Pak.j.stat.oper.res. Vol.19 No. 4 2023 pp 765-776 DOI: http://dx.doi.org/10.18187/pjsor.v19i4.4185

Pakistan Journal of Statistics and Operation Research

Estimation of the Parameters of the Modified Weibull Distribution with Bathtub-shaped Failure Rate Function

Adam Abdelrahman Hussein Adam[1], Hakan SavaÅŸ Sazak[2*]

* Corresponding Author

1. Faculty of Economics and Administrative Sciences, Department of Applied Statistics and Population Studies, Portsudan Town, Red Sea University, Sudan, adamabdelrahmanhusseinadam@gmail.com
2. Department of Statistics, Bornova, İzmir, Ege University, Turkey, hakan.savas.sazak@ege.edu.tr

Abstract

In this study, we propose two estimators called the 3-step modified maximum likelihood (MML) and the combined estimators of the parameters of the modified Weibull distribution which is used in reliability models with bathtub-shaped failure rate function. The simulations show the superiority of both estimators over the graphical estimators. Particularly, the combined estimators are the better of the two. Two real-life data applications also show the superiority of the proposed estimators compared to the graphical estimators.

Key Words: 3-step modified maximum likelihood; Combined estimators; Graphical method; Hazard function; Reliability.

Mathematical Subject Classification: 62F10; 62J05; 62N05

1. Introduction

The Weibull distribution has been commonly used for lifetime distributions owing to its flexibility in modelling different failure rate functions (decreasing, constant, increasing) depending on the value of its shape parameter (Xie and Lai 1996). Despite the nice features of the Weibull distribution, its failure rate function is monotonic which makes it very hard to model the complex systems which need to be modelled by using a non-monotonic failure rate function (Xie and Lai 1996; Bebbington et al. 2007). It was shown that the bathtub failure rate function is a very good choice of this type. Main reason of this fact is, in general, a material has a warm-up period with high failure rate in the beginning, then, its failure rate decreases and stabilized in the middle and because of the wearing out of the material, its failure rate increases as the time goes on (see Xie and Lai (1996), Xie et al. (2002), Lai et al. (2003) and Bebbington et al. (2007) for details). This fact has been a motivation for the researchers to find new distributions to model such situations. Xie and Lai (1996) proposed a new model which can be used for data having bathtub failure rate function. They also studied the estimation of the parameters related to the proposed model. A new two-parameter distribution with increasing or bathtub failure rate function was introduced by Chen (2000). Exact confidence intervals and regions based on Type II censoring were also discussed. Xie et al. (2002) presented a 3-parameter modified Weibull distribution with bathtub failure rate function and studied on the estimation of the parameters of this distribution. Lai et al. (2003) proposed a simple 3-parameter modified Weibull distribution with bathtub failure rate function by suggesting three methods of estimation of the parameters of the proposed distribution. Bebbington et al. (2007) introduced a flexible two-parameter Weibull distribution to handle many cases including some suggestions for the estimation of the parameters belonging to this distribution. Another modified Weibull distribution was introduced by Almalki and Yuan (2013) which was obtained by combining the classical Weibull distribution and the modified Weibull distribution proposed by Lai et al. (2003). They also studied its properties and the estimation of the parameters based on maximum likelihood (ML). Peng and Yan (2014) proposed a new extended Weibull distribution with increasing and upside down failure rate functions. They also studied the estimation of the parameters by ML and Bayesian techniques for progressively right censored data.

In this paper, we propose two methods for the estimation of the parameters of the modified Weibull distribution proposed by Xie et al. (2002). Although there are some proposals for the estimation of the parameters of this distribution, only the estimators produced by the graphical method are explicit functions of the observations. Thus, we compared the efficiencies of the estimators produced by the proposed methods w.r.t. the estimators produced by the graphical method. In Section 2, we give the details of the new estimation methods. Section 3 includes the simulation results. We give two real-life data applications for the illustration of the proposed methods in Section 4. The final section includes conclusion and some suggestions.

2. The Estimation of the Parameters

The cumulative distribution function (cdf) of the modified Weibull distribution proposed by Xie et al. (2002) is

$$F(x) = 1 - \exp\left(\lambda\alpha\left(1 - e^{\left(\frac{x}{\alpha}\right)^{\beta}}\right)\right), \ \alpha, \lambda, \beta > 0, \ x \geq 0. \tag{1}$$

The probability density function (pdf) of this distribution is

$$f(x) = \lambda\beta\left(\frac{x}{\alpha}\right)^{\beta-1} \exp\left[\left(\frac{x}{\alpha}\right)^{\beta} + \lambda\alpha\left(1 - e^{\left(\frac{x}{\alpha}\right)^{\beta}}\right)\right], \ \alpha, \lambda, \beta > 0, \ x \geq 0. \tag{2}$$

Then, the reliability function is

$$R(x) = \exp\left(\lambda\alpha\left(1 - e^{\left(\frac{x}{\alpha}\right)^{\beta}}\right)\right), \ \alpha, \lambda, \beta > 0, \ x \geq 0. \tag{3}$$

The corresponding failure rate function is

$$r(x) = \lambda\beta\left(\frac{x}{\alpha}\right)^{\beta-1} \exp\left(\left(\frac{x}{\alpha}\right)^{\beta}\right), \ \alpha, \lambda, \beta > 0, \ x \geq 0. \tag{4}$$

2.1. The Graphical Method

When we take $\lambda\alpha = 1$, the cdf of the modified Weibull distribution given in Eq. (1) is simplified to

$$R(x) = \exp\left(1 - e^{\left(\frac{x}{\alpha}\right)^{\beta}}\right), \ \alpha, \beta > 0, \ x \geq 0. \tag{5}$$

Then, according to Xie et al. (2002), we apply a transformation similar to Weibull transformation from which the following quantities can be written for a random sample of size *n*

$$y_i = \ln\left[\ln(1 - \ln(R(x_i)))\right] = \beta z_i - \beta \ln\alpha, \ z_i = \ln x_i, i = 1, \ldots, n. \tag{6}$$

