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A New Pareto Model: Risk Application, Reliability MOOP and PORT Value-at-Risk Analysis

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Abstract
The paper introduces a new reliability Burr Pareto type-II model, showcasing its versatility and effectiveness in engineering applications, particularly in analyzing the failure and service times of aircraft windshields. The BUPII model's application in failure analysis offers insights into the probabilistic behavior of windshield failures, aiding in risk prediction and management. Similarly, its extension to service time analysis demonstrates its utility in optimizing maintenance schedules and operational efficiency. Moreover, the paper conducts a rigorous mean-of-order P analysis under both failure and service time datasets, validating the new model's reliability assessment capabilities. Furthermore, employing the peaks over random threshold value at risk analysis highlights the model's practical relevance in quantifying financial risks associated with extreme events. Overall, the novel probability distribution emerges as a valuable tool for engineers and researchers involved in reliability and risk analysis, promising advancements in understanding and managing the reliability of engineering systems. Future research could explore broader applications and refined methodologies to further enhance predictive capabilities and decision-making support.

Key Words: Mean-of-order P; Pareto type-II; peaks over random threshold value at risk; risk analysis; reliability data; optimal order of P.

1. Introduction

In the realm of reliability engineering, the development of robust probability distributions that accurately assess and predict failure probabilities plays a pivotal role in ensuring the safety and longevity of critical systems. In this paper, the Burr Pareto type II (BUPII) probability distribution is introduced as a promising addition to this field, offering innovative insights into the analysis of mean-of-order P (MOOP) and Peaks Over Random Threshold Value-at-Risk (PORT VaR). This probability distribution represents a significant advancement in reliability analysis, providing engineers and decision-makers with a powerful toolset to evaluate and mitigate risks associated with complex systems, such as aircraft windshields. The BUPII probability distribution distinguishes itself by its capability to handle diverse datasets related to reliability, including the complex failure and service times observed in aerospace components like aircraft windshields. Through its integration of advanced statistical methodologies and probabilistic frameworks, the BUPII probability distribution offers a comprehensive approach to quantify and manage risks associated with operational failures. This paper delves into the application of the BUPII probability distribution across MOOP and PORT VaR analyses, demonstrating its utility in enhancing decision-making processes within reliability engineering. Detailed case studies and empirical analyses showcased in this paper illustrate how the BUPII probability distribution enhances predictive accuracy and optimizes maintenance strategies in aerospace applications. By elucidating the probability distribution's theoretical foundations and practical applications, this introduction lays the groundwork for an in-depth exploration of its implications in improving reliability assessments across critical infrastructure systems. In essence, the BUPII probability distribution represents a significant leap forward in reliability engineering. It offers novel perspectives on MOOP and PORT VaR analyses, essential for ensuring the safety, efficiency, and reliability of modern technological systems, particularly within the aerospace industry. This paper aims to elucidate the transformative potential of the BUPII probability distribution in shaping the future of reliability analysis and fostering innovation in engineering practices (for more details see Gomes et al. (2015), Figueiredo et al. (2017), Duthinh et al. (2017) and Caeiro et al. (2021)).

In the literature review, the MOOP and PORT VaR are used in many practical fields such as insurance, re-insurance, medicine, loss analysis, risk analysis, among others. Klugman et al. (2012) and Embrechts et al. (2013) offer fundamental insights into loss modeling and extremal events, respectively, laying the groundwork for understanding risk assessments like PORT VaR. Jansen and de Vries (1991) and Poon and Rockinger (2003) explore the frequency and distribution of extreme events in financial contexts, relevant to both MOOP and PORT VaR analyses. McNeil et al. (2015) and Beirlant et al. (2004) provide advanced methodologies and applications in risk management and extreme value theory, crucial for applying MOOP and PORT VaR in practice. Hosking and Wallis (1987) contribute further by detailing heavy-tail phenomena and quantile estimation methods essential for assessing tail risks in reliability engineering. Together, these references offer a robust foundation for understanding and implementing MOOP and PORT VaR in reliability engineering and risk management contexts. Recent advancements in statistical modeling and risk analysis have led to the development of several innovative approaches to assess and manage risks in various domains, including insurance claims, reliability engineering, and financial forecasting. Models such as the odd log-logistic Weibull family (Rasekhi et al. (2020 and 2022)) and the compound Lomax probability distribution (Hamed et al. (2022)) have expanded the repertoire of tools available for modeling complex data distributions and predicting extreme events. These probability distributions incorporate copulas and asymmetric density functions (Shrahili et al. (2021); Mohamed et al. (2024)) to enhance accuracy in risk estimation and improve decision-making processes. Additionally, novel approaches like the reciprocal Weibull extension (Yousof et al. (2023d)) and the Lindley Extension (Hashempour et al., 2023) offer robust frameworks for analyzing extreme values and assessing risk under specific data conditions. Furthermore, the application of these probability distributions extends beyond traditional risk assessment to include validation testing (Elbatal et al. (2024)), survival analysis (Loubna et al. (2024)), and entropy analysis (Elbatal et al. (2024)), providing comprehensive insights into the dynamics of risk in diverse datasets. Collectively, these contributions underscore the evolving landscape of risk analysis methodologies, highlighting their significance in enhancing predictive capabilities and informing strategic decisions across various sectors.

The BUPII probability distribution extends the well-established PII probability distribution, known for its robust right-heavy-tail properties and wide applicability across various fields such as business, actuarial science, biology, engineering, economics, and more. The PII probability distribution, which is a special case of the Pearson type VI distribution and can be seen as a mixture of exponential and gamma distributions, is used in studies on income and wealth (Harris, 1968; Atkinson and Harrison, 1978), firm size (Corbellini et al., 2007), reliability (Hassan and Al-Ghamdi, 2009), and Hirsch statistics (Glanzel, 2008). It is noted for its "decreasing" hazard rate function and is a heavy-tailed alternative to exponential, Weibull, and Gamma distributions (Bryson, 1974). Further insights on its relation to the Burr family and Compound Gamma probability distributions can be found in Tadikamalla (1980) and Durbey (1970). This study aims to enhance the PII probability distribution's flexibility through the Burr-G (BU-G) family, as proposed by Alizadeh et al. (2017). This extension seeks to enhance the modeling capabilities of the PII distribution, catering to a broader range of data characteristics and practical scenarios. A random variable (RV) $\boldsymbol{\mathcal{Y}}$ has the PII distribution with parameter ξ_3 if it has cumulative distribution function (CDF) (for $\boldsymbol{\mathcal{Y}} > 0$) given by

$$U_{\xi_3}(\boldsymbol{y}) = 1 - (\boldsymbol{y} + 1)^{-\frac{2}{\xi_3}}, \tag{1}$$

where $\xi_3 > 0$ refers to the shape parameter. Then the corresponding probability density function (PDF) of (1) is

$$u_{\xi_3}(\boldsymbol{y}) = 2\frac{1}{\xi_3}(\boldsymbol{y} + 1)^{-\frac{2}{\xi_3}-1}. \tag{2}$$

