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Remarks on the Papers by Coelho-Barros et al. (2017), Usman et al. (2021) and Obeid and Kadry (2022)

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Abstract

We would like to point out that the formula for the cumulative distribution given in Coelho-Barros et al. (2017) and a similar version of it given in Usman et al. (2021) are not cumulative distribution functions as these functions do not satisfy the one or more necessary and sufficient conditions for a function to be a cumulative distribution function. We would also like to mention that formulas for the cumulative distribution functions of product and ratio of two independent Pareto and Exponential random variables given by Obeid and Kadry (2022) are not cumulative distribution functions either. We do not believe that these formulas can be fixed to be cumulative distribution functions. In this short article, we provide mathematical justification in support of these claims.

Key Words: Cumulative distribution function; Distribution of the product and ratio; Monotonicity.

Mathematical Subject Classification: 60E10, 62E15, 62F10.

1. Introduction

The following formula is given in Coelho-Barros et al. (2017), equation (2.4) on page 153

$$F(x;\mu,\alpha,\lambda,p)=(1-p)\left\{1-\left[1+\lambda\left(\frac{x}{\mu}\right)^{\alpha}\right]^{-\frac{1}{\lambda}}\right\},\quad x\geq 0, \tag{1}$$

where $\mu>0,\alpha>0,\lambda>0,p\in(0,1)$ are parametrs.

Using the formula (1), Usman et al. (2021), proposed the following formula, equation (9) on page 697

$$F(x;\beta,\lambda,p)=(1-p)\left\{1-\exp\left[1-(1+\beta x)^{\lambda}\right]\right\},\qquad x\geq 0, \tag{2}$$

where $\beta>0,\lambda>0,p\in(0,1)$ are parameters.

Obeid and Kadry (2022) proposed the following formulas for the cumulative distribution functions of the product and the ratio of two independent Pareto and Exponential random variables, respectively

$$\begin{aligned} & F_P(x;a,c,\lambda) \\ = \quad & 1-\frac{a^c}{x^c\lambda^c}\Gamma(c+1)+\frac{a^c}{x^c\lambda^c}\Gamma\left(c+1,\frac{\lambda x}{a}\right)-e^{-\frac{\lambda x}{a}}, \quad x\geq 0, \end{aligned} \tag{3}$$

$$F_R(x;a,c,\lambda)=\begin{cases} e^{-\lambda a/x}-\frac{a^c\lambda^c}{x^c}\Gamma\left(1-c,\frac{\lambda a}{x}\right), & c<1 \\ e^{-\lambda a/x}-\frac{\lambda}{x}E_c(\lambda a/x) & , \; c=1,2,\ldots \end{cases}, \qquad x\geq 0, \tag{4}$$

where a,c,λ are all positive parameters and $E_c(\lambda a/x)=\left(\frac{\lambda a}{x}\right)^{c-1}\Gamma\left(c+1,\frac{\lambda x}{a}\right)$.

2. Remarks

(I) For the function in Eq. (1), clearly $\lim_{x\to\infty}F(x;\mu,\alpha,\lambda,p)=1-p\neq 1$ and $\lim_{x\to\infty}F(x;\beta,\lambda,p)=1-p\neq 1$. We believe there is no way that the formulas (1) and (2) can be fixed to make them cumulative distribution functions.

(II) For the function in Eq. (2), clearly $\lim_{x\to 0}F_P(x;a,c,\lambda)=1-\lim_{x\to 0}\frac{a^c}{x^c\lambda^c}\Gamma(c+1)+\frac{a^c}{\lambda^c}\lim_{x\to 0}\frac{\Gamma\left(c+1,\frac{\lambda x}{a}\right)}{x^c}-1=-\lim_{x\to 0}\frac{a^c}{x^c\lambda^c}\Gamma(c+1)+\frac{a^c}{\lambda^c}\lim_{x\to 0}\frac{\Gamma\left(c+1,\frac{\lambda x}{a}\right)}{x^c}=-\lim_{x\to 0}\frac{a^c}{x^c\lambda^c}\Gamma(c+1)+\frac{a^c}{\lambda^c}\left(\frac{0}{0}\right)$.
Using L'Hopital's rule, $\lim_{x\to 0}\frac{\Gamma\left(c+1,\frac{\lambda x}{a}\right)}{x^c}=0$, so $\lim_{x\to 0}F_P(x;a,c,\lambda)=-\infty$ instead of 0 and hence $F_P(x;a,c,\lambda)$ is not a cdf.

(III) For the function in Eq. (3), $\lim_{x\to 0}F_D(x;a,c,\lambda)=0-\lim_{x\to 0}\frac{a^c}{x^c\lambda^c}\Gamma(1-c,\frac{\lambda a}{x})=-\frac{a^c}{\lambda^c}\Gamma(1-c)\lim_{x\to 0}\frac{1}{x^c}=-\infty$. So, $F_D(x;a,c,\lambda)$ is not a cdf for $c<1$. Similarly for $c=1,2,3,\ldots$
Again, for the function in Eq. (3), one may also observe the fact that (for any set of $(a,c,\lambda)>0$), and for any $x_1<x_2$ for $(x_1,x_2)\in S(X)$

$$\frac{a^c}{(\lambda x_1)^c}\Gamma\left(c+1,\frac{\lambda x_1}{a}\right)\not<\not>\frac{a^c}{(\lambda x_2)^c}\Gamma\left(c+1,\frac{\lambda x_2}{a}\right). \tag{1}$$

In other words, the function as defined in Eq. (3), is not monotonically increasing always, another necessary condition to be a cdf.

Again, for Eq. (4), one may observe the following as well (after taking appropriate limits)

(a) For any $c<1$, $\lim_{x\to\infty}F_R(x;a,c,\lambda)\neq 1$.

(b) Similarly, for any $c=1,2,\cdots$ $\lim_{x\to\infty}F_R(x;a,c,\lambda)\neq 1$.

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