

8.2 Uncensored applications under the NURR statistics Y^2

8.2.1 Example 1: Uncensored heat exchanger tube crack data

The crack data utilized in this analysis is derived from the comprehensive study conducted by Meeker and Escobar in 1998. This dataset captures the results of meticulous inspections carried out at eight discrete time intervals, each preceding the occurrence of cracks in a total of 167 identical turbine parts.

Time of inspection	186	606	902	1077	1209	1377	1592	1932
Number of fans found to have cracks	5	16	12	18	18	2	6	17

Utilizing the NURR Statistics obtained earlier, we proceed to conduct hypothesis testing to ascertain whether the data conforms to our proposed IBX-Lx distribution. The null hypothesis under consideration posits that the observed data aligns well with the characteristics and parameters of the IBX-Lx distribution. Utilizing R programming and the BB algorithm (see Ravi (2009)), we determine the .MLE $\hat{\mathcal{O}} = 14.5267$ and $\hat{\rho} = 0.9507$. At that point, the estimated Fisher information matrix is:

$$\mathbf{I}(\hat{\boldsymbol{\Xi}}) = \begin{pmatrix} 0.59287 & 1.847622 \\ 1.847622 & 13.09948 \end{pmatrix}.$$

Then, we derive the value of $Y^2 = 20.98651$. For significance level $\epsilon = 0.05$ and the critical value $\chi^2_{0.01}(12) = 21.02607$. The NURR statistic for this model (Y^2) is smaller than the critical value, which allows us to say that these data appropriately correspond to the IBX-Lx model.

8.2.2 Example 2: Uncensored strengths of glass fibers

This dataset comprises 100 measurements of carbon fiber fracture stresses, expressed in gigabars (Gba), as documented by Nichols and Padgett in 2006. Assuming that our IBX-Lx model can fit the strength data of $1.5cm$ glass fiber, we can use the BB algorithm to find the MLE value of the parameter $\boldsymbol{\Xi}$ vector:

$$\hat{\boldsymbol{\Xi}} = \left(\hat{\mathcal{O}}, \hat{\rho}\right)^T = (2.19753,3.55766)^T.$$

Using the $\hat{\boldsymbol{\Xi}}$ value, we can estimate and give the Fisher information matrix cas follow:

$$\mathbf{I}(\widehat{\Xi}) = \begin{pmatrix} 0.500497 & 0.400875 \\ 0.400875 & 3.866549 \end{pmatrix}.$$

After the calculation, we performed the N.R.R statistical test, and the critical values were:

$$Y^2 = 12.08823 \text{ and } \chi^2_{0.05}(7-1) = 12.59159.$$

What we can be sure of is that the 1.5 cm glass fiber data can be modeled satisfactorily with our IBX-Lx distribution.

9 Censored distributional validation

We apply the statistic type test based on a version of the NURR statistic given by Bagdonavičius and Nikulin (2011a,b) and Bagdonavičius et al. (2013) to confirm the sufficiency of the IBX-Lx model when the parameters are unknown and the data are censored. We adapt this test for a IBX-Lx model (the failure rate $\mathcal{Y}_i$ follows an IBX-Lx distribution). Let us consider the composite hypothesis

$$H_0: F(\mathcal{Y}) \in F_0 = \{F_0(\mathcal{Y}, \Xi), \mathcal{Y} \in R^1,\ \Xi \in \Theta \subset R^{\mathcal{S}}\},$$

the survival function and the cumulative hazard function of the IBX-Lx distribution are:

$$S_{I\mathcal{B}X-L\mathcal{Y}}(\mathcal{Y}, \Xi) = 1 - F_{I\mathcal{B}X-L\mathcal{Y}}(\mathcal{Y}; \mathcal{O}, \rho) = (1 - \exp\{-[(1+\mathcal{Y})^{\rho} - 1]^{-2}\})^{\mathcal{O}}.$$

$$\mathbf{\Lambda}_{I\mathcal{B}X-L\mathcal{Y}}(\mathcal{Y}, \Xi) = -\ln S_{I\mathcal{B}X-L\mathcal{Y}}(\mathcal{Y}, \Xi) = -\mathcal{O}\ln(1 - \exp\{-[(1+\mathcal{Y})^{\rho} - 1]^{-2}\}).$$