According to Alizadeh et al. (2017), the cumulative distribution function (CDF) of the Burr-G (BU-G) family is formulated as follows

$$G_{\xi_1,\xi_2,\underline{\xi}}(\boldsymbol{y}) = 1 - \frac{\overline{U}_{\underline{\xi}}(\boldsymbol{y})^{\xi_1\xi_2}}{\left[U_{\underline{\xi}}(\boldsymbol{y})^{\xi_1} + \overline{U}_{\underline{\xi}}(\boldsymbol{y})^{\xi_1}\right]^{\xi_2}}, \tag{3}$$

where $\overline{U}_{\underline{\xi}}(\boldsymbol{y}) = 1 - U_{\underline{\xi}}(\boldsymbol{y})$. The PDF corresponding to (3) is given by

$$g_{\xi_1,\xi_2,\underline{\xi}}(\boldsymbol{y}) = \xi_1\xi_2 u_{\underline{\xi}}(\boldsymbol{y}) U_{\underline{\xi}}(\boldsymbol{y})^{\xi_1-1} \frac{\overline{U}_{\underline{\xi}}(\boldsymbol{y})^{\xi_1\xi_2-1}}{\left[U_{\underline{\xi}}(\boldsymbol{y})^{\xi_1} + \overline{U}_{\underline{\xi}}(\boldsymbol{y})^{\xi_1}\right]^{1+\xi_2}}. \tag{4}$$

The BUPII CDF is given by:

$$F_{\underline{\psi}}(\boldsymbol{y}) | \left(\underline{\boldsymbol{\psi}} = \xi_1, \xi_2, \xi_3\right) = 1 - \frac{(\boldsymbol{y}+1)^{-\xi^*}}{\left\{\left[1-(\boldsymbol{y}+1)^{-\frac{2}{\xi_3}}\right]^{\xi_1} + (\boldsymbol{y}+1)^{-2\frac{\xi_1}{\xi_3}}\right\}^{\xi_2}}, \tag{5}$$

where $\xi^* = 2\xi_1\xi_2\frac{1}{\xi_3}$. The PDF corresponding to (5) is given by

$$f_{\underline{\psi}}(\boldsymbol{y}) = \xi^*(\boldsymbol{y}+1)^{-(1+\xi^*)} \frac{\left[1-(\boldsymbol{y}+1)^{-\frac{2}{\xi_3}}\right]^{\xi_1-1}}{\left\{\left[1-(\boldsymbol{y}+1)^{-\frac{2}{\xi_3}}\right]^{\xi_1} + (\boldsymbol{y}+1)^{-\xi^*}\right\}^{1+\xi_2}}. \tag{6}$$

The asymptotes of the CDF, PDF and HRF as $\boldsymbol{y} \to 0$ are given by

$$F_{\underline{\psi}}(\boldsymbol{y}) \sim \xi_2\left[1-(\boldsymbol{y}+1)^{-\frac{2}{\xi_3}}\right]^{\xi_1} | \boldsymbol{y} \to 0, f_{\underline{\psi}}(\boldsymbol{y}) \sim \xi_1\xi_2\frac{2}{\xi_3}(\boldsymbol{y}+1)^{-\left(\frac{1}{\xi_3}+1\right)}\left[1-(\boldsymbol{y}+1)^{-\frac{2}{\xi_3}}\right]^{\xi_1-1} | \boldsymbol{y} \to 0,$$

and

$$h_{\underline{\psi}}(\boldsymbol{y}) \sim \xi^*(\boldsymbol{y}+1)^{-\left(\frac{2}{\xi_3}+1\right)}\left[1-(\boldsymbol{y}+1)^{-\frac{2}{\xi_3}}\right]^{\xi_1-1} | \boldsymbol{y} \to 0.$$

The asymptotes of CDF, PDF and HRF as $\boldsymbol{y} \to \infty$ are given by

$$1 - F_{\underline{\psi}}(\boldsymbol{y}) \sim \xi_1^{\xi_2}(\boldsymbol{y}+1)^{-2\frac{\xi_2}{\xi_3}} | \boldsymbol{y} \to \infty, f_{\underline{\psi}}(\boldsymbol{y}) \sim \xi_2\xi_1^{\xi_2}\frac{2}{\xi_3}(\boldsymbol{y}+1)^{-2\frac{\xi_1}{\xi_3}-1} | \boldsymbol{y} \to \infty$$

and

$$h_{\underline{\psi}}(\boldsymbol{y}) \sim \xi_2\frac{2}{\xi_3}(\boldsymbol{y}+1)^{-1} | \boldsymbol{y} \to \infty.$$

See Gupta and Gupta (2007) and Gleaton and Lynch (2006, Aboraya et al. (2020), Chesneau et al. (2021), Ibrahim et al. (2021) and Yousof et al. (2021) for more details. For simulation of this new probability distribution, we obtain the quantile function (QF) of $\boldsymbol{\mathcal{Y}}$ (by inverting (5)), say $\boldsymbol{y}_u = F^{-1}(u)$, as

$$\boldsymbol{y}_u = \left\{\left[1 - \frac{u(\xi_1,\xi_2)}{u_*^{\frac{1}{\xi_1\xi_2}} + u(\xi_1,\xi_2)}\right]^{-\xi_3} - 1\right\} | u_* = 1 - u, \tag{7}$$

where $u(\xi_1,\xi_2) = \left(1 - u_*^{\frac{1}{\xi_2}}\right)^{\frac{1}{\xi_1}}$. Equation (7) is used for simulating the new probability distribution.

