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Amputated Life Testing for Weibull Reciprocal Weibull Percentiles: Single, Double and Multiple Group Sampling Inspection Plans with Applications

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Abstract

When a life test is terminated at a predetermined time to decide whether to accept or refuse the submitted batches, the types of group sampling inspection plans (single, two, and multiple stages) are introduced. The tables in this study give the optimal number of groups for various confidence levels, examination limits, and values of the ratio of the determined experiment time to the fixed percentile life. At various quality levels, the operating characteristic functions and accompanying producer's risk are derived for various types of group sampling inspection plans. At the determined producer's risk, the optimal ratios of real percentile life to a fixed percentile life are obtained. Three case studies are provided to illustrate the processes described here. Comparisons of single-stage and iterative group sampling plans are introduced. The first, second, and third sample minimums must be used to guarantee that the product's stipulated mean and median lifetimes are reached at a certain degree of customer trust. The suggested sample plans' operational characteristic values and the producer's risk are given. In order to show how the suggested approaches based on the mean life span and median life span of the product may function in reality, certain real-life examples are examined.

Key Words: Amputated life test; Average group number; Consumer's risk; Operating characteristic; Single-stage group sampling inspection plans; Two-stage group sampling inspection plans; Multiple group sampling inspection plans with three-stage; Producer's risk; Weibull Reciprocal Weibull distribution.

1. Introduction

The organizational structures used in the industrial processes are quite complicated. As a result, the unpredictable events that impact the system usually result in some failures and disruptions. Generally, decreasing production system failures is more important for all companies. At this time, process control is prioritized over quality control as a means of increasing productivity and performance. Another method used to guarantee a given standard of quality in a good or service is quality control. It could include whatever steps a company believes are necessary to ensure the control and verification of a product's or service's features. The main objective of quality control is to ascertain if the monitored process or product will exhibit these qualities or not, to make sure that the offered goods, services, or processes are reliable and satisfactory and meet strict standards. We cannot test every test unit if there are many of them. To address this issue, we use destructive testing and statistical quality control techniques. The quality assurance managers employ the acceptance sampling plan, a crucial inspection tool, to decide whether to accept or reject a product based on pre-established quality requirements. Acceptance sampling plans are frequently employed when inspecting many goods, which is expensive. A simple random sample is taken from a particular lot as part of the inspection process for a single acceptance sampling plan, and the manager should decide whether to reject the lot if some of the items don't satisfy the pre-assigned quality requirements.

Plans for acceptance sampling allow you to examine a representative sample from a batch of goods to determine whether or not to accept the batch, depending on the intended level of quality. The decision can be made to accept, reject, or, in the case of multiple and sequential sampling plans, draw another sample before repeating the decision operation, depending on the quantity of failure items in a specimen. It should be noted that if the observed number of failures does not reach the examination limit prior to the test's termination time, the batch is deemed acceptable. To determine the sample size from a batch being evaluated, inspection sampling plans for amputated life tests are typically used. Normally, just one item is inspected when employing an acceptance sampling strategy, but numerous things might be checked at once to save money and time. Group acceptance sampling is the name given to this procedure. The group sampling inspection plan (GSIP) is what we'll name the acceptance sampling plan based on these groupings of objects. Calculating the sample size is the same as figuring out how many groups there are in this kind of test. Testing for unexpected death typically employs this kind of tester. Many manufacturers use sudden death testing to shorten testing times. This approach was put forward by a number of scholars, including Balasooriya (1995), Pascual and Meeker (1998), and Vlcek et al. (2004). The sudden death test was most recently developed by Jun et al. (2006) on the presumption that objects' lifespans follow the Weibull distribution with specified shape parameters. In the sudden death tests, they created the single and double sampling variable schemes.

A GSIP is sometimes preferable to a single sampling strategy because it may examine a greater number of items in a certain test time. In the group inspection plan, the number of items to be examined is specified, and the number of groups is determined. The parameters here are the determining group size and the examination limit. According to (Aslam et al. (2009)), the selected specimen size n is divided into ℓ (the number of groups) groups and k (group size) items are put on test in each group so that $n = k.\ell$. The ℓ items in a group are tested simultaneously on each different tester for a pre-assigned time. The experiment is amputated if more than the examination limit a_c of failures occurs in any group during the experiment time. This method is used by many researchers (Aslam and Jun (2009 a, b)), (Rao (2009, 2010 and 2011)), (Aslam et. al. (2011)), (Aslam et. al. (2013)), (Mughal and Ismail (2013)), (Rao et al. (2014)), (Mughal, et al. (2015)), (Rosaiah et al. (2016)), (Aziz, et al. (2020)), (Yiğiter et al. (2021)), (Almarashi (2021)) Saber et al. (2022), Ali et al. (2020) and (Ahmed and Yousof (2022)). However, very useful applications in the quality of the insurance outcomes and actuarial works are available in Mohamed et al. (2022,a,b,c). Numerous studies analyzed and examined the problem of creating statistical sampling plans with specifics and using them to monitor and control the quality of industrial products in the context of the producer-consumer relationship by looking at specific production quantities based on pre-defined criteria in order to determine whether the batch was able to achieve the acceptable quality level or not.

All acceptance sampling begins with the single-sample plan. Single sampling by characteristics, the most basic variation of such a strategy, is applicable to dichotomous circumstances, i.e., those in which inspection findings may be divided into just two classes of outcomes. In addition to other categories, such as measures in or out requirements, this also contains go, no-go gauging approaches. The characteristics single-sampling plan, which is applicable to all sampling circumstances, has evolved into the standard by which other sampling plans are measured. It is used in inspection to determine the fraction of processes, the big lots, the individual lots, or the number of defects detected in the sample (Poisson distribution) that are faulty (hypergeometric distribution). Of all the sampling techniques, single sampling is unquestionably the most popular (see Schilling and Neubauer (2009) for more details and more examples).

Plans for triple and double sampling reflect the propensity of many seasoned inspectors to give a suspect lot a second opportunity. Consequently, in the case of twofold sampling, if the findings of the first sample do not clearly indicate whether the lot should be accepted or rejected, a second sample is collected, and the disposition of the lot is decided after that. This method makes sense due to the procedure's mathematical features as well as the results of experience. For starters, the average sample number (ASN) for a double-sampling plan may often be set to be lower than for a single-sampling plan with the same protection. Allowing for the collection of more samples in order to gain even greater distinction in a lot's disposition is a logical extension of double sampling. When the last sample is built to compel a choice at that time, as with double sampling, these techniques are known as multiple sampling plans.

This paper's goal is to provide a novel method for comparing group sample inspection plans (single, two, and multiple stages) based on amputation life tests where a product's lifetime follows the Weibull reciprocal Weibull (WRW) distribution with a specified shape parameter. The significance of the study stems from how crucial it is to have high-quality goods available at competitive costs that can satisfy customers' needs and compete in both domestic and international markets. The research is significant because it directly applies the idea of statistical sampling to the goods

produced by businesses and factories, whether they are finished goods, raw materials, or semi-manufactured materials. In fact, there are a lot of elastic probability distributions that deserve further study and analysis using the examination of acceptance plans and related theories. Specifically, the Reciprocal Weibull distribution has many useful extensions. See, for example: Korkmaz et al. (2017), Haq et al. (2017), Al-Babtain et al. (2020), Elsayed and Yousof, H. M. (2020), Jahanshahi et al. (2020), Salah et al. (2020), Yousof et al. (2016, 2018a,b, 2019, 2020), Hamedani et al. (2022), Ibrahim et al. (2022) and Salem et al. (2022).

The remainder of the work is structured as follows: The amputated life test for the WRW distribution is presented in Section 2. The single-stage group inspection plans are designed in Section 3. The design of the two-stage group sample inspection strategy is introduced in Section 4. The design of the three-stage multiple group sample inspection strategy is suggested in Section 5. With the use of industry illustrative scenarios, Section 6 explains how the proposed forms of group sample inspection plans might be applied. Comparisons of the one-stage and iterative group sampling plans are shown in Section 7.

2. The amputated life test for the WRW distribution

This kind of abbreviated life test has been proposed by several scholars. In all earlier studies, the authors often build acceptance sampling plans (ASP) with the presumption that the product's lifetime follows a certain lifetime distribution, such as exponential distribution, gamma distribution, Weibull distribution, and so forth. Take into consideration the lifetime of an item, symbolized by μ_ρ, as its quality property. If the lifetime of an item is more than the determined value of as μ_ρ^0, then the product complies with the standard. Alternatively, if $\mu_\rho \geq \mu_\rho^0$, then the item is conforming and if $\mu_\rho < \mu_\rho^0$, then the item is not conforming. Among the constraints in most life testing experiments, are limits on time spent on life testing. If the product life is anticipated to be long, waiting until all items have failed is extremely time-consuming. Consequently, it is better to stop the life test at a pre-set time and record the number of nonconforming items (Aslam et al. (2016)). This kind of test for assessing a product's lifetime is known as an amputated life test. The following facets are researched in amputated life tests:

i. The suitable distribution for the item's lifetime,
ii. The sampling plan employed for this kind of test,
iii. Analyzing the mean/median or quantile lifetime of the item.

