An EPQ Model for Delayed Deteriorating Items with Reliability Consideration, Quadratic Demand and Shortages

Dari Sani

* Corresponding Author

1. Department of Mathematical Sciences, Kaduna State University, Kaduna, Nigeria. sanisdari@yahoo.com

Abstract

In this paper, an EPQ model for items that exhibit delay in deterioration with partial backordered and reliability consideration is developed. It is assumed that there is no demand and deterioration during production buildup period. It is also supposed that the cost of a unit product is inversely related to the rate of demand and directly related to the process reliability. The demand before deterioration sets in is quadratic time dependent while demand after deterioration sets in is a constant. Shortages are allowed and partially backordered. The model developed will determine the best cycle length, EPQ and the total variable cost of the system. A numerical example will be given to see the applicability of the model and sensitivity analysis will also be carried out to see the effect of changes on some of the system parameters.

Key Words: Delayed deterioration; reliability consideration; quadratic demand; shortages.

Mathematical Subject Classification: 90B05

1.0 INTRODUCTION

Allowing for shortages has always been a standard operating procedure for some stockiest and production managers. The shortages either become lost sales or backordered. In the case of backorders, whenever the inventory runs out, the stockiest would record the backorders and the customer would simply have to wait for the item to be in-stock again. The backorders can either be full or partial, in which case a fraction of the shortages become lost sales. Many EOQ and EPQ policies were developed on the basis that the demand rate is always constant, but this is not always the case in real life situations. New products in the market and some passion goods are some of the items with their demand as either linear, quadratic or some other polynomial function of time.

Decisions made during the design and manufacture of a product influence the effectiveness of the product (product reliability). Dependence can be viewed as a link to integrate different stages of design, engineering, production, sales and after-sales service into an integrated process Cheng (1989). Hence, it is important to consider reliability consideration in studying inventory models. Many EOQ/EPQ models with deterioration were established on the supposition that the deterioration is instantaneous. However, this is not true in some situations. Apple, orange, cake, grains, vegetables and so on are examples of items with non-instantaneous deterioration. This paper presents an EPQ model for items with delay in deterioration, reliability consideration, quadratic demand and shortages are allowed but partially backordered. It is supposed that the cost of producing a unit item is inversely related to the demand of such item and directly related to the trustworthiness (reliability) of such item (as assumed by Tripathy et al. (2015) and modified by Dari and Sani (2015)). The EPQ model advanced in this paper will determine the optimum total variable cost, the inventory replenishment cycle, the total backorder quantity and the amount of economic production quantity (EPQ). Numerical example will be given and sensitivity analysis will also be carried out to see the effect of changes on some of the system parameters. The following table gives a summary of some related literature from 2012.
2.0 Literature Review
Zhang et al. (2016) invented a new order policy with partial order when ordered customers delayed shopping after the return of stock. They have created a model with the order of the parties, considering the delay in purchasing the ordered customers with the assumption that the number of previous requests is the same as the other orders. Mainly, they model the problem by presenting a new inventory cost section of holding the backordered items. Eduardo and Barron (2001), Chang (2004), Mahapatra et al. (2012), Pal and Chandra (2014), Sahoo and Tripathy (2018) also worked with inventory model with shortages.
Leung (2007) established a more general results by means of arithmetic-geometric mean inequality in which a general power function is projected to model the connection between production set-up cost and quality assurance. These two were considered as the independent variables while interest and depreciation cost was considered as dependent variable. The objective was to minimize the long-run expected average annual cost function. Porteus (1986), Charles (1987), Cheng (1989), Lee et al. (1996), Mettas (2000), Salameh and Jaber (2000), Huang (2004), Jaber et al. (2008), Jaber et al. (2009), Sakar et al. (2010), Dari and Sani (2013) and Dari and Sani (2020) also worked on inventory model with reliability consideration.
This paper presents an EPQ model for items with delay in deterioration, reliability consideration, quadratic demand and shortages are allowed but partially backorder. It is supposed that the cost of producing a unit item is inversely related to the trustworthiness (reliability) of such item (as assumed by Tripathy et al. (2015) and modified by Dari and Sani (2015)).
The following table gives a summary of some related literature from 2012.