Under such choice of intervals we have a constant value of $e_{j} = E_k/k$ for any j. There is no explicit form of the inverse hazard function of IBX-Lx distribution, so we can estimate intervals by iterative method. Let us dividing a finite time interval $[0, \tau]$ into $k > \mathcal{S}$ smaller intervals $\mathbf{I}_{j} = (a_{j-1}, a_{j}]$, where τ is the maximum time of the study and $0 =< a_0 < a_1 \ldots < a_{k-1} < a_k = +\infty$. If $\mathbf{\Lambda}^{-1}$ is the inverse of cumulative hazard function $\mathbf{\Lambda}$, $\widehat{\Xi}$ is the maximum likelihood estimator of the parameter Ξ and $\mathcal{Y}_{(i)}$ is the $i^{\mathcal{Y}h}$ element in the ordered statistics $(\mathcal{Y}_{(1)}, , , \mathcal{Y}_{(n)})$, we can give the estimated $\hat{a}_{j}$ as:

$$\hat{a}_{j} = \mathbf{\Lambda}^{-1}\left((E_{j} - \sum_{l=1}^{i-1} \mathbf{\Lambda}(\mathcal{Y}_{(l)}, \widehat{\Xi}))/(n - i + 1), \hat{\theta}\right), \quad \widehat{a_k} = \mathcal{Y}_{(n)}|_{(j=1,\ldots,k)},$$

where

$$E_{j} = (n - i + 1)\mathbf{\Lambda}(\widehat{a_{j}}, \widehat{\Xi}) + \sum_{l=1}^{i-1} \mathbf{\Lambda}(\mathcal{Y}_{(l)}, \widehat{\Xi}) = \sum_{i:\mathcal{Y}_i > a_{j}} (\mathbf{\Lambda}(a_{j} \wedge \mathcal{Y}_i, \widehat{\Xi}) - \mathbf{\Lambda}(a_{j-1}, \widehat{\Xi}),$$

$$E_k = \sum_{i=1}^{n} \mathbf{\Lambda}(\mathcal{Y}_i, \widehat{\Xi}),$$

and a_{j} are random data functions such as the k intervals chosen have equal expected numbers of failures e_{j}. For hypothesis H_0, the test can be based on the statistic

$$Y_n^2 = \mathbf{Z}^T \widehat{\mathbf{\Sigma}}^{-1} \mathbf{Z},$$

where

$$\mathbf{Z} = (Z_1, \ldots, Z_k)^{\mathcal{Y}}, \quad Z_{j} = \frac{1}{\sqrt{n}}(\mathbf{U}_{j} - e_{j})|_{(\ j=1,2,\ldots,k)}$$

and $\mathbf{U}_{j}$ represent the numbers of observed failures in these intervals. The test statistic of Bagdonavičius and Nikulin (2011a,b) and Bagdonavičius et al. (2013) can be written as:

$$Y_n^2 = \sum_{j=1}^{k} \frac{1}{\mathbf{U}_{j}}(\mathbf{U}_{j} - e_{j})^2 + \mathbf{Q}(Y_n^2),$$

where

$$\widehat{\mathbf{\Sigma}}^{-1} = \widehat{\mathcal{A}}^{-1} + \hat{\mathbb{c}}^{-1}\widehat{\mathcal{A}}^{\mathcal{Y}}\widehat{\mathbf{G}}^{-1}\hat{\mathbb{c}}\widehat{\mathcal{A}}^{-1}, \widehat{\mathbf{G}} = [\hat{g}_{ll'}]_{\mathcal{S}\times\mathcal{S}} = \hat{\imath} - \hat{\mathbb{c}}\widehat{\mathcal{A}}^{-1}\hat{\mathbb{c}}^{\mathcal{Y}},$$

$$\hat{\mathbb{c}}_{lj} = \frac{1}{n}\sum_{i:\mathcal{Y}_i \in \mathbf{I}_{j}} \rho_i \frac{\partial}{\partial \Xi} \ln[\lambda_i(\mathcal{Y}_i, \widehat{\Xi})], \mathbf{U}_{j} = \sum_{i:\mathcal{Y}_i \in \mathbf{I}_{j}} \rho_i, \quad \widehat{\mathcal{A}}_{j} = n^{-1}\mathbf{U}_{j},$$

$$\mathbf{Q}(Y_n^2) = \widehat{\mathcal{W}}^{\mathcal{Y}}\widehat{\mathbf{G}}^{-1}\widehat{\mathcal{W}}, \widehat{\mathcal{W}}_l = \sum_{j=1}^{k} \hat{\mathbb{c}}_{lj}\widehat{\mathcal{A}}_{j}^{-1}\mathbf{Z}_{j}, \qquad l, l' = 1, \ldots, \mathcal{S},$$