Xie et al. (2002) used $1 - i/(n+1)$ as the estimated reliability function $\hat{R}(x_i)$ in Eq. (6). Afterwards, taking y_i as a dependent variable and z_i as an explanatory variable in simple linear regression analysis, the slope will be the graphical estimator of β which is given as follows

$$\hat{\beta}_{gr} = \frac{S_{zy}}{S_{zz}} \tag{7}$$

where

$$S_{zy} = \sum_{i=1}^{n} z_i y_i - \frac{\sum_{i=1}^{n} z_i \sum_{i=1}^{n} y_i}{n}, \ S_{zz} = \sum_{i=1}^{n} z_i^2 - \frac{\left(\sum_{i=1}^{n} z_i\right)^2}{n}.$$

Then, the graphical estimator of α is a function of the intercept and the slope which is

$$\hat{\alpha}_{gr} = \exp\left(\frac{\bar{y} - \hat{\beta}_{gr}\bar{z}}{-\hat{\beta}_{gr}}\right). \quad (8)$$

Finally, the graphical estimator of λ is

$$\hat{\lambda}_{gr} = 1/\hat{\alpha}_{gr} . \quad (9)$$

2.2. The 3-step MML Method

In order to derive the 3-step modified maximum likelihood (MML) estimators of the parameters of the modified Weibull distribution, we first assume that the shape parameter (β) is known. Then, we will explain the procedure to find the 3-step MML estimators of α , λ and β .

2.2.1. The MML Estimators of α and λ

If we obtain a random sample of size *n* from the pdf given in Eq. (2), the loglikelihood of this sample is

$$\ln L = n\ln\lambda + n\ln\beta + (\beta-1)\sum_{i=1}^{n}\ln\left(\frac{x_i}{\alpha}\right) + \sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta} + n\lambda\alpha - \lambda\alpha\sum_{i=1}^{n}\exp\left(\frac{x_i}{\alpha}\right)^{\beta} . \quad (10)$$

Then, the ML equations are

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n}{\lambda} + n\alpha - \alpha\sum_{i=1}^{n} e^{\left(\frac{x_i}{\alpha}\right)^{\beta}} = 0 , \quad (11)$$

$$\frac{\partial \ln L}{\partial \alpha} = n\lambda - \frac{n}{\alpha}(\beta-1) - \frac{\beta}{\alpha}\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta} + \lambda\beta\sum_{i=1}^{n} e^{\left(\frac{x_i}{\alpha}\right)^{\beta}}\left(\frac{x_i}{\alpha}\right)^{\beta} - \lambda\sum_{i=1}^{n} e^{\left(\frac{x_i}{\alpha}\right)^{\beta}} = 0 , \quad (12)$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\ln\left(\frac{x_i}{\alpha}\right) + \sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\ln\left(\frac{x_i}{\alpha}\right) - \lambda\alpha\sum_{i=1}^{n} e^{\left(\frac{x_i}{\alpha}\right)^{\beta}}\left(\frac{x_i}{\alpha}\right)^{\beta}\ln\left(\frac{x_i}{\alpha}\right) = 0 . \quad (13)$$

The ML equations do not admit explicit solutions. Solving them by iterations is indeed problematic for reasons of (i) multiple roots, (ii) non-convergence of iterations, and (iii) convergence to wrong values; see, for example, Puthenpura and Sinha (1986) and Vaughan (1992, 2002). In order to alleviate these problems, the MML equations are used by linearizing the intractable terms in the likelihood equations. The MML estimators are asymptotically equivalent to the ML estimators under some very general regularity conditions (Vaughan and Tiku 2000).

In order to derive the MML estimators of α and λ (for known β), we first write Eqs. (11) and (12) in terms of $g_1(z_i) = e^{z_i^{\beta}}$ and $g_2(z_i) = z_i^{\beta}$ ($z_i = x_i/\alpha$), then rewrite these functions in terms of the ordered statistics as $g_1(z_{(i)})$ and $g_2(z_{(i)})$, and replace them by the linear approximations of the first two terms of Taylor series expansion around $t_{(i)} = E(z_{(i)})$ to obtain the MML equations as follows

$$g_1(z_{(i)}) \cong C_{1i} + S_{1i}z_{(i)} \text{ and } g_2(z_{(i)}) \cong C_{2i} + S_{2i}z_{(i)}$$

where

$$C_{1i} = e^{t_{(i)}^{\beta}}(1-\beta)t_{(i)}^{\beta},\ C_{2i} = (1-\beta)t_{(i)}^{\beta},\ S_{1i} = \beta t_{(i)}^{\beta} e^{t_{(i)}^{\beta}} ,\ S_{2i} = \beta t_{(i)}^{\beta-1}.$$

The approximate values of $t_{(i)}$ can be obtained from the following equation

$$t_{(i)} \cong F^{-1}(i/(n+1), \beta),\ i = 1,\ldots,n \quad (14)$$

where $F(.,\beta)$ is the cdf of the standard Weibull distribution with shape parameter β (see Sürücü and Sazak (2009) for details). Solving the MML equations gives the following MML estimators

$$\hat{\lambda} = \frac{n}{\hat{\alpha}D + U - n\hat{\alpha}}, \tag{15}$$

$$\hat{\alpha} = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{16}$$

where

$$a = n^2 + Hn + Kn - Dn - HD,\ b = Fn + Pn + Wn - Un - DF - UH,\ c = Mn - UF,$$
$$D = \sum_{i=1}^{n} C_{1i},\ H = \sum_{i=1}^{n} C_{2i},\ K = \sum_{i=1}^{n} C_{1i}C_{2i},\ F = \sum_{i=1}^{n} S_{2i}x_{(i)},\ P = \sum_{i=1}^{n} C_{2i}S_{1i}x_{(i)},$$
$$W = \sum_{i=1}^{n} C_{1i}S_{2i}x_{(i)},\ U = \sum_{i=1}^{n} S_{1i}x_{(i)},\ M = \sum_{i=1}^{n} S_{1i}S_{2i}x_{(i)}^2.$$

2.2.2. The 3-step MML Estimators of α, λ and β

We will use a 3-step iteration technique to derive the MML estimators of α, λ and β. In the first iteration, by using the estimator of β produced by the graphical method, we obtain the first step MML estimators of α and λ which are given in Eqs. (15) and (16), and plug them in Eq. (17) which can be obtained by withdrawing β in Eq. (3) and replacing $R(x_i)$ by its estimator $\hat{R}(x_i)$ (we used $1 - i/(n+1)$ as the estimator of $R(x_i)$ depending on Xie, Tang, and Goh (2002)).