Many researchers use an acceptance sampling plan based on an amputated life test with a single-item group using various statistical lifetime distributions, including Goode and Kao (1961), Gupta and Groll (1961), Fertig and Mann (1980), Kantam and Rosaiah (1998), Tsai and Wu (2006), Balakrishnan et al. (2007), Lio et al. (2010), Gui and Aslam (2017), and Ahmed et al. (2022). Afify et al. (2016) introduced the WRW distribution using the Weibull-G family pioneered by Bourguignon et al. (2014). The WRW distribution can be used to efficiently examine lifetime data, engineering, medicine, and other fields of study. The cumulative distribution function (CDF) and the probability density function (PDF) of the WRW distribution are given as

$$F_{\beta,\gamma,\sigma,\varphi}(z) = 1 - \exp\left[-\beta\left\{\exp\left[\left(\frac{\varphi}{z}\right)^{\gamma}\right] - 1\right\}^{-\sigma}\right] \mid z > 0, \tag{1}$$

and

$$f_{\beta,\gamma,\sigma,\varphi}(z) = \beta\sigma\gamma\varphi^{\gamma} z^{-(\gamma+1)} \exp\left[-\sigma\left(\frac{\varphi}{z}\right)^{\gamma}\right] \frac{\exp\left\{-\beta\left[exp\left(\frac{\varphi}{z}\right)^{\gamma} - 1\right]^{-\sigma}\right\}}{\left\{1 - exp\left[-\left(\frac{\varphi}{z}\right)^{\gamma}\right]\right\}^{\sigma+1}} \mid z > 0, \tag{2}$$

respectively, where $\varphi > 0$ is the scale parameter and β, γ and $\sigma > 0$ are shape parameters. In general, $Z \sim WF(\beta, \gamma, \sigma, \varphi)$ denotes a random variable with a CDF of (1). Hence, the WRW distributed item has the following percentile time:

$$\mu_\rho = \varphi\left(\log\left\{1 + \left[\left(-\frac{1}{\beta}\right)\log(1-\rho)\right]^{-\frac{1}{\sigma}}\right\}\right)^{-\frac{1}{\gamma}} | 0 < \rho < 1. \tag{3}$$

The median of μ_ρ is obtained by substituting $\rho = 0.5$ in Eq. (3). For further information about the WRW distribution, see Afify et al. (2016). Henceforward, unless otherwise stated, we consider μ_ρ as the quality parameter. From Eq. (3) it is evident that for fixed $\beta = \beta_0$, $\sigma = \sigma_0$, $\gamma = \gamma_0$, $\mu_\rho \geq \mu_\rho^0 \Leftrightarrow \varphi \geq \varphi_0$, where

$$\varphi_0 = \mu_\rho^0 \left(\log\left\{ 1 + \left[\left(-\frac{1}{\beta} \right) \log(1-\rho) \right]^{-\frac{1}{\sigma}} \right\} \right)^{-\frac{1}{\gamma}}. \tag{4}$$

Note that φ_0 depends on β_0, σ_0 and γ_0. In order to develop GSIPs for the WRW distribution, it is determined that μ_ρ surpasses μ_ρ^0 equivalently φ exceeds φ_0. The shape parameters are assumed to be known in acceptance sampling plans. If the shape parameters are not known, the suggested sampling plan is used for an estimated value of the shape parameters depending on the previous failure data. Manufacturers usually retain the estimated parameters of the life distribution for their product (see Aslam et al. (2009)). The GSIPs for the WRW distribution is developed to make sure that the ρ^{th} percentile lifetime of the product under examination exceeds a pre-specified quality given by the consumer say μ_ρ, equivalent φ exceeds φ_0, with probability ε^*. It is better to design an acceptance sampling plan to make sure that the lifetime ρ^{th} percentile of the product is at least μ_ρ^0. The batch is accepted if there is enough evidence that $\mu_\rho \geq \mu_\rho^0$ at a certain level of confidence for the consumers. To make things easier, it is presumed that the stop time of the test, z_0, for each item is a coefficient of the specified value of the lifetime ρ^{th} percentile, μ_ρ^0, when $z_0 = \mathcal{u}_\rho \mu_\rho^0$, where $\mathcal{u}_\rho$ is a factor of the coefficient. For instance, to test if the lifetime ρ^{th} percentile of the product exceeds 5,000 h and set $\mathcal{u}_\rho = 0.8$, the test for each item is terminated after 4,000 h from the start of the test. For the specified value of test termination time, z_0, the probability of failure for each item, π, is equal to the CDF of the WRW distribution at z_0, as

$$\pi = \Pr(\mathcal{Z} < z_0) = 1 - \exp\left(-\beta \left\{ \exp\left[\mathcal{Q} \left(\frac{\ell}{\mathcal{u}_\rho} \right)^\gamma \right] - 1 \right\}^{-\sigma} \right), \tag{5}$$

where

$$\mathcal{Q} = \left(\log\left\{ 1 + \left[\left(-\frac{1}{\beta} \right) \log(1-\rho) \right]^{-\frac{1}{\sigma}} \right\} \right)^{-\frac{1}{\gamma}},$$

and $\ell = \mu_\rho / \mu_\rho^0$, when the shape parameters, coefficient factor $\mathcal{u}_\rho$, and ratio ℓ are determined, the proportion π can be evaluated.

3. Design of the single-stage group inspection plan

In this part, the design single-group inspection plan is introduced to make sure that the $\mathcal{Z}_\rho$ percentiles life of an item in a batch is more than the specified life, i.e., $\mu_\rho \geq \mu_\rho^0$ under the assumption that the lifetime of an item follows the WRW distribution with known shape parameters. A batch of an item is recognized to be conforming if the real percentiles life μ_ρ is more than the specified life μ_ρ^0. A batch will be accepted if $\mu_\rho \geq \mu_\rho^0$ at a certain level of consumer's risk, otherwise, it will be rejected. A group inspection plans based on amputated life test comprises four parameters:

- ***i.*** The Number of groups ℓ,
- ***ii.*** The group size $\mathcal{k}$. When $\mathcal{k}$ = 1 the single-stage GSIPs (S-SGSIP) returns to the ordinary sampling plan.
- ***iii.*** Examination limit a_c. For instance, if during the pre-determined experiment time $\mathcal{Z}_0$, a_c or fewer failure happened, the lot is accepted.
- ***iv.*** The ratio $\mathcal{u}_\rho = \mathcal{Z}_0 / \mu_\rho^0$ where $\mathcal{Z}_0$ is the maximum testing time and μ_ρ^0 is the specified median or percentile life.

As a result, the S-GSIP can be defined by the parameters $(\ell, \mathcal{k}, a_c, \mathcal{u}_\rho)$. All groups are tested simultaneously, and the failure number in each group is recorded. The batch is accepted if the failure number in each group during the time $\mathcal{Z}_0$ is equal to a_c at most a_c, and if it is more than a_c in any group, then the batch is rejected

We are concerned with deciding the ℓ needed for the WRW distribution, whereas a_c, $\mathcal{k}$, and $\mathcal{Z}_0$ are assumed to be determined.

Both the consumer and the maker are exposed to risk as their desires are in conflict. The consumer desires the acceptance of low-quality batches with a small probability of ξ and the maker desires the acceptance of high-quality batches with a high probability $(1-\delta)$. The refusal probability of a high-quality batch is called the producer's risk δ, and the probability of acceptance to a low-quality batch is called the consumer's risk ξ. The consumer's risk is determined by the consumer's confidence level. If the consumer's confidence level is ε^*, then the consumer's risk will be $\xi = 1-\varepsilon^*$. The consumer's risk is fixed, i.e., the one for which the real percentile life μ_ρ is less than the specified μ_ρ^0, not exceed $1-\varepsilon^*$. The probability distribution of the failure number might be assumed to be the hypergeometric distribution. However, the binomial or Poisson distributions can also be used if a batch size N is big enough to be considered infinite for example $\beta/N \leq 0.10$ and if a decision about the batch is either accepted or rejected (see Stephens (2001)). In such cases, the binomial distribution can be determined and design the parameters of inspection plans in this study. Consequently, the acceptance and rejection standards for the batch are tantamount to judgments of accepting or rejecting the hypothesis $\mu_\rho \geq \mu_\rho^0$. Now, we need to determine the optimal number of groups ℓ such that

$$\left[\sum_{v=0}^{a_c}\binom{\hbar}{v}\pi^v(1-\pi)^{\hbar-v}\right]^{\ell} \leq 1-\varepsilon^*, \tag{6}$$

where π is the nonconforming proportion in the group prior to termination time $\mathcal{Z}_0$ if the real percentile life equals the fixed percentile life (or $\mu_\rho = \mu_\rho^0$) and is given by:

$$\pi = 1-\exp\left[-\beta\left\{\exp\left(\mathcal{Q}\left(\mathcal{u}_\rho\right)^{-\gamma}\right)-1\right\}^{-\sigma}\right]. \tag{7}$$

It is evident that π relies on the ratio $\mathcal{u}_\rho = \frac{\mathcal{Z}_0}{\mu_\rho^0}$ and is monotonically increasing function of the ratio. Therefore, just this ratio needs to be specified in the experiment. If the number of recorded defectives is at most a_c, from Eq. (6) we can establish with probability ε^* that $F_{\beta,\gamma,\sigma,\varphi}(z) \leq F_{\beta,\gamma,\sigma,\varphi_0}(z)$, which implies $\varphi \geq \varphi_0$ (or $\mu_\rho \geq \mu_\rho^0$). Thus, the percentile lifetime of the items can be assured to be at least equal to their specified value with probability ε^*. The optimal values of ℓ satisfying inequality (6) for the S-SGSIP of the WRW distribution with β = 0.8, σ = 0.2 and γ = 4 according to a_c= 0 and 1 ; ε^*= 0.75, 0.90, 0.95 0.99; $\mathcal{u}_\rho$ = 0.6, 1.0, 2.15, 3.8, 5.5 at ρ = 0.25 when $\hbar$ = 2 and 4 are presented in Table 1. From Table 1, we see that the ℓ decreases as the $\hbar$ and $\mathcal{u}_\rho$ increase. It can be observed that the ℓ and average group number (AGN) increase as the a_c increases from zero to one. Furthermore, as the ε^* increases, so does the AGN. In spite of the S-SGSIP with a_c = 0 leads to a smaller AGN, it also exhibits undesirable OCs, as we will see later. As a result, we consider the S-SGSIP with a_c = 1 as well.