The HRF for the new probability distribution can be derived from $f_{\underline{\psi}}(\boldsymbol{y}) / \left[1 - F_{\underline{\psi}}(\boldsymbol{y})\right]$. Numerous extensions and variants of the Pareto type II (PII) distribution have been developed by researchers to cater to diverse modeling needs across various fields. These extensions reflect the flexibility and applicability of the PII distribution in addressing specific characteristics of real-world data. Here is an expanded overview of some notable PII extensions found in the literature, including Weibull PII distribution by Tahir et al. (2015), which integrates the PII framework with the

Weibull distribution. One parameter PII system of densities proposed by Cordeiro et al. (2018), offering a simplified version of the PII probability distribution. Odd log-logistic PII and Zografos-Balakrishnan PII distribution from Altun et al. (2018a), introducing alternative forms based on specific skewed and symmetric distributions. Weibull generalized PII, Rayleigh generalized PII, and Exponential generalized PII distributions developed by Elbiely and Yousof (2018), which extend the PII probability distribution using these well-known distributions. Topp Leone Poisson PII distribution by Yousof et al. (2019), blending the PII framework with Poisson processes. PII inverse Rayleigh by Goual and Yousof (2019), offering an inverse Rayleigh distribution within the PII family. Topp-Leone generated PII probability distribution by Yousof et al. (2019b), exploring the application of Topp-Leone distributions in the context of PII. Burr type XII PII, PII Burr type XII, and PII PII distributions by Gad et al. (2019), introducing variations based on the Burr distribution type XII. New zero-truncated version of the Poisson PII distribution by Yousof et al. (2019a), addressing truncated data scenarios within the PII framework. Topp Leone PII distribution by Yadav et al. (2020), focusing on the integration of Topp Leone distributions with the PII probability distribution. Poisson Burr X generalized PII and Poisson Rayleigh generalized PII distributions by Ibrahim and Yousof (2020), extending the PII probability distribution using generalized forms of Poisson, Burr X, and Rayleigh distributions. Extended Poisson Generalized PII distribution by Elsayed Yousof (2021), further expanding the PII probability distribution to incorporate generalized Poisson distributions. These extensions and variants highlight the ongoing development and refinement of the PII distribution, enhancing its utility and versatility in statistical modeling across diverse applications. Each variant offers specific advantages suited to different types of data characteristics and modeling requirements in fields such as economics, engineering, actuarial science, and beyond.

In modeling failure time data, the BUPII probability distribution stands out favorably when compared to various established extensions of the PII probability distribution. These include the exponentiated PII, odd log-logistic PII, transmuted Topp-Leone PII, Kumaraswamy PII, Gamma PII, special generalized mixture PII, Burr Hatke PII, and proportional reversed hazard rate PII extensions. The probability distribution's performance is evaluated using criteria such as the consistent-information criterion, Akaike information criterion, Bayesian information criterion, and Hannan-Quinn information criterion. Similarly, when applied to service time data, the BUPII probability distribution is assessed against the same PII extensions using these evaluation metrics. This comparative analysis highlights the BUPII probability distribution’s robustness and flexibility, demonstrating its effectiveness as a versatile tool in statistical modeling and analysis across different types of datasets.

2. Properties

Due to Alizadeh et al. (2017), the PDF in (6) can be expressed as

$$f(\boldsymbol{y}) = \sum_{a=0}^{\infty} \varsigma_a \, w_{1+a,\xi_3}(\boldsymbol{y}), \tag{8}$$

where

$$\varsigma_a = \frac{\xi_1 \xi_2}{1+a} \sum_{\varkappa_1,\varkappa_2=0}^{\infty} \sum_{\varkappa_3=a}^{\infty} (-1)^{\varkappa_2+\varkappa_3+a} \binom{-(1+\xi_2)}{\varkappa_1} \binom{-[\xi_1(1+\varkappa_1)+1]}{\varkappa_2} \binom{\xi_1(1+\varkappa_1)+\varkappa_2+1}{\varkappa_3} \binom{\varkappa_3}{a},$$

and $w_{1+a,\xi_3}(\boldsymbol{y})$ is the PDF of the PII probability distribution with power parameter $1+a$. By integrating Equation (8), the CDF of $\boldsymbol{\mathcal{Y}}$ becomes

$$F(\boldsymbol{y}) = \sum_{a=0}^{\infty} \varsigma_a \, W_{1+a,\xi_3}(\boldsymbol{y}), \tag{9}$$

where $W_{1+a,\xi_3}(\boldsymbol{y})$ is the CDF of the PII distribution with power parameter $1+a$. The c^{th} ordinary moment of $\boldsymbol{\mathcal{Y}}$ is given by

$$\mu'_{c,\boldsymbol{y}} = E(\boldsymbol{\mathcal{Y}}^c) = \int_{-\infty}^{\infty} \boldsymbol{y}^c f(\boldsymbol{y}) d\boldsymbol{y},$$

then we obtain

$$\mu'_{c,\boldsymbol{y}} = \sum_{a=0}^{\infty} \sum_{b=0}^{c} \varsigma_a \,(1+a)(-1)^b \binom{c}{a} \boldsymbol{B}\big((1+a), 1+2\xi_3(b-c)\big)| \frac{2}{\xi_3} > c, \tag{10}$$

where $\boldsymbol{B}(\tau_1,\tau_2)=\int_0^1 t^{\tau_1-1}(1-t)^{\tau_2-1}dt$. Setting $c=1,2,3$and 4 in (10), we have

$$E(\boldsymbol{\mathcal{Y}})=\sum_{a=0}^{\infty}\sum_{b=0}^{1}\varsigma_a\,(1+a)(-1)^b\binom{1}{b}\boldsymbol{B}\big((1+a),1+2\xi_3(b-1)\big)|\frac{2}{\xi_3}>1,$$

$$E(\boldsymbol{\mathcal{Y}}^2)=\sum_{a=0}^{\infty}\sum_{b=0}^{2}\varsigma_a\,(1+a)(-1)^b\binom{2}{b}\boldsymbol{B}\big((1+a),1+2\xi_3(b-2)\big)|\frac{2}{\xi_3}>2,$$

$$E(\boldsymbol{\mathcal{Y}}^3)=\sum_{a=0}^{\infty}\sum_{b=0}^{3}\varsigma_a\,(1+a)(-1)^{ba}\binom{3}{b}\boldsymbol{B}\big((1+a),1+2\xi_3(b-3)\big)|\frac{2}{\xi_3}>3,$$

and

$$E(\boldsymbol{\mathcal{Y}}^4)=\sum_{a=0}^{\infty}\sum_{b=0}^{4}\varsigma_a\,(1+a)(-1)^b\binom{4}{b}\boldsymbol{B}\big((1+a),1+2\xi_3(b-4)\big)|\frac{2}{\xi_3}>4,$$