$$\hat{\beta}_i = \frac{\ln\left[\ln\left(1 - \frac{\ln(\hat{R}(x_i))}{\hat{\lambda}\hat{\alpha}}\right)\right]}{\ln\left(\frac{x_i}{\hat{\alpha}}\right)},\ i = 1,\ldots,n\ . \tag{17}$$

Then, the arithmetic mean of the values of $\hat{\beta}_i$ will be the first step MML estimator of β which is given below

$$\hat{\beta} = \frac{\sum_{i=1}^{n} \hat{\beta}_i}{n}. \tag{18}$$

We repeat this algorithm two more times to obtain the 3-step MML estimators of α, λ and β.

2.2.3. Some computational problems and solutions

We face some problematic situations in the calculation of the 3-step MML estimators. These situations happen rarely and thus the suggested modifications do not affect the general properties of the MML estimators. Here are the problematic situations and solutions:

(i) We face the minus square root domain problem when $b^2 - 4ac < 0$ or $a < 0$ which prevents the calculation of $\hat{\alpha}$. In such situations we use $\hat{\alpha}$ found in the previous step.

(ii) We very rarely obtain negative $\hat{\alpha}$ values. In such situations we again use $\hat{\alpha}$ found in the previous step.

(iii) The values of $\hat{\beta}_i$ rarely happen to be less than zero which may end up with negative $\hat{\beta}$ values. This causes a minus logarithm domain problem. Thus, when $\hat{\beta}_i < 0$, we equate it to zero, $\hat{\beta}_i = 0$.

(iv) When $\hat{\beta}_i$ is greater than 1.3, the value of some components become extremely big for the computer to handle the computations. Thus, when $\hat{\beta}_i > 1.3$, similar to the situation (iii), we equate it to zero, $\hat{\beta}_i = 0$.

(v) When more than 10 % of the $\hat{\beta}_i$ values are equated to zero, we rarely face a problem of a non-integer power of a negative value domain error. Thus, when more than 10 % of the $\hat{\beta}_i$ values are equated to zero, we use $\hat{\beta}$ found in the previous step.
(vi) When C_{1i} is greater than 1.9, the variances of the MML estimators become very large. Thus, when $C_{1i} > 1.9$, we equate it to zero, $C_{1i} = 0$.

2.3. The combined method

In order to increase the efficiency of the 3-step MML estimators, we propose the following combined estimators which are the arithmetic means of the 3-step MML and the graphical estimators

$$\hat{\theta}_{i,3} = \frac{\left(\hat{\theta}_{i,1} + \hat{\theta}_{i,2}\right)}{2}, \; i = 1,2,3. \tag{19}$$

Here, $\hat{\theta}_{1,j} = \hat{\alpha}_j$, $\hat{\theta}_{2,j} = \hat{\lambda}_j$, $\hat{\theta}_{3,j} = \hat{\beta}_j$; $j = 1$: 3-step, 2: Graphical, 3: Combined.

2.4. The Fisher information matrix and the Cramer-Rao lower bound

The Fisher information matrix for the modified Weibull distribution is (Kay 1993)

$$I = -E\begin{bmatrix} \frac{\partial^2 \ln L}{\partial \alpha^2} & \frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} & \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} \\ \frac{\partial^2 \ln L}{\partial \lambda \partial \alpha} & \frac{\partial^2 \ln L}{\partial \lambda^2} & \frac{\partial^2 \ln L}{\partial \lambda \partial \beta} \\ \frac{\partial^2 \ln L}{\partial \beta \partial \alpha} & \frac{\partial^2 \ln L}{\partial \beta \partial \lambda} & \frac{\partial^2 \ln L}{\partial \beta^2} \end{bmatrix} \tag{20}$$

It is extremely difficult to take the expected value of the second derivatives of the loglikelihood of the modified Weibull distribution since it includes integrations of many nonlinear functions of the observations. Instead, we took numerical integration of these functions by using Matlab. The diagonal elements of the inverse of this matrix give the Cramer-Rao lower bound (CRLB) values of the estimators under the modified Weibull distribution. We will give these values with the relevant tables for the comparison of the mean square error (mse) values of the estimators included in the study.

3. The simulation results

In this section, we conducted a simulation study to compare the performances of the estimators produced by the 3-step MML, the graphical and the combined methods for several situations. In this study, the simulations were conducted with sample sizes *n*=20, 50 and 100 for $\alpha = 50$, 200, 350, $\lambda = 0.005$, 0.01, 0.02, 0.04 and $\beta = 0.5$, 0.7, 1.1 for *nn*=[100000/*n*] Monte Carlo runs. In the simulation results, we also indicate the $\lambda\alpha$ values. For the mentioned values of α and λ, 10 different values of $\lambda\alpha$ are obtained which are 0.25, 0.50, 1, 1.75, 2, 3.5, 4, 7, 8 and 14. For each simulation run, we entered the mentioned values of the parameters and simulated samples from the modified Weibull distribution with these values through inverse cdf method and produced $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\beta}$ for the 3-step MML, the graphical and the combined methods by using the data values belonging to these samples and obtained their simulated means, biases, variances and mse's, and calculated the simulated relative efficiency (RE) and joint relative efficiency (JRE) of the estimators w.r.t. the estimators produced by the graphical method. The RE of $\hat{\theta}_{i,j}$ w.r.t. $\hat{\theta}_{i,k}$, both being the estimators of θ, can be given by the following formula

$$\mathrm{RE}(\hat{\theta}_{i,j} \mid \hat{\theta}_{i,k}) = 100\frac{mse(\hat{\theta}_{i,k})}{mse(\hat{\theta}_{i,j})},\ i=1,2,3;\ j,k=1,2,3. \tag{21}$$

Here, $mse(\hat{\theta}_{i,j}) = E(\hat{\theta}_{i,j}-\theta_{i,.})^2$, $i=1,2,3$; $\theta_{1,.}=\alpha$, $\theta_{2,.}=\lambda$, $\theta_{3,.}=\beta$; $\hat{\theta}_{1,j}=\hat{\alpha}_j$, $\hat{\theta}_{2,j}=\hat{\lambda}_j$, $\hat{\theta}_{3,j}=\hat{\beta}_j$; $j=1$: 3-step, 2: Graphical, 3: Combined.