Table 1: The number of groups in the S-SGSIP that must be examined for a time $\mathcal{Z}_0$ in order to affirm with probability ε^* that $\mu_\rho \geq \mu_\rho^0$ is given by the entries in this table.

ε^*	$\mathcal{u}_{0.25}$	$\hbar = 2$				$\hbar = 4$			
		Single with a_c= 0		Single with a_c= 1		Single with a_c= 0		Single with a_c= 1	
		ℓ	AGN	ℓ	AGN	ℓ	AGN	ℓ	AGN
0.75	0.6	9	18	120	240	5	20	23	92
	1.0	5	10	28	56	3	12	6	24
	2.15	3	6	9	18	2	8	2	8
	3.8	2	4	6	12	1	4	1	4
	5.5	2	4	5	10	1	4	1	4
0.90	0.6	17	34	199	398	7	28	38	152
	1.0	7	14	47	94	4	16	10	40
	2.15	3	6	15	30	2	8	4	16
	3.8	2	4	10	20	2	8	2	8
	5.5	2	4	8	16	1	4	2	8
0.95	0.6	20	40	259	518	9	36	49	196
	1.0	10	20	61	122	4	16	13	52
	2.15	4	8	20	40	2	8	5	20
	3.8	3	6	13	26	2	8	3	12
	5.5	3	6	10	20	2	8	3	12
	0.6	27	54	399	798	14	56	75	300

0.99	1.0	12	24	93	186	6	24	19	76
	2.15	6	12	30	60	4	16	7	28
	3.8	4	8	19	38	3	12	5	20
	5.5	4	8	16	32	2	8	4	8

The performance of the inspection plan is interpreted by obtaining the operation characteristic (OC) function. It gauges the effectiveness of a statistical hypothesis test used to accept or refuse a batch. The OC function of the S-SGSIP $(\ell, \kappa, a_c, u_\rho)$ gives the probability of accepting the batch and is represented by

$$\mathrm{P}_1 = \mathrm{P}_{\ell,\pi} = \left[\sum_{v=0}^{a_c} \binom{\kappa}{v} \pi^v (1-\pi)^{\kappa - v}\right]^{\ell}, \tag{8}$$

where v is the number of nonconforming items in each group and $\pi = F_{\beta,\gamma,\sigma,\mu_\rho}(z)$ is decreasing function of $\mu_\rho \geq \mu_\rho^0$. The OC function P_1 is decreasing function of π for fixed time z_0. The OC function of an inspection plan is computed using the formula (8). For given κ, ε^* and u_ρ the selection of a_c and ℓ will be made on the fundament of the OC function. Values of OC functions based on formula (8) for the S-SGSIP of the WRW distribution with β = 0.8, σ = 0.2 and γ = 4 for various values of percentile ratios $\mu_{0.25}/\mu_{0.25}^0$, ε^*, $u_{0.25}$ according to a_c= 0 and 1 are displayed in Table 2.

Table 2: OC values of inspection plan $(\ell, \kappa, a_c, u_{0.25})$ for a given ε^*, under the WRW distribution

		a_c = 0 , κ = 2					a_c = 0 , κ = 4				
		$\mu_{0.25}/\mu_{0.25}^0$									
ε^*	$u_{0.25}$	2	4	6	8	10	2	4	6	8	10
0.75	0.6	0.749	0.994	1	1	1	0.725	0.994	1	1	1
	1.0	0.464	0.929	0.993	0.999	1	0.437	0.924	0.992	0.999	1
	2.15	0.222	0.486	0.784	0.922	0.973	0.199	0.582	0.834	0.941	0.98
	3.8	0.175	0.394	0.605	0.763	0.864	0.117	0.318	0.539	0.716	0.835
	5.5	0.117	0.255	0.41	0.559	0.684	0.105	0.238	0.392	0.543	0.671
0.90	0.6	0.579	0.989	1	1	1	0.640	0.991	1	1	1
	1.0	0.342	0.902	0.99	0.999	1	0.336	0.901	0.99	0.999	1
	2.15	0.199	0.582	0.834	0.941	0.97	0.106	0.47	0.776	0.918	0.972
	3.8	0.147	0.359	0.575	0.742	0.851	0.069	0.24	0.463	0.66	0.799
	5.5	0.105	0.238	0.392	0.543	0.671	0.048	0.143	0.282	0.438	0.583
0.95	0.6	0.526	0.987	1	1	1	0.559	0.988	1	1	1
	1.0	0.216	0.863	0.986	0.999	1	0.277	0.884	0.988	0.999	1
	2.15	0.116	0.486	0.784	0.922	0.973	0.076	0.422	0.748	0.907	0.968
	3.8	0.073	0.247	0.471	0.666	0.803	0.059	0.221	0.443	0.645	0.789
	5.5	0.062	0.169	0.314	0.47	0.61	0.039	0.125	0.258	0.413	0.562
0.99	0.6	0.424	0.983	1	1	1	0.413	0.982	1	1	1
	1.0	0.152	0.835	0.983	0.998	1	0.181	0.849	0.984	0.998	1
	2.15	0.044	0.351	0.703	0.888	0.961	0.023	0.281	0.653	0.867	0.953
	3.8	0.026	0.141	0.348	0.566	0.736	0.012	0.096	0.282	0.505	0.692
	5.5	0.019	0.08	0.192	0.341	0.495	0.015	0.069	0.176	0.321	0.477
		a_c = 1 , κ = 2					a_c = 1 , κ = 4				
0.75	0.6	0.97	1	1	1	1	0.967	1	1	1	1
	1.0	0.858	0.998	1	1	1	0.839	0.998	1	1	1
	2.15	0.592	0.934	0.992	0.999	1	0.563	0.918	0.989	0.999	1
	3.8	0.46	0.773	0.921	0.975	0.993	0.438	0.739	0.901	0.967	0.99
	5.5	0.411	0.662	0.826	0.917	0.962	0.394	0.627	0.793	0.895	0.949
0.90	0.6	0.951	1	1	1	1	0.945	1	1	1	1
	1.0	0.775	0.998	1	1	1	0.748	0.997	1	1	1
	2.15	0.419	0.893	0.987	0.998	1	0.385	0.867	0.982	0.998	1
	3.8	0.276	0.652	0.873	0.959	0.988	0.254	0.605	0.841	0.946	0.983
	5.5	0.229	0.504	0.728	0.866	0.938	0.213	0.46	0.68	0.831	0.917

0.95	0.6	0.936	1	1	1	1	0.93	1	1	1	1
	1.0	0.718	0.997	1	1	1	0.685	0.996	1	1	1
	2.15	0.323	0.862	0.982	0.998	1	0.289	0.831	0.977	0.997	1
	3.8	0.187	0.574	0.838	0.947	0.984	0.168	0.52	0.799	0.93	0.978
	5.5	0.147	0.41	0.661	0.829	0.92	0.133	0.364	0.606	0.786	0.894
0.99	0.6	0.904	1	1	1	1	0.894	1	1	1	1
	1.0	0.601	0.995	1	1	1	0.559	0.994	1	1	1
	2.15	0.176	0.797	0.973	0.997	1	0.148	0.752	0.964	0.996	1
	3.8	0.076	0.426	0.762	0.92	0.975	0.065	0.366	0.708	0.894	0.966
	5.5	0.052	0.254	0.529	0.749	0.879	0.045	0.212	0.463	0.691	0.842

From Table 2, it is noted that, the OC values decrease as the u_ρ increase, but the OC values increase to one at a higher value of the quality ratio. The OC values decrease as the group size (k) increase from 2 to 4 at the value of a_c. Also, the OC values increase as the a_c increase from zero to one at the same value of group size (k). The manufacturer may be concerned with improving the product's quality so that the acceptance probability exceeds a certain level. For a given value of the producer's risk, say 5%, the value of μ_ρ/μ_ρ^0 that ensures the producer's risk is 5% or less can be obtained by satisfying the following inequality:

$$\left[\sum_{v=0}^{a_c}\binom{k}{v}\pi^v(1-\pi)^{k-v}\right]^{\ell} \le 0.05, \tag{9}$$

where π is a function as shown in (5). For a given the parameter of S-SGSIP (ℓ, k, a_c, u_ρ), at a specified probability ε^*, the values of μ_ρ/μ_ρ^0 satisfying (9) are presented in Table 3.

Table 3. For a given S-SGSIP $(\ell, k, a_c, u_{0.25})$, the entries in this table give the optimal ratio of μ_ρ/μ_ρ^0 in order that the lot be accepted with producer's risk of 5%.