where $E(\boldsymbol{\mathcal{Y}})=\mu_1'$ is the mean of $\boldsymbol{\mathcal{Y}}$. The c^{th} incomplete moment, say $I_c(t)$, of $\boldsymbol{\mathcal{Y}}$ can be expressed, from (9), as

$$I_{c,\boldsymbol{y}}(t)=\int_{-\infty}^{t}\boldsymbol{y}^c f(\boldsymbol{y})d\boldsymbol{y}=\sum_{a=0}^{\infty}\varsigma_a\int_{-\infty}^{t}\boldsymbol{y}^c w_{(1+a),\xi_3}(\boldsymbol{y})d\boldsymbol{y}$$

then

$I_{c,\boldsymbol{y}}(t)=\sum_{a=0}^{\infty}\sum_{b=0}^{c}\varsigma_a\,(1+a)(-1)^b\binom{c}{b}\boldsymbol{B}_t\big((1+a),1+2\xi_3(b-c)\big)|\frac{2}{\xi_3}>c,$ (11)

where $\boldsymbol{B}_{\boldsymbol{y}}(\tau_1,\tau_2)=\int_0^{\boldsymbol{y}} t^{\tau_1-1}(1-t)^{\tau_2-1}dt$. The first incomplete moment given by (11) with $c=1$ is

$$I_{1,\boldsymbol{y}}(t)=\sum_{a=0}^{\infty}\sum_{b=0}^{1}\varsigma_a\,(1+a)(-1)^b\binom{1}{b}\boldsymbol{B}_t\big((1+a),1+2\xi_3(b-1)\big)|\frac{1}{\xi_3}>1.$$

The moment generating function (MGF) can be derived using (8) as

$$M_{\boldsymbol{y}}(t)=\sum_{a=0}^{\infty}\sum_{c=0}^{\infty}\sum_{b=0}^{c}\frac{t^c}{c!}\varsigma_a(1+a)(-1)^b\binom{c}{b}\boldsymbol{B}\big((1+a),1+2\xi_3(b-c)\big)|\frac{1}{\xi_3}>c.$$

The first c derivatives of $M_{\boldsymbol{y}}(t)$, with respect to t at $t=0$, yield the first c moments about the origin, i.e.,

$$\mu_{c,\boldsymbol{y}}'=E(\boldsymbol{\mathcal{Y}}^c)=\frac{d^c}{dt^c}M_{\boldsymbol{y}}(t)|_{(t=0\text{ and }c=1,2,3,\ldots)}.$$

The cumulant generating function CGF is the logarithm of the MGF. Thus, c^{th} cumulant, say $\kappa_{c,\boldsymbol{y}}$, can be obtained from

$$\kappa_{c,\boldsymbol{y}}=\frac{d^c}{dt^c}log\left[\sum_{a=0}^{\infty}\sum_{c=0}^{\infty}\sum_{b=0}^{c}\frac{t^c}{c!}\varsigma_a(1+a)(-1)^b\binom{c}{b}\boldsymbol{B}\big((1+a),1+2\xi_3(b-c)\big)\right]|t=0,\text{ and }c=1,2,3,\ldots$$

3. Comparing probability distributions

The first real dataset (Data Set I) comprises failure times for 84 aircraft windshields, as documented by Murthy et al. (2004). This dataset provides valuable insights into the durability and failure characteristics of aircraft components. The second dataset (Data Set II) includes service time data for 63 aircraft windshields, also reported by Murthy et al. (2004). This dataset is crucial for understanding the operational lifespan and performance of the same components in service conditions. Additionally, several other significant real-life datasets are available for research and analysis. These datasets are provided in the works of Nofal et al. (2016), Aryal et al. (2017), Yousof et al. (2016, 2018a,b), Elbiely and Yousof (2018), Korkmaz et al. (2017, 2019, 2022), Ibrahim and Yousof (2020), Yadav et al. (2020), Mansour et al. (2020e), and Goual et al. (2020). They offer a wide range of data suitable for various analytical and research applications, expanding the scope of studies that can benefit from such empirical evidence. In this section, we demonstrate the versatility and efficacy of the BUPII probability distribution by applying it to these specific datasets. We perform a comparative analysis of the BUPII probability distribution against several established probability distributions, including:

I. Exponentiated PII (ExpPII): Introduced by Gupta et al. (1998), this probability distribution extends the basic PII distribution to account for more complex data patterns.
II. Beta PII (BetaPII): Proposed by Lemonte and Cordeiro (2013), this probability distribution integrates the Beta distribution to enhance the modeling of data with varying shapes.
III. Gamma PII (GamPII): Developed by Cordeiro et al. (2015), this probability distribution incorporates the Gamma distribution for applications where data exhibit gamma-like behavior.
IV. Transmuted Topp-Leone PII (TTLPII) and its reduced form (RTTLPII): Introduced by Yousof et al. (2017), these probability distributions provide flexibility in handling data with different tail behaviors.
V. Odd Log-Logistic PII (OLLPII) and its reduced version (ROLLPII): Presented by Altun et al. (2018b), these probability distributions are designed for data with log-logistic characteristics.
VI. Reduced Burr-Hatke PII (RBHPII): Proposed by Yousof et al. (2018), this probability distribution offers a variation of the Burr-Hatke distribution for improved fit.
VII. Proportional Reversed Hazard Rate PII (PRHRPII): This probability distribution accounts for data with proportional reversed hazard rates.
VIII. Special Generalized Mixture PII (SGMPII): As described by Chesneau and Yousof (2021), this probability distribution allows for complex data patterns through a mixture approach.

To evaluate the performance and appropriateness of these probability distributions, we use various statistical tools:
I. Quantile-Quantile (Q-Q) Plots: These plots assess the normality of the datasets, comparing the quantiles of the observed data with the quantiles of a theoretical normal distribution.
II. Total Time Test (TTT) Plots: TTT plots are used to explore the hazard rate function (HRF) of the data, providing insights into the risk of failure over time.
III. Nonparametric Kernel Density Estimation (NKDE): NKDE is employed to examine the initial density shape of the data without assuming a specific parametric form.