The JRE of $\hat{\theta}_{.,j}$ w.r.t. $\hat{\theta}_{.,k}$ is

$$\mathrm{JE}(\hat{\theta}_{.,j} \mid \hat{\theta}_{.,k}) = \mathrm{RE}(\hat{\theta}_{1,j} \mid \hat{\theta}_{1,k}) + \mathrm{RE}(\hat{\theta}_{2,j} \mid \hat{\theta}_{2,k}) + \mathrm{RE}(\hat{\theta}_{3,j} \mid \hat{\theta}_{3,k}),\ j,k=1,2,3. \tag{22}$$

The steps of the simulation process are given as follows:

1. Create $U_{w,v} \sim \mathrm{U}(0,1)$, $w=1,..,n; v=1,..,nn.$
2. For given values of α,λ and β, the following sample values are obtained

$$X_{w,v} = \alpha\left(\ln\left(1-\frac{\ln(1-U_{w,v})}{\lambda\alpha}\right)\right)^{1/\beta},\ w=1,..,n; v=1,..,nn.$$

3. For each sample, the following estimators are calculated by using $X_{w,v}$

$\hat{\theta}_{i,j,v}$, $i=1,2,3;\ j=1,2,3; v=1,..,nn.$

In the simulation results, we observe that some mse values are less than the CRLB values because CRLB is based on the estimators which are unbiased but the estimators covered in this study are mostly biased. It is known that the biased estimators can have less mse values than the unbiased ones such as the estimators in ridge regression (Sazak 2019). There are also some simulation errors.

Although we simulated all the situations including three values of α, four values of λ and three values of β for three sample sizes *n*=20, 50 and 100 which makes totally 108 situations, for the sake of conciseness, we picked and gave the simulation results for 12 situations in Tables 1-4 to give a general idea about all the situations covered in the simulations. The picked situations include the cases of $\lambda\alpha$ with values 0.50, 1, 2 and 8 with three sample sizes *n*=20, 50 and 100. One can find the simulation results for all situations in Hussein Adam (2017).

Table 1 includes the simulation results for $\alpha=50$, $\lambda=0.01$ and $\beta=1.1$ ($\lambda\alpha=0.5$). In this situation, both the 3-step MML and the combined estimators of λ show very poor performance compared to the graphical estimator $\hat{\lambda}_{gr}$. The result is similar in the estimation of α although their performance is not that bad. On the contrary, the estimators of β for both are much better than the graphical estimator $\hat{\beta}_{gr}$. Here, we do not observe any improvement in the efficiency of the combined estimator $\hat{\beta}_c$ over the 3-step MML estimator $\hat{\beta}_{3-step}$. In this situation, the graphical estimators seem to get better as the sample size increases. We observe that all the estimators of α, λ and β produced upward bias but the bias produced by α is much greater. Additionally, in the estimation of λ, the bias produced by the graphical estimator $\hat{\lambda}_{gr}$ is much less than the others. The situation is similar for β but the bias of the graphical estimator $\hat{\beta}_{gr}$ is closer to the others.

Table 1: Simulation results for α=50, λ=0.005, β=0.5 (λα=0.25), *n*=20, 50 and 100.

		n=20			*n*=50			*n*=100		
	Meth.	α	λ	β	α	λ	β	α	λ	β
Mean	Grap.	319.70923	0.00380	0.56344	79.68130	0.01278	1.10968	76.85372	0.01312	1.14526
	3-MML	450.74091	0.00519	0.73117	99.59440	0.01941	1.14330	96.65137	0.01924	1.16480
	Comb.	385.22549	0.00449	0.64731	89.63782	0.01610	1.12649	86.75252	0.01618	1.15503
Bias	Grap.	269.70923	-0.00120	0.06344	29.68130	0.00278	0.00968	26.85372	0.00312	0.04526
	3-MML	400.74091	0.00019	0.23117	49.59440	0.00941	0.04330	46.65137	0.00924	0.06480
	Comb.	335.22549	-0.00051	0.14731	39.63782	0.00610	0.02649	36.75252	0.00618	0.05503
nxvar	Grap.	1716972	0.00003519	0.69287	7332	0.00013092	2.48088	5191	0.00013302	2.57529
	3-MML	2224240	0.00006780	0.78953	6626	0.00015227	0.79478	8079	0.00017101	1.17117
	Comb.	1647681	0.00004069	0.66178	5808	0.00010250	1.21488	5379	0.00009967	1.41861
nxmse	Grap.	3171834	0.00006401	0.77337	51381	0.00051811	2.48556	77303	0.00110552	2.78013
	3-MML	5436106	0.00006849	1.85828	129606	0.00457912	0.88851	225714	0.00871051	1.59107
	Comb.	3895204	0.00004583	1.09576	84366	0.00196062	1.24995	140454	0.00391855	1.72144
RE	3-MML	58.35	93.47	41.62	39.64	11.31	279.74	34.25	12.69	174.73
	Comb.	81.43	139.66	70.58	60.90	26.43	198.85	55.04	28.21	161.50
JRE	3-MML	193.43			330.70			221.67		
	Comb.	291.67			286.18			244.75		
CRLB		99221	0.00265404	9.79574						

The simulation results for $\alpha = 200$, $\lambda = 0.005$ and $\beta = 0.5$ ($\lambda\alpha = 1$) are given in Table 2. First we can see that the combined estimators show a significant improvement over the 3-step MML estimators but here there is more than that. Here, they are worse than the graphical estimators just in the estimation of α. Unfortunately, their joint efficiency dropped below 300 % because of the extremely low efficiency of $\hat{\alpha}_c$. All the estimators of α have upward bias similar to the previous situation but here it is interesting to observe that all the estimators of λ and β are almost unbiased.