	$k=2$									
	$u_{0.25}$									
	0.6	1.0	2.15	3.8	5.5	0.6	1.0	2.15	3.8	5.5
ε^*	Single with a_c= 0					Single with a_c= 1				
0.75	2.885	4.307	8.851	13.396	19.390	1.866	2.480	4.275	6.795	9.310
0.90	3.108	4.594	8.323	13.705	19.618	1.997	2.701	4.756	7.651	10.555
0.95	3.243	4.747	8.851	14.711	20.62	2.064	2.815	5.003	8.091	11.195
0.99	3.441	4.992	9.532	15.801	22.275	2.175	3.000	5.407	10.09	12.235
$k=4$										
0.75	2.939	4.373	8.923	14.067	19.618	1.891	2.555	4.509	7.264	10.028
0.90	3.210	4.606	8.93	14.779	21.035	2.023	2.78	5.004	8.150	11.318
0.95	3.294	4.898	9.177	14.96	21.354	2.092	2.896	5.258	8.603	11.979
0.99	3.457	5.075	9.883	16.392	22.527	2.203	3.084	5.669	12.094	13.047

Table 3 shows that as the termination time ratio and confidence level increase, so does the required optimal ratio. Furthermore, the optimal ratio decreases as a_c increase from 0 to 1 at the same group size (k). At the same value of a_c, the optimal ratio increases as the group size (k) increase from 2 to 4.

4. Design of the two-stage GSIP

Assuming that v_1 and v_2 denote the number of non-conforming items in the first and second sample. The rules for two-stage GSIP (T-SGSIP) are: Take a first sample of size is β_1 from a batch, assign k item to each of ℓ_1 groups; so that $n_1 = k\ell_1$, accept the batch if $v_1 \le a_{c_1}$, reject the batch if $v_1 \ge r_{e_1}$ and take a second sample of size n_2, assign k item to each of ℓ_2 groups; so that $n_2 = k\ell_2$ if $a_{c_1} < v_1 < r_{e_1}$; accept the batch if $v_1 + v_2 \le a_{c_2}$ and reject otherwise. Despite the T-SGSIP is important, yet the number of studies in this area is fewer than the number of studies in the S-SGSIP such as Aslam, et al. (2010)), (Aslam, et al. (2011)), (Mughal, et al. (2011)), (Rao (2013)), (Rao, et al. (2014)), (Aslam, et al. (2021)), (Azam, et al. (2015)), (Rao and Rao (2016)), (Prasad, et al. (2018)), (Rao, et al. (2019)) and (Kanaparthi (2020). When $a_{c_1} = a_{c_2}$, the T-SGSIP turns into S-SGSIP. The two-stage group sampling plan (T-SGSIP) tends to be an ordinary double sampling plan when $k = 1$. The number of groups in each of the T-

SGSIP will be the key design parameter of the suggested plan. It is to be noticed that the group size κ will be given by the type of testers to be used. The proposed plan involves the parameters of ℓ_1, ℓ_2, a_{c_1} and a_{c_2} where $a_{c_1} < a_{c_2}$, then, the OC function for T-SGSIP is determined by

$$P_2 = \left[\sum_{v_1=0}^{a_{c_1}} \binom{\kappa}{v_1} \pi^{v_1}(1-\pi)^{\kappa-v_1}\right]^{\ell_1} + \left[\sum_{v_1=a_{c_1}+1}^{a_{c_2}} \binom{\kappa}{v_1} \pi^{v_1}(1-\pi)^{\kappa-v_1}\right]^{\ell_1} \left[\sum_{v_2=0}^{a_{c_2}-v_1} \binom{\kappa}{v_2} \pi^{v_2}(1-\pi)^{\kappa-v_2}\right]^{\ell_2}, \quad (10)$$

where π is the probability that any item in any group will not work out before the termination time $\mathcal{Z}_0$. The first part in Eq. (10) provides the batch acceptance probability from the first stage of the T-SGSIP, while the second part indicates the acceptance probability for the second sample. When zero and two failure techniques, i.e., a_{c_1}= 0 and a_{c_2}, = 2, the probability that the batch is accepted, can be gotten as

$$P_2 = (1-\pi)^{\kappa.\ell_1}\left[\frac{\kappa(\kappa-1)\pi^2(1-\pi)^{\kappa-2}}{2} + \kappa\pi(1-\pi)^{\kappa-1}\right]^{\ell_1}[(1-\pi)^{\kappa} + \kappa\pi(1-\pi)^{\kappa-1}]^{\ell_2}.$$

Consumers favor a sampling technique with fewer examination limits, thus we are interested in the status when a_{c_1}= 0 and a_{c_2}= 1. Consequently, the batch acceptance probability of Eq. (10) becomes

$$P_2 = (1-\pi)^{\kappa.\ell_1} + (\kappa\pi)^{.\ell_1}(1-\pi)^{(\kappa-1).\ell_1+\kappa\ell_2}. \quad (11)$$

Then, the optimal number of groups ℓ_1 and ℓ_2 ensuring $\mu_\rho \geq \mu_\rho^0$ at the consumer's confidence level ε^* can be established by disbanding the inequality below

$$P_2 \leq 1-\varepsilon^*, \quad (12)$$

where π is the failure probability at $\mu_\rho = \mu_\rho^0$ and is given by Eq. (7). There may be numerous solutions for the number of groups from fulfilling Eq. (12), so it must be reducing the AGN to determine them by putting the condition $\ell_1 \geq \ell_2$. The AGN is the average number of group units inspected per lot to access the decision to accept or refuse. It is noted that the AGN in the S-SGSIP with parameters κ, ℓ, a_c is $AGN = \kappa.\ell$. For a T-SGSIP, the AGN is given by

$$AGN = \kappa\ell_1 + \kappa\ell_2(1-\psi), \quad (13)$$

where ψ is the probability of making a decision on the first sample. Note that for given group size and number of groups, the number of units examined before making a decision is either $\kappa\ell_1$ or $(\kappa\ell_1 + \kappa\ell_2)$. There is a probability ψ that a decision will be made after examining $\kappa\ell_1$ units; the probability of examining $(\kappa\ell_1 + \kappa\ell_2)$ units prior to making a decision is $(1-\psi)$. This is the rationale behind the form of Eq. (13). The probability ψ can be expressed as

$\psi = P$ (batch accepted on first sample) + P (batch rejected on first sample),

then,

$$\psi = P(v \leq a_{c_1}) + P(v \geq r_{e_1}) = 1 - P(a_{c_1} < v_1 \leq a_{c_2}),$$

which can be expressed as

$$\psi = P(v \leq a_{c_1}) + P(v \geq r_{e_1}) = 1 - \left[\sum_{v_1=a_{c_1}+1}^{a_{c_2}} \binom{\kappa}{v_1} \pi^{v_1}(1-\pi)^{\kappa-v_1}\right]^{\ell_1},$$

for $a_{c_1} = 0$ and $a_{c_2} = 1$, Eq. (13) becomes

$$AGN = \kappa\ell_1 + \kappa\ell_2\left[\kappa\pi(1-\pi)^{\kappa-1}\right]^{\ell_1}.$$

Consequently, the optimal number of groups ℓ_1 and ℓ_2 for $a_{c_1} = 0$ and $a_{c_2} = 1$ in T-SGSIP will be determined by solving the following optimization problem:

$$\text{Minimize } AGN = \kappa\ell_1 + \kappa\ell_2\left[\kappa\pi(1-\pi)^{\kappa-1}\right]^{\ell_1}, \quad (14)$$

subject to

$$P_2 \leq 1-\varepsilon^*, \quad (15)$$

$$\ell_1 \geq \ell_2 \geq 1, \quad (16)$$

ℓ_1 and ℓ_2 positive integers.

The constraint in Eq. (16) is specified because it may not be desirable if the number of groups in the second stage is large than that in the first stage. Table 4 shows the optimal number of groups required for the two-stage group sampling plan under the WRW distribution with β = 0.8, σ = 0.2 and γ = 4; ε^*= 0.75, 0.90, 0.95 and 0.99; $u_\rho = 0.6$, 1.0, 2.15, 3.8 and 5.5 at ρ = 0.25 when k = 2 and 4, as mentioned earlier for a_{c_1}= 0 and a_{c_2} = 1 in all cases.

Table 4: The optimal number of groups in T-SGSIP with $a_{c1} = 0$ and $a_{c2} = 1$ for the WRW distribution with β = 0.8, σ = 0.2 and γ = 4.

ε^*	$u_{0.25}$	$k = 2$			$k = 4$		
		ℓ_1	ℓ_2	AGN	ℓ_1	ℓ_2	AGN
0.75	0.6	8	3	15.43	4	2	16.06
	1.0	4	2	8.60	3	2	10.74
	2.15	3	2	5.57	2	2	8.726
	3.8	2	2	4.96	1	1	6.23
	5.5	2	1	4.52	1	1	5.51
0.90	0.6	11	7	21.92	6	4	24.05
	1.0	6	5	11.09	3	2	12.88
	2.15	3	3	6.39	2	2	8.81
	3.8	2	2	5.25	2	1	7.05
	5.5	2	1	5.28	1	1	6.16
0.95	0.6	15	9	30.45	8	7	31.57
	1.0	7	5	13.77	4	4	14.95
	2.15	4	3	7.43	2	2	9.35
	3.8	3	3	6.23	2	1	6.98
	5.5	3	1	6.06	2	1	7.10
0.99	0.6	18	9	47.70	12	12	49.28
	1.0	10	6	19.492	5	5	20.76
	2.15	6	5	11.399	3	3	12.64
	3.8	4	4	8.518	2	2	9.41
	5.5	4	2	8.065	2	1	8.98

After obtaining the number of groups ℓ_1 and ℓ_2, one may be concerned with determining the probability of accepting the batch when the quality of an item is sufficient. As previously stated, an item is considered conforming if the $\mu_\rho \geq \mu_\rho^0$. Table 5 displays the OC values according to Eq. (11) for a given proposed T-SGSIP $(k, \ell_1, \ell_2, u_\rho)$ and ε^*.