Figures 1 and 2 present various visualizations, including Q-Q plots, box plots, TTT plots, and NKDE plots for both datasets. Observations from these figures are as follows:
I. Figures 7(a) and 8(a): Show near-normality in the datasets, indicating that the data distribution approximates a normal distribution.
II. Figures 1(a, b) and 2(a, b): Reveal the absence of significant outliers, suggesting that the data are relatively clean and free from extreme deviations.
III. Figures 1(c) and 2(c): Depict a monotonically increasing hazard rate function, indicating that the risk of failure increases over time.
IV. Figures 1(d) and 2(d): Show a bimodal and nearly symmetric density shape, highlighting the complexity and diversity in the data distribution.

This comprehensive analysis underscores the BUPII probability distribution’s robustness and adaptability, validating its effectiveness as a versatile tool for modeling and analyzing real-life data across different contexts.

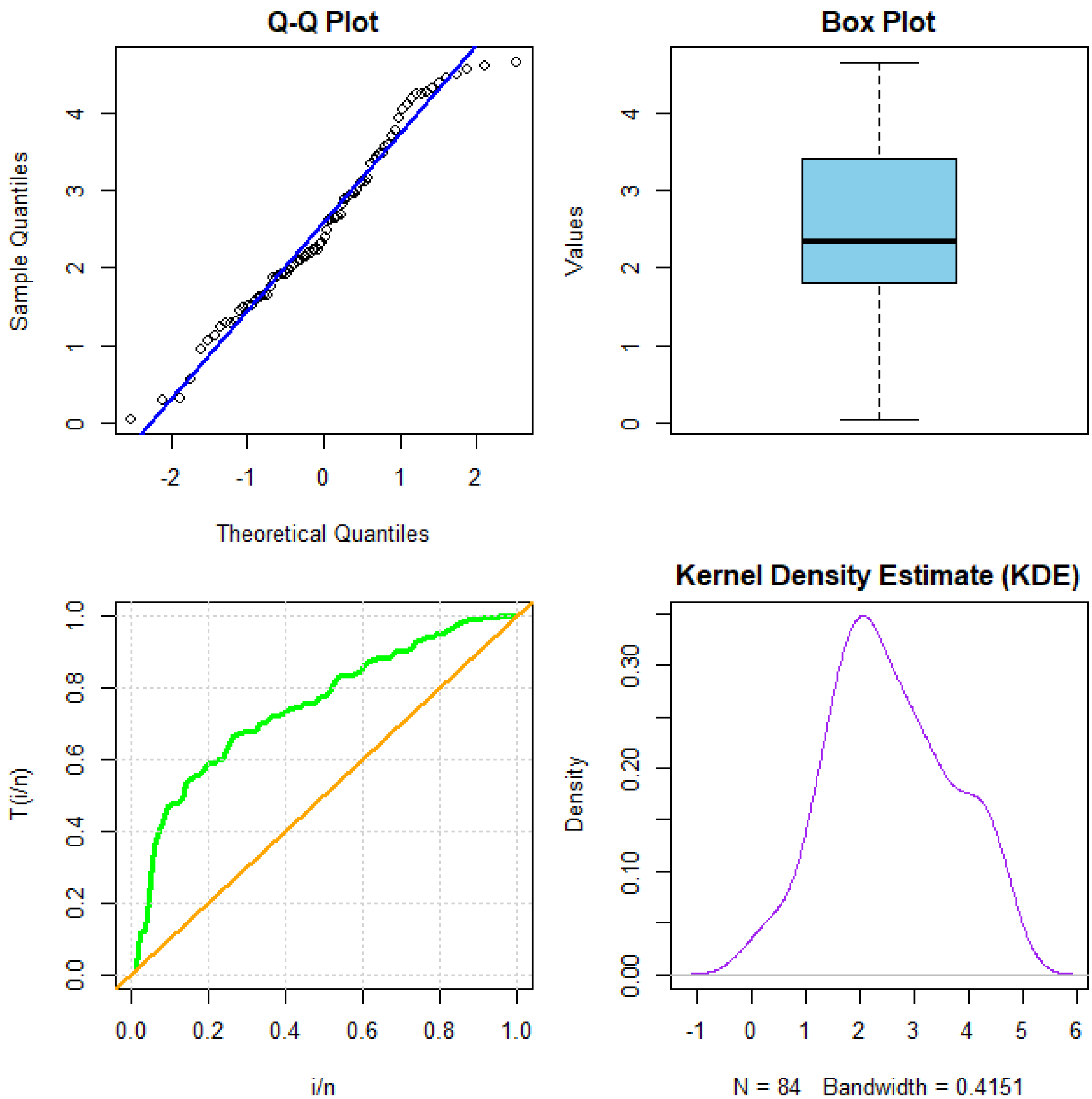


Figure 1: Q-Q plot, box plot, TTT plot and NKDE plot for the failure times.