Table 2: Simulation results for α=200, λ=0.005, β=0.5 (λα=1), *n*=20, 50 and 100.

		n=20			*n*=50			*n*=100		
	Meth.	α	λ	β	α	λ	β	α	λ	β
Mean	Grap.	277.25665	0.00442	0.44334	241.50327	0.00449	0.46038	222.68321	0.00468	0.47177
	3-MML	504.17465	0.00450	0.53898	456.18387	0.00419	0.51856	416.24667	0.00423	0.51256
	Comb.	390.71558	0.00446	0.49116	348.84415	0.00434	0.48947	319.46509	0.00446	0.49217
Bias	Grap.	77.25665	-0.00058	-0.05666	41.50327	-0.00051	-0.03962	22.68321	-0.00032	-0.02823
	3-MML	304.17465	-0.00050	0.03898	256.18387	-0.00081	0.01856	216.24667	-0.00077	0.01256
	Comb.	190.71558	-0.00054	-0.00884	148.84415	-0.00066	-0.01053	119.46509	-0.00054	-0.00783
nxvar	Grap.	618222	0.00006934	0.27917	320206	0.00007080	0.33399	248429	0.00008471	0.33877
	3-MML	1494509	0.00007881	0.34002	1699147	0.00007065	0.39684	1848258	0.00007549	0.46432
	Comb.	684662	0.00005852	0.24164	645761	0.00005285	0.28366	693725	0.00006306	0.32259
nxmse	Grap.	737594	0.00007618	0.34339	406332	0.00008382	0.41246	299881	0.00009493	0.41845
	3-MML	3344954	0.00008372	0.37041	4980655	0.00010326	0.41407	6524521	0.00013433	0.48010
	Comb.	1412111	0.00006435	0.24320	1753490	0.00007456	0.28921	2120916	0.00009259	0.32872
RE	3-MML	22.05	91.00	92.70	8.16	81.18	99.61	4.60	70.67	87.16
	Comb.	52.23	118.39	141.20	23.17	112.42	142.62	14.14	102.53	127.2
JRE	3-MML	205.75			188.95			162.42		
	Comb.	311.82			278.21			243.96		
CRLB		6514301	0.00017747	1.20894						

We give the simulation results for $\alpha = 200$, $\lambda = 0.01$ and $\beta = 0.7$ ($\lambda\alpha = 2$) in Table 3. It is very interesting to see that all the estimators of α possess downward bias which is very different from the previous situations while all the estimators of λ seem to be almost perfectly unbiased. In the estimation of β, although all the estimators produced downward bias, the graphical estimator produced the largest bias among the three. In this situation, both the 3-step MML and the combined estimators are better in almost every case. The only exception is the 3-step MML estimator of λ. Particularly, the 3-step MML and the combined estimators are extremely efficient w.r.t. the graphical estimators in the estimation of α. We also observe the positive effect of the increasing sample size on the 3-step MML and the combined estimators.

Table 3: Simulation results for α=200, λ=0.01, β=0.7 (λα=2), *n*=20, 50 and 100.

		n=20			*n*=50			*n*=100		
	Meth.	α	λ	β	α	λ	β	α	λ	β
Mean	Grap.	124.34526	0.00909	0.57969	110.80714	0.00944	0.60446	106.38354	0.00960	0.61510
	3-MML	214.18542	0.01047	0.68062	178.15956	0.01044	0.67833	166.10443	0.01047	0.68466
	Comb.	169.26541	0.00978	0.63015	144.48337	0.00994	0.64139	136.24399	0.01004	0.64988
Bias	Grap.	-75.65474	-0.00091	-0.12031	-89.19286	-0.00056	-0.09554	-93.61646	-0.00040	-0.08490
	3-MML	14.18542	0.00047	-0.01938	-21.84044	0.00044	-0.02167	-33.89557	0.00047	-0.01534
	Comb.	-30.73459	-0.00022	-0.06985	-55.51663	-0.00006	-0.05861	-63.75601	0.00004	-0.05012
nxvar	Grap.	67993	0.00017882	0.43132	31308	0.00019152	0.55560	25787	0.00019066	0.51159
	3-MML	265855	0.00031271	0.59388	169500	0.00027588	0.68556	94210	0.00023152	0.65953
	Comb.	112242	0.00019978	0.44520	70626	0.00018893	0.51337	41704	0.00015711	0.43507
nxmse	Grap.	182466	0.00019548	0.72083	429076	0.00020701	1.01196	902191	0.00020641	1.23247
	3-MML	269879	0.00031705	0.60139	193351	0.00028542	0.70905	209101	0.00025398	0.68306
	Comb.	131134	0.00020078	0.54276	224731	0.00018911	0.68510	448187	0.00015726	0.68629
RE	3-MML	67.61	61.66	119.86	221.92	72.53	142.72	431.46	81.27	180.43
	Comb.	139.14	97.36	132.81	190.93	109.47	147.71	201.30	131.25	179.58
JRE	3-MML	249.13			437.17			693.17		
	Comb.	369.31			448.11			512.14		
CRLB		4230824	0.00018173	1.59336						

Table 4 includes the simulation results for $\alpha = 200$, $\lambda = 0.04$ and $\beta = 1.1$ ($\lambda\alpha = 8$). In this situation, both the 3-step and the combined estimators are better than the graphical estimators in the estimation of all parameters and so for the joint efficiency. Additionally, the 3-step MML estimators are always better than the combined estimators. In this situation, all the estimators of α, λ and β possess downward bias.