Table 5: The OC values of the T-SGSIP for the WRW distribution with β = 0.8, σ = 0.2 and γ = 4.

ε^*	$u_{0.25}$	$k = 2$					$k = 4$				
		$\mu_{0.25}/\mu_{0.25}^0$									
		2	4	6	8	10	2	4	6	8	10
0.75	0.6	0.781	0.995	1	1	1	0.773	0.995	1	1	1
	1.0	0.519	0.939	0.994	0.999	1	0.479	0.929	0.993	0.999	1
	2.15	0.287	0.641	0.859	0.95	0.983	0.154	0.54	0.803	0.927	0.974
	3.8	0.206	0.421	0.614	0.762	0.861	0.134	0.393	0.642	0.798	0.885
	5.5	0.196	0.337	0.473	0.597	0.705	0.11	0.286	0.488	0.661	0.782
0.90	0.6	0.703	0.99	1	1	1	0.680	0.992	1	1	1
	1.0	0.428	0.922	0.992	0.999	1	0.397	0.914	0.991	0.999	1
	2.15	0.225	0.598	0.84	0.943	0.981	0.149	0.533	0.8	0.925	0.974
	3.8	0.182	0.401	0.603	0.756	0.857	0.085	0.302	0.541	0.716	0.828
	5.5	0.13	0.243	0.373	0.51	0.637	0.075	0.222	0.405	0.572	0.699
0.95	0.6	0.602	0.989	1	1	1	0.575	0.99	1	1	1
	1.0	0.335	0.884	0.988	0.999	1	0.276	0.9	0.99	0.999	1
	2.15	0.156	0.53	0.807	0.931	0.976	0.125	0.495	0.78	0.918	0.971
	3.8	0.106	0.297	0.514	0.695	0.821	0.089	0.309	0.549	0.723	0.833
	5.5	0.087	0.177	0.299	0.442	0.582	0.046	0.159	0.315	0.472	0.605

0.99	0.6	0.453	0.983	1	1	1	0.436	0.984	1	1	1
	1.0	0.224	0.867	0.986	0.999	1	0.207	0.86	0.986	0.999	1
	2.15	0.049	0.362	0.711	0.892	0.962	0.042	0.343	0.695	0.884	0.96
	3.8	0.031	0.156	0.365	0.58	0.745	0.024	0.145	0.356	0.567	0.731
	5.5	0.024	0.078	0.18	0.322	0.476	0.019	0.086	0.198	0.334	0.473

Consider that the manufacture now wishes to achieve what the optimal product quality level will be in order to maintain the producer's risk at the identified level. At the producer's risk of τ the optimal ratio μ_ρ/μ_ρ^0 can be obtained by solving

$$P_2 \leq \tau, \tag{17}$$

where π is given by Eq. (5), ℓ_1 and ℓ_2 are chosen according to the consumer's risk $1-\varepsilon^*$ when $\mu_\rho/\mu_\rho^0 = 1$. The optimal values of the ratio μ_ρ/μ_ρ^0 satisfying inequality (17) are computed and presented in Table 6.

Table 6: The optimal ratio of μ_ρ/μ_ρ^0 in order that the lot accepted with the producer's risk of 5% for the WRW distribution with β = 0.8, σ = 0.2 and γ = 4.

ε^*	$u_{0.25}, \hbar = 2$					$u_{0.25}, \hbar = 4$				
	0.6	1.0	2.15	3.8	5.5	0.6	1.0	2.15	3.8	5.5
0.75	2.806	4.174	8.000	13.066	17.172	2.825	4.311	8.743	13.53	19.36
0.90	2.986	4.393	8.671	13.609	19.927	3.033	4.478	8.775	14.528	20.354
0.95	3.172	4.613	8.771	14.375	21.122	3.217	4.748	8.95	14.435	21.552
0.99	3.4	4.876	9.476	15.657	22.546	3.425	4.92	9.602	15.953	23.065

5. Design of the multiple GSIP with three-stage

A normal extension of the double inspection plan is to permit more extra samples to be drawn to accomplish even greater discrimination in the disposition of a batch. Such treatments are called multiple sampling plans when, as with double sampling, the last sample is built to impose a decision at that stage. That is, for a specified last specimen (say, the xth sample), the arrangement is such that $r_{e_x} = a_{c_x} + 1$, where r_{e_x} is the rejection limit and $a_{c_x}$is the examination limit. Thus, double sampling plan is a special case of the multiple sampling plan where x = 2. Multiple-stage group sampling plans (M-SGSIP) provide even more elasticity and reduce the average number of groups even further than T-SGSIP, but they are frequently found to be difficult to administer due to the complexity of handling and recording all of the samples/groups required.

Many studies have been conducted to investigate the effects of S-SGSIP and T-SGSIP, but research into M-SGSIP is uncommon. As a result, a multiple GSIP with three stages based on an amputated life test is provided assuming that the lifetime of an item follows the WRW distribution with known shape parameters. Furthermore, zero, one and two failure techniques are provided. To accomplish this, the optimal number of groups are found at given group size $\hbar$, consumer's risk and time ratio. The procedure of this plan is explained below:

Stage one:

i. A random sample of size $n_1 = \hbar.\ell_1$ is extracted from the submitted batch by selected number of groups ℓ_1 and the predefined group size $\hbar$. Then, the items of each group are put in a tester.

ii. The life test for ℓ_1 groups is performed till the specified experiment time Z_0 is reached and count the number of defective, v_1, in this stage.

$$\begin{cases} \text{If} \quad v_1 \leq a_{c_1}, & \text{Accept the batch.} \\ \text{If} \quad v_1 > r_{e_1}, & \text{terminate the test and reject the batch.} \\ \text{If} \quad a_{c_1} < v_1 < r_{e_1} & \text{go to the stage two,} \end{cases}$$

Stage two:

i. A random sample of size $n_2 = \hbar.\ell_2$ is extract from the same batch by determined number of groups ℓ_2 and the assign group size $\hbar$. Then, the items of each group are put in a tester.

ii. The life test for ℓ_2 groups is performed till the specified experiment time Z_0 is reached and count the cumulative number of defective, V_1, in this stage.

$$\begin{cases} \text{If} & V_1 \le a_{c_2}, & \text{Accept the batch.} \\ \text{If} & V_1 > r_{e_2}, & \text{terminate the test and reject the batch.} \\ \text{If} & a_{c_2} < V_1 < r_{e_2} & \text{go to the stage three,} \end{cases}$$

Stage three:

i. A random sample of size $n_3 = k.\ell_3$ is extracted from the same batch by determined number of groups ℓ_3 and the assign group size k. Then, the items of each group are put in a tester.

ii. The life test for ℓ_3 groups is performed till the specified experiment time Z_0 is reached and count the cumulative number of defectives, V_2, in this stage.

$$\begin{cases} \text{If} & V_2 = \sum_{i=1}^{3} v_i \le a_{c_3} & \text{accept the batch.} \\ \text{If} & V_2 = \sum_{i=1}^{3} v_i \ge r_{e_3} & \text{reject the batch,} \end{cases}$$

where $r_{e_x} = a_{c_x} + 1$ and the cumulative number of defective, $V_1 = (v_1 + v_2)$. The x parameter is related to x stages of the sampling plan. Of course, $a_{c_1} < a_{c_2} < a_{c_3}$, and we may observe that if $a_{c_1} = a_{c_2} = a_{c_3}$, we have a S-SGSIP. In general, the proposed multiple GSIP is determined by $(\ell_x, k, a_{c_x}, r_{e_x})$ where $0 \le a_{c_1} < \cdots < a_{c_x}$ and $1 \le r_{e_1} < \cdots < r_{e_x}$. The OC function for multiple group inspection sampling is determined by

$$\begin{aligned} \mathrm{P}_{i,i=1,2,\ldots,x} = {} & B(a_{c_1}|k\ell_1) + \sum_{v_1=a_{c_1}+1}^{r_{e_1}-1} b(v_1|k\ell_1) \sum_{v_2=a_{c_2}+v_1+1}^{r_{e_2}-1} b(v_2|k\ell_2)\, B(a_{c_3-V_1}|k\ell_3) \\ & + \sum_{v_1=a_{c_1}+1}^{r_{e_1}-1} b(v_1|k\ell_1) \sum_{v_2=a_{c_2}+v_1+1}^{r_{e_2}-1} b(v_2|k\ell_2) \sum_{v_3=a_{c_3-V_1}+1}^{r_{e_3}-1} b(v_3|k\ell_3) B(a_{c_4-V_2}|k\ell_3) + \cdots \\ & + \sum_{v_1=a_{c_1}+1}^{r_{e_1}-1} b(v_1|k\ell_1) \sum_{v_2=a_{c_2}+v_1+1}^{r_{e_2}-1} b(v_2|k\ell_2) \sum_{v_3=a_{c_3-V_1}+1}^{r_{e_3}-1} b(v_3|k\ell_3) \ldots \sum_{v_j=a_{c_j}-V_{j-1}+1}^{r_{e_j}-1} b(v_j|k\ell_j)\, B(a_{c_{x-1}} - V_{x-2}|k\ell_{x-1}) \\ & + \sum_{v_1=a_{c_1}+1}^{r_{e_1}-1} b(v_1|k\ell_1) \sum_{v_2=a_{c_2}+v_1+1}^{r_{e_2}-1} b(v_2|k\ell_2) \sum_{v_3=a_{c_3-V_1}+1}^{r_{e_3}-1} b(v_3|k\ell_3) \\ & \quad \ldots \sum_{v_j=a_{c_j}-V_{j-1}+1}^{r_{e_j}-1} b(r_k|\beta_k) \sum_{r_{x-1}=a_{c_{x-1}}-R_{x-2}+1}^{r_{e_{x-1}}-1} b(v_{x-1}|k\ell_{x-1}) \left(B(a_{c_x} - V_{x-1}|k\ell_x)\right), \end{aligned} \tag{18}$$