Table 1. Summary of selected related literature from the year 2012

<table>
<thead>
<tr>
<th>Author(s) (Year of Publication)</th>
<th>Model Structure</th>
<th>Demand Pattern</th>
<th>Deterioration</th>
<th>Allowing shortages</th>
<th>Model Optimization</th>
<th>Reliability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panda et al. (2012)</td>
<td>EPQ</td>
<td>Quadratic</td>
<td>Constant</td>
<td>Yes</td>
<td>Cost</td>
<td>No</td>
</tr>
<tr>
<td>Dari and Sani (2013)</td>
<td>EPQ</td>
<td>Linear</td>
<td>Delayed</td>
<td>No</td>
<td>Cost</td>
<td>No</td>
</tr>
<tr>
<td>Dash (2014)</td>
<td>EPQ</td>
<td>Quadratic</td>
<td>Weibull</td>
<td>Yes</td>
<td>Profit</td>
<td>No</td>
</tr>
<tr>
<td>Singh and Rathore (2014)</td>
<td>EPQ</td>
<td>Exponential</td>
<td>Constant</td>
<td>No</td>
<td>Cost</td>
<td>Yes</td>
</tr>
<tr>
<td>Umakanta (2016)</td>
<td>EOQ</td>
<td>Quadratic</td>
<td>Weibull</td>
<td>Yes</td>
<td>Cost</td>
<td>No</td>
</tr>
<tr>
<td>Dari and Sani (2017)</td>
<td>EPQ</td>
<td>Quadratic</td>
<td>Delayed</td>
<td>No</td>
<td>Cost</td>
<td>No</td>
</tr>
<tr>
<td>Dari and Sani (2017)</td>
<td>EPQ</td>
<td>Quadratic</td>
<td>Delayed</td>
<td>Yes</td>
<td>Cost</td>
<td>No</td>
</tr>
<tr>
<td>Malik et al. (2017)</td>
<td>EOQ</td>
<td>Quadratic</td>
<td>Constant</td>
<td>No</td>
<td>Cost</td>
<td>No</td>
</tr>
<tr>
<td>Palanivel and Gowri (2018)</td>
<td>EPQ</td>
<td>Quadratic</td>
<td>Delayed</td>
<td>No</td>
<td>Cost</td>
<td>No</td>
</tr>
<tr>
<td>Sahoo and Tripathy (2018)</td>
<td>EOQ</td>
<td>Quadratic</td>
<td>Parabolic</td>
<td>Yes</td>
<td>Cost</td>
<td>No</td>
</tr>
<tr>
<td>Tripathi and Tomar (2018)</td>
<td>EOQ</td>
<td>Quadratic</td>
<td>Constant</td>
<td>No</td>
<td>Cost</td>
<td>No</td>
</tr>
<tr>
<td>Dari and Sani (2020)</td>
<td>EPQ</td>
<td>Quadratic</td>
<td>Delayed</td>
<td>No</td>
<td>Cost</td>
<td>No</td>
</tr>
<tr>
<td>Dari and Sani (2020)</td>
<td>EPQ</td>
<td>Quadratic</td>
<td>Delayed</td>
<td>No</td>
<td>Cost</td>
<td>Yes</td>
</tr>
<tr>
<td>Present Work</td>
<td>EPQ</td>
<td>Quadratic</td>
<td>Delayed</td>
<td>Yes</td>
<td>Cost</td>
<td>Yes</td>
</tr>
</tbody>
</table>

3.0 Model Formulation
The model is based on the following notation and assumptions:

3.1 Notation
\( Y(t) \) The inventory level during the first stage (production build up period)
\( Y_1(t) \) The inventory level during the second stage (period before deterioration sets in)
\( Y_2(t) \) The inventory level during the third stage (period after deterioration begins)
\( b(t) \) The inventory level during shortage time, which is to be backordered
3. The inventory cycle period
4. The period of first stage
5. The period of second stage
6. Shortage period
7. The total maximum inventory level at the end of production build up
8. The inventory level at the time deterioration begins
9. The rate of demand after production period and before deterioration sets in.
10. The rate of demand after deterioration sets in and before shortage period.
11. The deteriorating rate
12. Production rate of the items
13. The cost of unit of item (in Naira)
14. The cost of unit of backorder (in Naira)
15. Total backorder cost (Naira per production run)
16. The set-up cost (in Naira per production run)
17. The cost of holding a unit item
18. The inventory carrying charge
19. Total backorder (per production run)

3.2 Assumptions
1. The cost of production of an item ($C_o$) is assumed to be varies inversely to the demand rates ($d_1$ & $d_2$) but directly proportional to the reliability ($r$). We specifically assume the relationship to be:

$$C_o \propto r \left( d_1 \frac{n_2 - n_1}{n} + d_2 \frac{n_3 - n_2}{n} + \beta d_2 \frac{n - n_3}{n} \right)^{-1}$$

$$\therefore C_o = \alpha \left( 1 - r \right)^{-1} \left( d_1 \frac{n_2 - n_1}{n} + d_2 \frac{n_3 - n_2}{n} + \beta d_2 \frac{n - n_3}{n} \right)$$

where $\alpha > 0$ (as assumed by Tripathy et al. (2015) and modified by Dari and Sani (2015))

2. Shortages are allowed and partially backlogged with a fraction $\beta (0 < \beta < 1)$ of the demand, $d_2$

3. Unconstrained suppliers capital
4. All items are of good quality
5. Demand of product is always less than its supply
6. $n_1 \leq n_2 \leq n$
7. Instantaneous production
8. Demand before deterioration begins $d_1$, is assumed to be Quadratic and defined by

$$d_1 = \left( c_1 + c_2 t + c_3 t^2 \right), \text{ where } c_2 \& c_3 > 0$$

9. A constant demand, $d_2$ is considered after deterioration sets in
Inventory level

![Graphical representation of inventory system with shortages](image)

During the first stage of the production \(0 \leq t \leq n_1\) {i.e. production period}, it is assumed that no demand and no deterioration during this period. Thus, the inventory level depends only on the production rate \(p\) and equation (2) describes this situation.

\[
\frac{dY(t)}{dt} = p
\]  

(2)

Solving equation (2) by variable separable technics to obtain

\[
Y(t) = pt + L
\]  

(3)

From Fig 1, \(Y(t) = 0\), at \(t = 0\) and substituting into equation (3) to have

\(L = 0\)

Substituting the value of \(L\) into equation (3), we have

\(Y(t) = pt\)

(4)

We have from Fig 1 that \(Y(t) = I_o\) at \(t = n_1\). Replacing this into equation (4), we have

\[p = \frac{I_o}{n_1}\]

(5)

Substituting \(p\) from equation (5) into equation (4) to obtain

\[Y(t) = \frac{I_o}{n_1} t\]