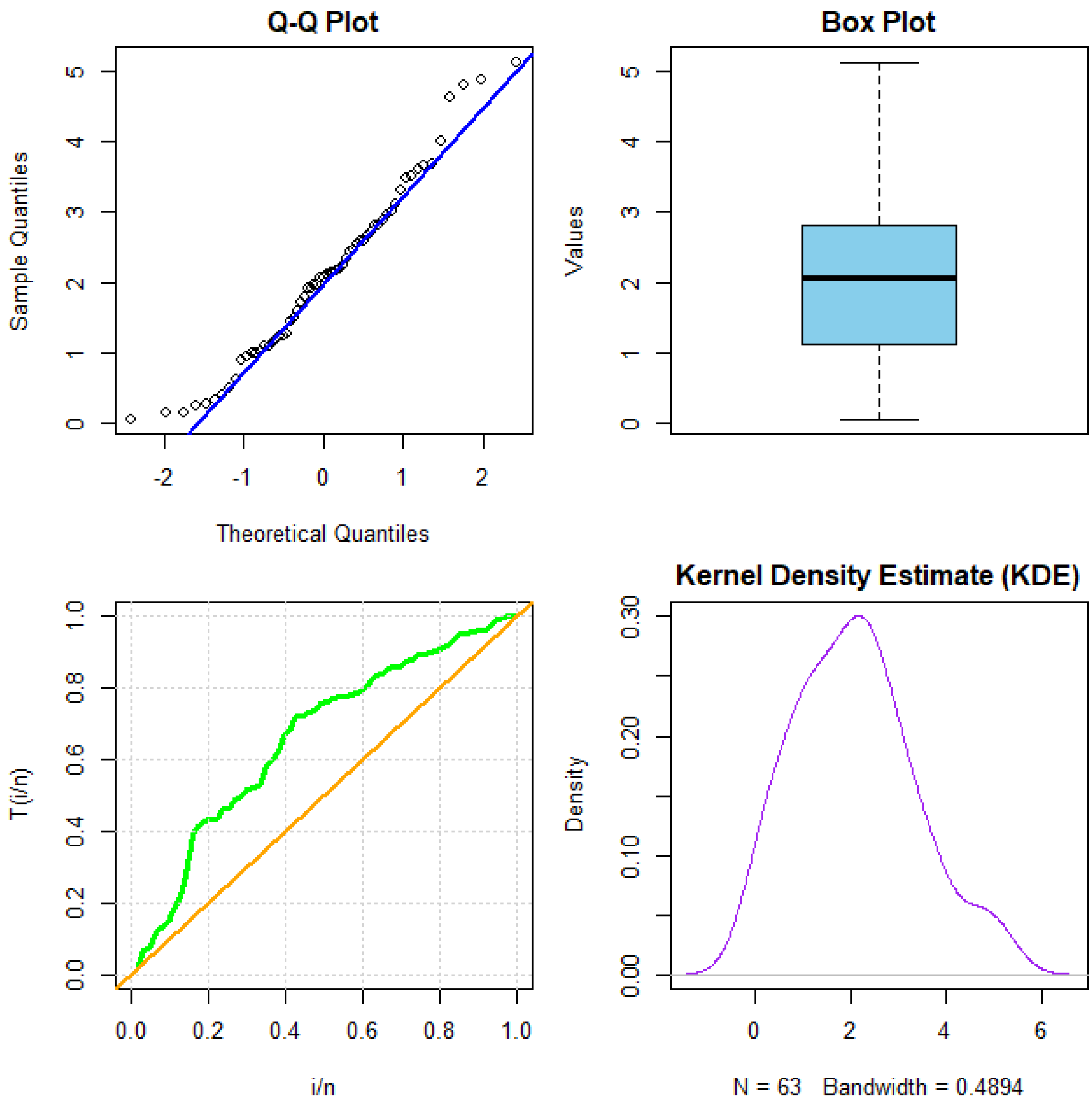


Figure 2: Q-Q plot, box plot, TTT plot and NKDE plot for the service times.

Parameter estimation for each probability distribution is performed via maximum likelihood using the "L-BFGS-B" method. The goodness-of-fit is evaluated using several criteria: Akaike information criterion (AK-IC), Consistent AK-IC (CAK-IC), Bayesian IC (BS-IC), Hannan-Quinn IC (HQN-IC), A^* and W^*. These metrics facilitate a comprehensive comparison of the probability distributions' performances, ensuring robust statistical inference and probability distribution selection tailored to the characteristics of the data sets.

Table 1 provides the MLEs for the parameters of the BUPII probability distribution applied to the failure times dataset. It also includes the standard errors associated with these estimates, which quantify the precision of the parameter values. The MLEs are the values that maximize the likelihood function, fitting the probability distribution to the observed failure times data. The standard errors help gauge the reliability and stability of these estimates, indicating how much the parameter values might vary with different samples from the same population. Table 2 reports various statistics used to assess how well the BUPII probability distribution fits the failure times data. It includes metrics such as the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC), among others, which help evaluate the probability distribution's performance. These statistics compare the BUPII probability distribution's fit against other potential probability distributions, providing insights into how well it captures the patterns and characteristics of the failure times data. Similar to Table 1, Table 3 presents the MLEs for the BUPII probability distribution parameters when applied to the service times dataset. It also lists the standard errors for these estimates, which reflect the precision and reliability of the parameter values. The MLEs provide the best estimates of the probability distribution parameters for the service times data, while the standard errors indicate the level of uncertainty associated with these estimates. Table 4 provides the goodness-of-fit statistics for the BUPII probability distribution applied to the service times data. It includes various fit indices, such as AIC and BIC, which are used to evaluate how well the probability distribution describes the service times data. These statistics help determine the adequacy of the BUPII probability distribution in capturing the data's underlying distribution compared to other probability distributions.

Based on results of Tables 2 and 4, it is noted that the BUPII probability distribution has the lowest values of AK-IC, CAK-IC, BS-IC, HQN-IC, A^* and W^*. For failure times data: $\hat{\ell} = -134.35841$, AK-IC=274.71692, CAK-IC=275.01693, BS-IC=282.00931, HQN-IC=277.64842, $A^* = 0.94444$ and $W^* = 0.10053$.For service times data: $\hat{\ell} = -104.42582$, AK-IC=214.85170, CAK-IC=215.25841, BS-IC=221.28114, HQN-IC=217.38047, $A^* = 1.28202$ and $W^* = 0.21151$. Conducting thorough probability distribution diagnostics, including residual analysis and sensitivity testing, can validate the assumptions underlying BUPII probability distribution. This step ensures robustness and enhances confidence in using these probability distributions for forecasting and risk assessment in aircraft windshield maintenance.

Aircraft windshield companies should prioritize BUPII probability distribution for its capability to accurately predict failure times. These probability distributions can support proactive maintenance strategies, minimize downtime, and enhance overall operational efficiency and safety of aircraft fleets. In conclusion, BUPII probability distribution offers strong statistical performance and practical utility for aircraft windshield companies seeking reliable methods to forecast failure times and optimize maintenance schedules. Figure 3 presents the fitted PDF and CDF plots for the failure data. Figure 4 gives the fitted PDF and CDF plots for the service data.

Table 1**:** MLEs and SEs for failure times data.