$$\hat{\imath}_{ll'} = n^{-1}\sum_{i=1}^{n} \rho_i \frac{\partial}{\partial \Xi_l}\ln\left[\lambda_i(\mathcal{Y}_i,\widehat{\Xi})\right]\frac{\partial}{\partial \Xi_{l'}}\ln\left[\lambda_i(\mathcal{Y}_i,\widehat{\Xi})\right]$$

and

$$\hat{g}_{ll'} = \hat{\imath}_{ll'} - \sum_{j=1}^{k} \hat{\mathbb{C}}_{lj}\hat{\mathbb{C}}_{l'j}\widehat{\mathcal{A}}_j^{-1}.$$

We calculate all the elements of the statistic Y_n^2 for the IBX-Lx model. The limit distribution of the statistic Y_n^2 is chi-square and its degree of freedom is df =rank($\mathbf{\Sigma}$) =trace($\mathbf{\Sigma}^{-1}\mathbf{\Sigma}$). If $\mathbf{G}$ is non-degenerate, then $df = k$. If $Y_n^2 > \chi_\epsilon^2(df)$ (where $\chi_\epsilon^2(df)$ is the quantile of chi-square with df degrees of freedom), then the approximate significance level ϵ is rejected Hypothesis. The principal element of the Y_n^2 statistic test of the IBX-Lx model is the matrix $\hat{\mathbb{C}}_{lj}$ given as

$$\hat{\mathbb{C}}_{lj} = \frac{1}{n}\sum_{i:\mathcal{Y}_i\in \mathbf{I}_j} \rho_i \frac{\partial}{\partial \Xi}\ln\left[\lambda(\mathcal{Y}_i,\widehat{\Xi})\right].$$

After calculating derivatives, we can give the elements of the matrix $\hat{\mathbb{C}}_{lj}$ as follows:

$$\ln\lambda(\mathcal{Y},\widehat{\Xi}) = \ln(2) + \ln(\mathcal{O}) + \ln(\rho) - [(1+\mathcal{Y})^\rho - 1]^{-2}$$
$$+(\rho-1)\ln(\mathcal{Y}+1) - 3\ln[(\mathcal{Y}+1)^\rho - 1] - \ln(1-\exp\{-[(1+\mathcal{Y})^\rho - 1]^{-2}\}).$$

So

$$\hat{\mathbb{C}}_{1j} = \frac{1}{n}\sum_{i.t_i\in \mathbf{I}_j} \rho_i[\mathcal{O}^{-1} + \ln(1-\exp\{-[(1+\mathcal{Y}_i)^\rho - 1]^{-2}\})],$$

and

$$\hat{\mathbb{C}}_{2j} = \frac{1}{n}\sum_{i.t_i\in \mathbf{I}_j} \rho_i \begin{bmatrix} \rho^{-1} + \dfrac{2\rho(\mathcal{Y}_i+1)^{\rho-1}}{((\mathcal{Y}_i+1)^\rho - 1)^3} + \ln(\mathcal{Y}_i+1) - 3\dfrac{\rho(\mathcal{Y}+1)^{\rho-1}}{(\mathcal{Y}+1)^\rho - 1} \\ -\dfrac{2\rho\exp\{-[(\mathcal{Y}+1)^\rho - 1]^2\}(\mathcal{Y}+1)^{\rho-1}}{(1-\exp\{-[(\mathcal{Y}+1)^\rho - 1]^2\})[(\mathcal{Y}+1)^\rho - 1]^3} \end{bmatrix}.$$

9.1 Censored simulation study under the NURR statistics Y^2

In order to test the sample belongs to the null hypothesis H_0 of the IBX-Lx model, it is assumed that the generated sample ($N = 15000$) is censored at 25% and $df = 5$ grouping intervals. For different theoretical levels ($\epsilon = 0.01, 0.02, 0.05, 0.1$), when $Y^2 \leq \chi_\epsilon^2(r-1)$, we calculate the average value of the non-rejection numbers of the null hypothesis. Table 11 displays the relevant theoretical and empirical levels. The computed empirical level value is quite similar to the matching theoretical level value, as can be seen in Table 11. As a result, we draw the conclusion that the customised test is ideal for the IBX-Lx model.

Table 11: Empirical levels and corresponding theoretical levels ($\epsilon = 0.01; 0.02; 0.05; 0.1$) and $N = 15000$.