(6)

During the second stage of the replenishment circle, it is assumed that no deterioration at that time and the inventory level at this stage will depend only on demand. Thus, the situation can be describes by:
\[ \frac{dY_1(t)}{dt} = -d_1 = -(c_1 + c_2t + c_3t^2) \]  

(7)

Solving equation (7) by variable separable technic, the solution obtain is

\[ Y_1(t) = -\left(c_1t + c_2 \frac{t^2}{2} + c_3 \frac{t^3}{3}\right) + v \]  

(8)

Also from Fig 1, we have \( Y_1(t) = I_o \) at \( t = n_1 \). Putting it into equation (8) to have

\[ v = I_o + \left(c_1n_1 + c_2 \frac{n_1^2}{2} + c_3 \frac{n_1^3}{3}\right) \]

Now, substituting \( v \) from the above equation into equation (8), we have

\[ Y_1(t) = c_1\left(n_1-t\right) + c_2\left(\frac{n_1^2}{2} - \frac{t^2}{2}\right) + c_3\left(\frac{n_1^3}{3} - \frac{t^3}{3}\right) + I_o \]  

(9)

Now, it can be observed from Fig 1 that we have the boundary condition \( Y_1(n_2) = I_1 \). By substituting this into equation (9), we have

\[ I_o = I_1 + c_1\left(n_2 - n_1\right) + c_2\left(\frac{n_2^2}{2} - \frac{n_1^2}{2}\right) + c_3\left(\frac{n_2^3}{3} - \frac{n_1^3}{3}\right) \]  

(10)

Putting equation (10) into equation (9), gives

\[ Y_1(t) = c_1\left(n_2-t\right) + c_2\left(\frac{n_2^2}{2} - \frac{t^2}{2}\right) + c_3\left(\frac{n_2^3}{3} - \frac{t^3}{3}\right) + I_1 \]  

(11)

During the third stage of the circle, it is assumed that the inventory level depends on rates demand and spoilage (deterioration). In this case, the following equation represents the situation:

\[ \frac{dY_2(t)}{dt} = -\lambda Y_2(t) - d_2 \]  

(12)

Solution of the above equation \( i.e \) (12) is

\[ Y_2(t) = -\frac{d_2}{\lambda} + fe^{-\lambda t} \]  

(13)

Also, applying the boundary condition \( Y_2(n_2) = I_1 \) into equation (13) to have

\[ f = \left(\frac{d_2}{\lambda} + I_1\right)e^{\lambda n_2} \]

Substituting \( f \) into equation (13) to obtain

\[ Y_2(t) = -\frac{d_2}{\lambda} + \left(\frac{d_2}{\lambda} + I_1\right)e^{\lambda(n_2-t)} \]  

(14)

From Fig 1, at \( t = n_3 \), \( Y_2(t) = 0 \). Substituting this into equation (14), gives

\[ I_1 = \frac{d_2}{\lambda} \left(e^{-\lambda(n_2-n_1)} - 1\right) \]  

(15)

Substituting \( I_1 \) into equation (10), we have
\[ I_o = \frac{d_2}{\lambda} \left( e^{-\lambda(n_2-n_1)} - 1 \right) + c_1 \left( n_2 - n_1 \right) + c_2 \left( \frac{n_2^2}{2} - \frac{n_1^2}{2} \right) + c_3 \left( \frac{n_2^3}{3} - \frac{n_1^3}{3} \right) \]  

(16)

Equation (16) is the maximum inventory produce during the production period.

Also, substituting \( I_1 \) into equation (11), we have

\[ Y_1(t) = c_1 \left( n_2 - t \right) + c_2 \left( \frac{n_2^2}{2} - \frac{t^2}{2} \right) + c_3 \left( \frac{n_2^3}{3} - \frac{t^3}{3} \right) + \frac{d_2}{\lambda} \left( e^{-\lambda(n_2-n_1)} - 1 \right) \]  

(17)

Now, substituting equation (16) into equation (6) to have

\[ Y(t) = \frac{d_2}{\lambda n_1} \left( e^{-\lambda(n_2-n_1)} - 1 \right) + c_1 \left( n_2 - n_1 \right) + \frac{c_2}{n_1} \left( \frac{n_2^2}{2} - \frac{n_1^2}{2} \right) + \frac{c_3}{n_1} \left( \frac{n_2^3}{3} - \frac{n_1^3}{3} \right) \]  

(18)

Also, substituting equation (15) into equation (14) to have

\[ Y_2(t) = \frac{d_2}{\lambda} \left( e^{\lambda(n_1-t)} - 1 \right) \]  

(19)

### 3.3 Backorders

During the shortage period \([n_3, n]\), the demand is partially backlogged with a fraction \( \beta (0 < \beta < 1) \) of the demand \( d_2 \). Therefore, the following equation represents the case:

\[ \frac{db(t)}{dt} = -\beta d_2 \]  

(20)

Solution of (19) is obtained as follows:

\[ b(t) = -\beta d_2 t + g \]  

(21)

Also, at \( t = n_3 \), \( b(t) = 0 \), so that equation(21) becomes

\[ g = \beta d_2 n_3 \]

Substituting \( g \) back into equation (21) gives

\[ b(t) = \beta d_2 (n_3 - t) \]

### 3.4 Number of Deteriorated Items

The total items that deteriorate \( d(n_2) \) is as follows:

\[ d(n_2) = I_1 - d_2 \left( n_3 - n_2 \right) = \frac{d_2}{\lambda} \left( e^{-\lambda(n_2-n_1)} - 1 \right) - d_2 \left( n_3 - n_2 \right) \]  