when the proposed multiple group M-SGSIP is characterized by $(\ell_i, k, a_{c_i}, r_{e_i})$, i = 1, 2, 3, the probability of acceptance is given by

$$\begin{aligned} \mathrm{P}_3 = {} & \left[\sum_{v_1=0}^{a_{c_1}} \binom{k}{v_1} \pi^{v_1}(1-\pi)^{k-v_1}\right]^{\ell_1} + \left[\sum_{v_1=a_{c_1}+1}^{r_{e_1}-1} \binom{k}{v_1} \pi^{v_1}(1-\pi)^{k-v_1}\right]^{\ell_1} \\ & \times \left(\begin{array}{c} \left[\sum_{v_2=0}^{a_{c_2}-v_1} \binom{k}{v_2} \pi^{v_2}(1-\pi)^{k-v_2}\right]^{\ell_2} \\ + \left\{ \begin{array}{c} \left[\sum_{v_1=a_{c_2}-v_1+1}^{r_{e_2}-1} \binom{k}{v_2} \pi^{v_2}(1-\pi)^{k-v_2}\right]^{\ell_2} \\ \times \left[\sum_{v_3=0}^{a_{c_3}-V_1} \binom{k}{v_3} \pi^{v_3}(1-\pi)^{k-v_3}\right]^{\ell_3} \end{array} \right\} \end{array} \right). \end{aligned} \tag{19}$$

The first expression in Eq. (19) reflects the acceptance probability from the first stage, while the second expression represents the acceptance probability combined from the second and third stages. As an illustration of the application of these formulas, consider the following techniques:

Stage	a_{c_i}	r_{e_i}
1	0	2

2	1	3
3	2	3

A probability tree, as shown in Figure 1, is perhaps the best way to describe the evaluation of M-SGSIP.

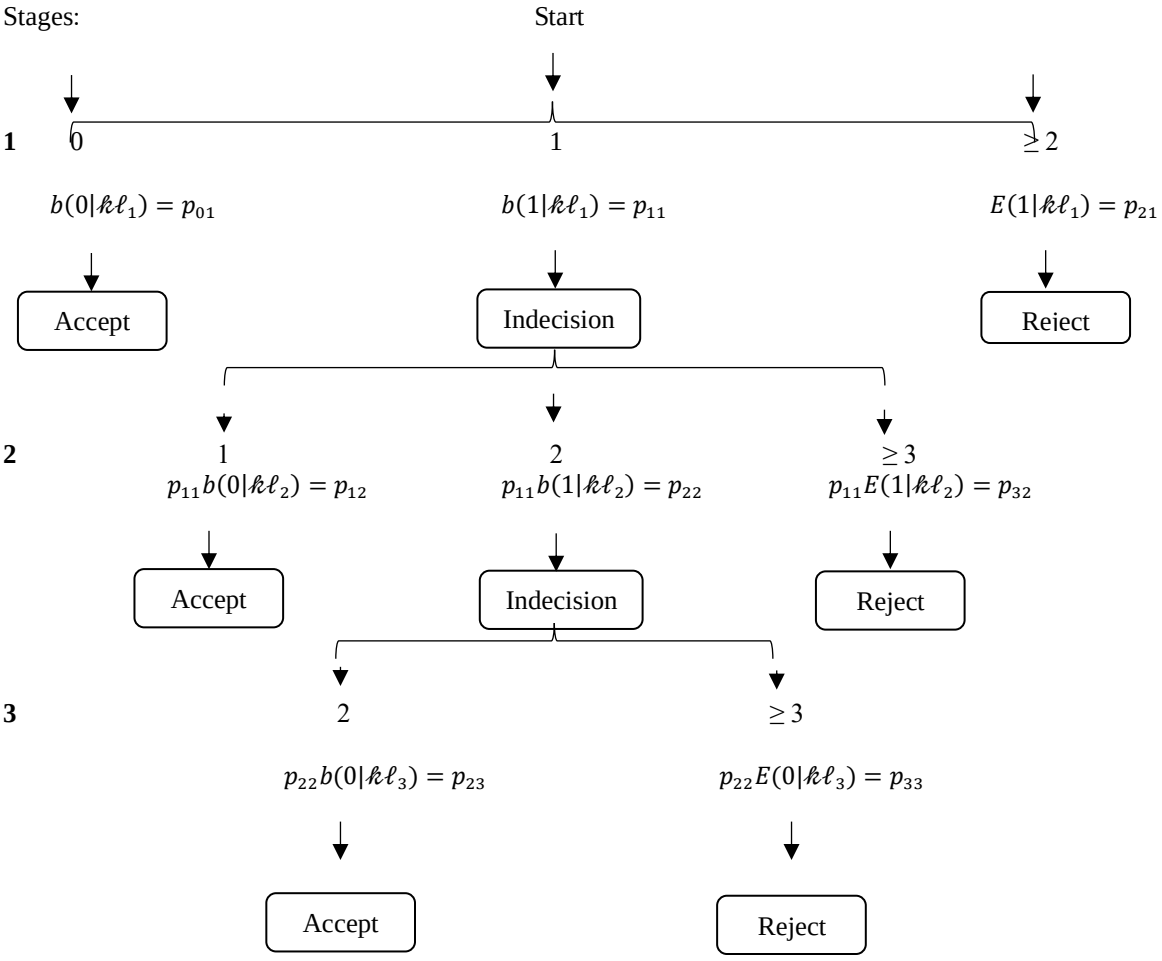

Figure 1: Control Chart

where

$$E(\emptyset|n_{(.)}) = [1 - B(\emptyset|n_{(.)})],$$

$$B(\emptyset|n_{(.)}) = \sum_{v=0}^{\emptyset} \binom{n_{(.)}}{v} \pi^{\emptyset}(1-\pi)^{n_{(.)}-\emptyset},$$

are defined as the probability that the binomially distributed random variable is larger than or equal to $\emptyset$. The results of Figure 1 give the following probabilities.

Table 7: Quality Control table

Stages	1	2	3
Accept, A_{c_i}	$A_{c_1} = p_{01}$	$A_{c_2} = p_{12}$	$A_{c_3} = p_{23}$
Reject, R_{e_i}	$R_{e_1} = p_{21}$	$R_{e_2} = p_{32}$	$R_{e_3} = p_{33}$
Terminate, T_i	$A_{c_1} + R_{e_1}$	$A_{c_2} + R_{e_2}$	$A_{c_3} + R_{e_3}$
No decision, D_i	$D_1 = p_{11}$	$D_2 = p_{22}$	0

Table 7 introduces the probabilities related to Figure 1 showing the probability of accepting (A_{c_i}), rejecting (R_{e_i}), terminating (T_i) and no decision (D_i) at each stage. The probabilities of acceptance and rejection are the sum of the probabilities shown in Table 7 leading to acceptance and rejection , and are defined by

$$P_{(A_{c_i})} = A_{c_1} + A_{c_2} + A_{c_3},$$

and

$$P_{(R_{e_i})} = R_{e_1} + R_{e_2} + R_{e_3},$$

the probability of making no decision is the sum of the probabilities displayed in Table 7.

$$P_{(D_i)} = D_1 + D_2 + D_3,$$

where $D_i = \sum_{v=a_{c_i+1}}^{r_{e_i}-1} \left[\binom{k}{v} \pi^v (1-\pi)^{k-v}\right]^{\ell_i}$, $i = 1, \ldots, x-1$, $D_0 = 1$ and $D_x = 0$. Accordingly, the probability of acceptance for multiple-group sampling with three- stage based on our techniques can be obtained as follows:

$$\mathrm{P}_3 = \left[(1-\pi)^k\right]^{\ell_1} + (k.\pi)^{\ell_1}(1-\pi)^{k.(\ell_1+\ell_2)-\ell_1} + \left[k.\pi(1-\pi)^{k-1}\right]^{\ell_1+\ell_2}\left[(1-\pi)^k\right]^{\ell_3}. \quad (20)$$