$n\downarrow \& \ \epsilon\rightarrow$	ϵ =0.01	ϵ =0.02	ϵ =0.05	ϵ =0.1
n=25	0.9868	0.9759	0.9544	0.9027
n=50	0.9879	0.9779	0.9523	0.9013
n=150	0.9893	0.9795	0.9514	0.9005
n=400	0.9912	0.9799	0.9505	0.9002
n=700	0.9974	0.9801	0.9502	0.9001

Based on these results, we find that the empirical significance level of the Y_n^2 statistics corresponds to the level of the theoretical level of the chi-square distribution on df degrees of freedom. For that reason, it can be said that the proposed test can rightly fit the censored data from the IBX-Lx distribution.

9.2 Censored applications under the NURR statistics Y^2

9.2.1 Example 1: Censored reliability data set

A set of data for simple reliablility analyses, taken from the book by Meeker and Escobar 1998. capacitor: Data from a factorial experiment on the life of glass capacitors as a function of voltage and operating temperature. There were 8 capacitors at each combination of temperature and voltage. Testing at each combination was terminated after the fourth failure, where $n = 64$ and censored items= 32.This data is available in the Sirvival package of R. Assuming that these data are distributed according to the IBX-Lx distribution, the maximum likelihood estimator $\widehat{\Xi}$ of the parameter vector Ξ is $\widehat{\Xi} = (\hat{\mathcal{O}}, \hat{\rho})^T = (2.3114,1.8376)^T$. We choose $df = 8$ a number of classes. The element of the statistic test Y_n^2 are presented as:

$\widehat{a_j}$	263.59	342.87	443.75	560.39	613.09	949.55	1.91.18	1110.83
$\widehat{\mathbf{U}_j}$	6	5	9	11	6	8	9	10
e_j	7.66493	7.66493	7.66493	7.66493	7.66493	7.66493	7.66493	7.66493

The estimated matrix $\widehat{\mathbb{C}_{lj}}$ and fisher's estimated matrix $\hat{\mathbf{I}}_{(2\times2)}$ are:

$$\widehat{\mathbb{C}_{lj}} = \begin{pmatrix} -0.45875 & 0.64871 & 0.61943 & -0.48912 & 0.39561 & 0.84751 & 0.48751 & -0.37918 \\ 0.84552 & -0.67822 & 0.74820 & 0.33761 & -0.63302 & 0.03784 & 0.30224 & 0.462130 \end{pmatrix}$$

and

$$\hat{\mathbf{I}}_{(2\times2)} = \begin{pmatrix} 1.094678 & 6.781964 \\ 6.781964 & 3.91536 \end{pmatrix}.$$

Then, we evaluate the value of the statistical test $Y_n^2 = 13.94067$. The critical value is $\chi^2_{0.05}(8) = 15.50731 > Y_n^2$. We can come to the conclusion that the life data of glass capacitors are adjusted with the IBX-Lx model.

9.2.2 Example 2: Censored lung cancer data set

The lung cancer data given by Loprinzi et al. (1994) from the North Central cancer treatment group, study the survival in patients ($n = 228$ and censored items= 63) with advanced lung cancer and their Performance scores rate how well the patient can perform usual daily activities. We can estimate the vector parameter $\widehat{\Xi}$ by using the maximum likelihood estimation method as: $\widehat{\Xi} = (\hat{\mathcal{O}}, \hat{\rho})^T = (5.03461,1.75352)^T$, if we suppose that this data are distributed according to IBX-Lx distribution. We use $df = 8$ as a number of classes. The test statistic $Y_n^{2\prime}$elements are presented as following:

$\widehat{a_j}$	60.548	109.403	168.094	201.536	267.088	374.179	651.007	1023.4391
$\widehat{\mathbf{U}_j}$	17	19	22	28	31	48	43	20
e_j	8.44905	8.44905	8.44905	8.44905	8.44905	8.44905	8.44905	8.44905

The estimated matrix $\widehat{\mathbb{C}_{lj}}$ and fisher's estimated matrix $\hat{\mathbf{I}}_{(2\times2)}$ are

$$\widehat{\mathbb{C}_{lj}} = \begin{pmatrix} -0.34121 & 0.448659 & 0.73914 & -0.80467 & -0.33064 & 0.27213 & 0.61678 & -0.36661 \\ 0.34511 & -0.68497 & -0.74588 & -0.96558 & 0.55428 & 0.73648 & 0.11765 & 0.462948 \end{pmatrix}$$

and

$$\hat{\mathbf{I}}_{(2\times2)} = \begin{pmatrix} 1.094678 & 6.781964 \\ 6.781964 & 3.91536 \end{pmatrix}.$$