(22)

### 3.5 The Items Sold out During the Second Stage (Period Before Deterioration Sets-In)

The items sold out during the second stage is given by

\[ d_1 = \int_{n_1}^{n_2} \left( c_1 + c_2 t + c_3 t^2 \right) dt = c_1 \left( n_2 - n_1 \right) + \frac{c_2 \left( n_2^2 - n_1^2 \right)}{2} + \frac{c_3 \left( n_2^3 - n_1^3 \right)}{3} \]  

(23)

### 3.6 The Holding Cost (Inventory Carrying Cost)

The total cost of keeping the whole inventory is given by:
\[ H = iC_0 \left[ \int_0^{n_1} Y(t) dt + \int_{n_1}^{n_2} Y_1(t) dt + \int_{n_2}^{n_3} Y_2(t) dt \right] \] (where \( i \) is the inventory carrying charge) \hspace{1cm} (24)

Substituting equation (1), equation (17) to equation (19) into equation (24) to have

\[ H = \frac{ia(1-r)^{-1}}{d_1 \left( n_2 - n_1 \right) + d_2 \left( n_3 - n_2 \right) + \beta d_3 \left( n_3 - n_2 \right)} \int_0^{n_1} \left( \frac{d_2}{\lambda n_1} \left( e^{-\lambda(n_2-n_1)} - 1 \right) + \frac{c_1}{n_1} \left( n_2 - n_1 \right) + \frac{c_2}{n_1} \left( \frac{n_2^2}{2} - \frac{n_1^2}{2} \right) + \frac{c_3}{n_1} \left( \frac{n_2^3}{3} - \frac{n_1^3}{3} \right) \right) dt \]

\[ + \int_{n_1}^{n_2} \left( c_1 \left( n_2 - t \right) + c_2 \left( \frac{n_2^2}{2} - \frac{t^2}{2} \right) + c_3 \left( \frac{n_2^3}{3} - \frac{t^3}{3} \right) + d_2 \left( e^{-\lambda(n_2-n_1)} - 1 \right) \right) dt \]

\[ + i \int_{n_2}^{n_3} \left( \frac{d_2}{\lambda} \left( e^{\lambda(n_2-n_1)} - 1 \right) \right) dt \]

\[ \therefore H = \frac{ia(1-r)^{-1}}{d_1 \left( n_2 - n_1 \right) + d_2 \left( n_3 - n_2 \right) + \beta d_3 \left( n_3 - n_2 \right)} \left( \frac{d_2}{2\lambda} + \frac{d_2}{2\lambda} + \frac{d_2}{\lambda} + \frac{d_2}{\lambda^2} - \frac{d_2}{\lambda^2} \right) \]

\[ + c_1 \left( \frac{n_2^2 - n_1^2}{2} \right) + c_2 \left( \frac{4n_2^3 - 3n_1n_2^2 - n_1^3}{12} \right) + c_3 \left( \frac{3n_2^4 - 2n_1n_2^3 - n_1^4}{12} \right) \] \hspace{1cm} (25)

### 3.7 The Backorder Cost

The backorder cost per cycle \( C_B \) is calculated as:

\[ C_B = c_b \int_{n_1}^{n} \left( -b(t) \right) dt = c_b \beta d_2 \int_{n_1}^{n} \left( t - n_3 \right) dt = c_b \beta d_2 \left( \frac{t^2}{2} - n_3 t \right)_{n_1}^{n} \]

\[ = c_b \beta d_2 \left( \frac{n_2^2}{2} - nn_3 \right) - \left( \frac{n_2^2}{2} - n_3^2 \right) = c_b \beta d_2 \left( \frac{n_2^2}{2} - nn_3 + n_3^2 \right) \]

\[ = c_b \beta d_2 \left( n_2^2 - 2nn_3 + n_3^2 \right) = \frac{c_b \beta d_2}{2} \left( n - n_3 \right)^2 \] \hspace{1cm} (26)

### 3.8 Total Variable Cost Per Unit Time

The total variable cost per unit time is given as

\[ Z(n) = \frac{A}{n} + c_1 \beta d_1 \left( n - n_1 \right)^2 + a(1-r)^{-1} \left( d_1 \left( n_2 - n_1 \right) + d_2 \left( n_3 - n_2 \right) + \beta d_3 \left( n_3 - n_2 \right) \right)^{-1} \left[ \frac{d_2}{\lambda} \left( e^{-\lambda(n_2-n_1)} - 1 \right) - d_2 \left( n_2 - n_1 \right) \right] \]

\[ + ia(1-r)^{-1} \left( d_1 \left( n_2 - n_1 \right) + d_2 \left( n_3 - n_2 \right) + \beta d_3 \left( n_3 - n_2 \right) \right)^{-1} \left[ \frac{d_2}{2\lambda} + \frac{d_2}{2\lambda} + \frac{d_2}{\lambda} + \frac{d_2}{\lambda^2} - \frac{d_2}{\lambda^2} \right] \]

\[ + c_1 \left( \frac{n_2^2 - n_1^2}{2} \right) + c_2 \left( \frac{4n_2^3 - 3n_1n_2^2 - n_1^3}{12} \right) + c_3 \left( \frac{3n_2^4 - 2n_1n_2^3 - n_1^4}{12} \right) \] \hspace{1cm} (27)