Table 4: Simulation results for α=200, λ=0.04, β=1.1 (λα=8), *n*=20, 50 and 100.

		n=20			*n*=50			*n*=100		
	Meth.	α	λ	β	α	λ	β	α	λ	β
Mean	Grap.	50.36630	0.02097	0.82993	46.87023	0.02181	0.87424	45.82138	0.02203	0.90066
	3-MML	71.67850	0.02958	0.93631	67.11907	0.02867	0.93986	66.48744	0.02785	0.94055
	Comb.	61.02241	0.02527	0.88312	56.99459	0.02524	0.90705	56.15443	0.02494	0.92060
Bias	Grap.	-149.63370	-0.01903	-0.27007	-153.12978	-0.01819	-0.22576	-154.17862	-0.01797	-0.19934
	3-MML	-128.32150	-0.01042	-0.16369	-132.88092	-0.01133	-0.16014	-133.51256	-0.01215	-0.15945
	Comb.	-138.97758	-0.01473	-0.21688	-143.00540	-0.01476	-0.19295	-143.84557	-0.01506	-0.17940
nxvar	Grap.	3199	0.00046440	0.71507	2591	0.00049853	0.83234	2057	0.00045302	0.93091
	3-MML	10453	0.00136878	0.46290	8712	0.00186814	0.71371	12898	0.00242925	1.06876
	Comb.	5160	0.00071264	0.47258	3903	0.00080194	0.54219	4732	0.00083060	0.65693
nxmse	Grap.	451004	0.00770988	2.17383	1175027	0.01704936	3.38069	2379162	0.03273828	4.90469
	3-MML	339781	0.00354052	0.99879	891579	0.00828369	1.99591	1795459	0.01719053	3.61118
	Comb.	391456	0.00505036	1.41334	1026430	0.01169579	2.40366	2073887	0.02350746	3.87524
RE	3-MML	132.73	217.76	217.65	131.79	205.82	169.38	132.51	190.44	135.82
	Comb.	115.21	152.66	153.81	114.48	145.77	140.65	114.72	139.27	126.56
JRE	3-MML	568.14			506.99			458.77		
	Comb.	421.68			400.90			380.55		
CRLB		8357509	0.05236480	2.44291						

When we examine all the situations covered in the simulations, we cannot observe a general behaviour of the estimators because we see different results in terms of biases and efficiencies for different combinations of parameter values, but we can say that for low $\lambda\alpha$ values, we sometimes observe very low efficiency results for the 3-step MML and the combined estimators in the estimation of α, λ and β, but it does not generally happen simultaneously for these parameters. We also observe that they generally have higher efficiency values w.r.t. the graphical estimators for high $\lambda\alpha$ values.

As we mentioned earlier, in fact we conducted simulations for 108 different situations which makes totally 324 cases for all the parameters (since there are 3 parameters). Overall, we observe that the 3-step MML estimators are better than the graphical estimators in most of the situations and cases. The combined estimators are even better. In Table 5 we give the summary of the proportions and the percentages of the number of the times where they are better than the graphical estimators. In terms of the JRE values, the 3-step MML estimators are better in 69 % of the situations whereas the combined estimators are better in 75 % of the situations. For the RE values, the 3-step MML estimators are better in 61 % of the cases and the combined estimators are better in 75 % of the cases. Thus, the combined estimators made important improvement over the 3-step MML estimators not only in means of the efficiency values but also in the number of the situations and cases where they are better than the graphical estimators.

Table 5: The summary of the proportions and the percentages of the number of the times where the 3-step MML and the combined estimators are better than the graphical estimators.

Method	JRE		RE	
	Proportion	Percentage	Proportion	Percentage
3-Step MML	75/108	69	198/324	61
Combined	81/108	75	242/324	75

4. The real-life data applications

In this section, we will work on two real-life data which were commonly used by many scientists working in the reliability area. The analysis of both real-life data sets shows the superiority of the proposed methods over the graphical method.

4.1. The real-life data application 1

Aarset (1987) worked on a real-life data set which consists of the lifetimes of 50 devices which were put on a life test starting at time 0. Xie et al. (2002) concluded that this data set can be modelled by the modified Weibull distribution quite well. They used the graphical and the ML methods for the estimation of the parameters of the modified Weibull distribution. They obtained the ML estimates by using iterations, thus, let us call it ML iteration. We obtained the estimates produced by the 3-step MML and the combined methods for the comparison of the methods available so far. Table 6 consists of the estimates and the corresponding loglikelihood (ln$L$) and Akaike information criterion (AIC) values of the four methods mentioned here.

Table 6: The estimates and the corresponding ln$L$ and AIC values of the four methods mentioned in the study for the application 1.

Method	α	λ	β	ln$L$	AIC
ML iteration	110.09	0.0141	0.8408	-236.25	478.49
Graphical	110.09	0.0091	0.5326	-241.83	489.65
3-step MML	117.45	0.0157	0.8310	-236.78	479.55
Combined	113.76	0.0124	0.6818	-237.45	480.89

The ln$L$ and the AIC values of the 3-step MML, the combined and the ML iteration methods show that they are much better than the graphical method. Thus, the graphical method should be the last choice in this case. The ln$L$ and the AIC values of the 3-step MML, the combined and the ML iteration methods are very close to each other but depending on these values, the best is the ML iteration method and the worst is the combined method. Please note the problems we mentioned before in solving the ML equations by using iterations. The methods proposed in this paper are in explicit forms and no such problems are encountered in obtaining them. That is why we can comfortably prefer using the 3-step MML or the combined estimates when there are just marginal differences between their ln$L$ and AIC values and those belonging to the ML iteration method.

Now, we give the quantile-quantile (Q-Q) plots using the estimates produced by the four methods mentioned here in Fig. 1. The Q-Q plot depending on the graphical method is far from constituting a linear line and so the worst of all. The Q-Q plots produced by the 3-step MML, the combined and the ML iteration methods are very similar. These results are very consistent with the results obtained by using the ln$L$ and the AIC values. We also obtained the Kolmogorov-Smirnov (K-S) statistics and the corresponding p-values for the mentioned methods (see Chakravarti, Laha, and Roy (1967) for the details about the K-S goodness of fit test). The K-S test statistics and the corresponding p-values for the graphical, 3-step MML, combined and the ML with iteration methods are respectively 0.24580 (p-value=0.003805622), 0.21944 (p-value=0.01355062), 0.18801 (p-value=0.05077438) and 0.18113 (p-value=0.06594697). Although none of the values are perfect fits for the data set, the values of the graphical method are definitely the worst of all. The ML iteration method seems to be the best but we mentioned about the possible problems in using iteration methods and there is no big difference between the values of the combined method and the ML iteration method. Among the explicit methods, the values of the combined methods seem to be the best with a big margin.