The critical value of the chi-squared test is $\chi^2_{0.05}(df = 8) = 15.50731$. Using the previous results, we find that the calculated statistic of the proposed test is $Y_n^2 = 15.00845$. Since the tabulated value of the Y_n^2 statistic is greater than the calculated value, then we can say that our hypothesis H_0 is accepted. Which leads us to conclude that the Lung cancer data can follow the IBX-Lx distribution with a 5% risk of error.

10 Conclusions and some future points

In this paper, we introduce an innovative extension of the Lomax distribution known as the inverted Burr X Lomax (IBX-Lx) model. Our focus is on providing a comprehensive understanding of this new distribution, emphasizing its practical applications while deriving key mathematical properties without unnecessary complexity. We explore three distinct approaches to characterize the IBX-Lx distribution: firstly, by utilizing two truncated moments; secondly, by examining the hazard function; and thirdly, by considering the conditional expectation of a function of the random variable. Through these characterizations, we aim to offer multiple perspectives on the distribution's behavior and structure. To estimate the parameters of the IBX-Lx distribution, we employ a variety of classical methods, including the maximum likelihood method, the Cramér-von Mises method, the Anderson-Darling method, and the right-tail Anderson-Darling method. Additionally, we delve into Bayesian estimation, comparing its performance with likelihood estimation using Pitman's proximity criterion. Within the Bayesian framework, we explore various loss functions, including the generalized quadratic, the Linex, and the entropy functions, providing detailed insights into their application and interpretation. We conduct thorough simulation experiments to assess the performance of all estimation methods under different scenarios, carefully documenting our findings within the paper. Furthermore, we present a detailed derivation and evaluation of the censored case maximum likelihood method, providing valuable insights into its utility in scenarios involving incomplete data. We compare the Bayesian technique with the censored maximum likelihood method using the BB algorithm, particularly focusing on process estimation under censored samples.

The paper also includes a comprehensive discussion on the construction of the NURR statistic for the IBX-Lx model under both uncensored and censored cases. We conduct simulation studies to assess the performance of the NURR statistic under each scenario and present two real data applications for both uncensored and censored cases. These applications include analyses of strengths of glass fibers, heat exchanger tube crack data, capacitor reliability data, and lung cancer medical data, showcasing the practical relevance and versatility of the IBX-Lx distribution in diverse domains.

Below, we provide some future points:

I. Explore and develop advanced Bayesian estimation methods for the NURR test that can handle complex censored and uncensored data scenarios more efficiently. Investigate computational improvements and convergence diagnostics for Bayesian parameter estimation.

II. Investigate the robustness of the proposed distribution under various model assumptions and data conditions. Analyze how deviations from the assumed model affect the reliability of statistical inferences and model validation.

III. Consider extensions or modifications to the NURR test to accommodate additional features or characteristics commonly encountered in practical applications. This could involve incorporating covariates, time-varying parameters, or mixture components.

IV. Develop methodologies for model selection and comparison between the NURR test and other competing models. Explore techniques like information criteria, cross-validation, or Bayesian model selection to assess the adequacy of the proposed model.

V. Apply the NURR test to diverse real-world applications beyond the scope of hydroelectric dams. Investigate its performance and usefulness in fields such as finance, environmental science, healthcare, and more.

VI. Extend the methodology to multivariate settings where data involve multiple correlated variables. Develop multivariate versions of the NURR test and investigate their applications in complex systems.

VII. Create user-friendly software packages or tools for implementing the proposed methodology, making it accessible to a wider audience of researchers and practitioners. Consider integration with popular statistical software environments.

VIII. Explore methods to enhance the interpretability of the model's parameters and their implications in real-world contexts. Visualizations and sensitivity analyses could aid in understanding the practical significance of the model's findings.

IX. Investigate data-driven approaches for model validation and parameter estimation, where the structure of the distribution is learned directly from the data, especially in cases where the underlying distribution may not be well-known a priori.Develop methodologies for prediction and forecasting using the NURR test, particularly in scenarios where future extreme events are of interest. Assess the model's predictive performance through simulation studies and empirical applications.

These future research points can help expand the understanding and applicability of the NURR test in statistical modeling and inference, making it a valuable tool in various fields of research and industry.

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