### 3.9 Optimality Condition

To obtain the best circle length \( L \) which will optimize the total cost function per unit time \( Z(n) \), the necessary condition for optimality need to be taken we taken \( (i.e \ \frac{dZ(n)}{dn} = 0) \). The root of that equation gives the minimum provided

\[ \frac{d^2 Z(n)}{dn^2} > 0 \] (sufficient condition). The derivative is given as:

\[ \frac{d^2 Z(n)}{dn^2} > 0 \]
\[
\frac{dZ(n)}{dn} = \frac{A}{n^2} + c_0 \beta d_z \left(n^2 - n_i^2\right)
\]

\[
- \left(\frac{a \beta d_z (1 - r)}{d_z (n_2 - n_1) + d_z (n_3 - n_2) + \beta d_z (n - n_3)}\right) - \left(\frac{d_z}{\lambda} \left(e^{-\lambda (n_2 - n_1)} - 1\right) - d_z \left(n_1 - n_2\right)\right) - d_z \left(n_1 - n_2\right)
\]

\[
+ i \left(\frac{d_z}{\lambda} \left(e^{-\lambda (n_2 - n_1)} - 1\right) - d_z \left(n_1 - n_2\right)\right) - d_z \left(n_1 - n_2\right)
\]

\[
\left(\frac{d_z}{\lambda} \left(e^{-\lambda (n_2 - n_1)} - 1\right) - d_z \left(n_1 - n_2\right)\right) - d_z \left(n_1 - n_2\right)
\]

Setting the derivative in (28) to zero to have

\[
- \left[\frac{c_0 \left(\beta d_z \right)^2}{2} n^4 - \left[2c_0 \left(\beta d_z \right)^2 \left(d_z (n_2 - n_1) + d_z (n_3 [1 - \beta] - n_2)\right)\right] n^3
\]

\[
+ \left[\frac{d_z}{\lambda} \left(e^{-\lambda (n_2 - n_1)} - 1\right) - d_z \left(n_1 - n_2\right)\right] n^2
\]

\[
+ \left(\frac{2 \beta d_z \left(d_z (n_2 - n_1) + d_z (n_3 [1 - \beta] - n_2)\right)}{2} \left[2A + c_0 \left(\beta d_z \right)n_i^2\right]\right) n
\]

\[
+ \left[\frac{d_z}{\lambda} \left(e^{-\lambda (n_2 - n_1)} - 1\right) - d_z \left(n_1 - n_2\right)\right] n^2
\]

\[
+ \left[\frac{2 \beta d_z \left(d_z (n_2 - n_1) + d_z (n_3 [1 - \beta] - n_2)\right)}{2} \left[2A + c_0 \left(\beta d_z \right)n_i^2\right]\right] n
\]

\[
+ \left[\frac{d_z}{\lambda} \left(e^{-\lambda (n_2 - n_1)} - 1\right) - d_z \left(n_1 - n_2\right)\right] n^2
\]

\[
= 0
\]

Equation (29) can be viewed as:

\[
-P n^4 - Q n^3 + R n^2 + S n + T = 0
\]

where

\[
P = \left[2c_0 \left(\beta d_z \right)^2 \left(d_z (n_2 - n_1) + d_z (n_3 [1 - \beta] - n_2)\right)\right]
\]

\[
Q = \left[2c_0 \left(\beta d_z \right)^2 \left(d_z (n_2 - n_1) + d_z (n_3 [1 - \beta] - n_2)\right)\right]
\]

\[
R = \left[\frac{d_z}{\lambda} \left(e^{-\lambda (n_2 - n_1)} - 1\right) - d_z \left(n_1 - n_2\right)\right]
\]

\[
S = \left[2 \beta d_z \left(d_z (n_2 - n_1) + d_z (n_3 [1 - \beta] - n_2)\right) \left[2A + c_0 \left(\beta d_z \right)n_i^2\right]\right]
\]

\[
T = \left[\frac{d_z}{\lambda} \left(e^{-\lambda (n_2 - n_1)} - 1\right) - d_z \left(n_1 - n_2\right)\right]
\]
Lemma 1: If \( d_1 = \frac{d_2(n_3-n_2[1-\beta])}{(n_2-n_1)} \), then \( Q = S = T = 0 \), and \( P > 0 \)

**Proof:** Since \( d_1 = \frac{d_2(n_2-n_3[1-\beta])}{(n_2-n_1)} \), then

\[
Q = \left[ 2c_6(\beta d_2)^2 \left( d_1(n_2-n_1)+d_2(n_3[1-\beta]-n_2) \right) \right] = \left[ 2c_6(\beta d_2)^2 \left( \frac{d_2(n_2-n_3[1-\beta])}{(n_2-n_1)} + d_2(n_3[1-\beta]-n_2) \right) \right]
\]

\[= \left[ 2c_6(\beta d_2)^2 \left( d_2(n_2-n_3[1-\beta]) + d_2(n_3[1-\beta]-n_2) \right) \right] = 0
\]

Also,

\[
S = \left[ 2\beta d_2 \left( d_1(n_2-n_1)+d_2(n_3[1-\beta]-n_2) \right) \left( 2A + c_6(\beta d_2)n_2^2 \right) \right] = \left[ 2\beta d_2 \left( \frac{d_2(n_2-n_3[1-\beta])}{(n_2-n_1)} + d_2(n_3[1-\beta]-n_2) \right) \right]
\]

\[= \left[ 2\beta d_2 \left( d_1(n_2-n_3[1-\beta]) + d_2(n_3[1-\beta]-n_2) \right) \right] = 0
\]