Probability distribution	**Estimates**			
BUPII(ξ_1, ξ_2, ξ_3)	3.54705433	30.6535444	4.1176853	
	(0.3112534)	(53.271554)	(1.662294)	
TTLPII (v, ξ_1, ξ_2, ξ_3)	-0.8075241	2.4766249	15608.2133	38628.32
	(0.1396013)	(0.541798)	(1602.3665)	(123.9362)
Beta PII (v, ξ_1, ξ_2, ξ_3)	3.6035923	33.638665	4.8307014	118.83731
	(0.618723)	(63.714513)	(9.2382024)	(428.9271)
PRHRPII (ξ_1, ξ_2, ξ_3)	3.744×10^{6}	4.708×10^{-1}	4.543×10^{6}	
	1.035×10^{6}	(0.0000129)	37.146484	
RTTLPII (ξ_1, ξ_2, ξ_3)	-0.847325	5.5206043	1.15682533	
	(0.1001143)	(1.184842)	(0.095974)	
SGMPII (ξ_1, ξ_2, ξ_3)	$-1.04\ 4\times 10^{-1}$	9.835×10^{6}	1.207×10^{7}	

	(0.1223142)	(4843.353)	(501.0453)
RBUPII (ξ_1, ξ_2, ξ_3)	3.5479249	30.6374234	0.24294543
	(0.314148)	(55.840454)	(0.102664)
OLLPII (ξ_1, ξ_2, ξ_3)	2.3264032	(7.187×10^5)	(2.344×10^6)
	(2.144×10^{-1})	(1.207×10^4)	(2.604×10^1)
GamPII (ξ_1, ξ_2, ξ_3)	3.58761423	52001.564	37029.742
	(0.5134239)	(7955.1643)	(81.16970)
ExpPII (ξ_1, ξ_2, ξ_3)	3.6261133	20074.5034	26257.743
	(0.623721)	(2041.8324)	(99.742)
ROLLPII (ξ_1, ξ_2)	3.89056342	0.57315948	
	(0.3652323)	(0.0194426)	
RBHPIIPII (ξ_1, ξ_2)	1080175.18	51367189.25	
	(983309.27)	(232322.25)	
PII (ξ_1, ξ_2)	51425.3529	131789.746	
	(5933.4945)	(296.12946)	

Table 2: $\hat{\ell}$ and goodness-of-fits statistics for failure times data.

Probability distribution	$-\hat{\ell}$	AK-IC	CAK-IC	BS-IC	HQN-IC	A*	W*
BUPII	134.35841	274.71692	275.01693	282.00931	277.64842	0.94444	0.10053
OLLPII	134.42354	274.84703	275.14709	282.13943	277.77854	0.94897	0.10099
ExpPII	141.39973	288.79949	289.09571	296.12734	291.74699	1.74357	0.21964
GamPII	138.40424	282.80835	283.10463	290.13635	285.75596	1.36666	0.16188
BetaPII	138.71773	285.43544	285.93548	295.20607	289.36544	1.40854	0.16880
PII	164.98848	333.97679	334.12303	338.86230	335.94173	1.39756	0.16645
ROLLPII	142.84529	289.69048	289.83854	294.55208	291.64475	1.95696	0.25554
SGMPII	143.08745	292.17475	292.47470	299.46726	295.10627	1.34687	0.15798
PRHRPII	162.87704	331.75404	332.05402	339.04644	334.68558	1.36752	0.16039
RTTLPII	153.98093	313.96185	314.26182	321.25426	316.89334	3.75297	0.55942
TTLPII	135.57009	279.14009	279.64641	288.86331	283.04872	1.12587	0.12710

RBHPII	168.60405	341.20814	341.35624	346.06976	343.16244	1.67151	0.20699

Table 3: MLEs and SEs for service times data.

Probability distribution	**Estimates**		
BUPII(ξ_1,ξ_2,ξ_3)	2.3584632	22.9719743	4.92505312
	(0.2419440)	(41.777686)	(3.2902342)
PRHRPII(ξ_1,ξ_2,ξ_3)	1.6044×10^{6}	3.933×10^{-1}	1.3146×10^{6}
	2.0257×10^{3}	0.0033×10^{-1}	0.9442×10^{6}
RTTLPII(ξ_1,ξ_2,ξ_3)	−0.671560	2.7449749	1.01238434
	(0.1874784)	(0.669874)	(0.1141242)
RBUPII(ξ_1,ξ_2,ξ_3)	2.3583644	23.139994	0.2024549
	(0.241343)	(41.181349)	(0.132959)
OLLPII(ξ_1,ξ_2,ξ_3)	1.6641944	6.343×10^{5}	2.0254×10^{6}
	(1.824×10^{-1})	(1.743×10^{4})	7.2354×10^{6}
ROLLPII(ξ_1,ξ_2)	2.3723345	0.69109134	
	(0.2682535)	(0.0449233)	
RBHPII(ξ_1,ξ_2)	1405552.33	53203423.46	
	(422.00545)	(28.523276)	
PII(ξ_1,ξ_2)	99269.7823	207019.365	
	(11863.515)	(301.23721)	

Table 4: $\hat{\ell}$ and goodness-of-fits statistics for the service times data.

Probability distribution	$-\hat{\ell}$	AK-IC	CAK-IC	BS-IC	HQN-IC	A^*	W^*
BUPII	104.42582	214.85170	215.25841	221.28114	217.38047	1.28202	0.21151
OLLPII	104.90414	215.80829	216.21505	222.23764	218.33693	0.94248	0.15456
ROLLPII	110.72873	225.45739	225.65753	229.74363	227.14311	2.34729	0.39089
PRHRPII	109.29865	224.59735	225.0045	231.02672	227.1260	1.12640	0.18613
RTTLPII	112.18555	230.37105	230.77777	236.80044	232.89974	2.68757	0.45326
PII	109.29883	222.59763	222.79767	226.88388	224.28345	1.12655	0.18613

RBHPII	112.60056	229.20119	229.40118	233.48726	230.88699	1.39843	0.23169

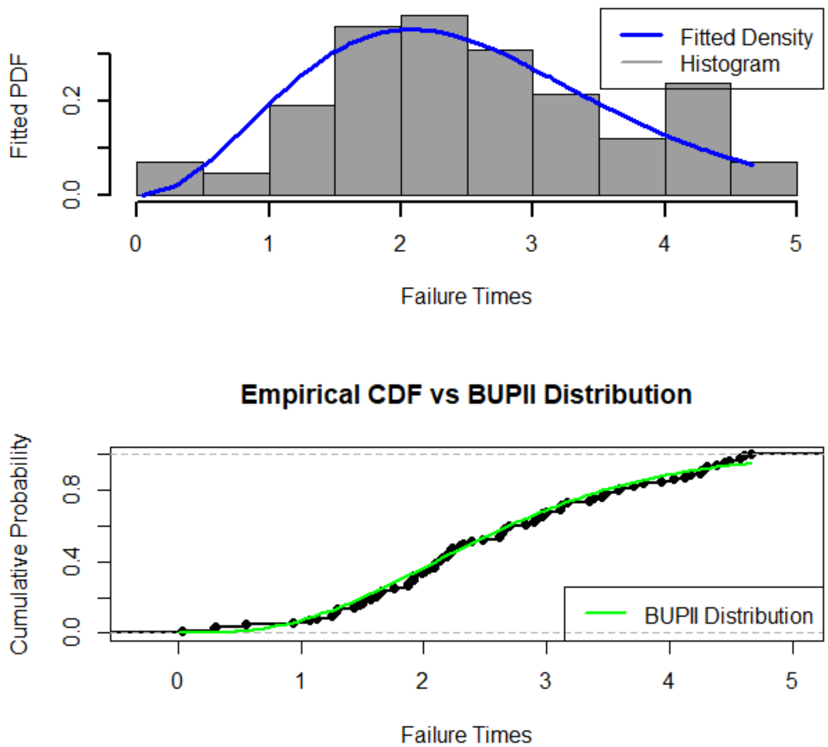


Figure 3: The fitted PDF and CDF plots for the failure data.