Using the following inequality, the optimal number of groups ℓ_1, ℓ_2 and ℓ_3 to ensure $\mu_\rho \geq \mu_\rho^0$ at the risk of the customer $(1-\varepsilon^*)$ may then be determined as

$$\left[(1-\pi)^k\right]^{\ell_1} + (k.\pi)^{\ell_1}(1-\pi)^{k.(\ell_1+\ell_2)-\ell_1} + \left[k.\pi(1-\pi)^{k-1}\right]^{\ell_1+\ell_2}\left[(1-\pi)^k\right]^{\ell_3} \leq 1-\varepsilon^*, \quad (21)$$

where π is the failure probability at $\mu_\rho = \mu_\rho^0$ and is given by Eq. (7). There may be many of solutions for the number of groups from fulfilling Eq. (21), so we want to reduce the AGN to acquire the suitable solution by imposing the condition $\ell_3 \leq \ell_2 \leq \ell_1$. Generally, the AGN for multiple-GSIP can be defined as follows:

$$AGN = \sum_{i=1}^{x}\sum_{j=1}^{i} n_j T_i,$$

where $T_i = \left(P_{(A_{c_i})} + P_{(R_{e_i})}\right)$ is the probability of termination on the ith stage and $n_j = k.\ell_j$. Simply, the AGN for multiple-GSIP with three stages for our techniques we can get immediately as follows

$$AGN = k\ell_1 + D_1[k\ell_2 + D_2(k\ell_3)], \quad (22)$$

where $D_i = (1-T_i)$ is the probability no decision on the ith stage. The probabilities D_1 and D_2 can be found respectively by using Figure 1 as follows:

$$D_1 = \left[k\pi(1-\pi)^{k-1}\right]^{\ell_1}, \quad (23)$$

and

$$D_2 = \left[k\pi(1-\pi)^{k-1}\right]^{\ell_1+\ell_2}. \quad (24)$$

Many sampling strategies described in the literature aim to reduce the AGN. in general, each sampling strategy with a minimum AGN would always be desirable because when the AGN is minimum, examination cost and examination time will be lessened. We also try to minimize the AGN of the proposed multiple-stage sampling inspection plan for the WRW distribution under amputated life tests. To specify the optimal number of groups for our techniques, we employ the following optimization problem that minimizes the AGN:

$$\text{Minimize } AGN = k\ell_1 + D_1[k\ell_2 + D_2(k\ell_3)]. \quad (25)$$

Subject to

$$\mathrm{P}_3 \leq 1-\varepsilon^* \quad (26)$$

$$\ell_1 \geq \ell_2 \geq \ell_3 \geq 1 \quad (27)$$

ℓ_1 , ℓ_2 and ℓ_3 positive integer

The optimal number of group for ℓ_1, ℓ_2 and ℓ_3 and AGN for the M-SGSIP under the WRW distribution with β = 0.8, σ = 0.2 and γ = 4; a_{c_1}= 0, a_{c_2}=1, a_{c_3}= 2 and r_{e_1}= 2, r_{e_2}= r_{e_3}= 3; ε^*= 0.75, 0.90, 0.95 0.99; u_ρ = 0.6, 1.0, 2.15, 3.8, 5.5 at ρ = 0.25 when k = 2 and 4 are show in Table 8.

Table 8: The optimal number of M-SGSIP with three-stage under a_{c_1}= 0, a_{c_2}=1, a_{c_3}= 2 and r_{e_1}= 2, r_{e_2}= r_{e_3}= 3 for the WRW distribution with β = 0.8, σ = 0.2 and γ = 4.

		$k=2$				$k=4$			
ε^*	$u_{0.25}$	ℓ_1	ℓ_2	ℓ_3	AGN	ℓ_1	ℓ_2	ℓ_3	AGN
	0.6	6	5	5	12.65	3	3	3	14.02
	1.0	4	2	2	7.13	2	2	2	10.24

0.75	2.15	2	2	1	5.19	2	1	1	7.26
	3.8	2	2	1	4.98	1	1	1	5.56
	5.5	2	2	1	4.65	1	1	1	5.33
0.90	0.6	10	10	6	20.68	6	6	3	22.99
	1.0	5	4	4	9.68	3	2	2	11.78
	2.15	3	3	3	5.97	2	2	2	7.49
	3.8	2	2	2	5.23	1	1	1	6.26
	5.5	2	2	1	5.09	1	1	1	5.38
0.95	0.6	13	6	6	26.54	7	3	3	27.15
	1.0	6	4	4	12.61	3	3	3	13.99
	2.15	3	3	3	7.23	2	2	2	9.16
	3.8	3	2	2	6.13	2	2	2	6.97
	5.5	3	2	1	5.88	2	2	1	6.78
0.99	0.6	21	12	12	42.22	11	11	4	45.50
	1.0	9	7	7	18.64	5	5	5	19.82
	2.15	5	4	4	10.15	3	3	3	11.41
	3.8	4	3	3	8.36	2	2	2	9.11
	5.5	4	2	2	7.74	2	2	1	8.04

From Tables 4 and 8 it can be observed that number of groups decrease as the group size $\hbar$ increase, whereas the AGN increase as the group size $\hbar$ increase. The number of groups decreases as the termination time ratio $u_{0.25}$ increases. Also, the number of groups increase as the consumer's confidence level ε^* increases. The OC values according to Eq. (20) for a given sampling plan $(\hbar, \ell_i, a_{c_x}, r_{e_x}, u_\rho)$, $i = 1, \ldots, x$, $x = 3$, μ_ρ/μ_ρ^0 and ε^* are exhibited in Table 9.

Table 9: The OC values of three-stage GSIP for the WRW distribution with $\beta = 0.8$, $\sigma = 0.2$ and $\gamma = 4$.

		$\hbar = 2$					$\hbar = 4$				
		$\mu_{0.25}/\mu_{0.25}^0$									
ε^*	$u_{0.25}$	2	4	6	8	10	2	4	6	8	10
0.75	0.6	0.816	0.996	1	1	1	0.801	0.996	1	1	1
	1.0	0.584	0.95	0.995	1	1	0.513	0.935	0.994	0.999	1
	2.15	0.352	0.688	0.878	0.957	0.985	0.283	0.677	0.865	0.948	0.981
	3.8	0.233	0.444	0.631	0.773	0.867	0.224	0.558	0.8	0.91	0.954
	5.5	0.215	0.377	0.522	0.644	0.743	0.138	0.351	0.583	0.758	0.864
0.90	0.6	0.717	0.993	1	1	1	0.691	0.993	1	1	1
	1.0	0.478	0.932	0.993	0.999	1	0.441	0.922	0.992	0.999	1
	2.15	0.263	0.628	0.854	0.948	0.982	0.259	0.661	0.86	0.946	0.981
	3.8	0.196	0.414	0.611	0.761	0.86	0.151	0.436	0.67	0.803	0.879
	5.5	0.158	0.304	0.45	0.585	0.698	0.133	0.343	0.573	0.748	0.854
0.95	0.6	0.653	0.991	1	1	1	0.647	0.991	1	1	1
	1.0	0.381	0.912	0.991	0.999	1	0.365	0.907	0.991	0.999	1
	2.15	0.169	0.544	0.814	0.933	0.977	0.139	0.513	0.788	0.921	0.972
	3.8	0.115	0.295	0.505	0.688	0.816	0.094	0.329	0.572	0.738	0.841
	5.5	0.094	0.201	0.334	0.477	0.611	0.051	0.177	0.355	0.523	0.655
0.99	0.6	0.508	0.986	1	1	1	0.482	0.985	1	1	1
	1.0	0.239	0.872	0.987	0.999	1	0.224	0.866	0.986	0.999	1
	2.15	0.069	0.406	0.739	0.903	0.966	0.063	0.392	0.725	0.897	0.964
	3.8	0.035	0.157	0.365	0.58	0.745	0.028	0.163	0.378	0.584	0.742
	5.5	0.027	0.089	0.199	0.344	0.497	0.022	0.101	0.237	0.393	0.538

From Tables 5 and 9, the main properties of OC function are (***i***) decreasing in u_ρ, (***ii***) decreasing in ℓ_i, (***iii***) increasing in μ_ρ/μ_ρ^0 , (iv) decreasing in $\hbar$, (***v***) increasing in a_{c_i} and r_{e_i}. The producer's risk is the probability of refusal of the batch when it is conforming $(\mu_\rho > \mu_\rho^0)$. For a given value of the producer's risk τ, We are concerned in learning what

value of μ_ρ/μ_ρ^0 will guarantee that the producer's risk is less than or equal to τ. Notice that π is a function of μ_ρ/μ_ρ^0 as indicated in Eq. (5), then μ_ρ/μ_ρ^0 is the smallest positive number for which π satisfies the following inequality

$$P_3 \leq \tau \,. \tag{28}$$

Thus, for a given the proposed plan $(\hbar, \ell_i, a_{c_x}, r_{e_x}, \mathcal{u}_\rho)$, i = 1, 2 and 3 at the specified value of ε^*, the values of μ_ρ/μ_ρ^0 satisfying inequality (28) are presented in Table 10.

Table 10: The optimal ratio of μ_ρ/μ_ρ^0 in order that the lot accepted with the producer's risk of 5% for the WRW distribution with β = 0.8, σ = 0.2 and γ = 4.

ε^*	$\mathcal{u}_{0.25}$, $\hbar = 2$					$\mathcal{u}_{0.25}$, $\hbar = 4$				
	0.6	1.0	2.15	3.8	5.5	0.6	1.0	2.15	3.8	5.5
0.75	2.705	4.006	7.72	13.382	18.686	2.749	4.238	8.092	9.727	13.996
0.90	2.956	4.275	8.072	13.542	19.415	3.01	4.387	8.141	13.436	14.527
0.95	3.084	4.504	8.545	14.464	20.7	3.095	4.547	8.882	14.284	20.744
0.99	3.321	4.838	9.256	15.659	22.272	3.359	4.878	9.379	15.815	22.237

From Tables 6 and 10, it can be seen that the optimal ratio of μ_ρ/μ_ρ^0 required for a given τ = 5% increases as the consumer's confidence level ε^* increases for a fixed termination time ratio. As the group size $\hbar$ increases from 2 up to 4 the optimal ratio increases.