Similarly,

\[
T = \left( d_1(n_2-n_1)+d_2(n_3[1-\beta]-n_2) \right)^2 \left( 2A + c_6(\beta d_2)n_2^2 \right) = \left( \frac{d_2(n_2-n_3[1-\beta])}{(n_2-n_1)} + d_2(n_3[1-\beta]-n_2) \right)^2 \left( 2A + c_6(\beta d_2)n_2^2 \right) = 0
\]

Finally, for

\[
R = \beta d_2 \left( \begin{array}{c}
-d_2 \left( n_2-n_1 \right)+d_2 \left( n_3[1-\beta]-n_2 \right) \\
+2a(1-r)^{-1} \\
+2a(1-r)^{-1} \\
+\frac{d_2}{\lambda} \left( e^{-\lambda(n_2-n_1)}-1 \right)
\end{array} \right)
\]

\[
= \beta d_2 \left( \begin{array}{c}
-d_2 \left( n_2-n_1 \right)+d_2 \left( n_3[1-\beta]-n_2 \right) \\
+2a(1-r)^{-1} \\
+2a(1-r)^{-1} \\
+\frac{d_2}{\lambda} \left( e^{-\lambda(n_2-n_1)}-1 \right)
\end{array} \right)
\]
Theorem 1: If the conditions in Lemma 1 are true, then the solution of Equation (30) is

\[ n^* = \sqrt{\frac{R}{P}} \]

Proof. The solution of Equation (30) is as follows

\[ -Pn^4 - Qn^3 + Rn^2 + Sn + T = (-Pn^2 + R)n^2 = 0 \quad \text{[From Lemma 1]} \]

\[ \Rightarrow \quad \text{either } n^* = 0, -\sqrt{\frac{R}{P}}, \text{ or } \sqrt{\frac{R}{P}} \]

But

\[ -\sqrt{\frac{R}{P}} < 0 \quad \text{[since } P, R > 0, \text{ from Lemma 1]} \]

Since the length of the cycle is positive, then

\[ n^* = \sqrt{\frac{R}{P}} \quad \text{(31)} \]

Theorem 2: With the hypotheses of Lemma 1, the total cost function \( Z(n) \) is a convex function of \( n \).

Proof: To check for the sufficient condition of optimality of \( Z(n) \), we look for \( \frac{d^2 Z(n)}{dn^2} \), that is

\[
\frac{d^2 Z(n)}{dn^2} = \frac{2A}{n^3} + \frac{c_3(\beta d_2)^2 n^2}{n^3} \left[ \frac{d_z}{\lambda} (e^{-\lambda (n-n_1)} - 1) - d_z (n_3 - n_2) \right] + i \left[ \frac{d_z}{\lambda} (e^{-\lambda (n-n_1)} - 1) - d_z (n_3 - n_2) \right] + \frac{2a(\beta d_2)^2 (1-r)^{-1}}{(d_z(n_2-n_1) + d_z(n_1-n_2) + \beta d_z (n-n_1))} \left[ \frac{d_z}{\lambda} (e^{-\lambda (n-n_1)} - 1) - d_z (n_3 - n_2) \right]
\]

\[
+ i \left[ \frac{d_z}{\lambda} (e^{-\lambda (n-n_1)} - 1) - d_z (n_3 - n_2) \right] + d_z (n_2) + d_z (n_3) + d_z (n_4) - \frac{d_z}{\lambda} + \frac{d_z}{\lambda^2} \left[ \frac{d_z}{\lambda} (e^{-\lambda (n-n_1)} - 1) - d_z (n_3 - n_2) \right]
\]

Applying Lemma 1 into equation (32), the equation becomes

\[
\frac{d^2 Z(n)}{dn^2} = \frac{2A}{n^3} + \frac{c_3(\beta d_2)^2 n^2}{n^3} \left[ \frac{d_z}{\lambda} (e^{-\lambda (n-n_1)} - 1) - d_z (n_3 - n_2) \right] + i \left[ \frac{d_z}{\lambda} (e^{-\lambda (n-n_1)} - 1) - d_z (n_3 - n_2) \right] + \frac{2a(\beta d_2)^2 (1-r)^{-1}}{(d_z(n_2-n_1) + d_z(n_1-n_2) + \beta d_z (n-n_1))} \left[ \frac{d_z}{\lambda} (e^{-\lambda (n-n_1)} - 1) - d_z (n_3 - n_2) \right]
\]

\[ + i \left[ \frac{d_z}{\lambda} (e^{-\lambda (n-n_1)} - 1) - d_z (n_3 - n_2) \right] + d_z (n_2) + d_z (n_3) + d_z (n_4) - \frac{d_z}{\lambda} + \frac{d_z}{\lambda^2} \left[ \frac{d_z}{\lambda} (e^{-\lambda (n-n_1)} - 1) - d_z (n_3 - n_2) \right]
\]

\[ > \frac{2A}{n^3} + \frac{c_3(\beta d_2)^2 n^2}{n^3} \geq 0 \]
\[ \therefore \quad \frac{d^2 Z(n)}{dn^2} > 0 \]

Therefore, we conclude that \( Z(n) \) is a minimization function of \( n \).