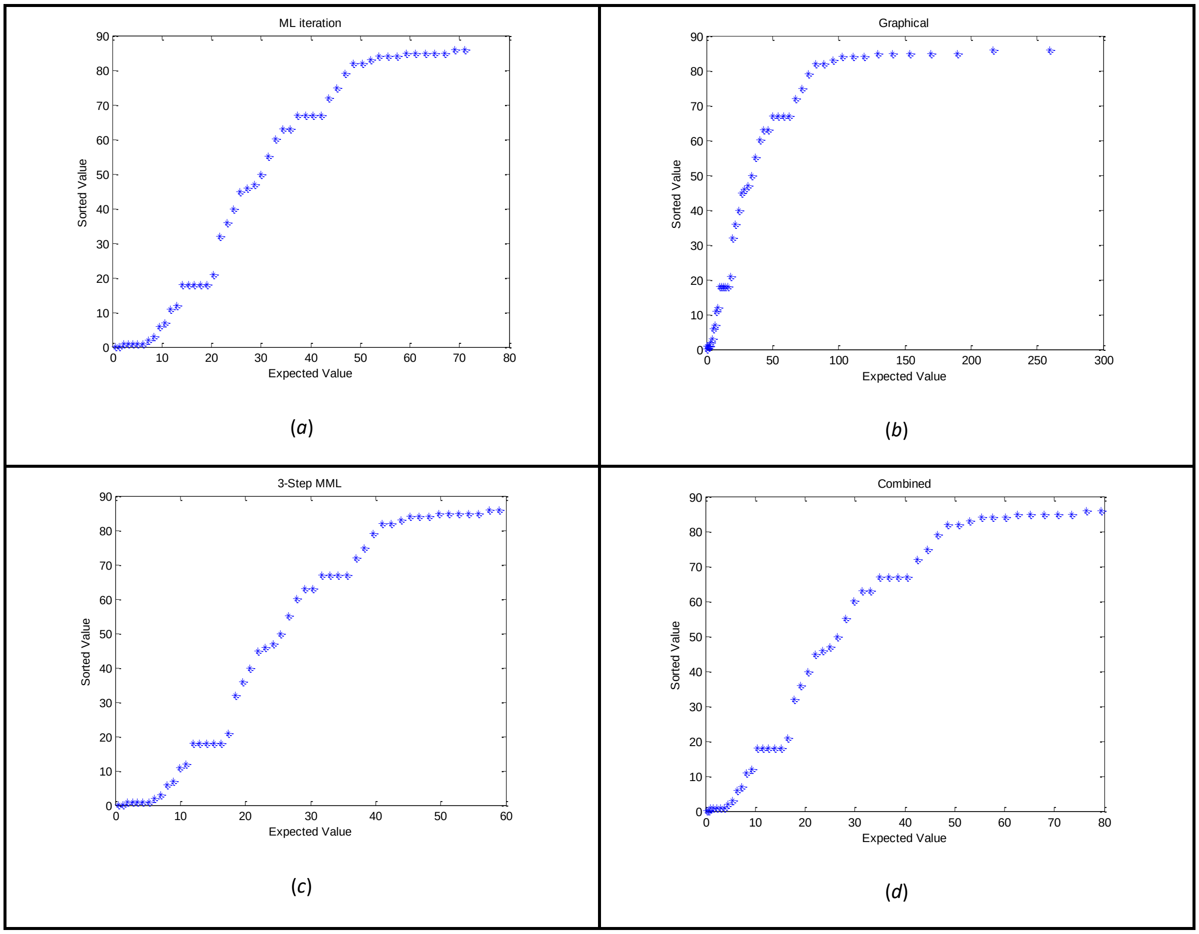

Figure 1: The Q-Q plots using the estimates produced by (*a*) the ML iteration method; (*b*) the graphical method; (*c*) the 3-step MML method; (*d*) the combined method for the application 1.

4.2. The real-life data application 2

As the second real-life data application we work on the data by Wang (2000) which consist of the failure times of 18 electronic devices. Xie et al. (2002) found that the modified Weibull distribution can comfortably be used for this data set. We obtained and gave the estimates produced by the 3-step MML, the graphical and the combined methods and the corresponding ln$L$ and AIC values in Table 7. Depending on the values of ln$L$ and AIC, the worst is the graphical method. The ln$L$ and the AIC values of the 3-step MML and the combined estimates are too close to make comments about.

Table 7: The estimates and the corresponding ln$L$ and AIC values of the three methods mentioned in the study for the application 2.

Method	α	λ	β	ln$L$	AIC
Graphical	349.15	0.0029	0.6802	-110.67	227.35
3-step MML	386.95	0.0042	1.0119	-109.50	225.01
Combined	368.05	0.0035	0.8440	-109.63	225.26

The Q-Q plots depending on the estimates of the 3-step MML, the graphical and the combined methods are given in Fig. 2. All the Q-Q plots look quite straight and thus acceptable but it is quite obvious that the Q-Q plot based on the graphical estimates is the worst. Although it is hard to discriminate between the other two, the Q-Q plot based on the combined estimates seems to be more straight which gives a clue that the combined estimates can be better. The K-S test statistics and the corresponding p-values for the graphical, 3-step MML and the combined methods are respectively 0.12301 (p-value=0.9178256), 0.11614 (p-value=0.9452766) and 0.08890 (p-value=0.9963178). Although the results of both methods are satisfactory, the best is the combined method by far and the graphical method is definitely the worst.