6. Numerical illustrations and assessment

The first example:
Suppose that the lifetime ($\mathcal{Z}_0$) of an item follows the WRW distribution with parameters β = 0.8, σ = 0.2 and γ = 4. A fitter intends to demonstrate that the real unknown 25th percentile lifetime for the mechanical components is at least 1000h (φ_0) with probability ε^*= 0.90 and that the life test will be terminated at $\mathcal{Z}_0$= 600h using testers equipped with 4 items each. This leads to the termination time ratio $\mathcal{u}_{0.25}$= 0.6. Then, for examination limit a_c= 0, the required ℓ is found in Table 1 to be 7. It should be noted that if the optimal sample size is desired, $\beta = \hbar.\ell$ can provide it. As result, we'll draw a random sample of size β = 28 items and allocate 4 items to each of the 7 groups to put on test for 600h. The batch is accepted, if no failure items occur during 600h, and the experimenter can assert with a confidence limit of 90% that the 25th percentile is at least 1000h. Otherwise, stop the experiment and reject the lot. For the S-SGSIP ($\hbar, \ell, a_c$ and $\mathcal{u}_{0.25}$) = (4, 7, 0, 0.6) at ε^*= 0.90, the OC values from Table 2 are as follows:

$\mu_{0.25}/\mu_{0.25}^0$	2	4	6	8	10
OC value	0.640	0.991	1	1	1

This shows that if the real 25th percentile is twice the fixed 25th percentile $\mu_{0.25}/\mu_{0.25}^0$ = 2 the producer's risk is about 0.36, while it is almost to be zero when $\mu_{0.25}/\mu_{0.25}^0 \geq 6$. Table 3 can be used to get the value of $\mu_{0.25}/\mu_{0.25}^0$ for various choices of ($\hbar$, a_c, and $\mathcal{u}_{0.25}$) such that the producer's risk may not exceed 5%. For example, the value of $\mu_{0.25}/\mu_{0.25}^0$ is 3.210 for $\hbar$ = 4, a_c = 0, $\mathcal{u}_{0.25}$ = 0.6 and ε^*= 0.90. This means that the product should have 25th percentile life of 3.210 times the specified 25th percentile life for the lot to be accepted with a confidence limit of ε^*= 0.90.

The second example:
Postulate the product lifespan follows the WRW distribution with β = 0.8, σ = 0.2 and γ = 4. Postulate that an inspector would like to apply a T-SGSIP to test the real unknown 25th percentile lifetime of the product under inspection is at least 1000 hours with confidence limit of ε^*= 0.90 under a_{c_1} = 0 and a_{c_2} = 1. An examiner wants to stop an experiment at 600 hours using testers (group size) equipped with 4 items. Based on this data, it is found that $\mathcal{u}_{0.25}$ = 0.60. Then, from Table 4, the optimum number of groups required are ℓ_1= 6 and ℓ_2 = 4. This plan will be implemented as follows: The first sample of size $n_1 = 24$ is drawn and distribute 4 items into each of 6 tester and the product will be accepted if no failure occurs during the experiment. The product is refused if more than one failure occurs before 600 hours. When there is exactly one failure observed, the second sample of size $n_2 = 16$ is drawn and distribute 4 items into each of 4 tester and put them on the same test. Accept the product if a total number of failures are one or fewer are recorded otherwise refuse the product. For a_{c_1} = 0 and a_{c_2} = 1, the OC values for T-SGSIP under the WRW distribution is ($\hbar, \ell_1, \ell_2$ and $\mathcal{u}_{0.25}$) = (4, 6, 4 and 0.60) with $\varepsilon^* = 0.90$ is as follows:

$\mu_{0.25}/\mu_{0.25}^0$	2	4	6	8	10
OC value	0.680	0.992	1	1	1

This implies that the batch is accepted with a probability of 68% if the real 25th percentile of the units in the batch is twice the identified 25th percentile life, and the producer's risk will be 0.32. The probability of accepting the batch increases up to 100% if the real 25th percentile life is 6 times the identified life. To know the optimal ratio corresponding to the producer's risk of 5%, can be found in Table 6. For instance, the optimal ratio of $\mu_{0.25}/\mu_{0.25}^{0}$ is 3.033. Therefore, the real 25th percentile life wanted of the product should be at least 3033h.

The third example:

Let's say a manufacturer wants to check the incoming batch of mechanical components using a multi-group sample plan with three stages. Let's say also that the life of this product follows WRW distribution with β = 0.8, σ = 0.2 and γ = 4. Multi-item testers with group size $\hbar$ = 4 will be used. It is known that the specified 25th percentile life of interested is 1000h. The test time was specified 600h with confidence limit of ε^*= 0.90 under examination limit a_{c_1} = 0, a_{c_2} = 1, a_{c_3}= 2 and rejection limit r_{e_1}= 2, r_{e_2}= r_{e_3}=3. This leads to the experiment termination multiplier, $\omega_{0.25}$ = 0.60. For the problem under consideration Table 8 gives the optimal number of three-stage group sampling plan with ℓ_1= 6, ℓ_2= 6 and ℓ_3 = 3. This plan is put into operation as follows: A sample of size $n_1 = 24$ is chosen and allocated to 6 testers. If no nonconforming items are found in stage one, the batch is accepted. If two or more nonconforming items are found, the batch is refused. If one nonconforming item is found, a second sample of size n_2 = 24 is extracted and distributed into 6 testers. If the combined number is one or less in stages one and two, the batch is accepted. If combined number of nonconforming items is three or more, the batch is refused. If the combined number of nonconforming items is two, a third sample of size n_3 = 12 is selected and assigned to 3 testers. If the combined number is two or less in stages two and three, the batch is accepted. If combined number of nonconforming items is three or more, the batch is refused.

For a_{c_1} = 0, a_{c_2} = 1 and a_{c_3}= 2, the OC values for three-stage GSIP under the WRW distribution is ($\hbar$, ℓ_1, ℓ_2, ℓ_3 and $\omega_{0.25}$) = (4, 6, 6, 3 and 0.60) with probability $\varepsilon^* = 0.90$ is as follows:

$\mu_{0.25}/\mu_{0.25}^{0}$	2	4	6	8	10
OC value	0.691	0.993	1	1	1

We find from the above-tabulated values of the OC function that the associated producer's risk is approximately 0.31 or less if the real 25th percentile life is twice the specified lifetime ($\mu_{0.25}/\mu_{0.25}^{0} = 2$), while it is about to be zero when ($\mu_{0.25}/\mu_{0.25}^{0} \geq 6$). As a result, the risk for the producer tends to be lower for greater values of the $\mu_{0.25}/\mu_{0.25}^{0}$. For a_{c_1} = 0, a_{c_2} = 1, a_{c_3}= 2 and rejection limit r_{e_1}= 2, r_{e_2}= r_{e_3}=3 and hence of ℓ_1, ℓ_2, ℓ_3 Table 10 gives the optimal ratio of $\mu_{0.25}/\mu_{0.25}^{0}$ in order that the producer's risk may not override 5%. Thus, the value of $\mu_{0.25}/\mu_{0.25}^{0}$ for $\hbar$ = 4, $\omega_{0.25}$ = 0.60 and ε^*= 0.90 is 3.01; this means that the item should have 25th percentile life of at least 3.01 times of the specified 25th percentile life of 1000h in order that the batch will be accepted with probability 0.90.

7. Comparisons of one-stage and iterative GSIP

The minimizing of the projected sample size under extreme (very good or very terrible) lot quality is the purpose of iterative sampling. One can compare the ASN functions of single and iterative plans that are matched to have roughly the same OC function for a straightforward quantitative comparison. In this section, a comparison is made between the S-SGSIP, T-SGSIP, and M-SGSIP. This comparison is made based on the AGN and OC values needed for types of group sampling plans. For example, if the product lifetime follows the WRW distribution, the next step would be to decide whether to use S-SGSIP, T-SGSIP or use M-SGSIP, which will have a maximum OC and a minimum AGN. Here we compare type of proposed plans for $\omega_{0.25}$= 0.6, ε^*= 0.90 and 25th percentile lifetime quality level μ_ρ/μ_ρ^0 = 2. Table 11 provides the values of OC and ASN for S-SGSIP, T-SGSIP and M-SGSIP, where the presumed group size was set as $\hbar$ = 4.

Table 11: Summarizing of OC and AGN values for types of GSIPs

Type of plan	OC	AGN
Single- stage with a_c= 0	0.640	28
Two- stage	0.680	24.05
Three- stage	0.691	22.99

It has been witnessed from the table that the multiple GSIPs with three-stage (M-SGSIP) are better than the S-SGSIP and two-stage GSIPs in terms of AGN and the OC values. Despite this, there are a number of drawbacks to multiple sampling: administrative challenges; system complicated and needs a great administrative effort; inspectors' inability to appropriately apply the algorithm; and multiple sampling costs are high. If the product quality history inspires data-

based confidence that batch quality is really on one of the extreme sides, should multiply sample methods be recommended over single sampling.

8. Conclusion

This research presents single-stage group sampling inspection plans; two-stage group sampling inspection plans; and multiple group sampling inspection plans with three stages for deciding upon the batch acceptance of manufactured parts under an amputated life test. Amputated testing is frequently adopted by parts manufacturers to reduce testing time. For some types of group sampling inspection plans, the optimal number of groups was determined for the WRW distribution with known shape parameters when the consumer's risk and other parameters are determined. The operating characteristic function values and related producer risk have also been evaluated. The minimum sample sizes for the first, second, and third samples that must be used to make sure that the product's specified mean and median lifetimes are attained at a particular level of customer confidence. The operational characteristic values and producer risk for the indicated example plans are provided. Some few real-life examples have been looked at in order to demonstrate how the recommended methodologies based on the mean and median life spans of the product may operate in exercise. For designing the two-stage group sampling inspection plans and multiple group sampling inspection plans, a nonlinear optimization problem of minimizing the average group number is considered with constraints on acceptance probabilities. To reduce the number of groups, the multiple group sampling inspection plans are preferred to the single-stage group sampling inspection plans and two-stage group sampling inspection plans. Types of proposed plans are explained with the help of numerical cases and tables.

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