### 3.10 EPQ of the Model

The \( EPQ \) of the corresponding best cycle length \( n^* \) will be computed from:

\[
EPQ^* = \text{total demand before deterioration starts} + \text{total demand after deterioration starts} + \text{total number of deteriorated items} + \text{Total demand during shortage period}
\]

\[
= c_1 (n_2 - n_1) + \frac{c_2 (n_2^2 - n_1^2)}{2} + \frac{c_3 (n_2^3 - n_1^3)}{3} + d_2 (n_3 - n_2) + \frac{d_2}{\lambda} \left( e^{-\lambda(n_3-n_1)} - 1 \right) - d_2 (n_3 - n_2) + \frac{\beta d_2}{2} (n - n_3)^2
\]

\[
\Rightarrow \quad EPQ^* = c_1 (n_2 - n_1) + \frac{c_2 (n_2^2 - n_1^2)}{2} + \frac{c_3 (n_2^3 - n_1^3)}{3} + \frac{d_2}{\lambda} \left( e^{-\lambda(n_3-n_1)} - 1 \right) + \frac{\beta d_2}{2} (n - n_3)^2
\]  

### 4.0 Numerical Example

The parameters used in this example are improved from Dari and Sani (2020) by adding the value for the parameter \( \beta \) which was not reflected in their Model.

- \( d_2 = 2100 \text{ units per unit time} \)
- \( A = \text{₦}3300 \text{ per production run} \)
- \( i = 0.1 \text{ per unit per unit time} \)
- \( \lambda = 0.2 \text{ in a unit time} \)
- \( \beta = 0.9 \text{ in a unit time} \)
- \( n_1 = 0.05753 \text{ year} \)
- \( n_2 = 0.08767 \text{ year} \)
- \( n_3 = 0.13699 \text{ year} \)
- \( a = 200 \)
- \( r = 0.7 \)
- \( c_1 = 8400 \)
- \( c_2 = 4 \)
- \( c_3 = 3 \)

Replacing these values into Equations (27), (31), (26) and (33), we get the best variable price per year, the best cycle length, the cost of backorder and the \( EPQ \) value per year respectively as

\[
Z = \text{₦}17907.54 \text{ per year}, \quad n^* = 0.23288 \text{ year}, \quad C_b = \text{₦}868.9 \text{ per year and} \quad EPQ = 365.93 \text{ units}
\]
5.0 Sensitivity Analysis

We examine the impact of changes in certain parameters of the system in example 1, and we obtain the following results.

**Table 1: Sensitivity analysis to see the effect of changes in some of the parameters**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>% change in parameter</th>
<th>% change in $Z$</th>
<th>% change in $n^*$</th>
<th>% change in $C_B$</th>
<th>% change in $EPQ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>+50</td>
<td>36.9649616</td>
<td>14.11764737</td>
<td>80.32653063</td>
<td>1.907449432</td>
</tr>
<tr>
<td></td>
<td>+25</td>
<td>19.12906405</td>
<td>7.058823469</td>
<td>37.22448982</td>
<td>0.88392865</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>-25</td>
<td>-20.75991819</td>
<td>-9.41764625</td>
<td>-40.48979592</td>
<td>-0.961458579</td>
</tr>
<tr>
<td></td>
<td>-50</td>
<td>-43.65084226</td>
<td>-18.82352925</td>
<td>-70.53061224</td>
<td>-1.674832315</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>+50</td>
<td>0.003302465</td>
<td>0</td>
<td>0</td>
<td>0.070369957</td>
</tr>
<tr>
<td></td>
<td>+25</td>
<td>0.001649864</td>
<td>0</td>
<td>0</td>
<td>0.035143986</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>-25</td>
<td>-0.001647184</td>
<td>0</td>
<td>0</td>
<td>-0.03508933</td>
</tr>
<tr>
<td></td>
<td>-50</td>
<td>-0.003291631</td>
<td>0</td>
<td>0</td>
<td>-0.070124004</td>
</tr>
<tr>
<td>$r$</td>
<td>+50</td>
<td>0.22205</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>+25</td>
<td>0.044421</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>-25</td>
<td>-0.01169</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>-50</td>
<td>-0.01709</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$d_2$</td>
<td>+50</td>
<td>7.788385</td>
<td>-11.7647</td>
<td>-23.4694</td>
<td>13.66344</td>
</tr>
<tr>
<td></td>
<td>+25</td>
<td>4.413517</td>
<td>-7.05882</td>
<td>-14.1837</td>
<td>6.77358</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>-25</td>
<td>-6.0849</td>
<td>9.411765</td>
<td>13.20408</td>
<td>-6.76968</td>
</tr>
<tr>
<td>$i$</td>
<td>+50</td>
<td>0.012603</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>+25</td>
<td>0.006302</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>-25</td>
<td>-0.0063</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>-50</td>
<td>-0.0126</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n_1$</td>
<td>+50</td>
<td>-0.01495</td>
<td>0</td>
<td>0</td>
<td>-66.0392</td>
</tr>
<tr>
<td></td>
<td>+25</td>
<td>-0.00723</td>
<td>0</td>
<td>0</td>
<td>-33.0194</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>-25</td>
<td>0.006516</td>
<td>0</td>
<td>0</td>
<td>33.0192</td>
</tr>
<tr>
<td></td>
<td>-50</td>
<td>0.012162</td>
<td>0</td>
<td>0</td>
<td>66.03817</td>
</tr>
<tr>
<td>$n_2$</td>
<td>+50</td>
<td>0.034018</td>
<td>0</td>
<td>0</td>
<td>75.33797</td>
</tr>
<tr>
<td></td>
<td>+25</td>
<td>0.012408</td>
<td>0</td>
<td>0</td>
<td>37.64096</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>-25</td>
<td>-0.00487</td>
<td>0</td>
<td>0</td>
<td>-37.5847</td>
</tr>
<tr>
<td></td>
<td>-50</td>
<td>-0.18471</td>
<td>0</td>
<td>0</td>
<td>-74.6883</td>
</tr>
<tr>
<td>$n_3$</td>
<td>+50</td>
<td>-23.6602</td>
<td>18.82353</td>
<td>-44.8163</td>
<td>38.90614</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>-25</td>
<td>16.68105</td>
<td>-7.05882</td>
<td>40.59184</td>
<td>-18.8169</td>
</tr>
<tr>
<td></td>
<td>-50</td>
<td>37.78834</td>
<td>-14.1176</td>
<td>88.08163</td>
<td>-37.3349</td>
</tr>
<tr>
<td>$c_b$</td>
<td>+50</td>
<td>7.793762</td>
<td>-11.7647</td>
<td>-23.4694</td>
<td>-1.16306</td>
</tr>
<tr>
<td></td>
<td>+25</td>
<td>4.416491</td>
<td>-7.05882</td>
<td>-14.1837</td>
<td>-0.74436</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>-25</td>
<td>-6.08878</td>
<td>9.411765</td>
<td>13.20408</td>
<td>1.209598</td>
</tr>
<tr>
<td></td>
<td>-50</td>
<td>-15.1832</td>
<td>28.23529</td>
<td>42.08163</td>
<td>4.373158</td>
</tr>
</tbody>
</table>
From Table 1, these can be observed:

i. As $A$ rises, $Z$, $n^*$, $C_B$ and $EPQ$ all rise. Naturally, any increment in the set-up cost will result to an increase to the total variable cost. Therefore, the model will try to increase the best cycle length and the $EPQ$ so as to minimize the set-up cost.

ii. As $\lambda$ rises, $Z$ rises, $EPQ$ declines whereas $n^*$ and $C_B$ remain constant. This is usually expected because if the rate of deterioration high, the model seeks to produce little items to avoid the spoilage. Thus, many items are going to be produced ($EPQ$) and so $Z$ is increased.

iii. As $r$ rises, $Z$ rises whereas $EPQ$, $n^*$ and $C_B$ remain constant. Naturally, the higher the reliability of an item, the higher the cost of such item, thus the total variable cost will be high. Therefore, to reduce the cost, the model seeks to maintain the optimal cycle length, the optimal total backorder cost and the optimal $EPQ$ so as to increase the revenue.

iv. As $d_2$ rises, $Z$ and $EPQ$ rise whereas $n^*$ and $C_B$ decline. This is obvious case because if the rate of demand of an item is high, the producer tends to produce more ($EPQ$) and $Z$ will ultimately increase. Therefore, the model pursues to maintain a smaller cycle size and the backorder cost will eventually reduce.

v. As $i$ rises, $Z$ rises while $EPQ$, $n^*$ and $C_B$ remain constant. That must be the case since if $i$ is high, it will amount to an increase in the total variable cost. Consequently, to lessen the cost, the model pursues a constant cycle size, total backorder cost and $EPQ$.

vi. As $n_2$ rises, $Z$ and $EPQ$ decline while $n^*$ and $C_B$ stay constant (may be due to the value of the range we considered in our study). This is so probably because if the total variable cost per unit time is small, the producer will spend more time in producing his goods (few items with high reliability). Therefore, $EPQ$ will also be smaller while the total cost of backorder will be a constant.

vii. As $n_3$ rises, $Z$ and $EPQ$ rise while $n^*$ and $C_B$ stay constant. This is so probably because if the total variable cost per unit time is high, the producer will produce more goods (with high reliability). Therefore, the model seeks to maintain the optimal cycle length and the best total backorder cost so as to increase the revenue.

viii. As $n_3$ rises, $Z$ and $C_B$ decline while $L^*$ and $EPQ$ rise. This is probably since as the period of backorder $n$ is high, more items are going to be produced so as to reduce the total variable cost. Hence, the total variable cost and the backorder cost will reduce.

ix. As $c_B$ rises, $Z$ rises, $n^*$, $C_B$ and $EPQ$ decline. This is obvious because if the unit cost of a backorder is increases, the $Z$ will eventually increases. Therefore, the model pursues a smaller cycle length, lower total backorder cost and fewer items are going to be produce so as to reduce the total variable cost.

6.0 Conclusions

In this paper, an EPQ model for items with non-instantaneous deterioration, reliability consideration and shortages is developed. A numerical example and a careful study were conducted to see the impact of the changes on some of the system parameters. From the results of the study, it has been shown that:

I. The decision variables $C_B$, $Z$ and $n^*$ are more affected by $A$, $d_2$, $n_3$ and $c_B$ while they are less affected by $r$, $n_1$, $n_2$, $i$ and $\lambda$.

II. On the other hand, the decision variable $EPQ$ is more sensitive to the changes in $n_1$, $n_2$, $n_3$ and $d_2$ while they are less sensitive to changes in $r$, $A$, $i$ and $\lambda$.

III. $EPQ$, $C_B$, $Z$ and $n^*$ all rise with rises in $A$, but $Z$ rises and $n^*$ declines with rise in both $d_2$ and $c_B$. 

An EPQ Model for Delayed Deteriorating Items with Reliability Consideration, Quadratic Demand and Shortages

487
REFERENCES