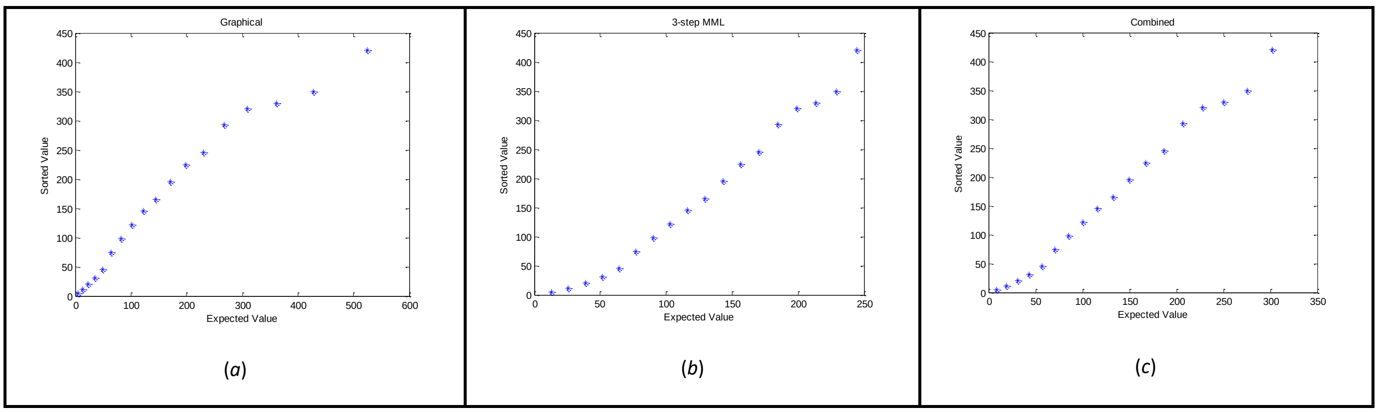

Figure 2: The Q-Q plots using the estimates produced by (a) the graphical method; (b) the 3-step MML method; (c) the combined method for the application 2.

5. Conclusion and suggestions

In this study, we have proposed two estimation methods called the 3-step MML and the combined methods for the parameters of the modified Weibull distribution with bathtub-shaped failure rate function. We conducted a very extensive simulation study including many parameter values with three sample sizes. The simulated efficiency results are very different for different combinations of the values of the parameters and so it is hard to make general inferences about the results, but we observed the superiority of the 3-step MML and the combined estimators over the graphical estimators in many situations. Additionally, the combined estimators showed a significant improvement over the 3-step MML estimators as it was planned. We also observed that the two are better than the graphical estimators in most of the situations and cases covered in this study. Moreover, similar to the results of the relative efficiencies, the combined estimators also showed significant improvement over the 3-step MML estimators in terms of the number of situations and cases where they are better than the graphical estimators. The graphical estimators are superior over the others just in the situations where $\lambda\alpha$ is less than 1. Thus, if there is a strong evidence that $\lambda\alpha$ is less than 1, one can prefer the usage of the graphical estimators. Otherwise, we suggest the general usage of the combined estimators. Surely, for a specific sample one has to check many other criteria since all data are original on their own. It is also hard to observe a general behaviour in the bias produced by the estimators, but in general the estimators of α seem to be more biased than the estimators of λ and β. Additionally, we did not observe any superiority of any method w.r.t. the bias they produced. At the end of the study, we gave two real-life data applications which show the clear superiority of both the 3-step MML and the combined estimators over the graphical estimators. In the first real-life data application, the ML iteration method gave just marginally better results than the 3-step MML and the combined methods. For this reason, one can comfortably prefer using the estimates produced by the 3-step MML or the combined methods taking also the problems arising in solving the ML equations by iterations into account. In the second real-life data application, the 3-step MML and the combined estimators gave very close results depending on the ln$L$ and the AIC values and the Q-Q plots, and thus, both are acceptable but the results of the K-S test statistic for the combined method were better, thus, we can suggest the usage of the estimates produced by the combined estimators for this data set.

Acknowledgments

We are thankful for the positive attitude of the editor and for the helpful comments of the reviewer which significantly improved the quality of the paper.

References

1. Aarset, M. V. (1987). How to identify a bathtub hazard rate. IEEE Transactions on Reliability, 36(1), 106-108.
2. Almalki, S. J., & Yuan, J. (2013). A new modified Weibull distribution. Reliability Engineering and System Safety, 111(C), 164-170.
3. Bebbington, M., Lai, C. D., & Zitikis, R. (2007). A flexible Weibull extension. Reliability Engineering and System Safety, 92(6), 719-726.
4. Chakravarti, I. M., Laha, R. G., & Roy, J. (1967). Handbook of Methods of Applied Statistics, vol I, 392-394. John Wiley and Sons, Hoboken.
5. Chen, Z. (2000). A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function. Statistics and Probability Letters, 49(2), 155-161.
6. Hussein Adam, A. A. (2017). Estimation in Reliability Models with Distribution Having Bathtub Shaped Hazard Function. PhD thesis, İzmir, Ege University.
7. Kay, S. M. (1993). Fundamental of Statistical Signal Processing Estimation Theory, Prentice Hall, New Jersey.
8. Lai, C.D., Xie, M., & Murthy, D. N. P. (2003). A modified Weibull distribution. IEEE Transactions on Reliability, 52(1), 33-37.
9. Peng, X., & Yan, Z. (2014). Estimation and application for a new extended Weibull distribution. Reliability Engineering and System Safety, 121, 34-42.
10. Puthenpura, S., & Sinha, N. K. (1986). Modified maximum likelihood method for the robust estimation of system parameters from noisy data. Automatica, 22(2), 231-235.
11. Sazak, H. S. (2019). The combined dynamically weighted modified maximum likelihood estimators of the location and scale parameters. Journal of Statistical Computation and Simulation, 89(5), 751-762.
12. Sürücü, B., & Sazak, H. S. (2009). Monitoring reliability for a three-parameter Weibull distribution. Reliability Engineering and System Safety, 94(2), 503-508.
13. Vaughan, D. C. (1992). On the Tiku-Suresh method of estimation. Communications in Statistics - Theory and Methods, 21(2), 451-469.
14. Vaughan, D. C. (2002). The generalized secant hyperbolic distribution and its properties. Communications in Statistics - Theory and Methods, 31(2), 219-238.
15. Vaughan, D. C., & Tiku, M. L. (2000). Estimation and hypothesis testing for a non-normal bivariate distribution and applications. Mathematical and Computer Modelling, 32(1-2), 53-67.
16. Wang, F. K. (2000). A new model with bathtub-shaped failure rate using an additive Burr XII distribution. Reliability Engineering and System Safety, 70(3), 305-321.
17. Xie, M., & Lai C. D. (1996). Reliability analysis using an additive Weibull model with bathtub-shaped failure rate function. Reliability Engineering and System Safety, 52(1), 87-93.
18. Xie, M., Tang, Y., & Goh, T. N. (2002). A modified Weibull extension with bathtub-shaped failure rate function. Reliability Engineering and System Safety, 76(3), 279-285.