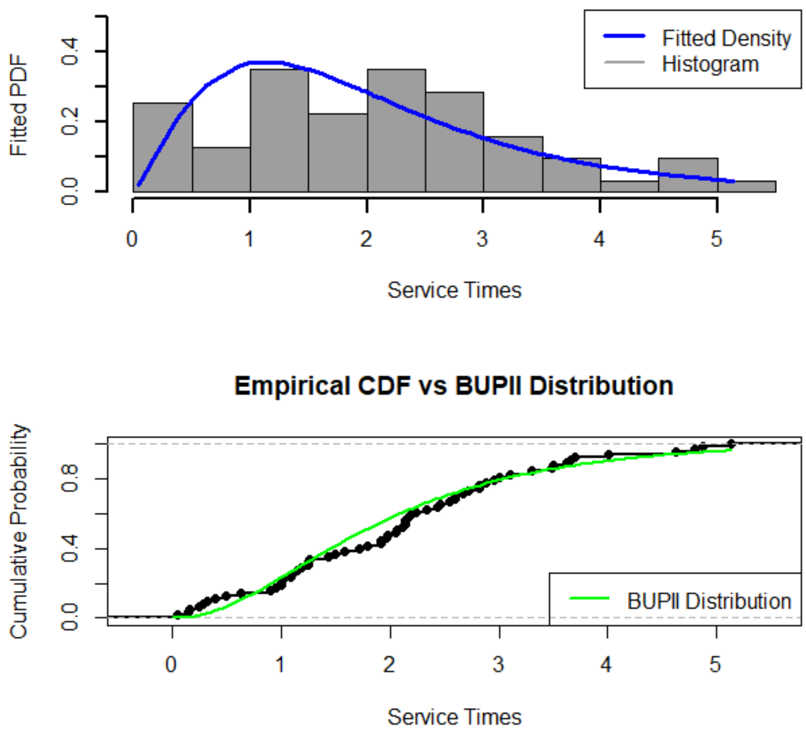


Figure 4: The fitted PDF and CDF plots for the service data.

4. MOOP analysis

The MOOP analysis holds significant importance for aircraft windshield companies in understanding and managing the service and failure times of their products. The MOOP analysis helps in identifying the mean service and failure times across different order levels (P). This insight is crucial for aircraft windshield companies to understand the typical lifespan of their products before failure. By analyzing MOOP over time, companies can establish early warning systems for potential failures. This allows proactive maintenance and replacement scheduling, reducing operational disruptions and safety risks. The MOOP analysis facilitates predictive maintenance strategies. By knowing the average service life at various P levels, companies can schedule maintenance interventions effectively, minimizing downtime and associated costs. Aircraft windshields must meet stringent safety standards. MOOP analysis supports compliance by providing data-driven insights into product reliability and lifespan, crucial for regulatory approvals.

4.1 MOOP for the failure times

By understanding failure times, companies can mitigate safety risks associated with windshield failures during flight operations, ensuring passenger and aircraft safety. Table 5 below presents the MOOP analysis under the failure times data for P=1,2,3,4 and 5.

Table 5: MOOP analysis under the failure times data.

P	1	2	3	4	5
TRM			2.557452		
MOOP	0.04	0.1705	0.2166667	0.30175	0.43
MSE	6.337566	5.697542	5.479278	5.088193	4.526054
Bias	2.517452	2.386952	2.340786	2.255702	2.127452

Based on Table 5, it is seen that the true mean failure time of the aircraft windshields is approximately 2.557452 units. The initials for the MOOP, MSE, and Bias are provided for $P=1$ to $P=5$. MOOP represents the average of the smallest P ordered values of the failure times. As P increases from 1 to 5, MOOP increases gradually, indicating that the average of the smallest ordered values increases with more data points considered. MSE quantifies the average squared difference between the estimated mean (MOOP) and the true mean. As P increases from 1 to 5, MSE decreases, suggesting that the estimation of the mean becomes more accurate as more data points are included. Moreover, Bias measures the deviation between the estimated mean (MOOP) and the true mean. Like MSE, the Bias decreases as P increases, indicating that the estimation of the mean becomes less biased with more data points.

Recommendations for aircraft windshield companies:

- **I.** Companies should recognize that the true mean failure time of aircraft windshields is around 2.557452 units. This value serves as a benchmark for evaluating the reliability and longevity of their products
- **II.** Monitoring MOOP across different P values (from 1 to 5) provides insights into the average failure times of aircraft windshields under various scenarios.
- **III.** Companies can use MOOP to assess the average performance of their windshields in terms of failure times and compare it against the true mean for accuracy.
- **IV.** The decrease in MSE and Bias with increasing P indicates improved estimation of the true mean.
- **V.** Companies should aim to collect sufficient data points (increasing P) to reduce the variance in estimating the mean failure time, thereby enhancing reliability assessments.
- **VI.** Based on the observed MOOP, MSE, and Bias trends, companies could refine their design and testing protocols for aircraft windshields.
- **VII.** This includes incorporating data-driven insights to optimize material selection, manufacturing processes, and maintenance schedules to potentially extend the failure times beyond the current average.
- **VIII.** Implementing continuous monitoring and analysis of failure times using similar statistical techniques can provide ongoing feedback on the performance and reliability of aircraft windshields.
- **IX.** This proactive approach helps in identifying potential issues early, improving product quality, and enhancing customer satisfaction and safety.

By leveraging these insights and recommendations, aircraft windshield companies can strengthen their product development strategies, enhance reliability assessments, and potentially extend the operational lifespan of their products. These actions contribute to overall safety and cost-effectiveness in the aviation industry.

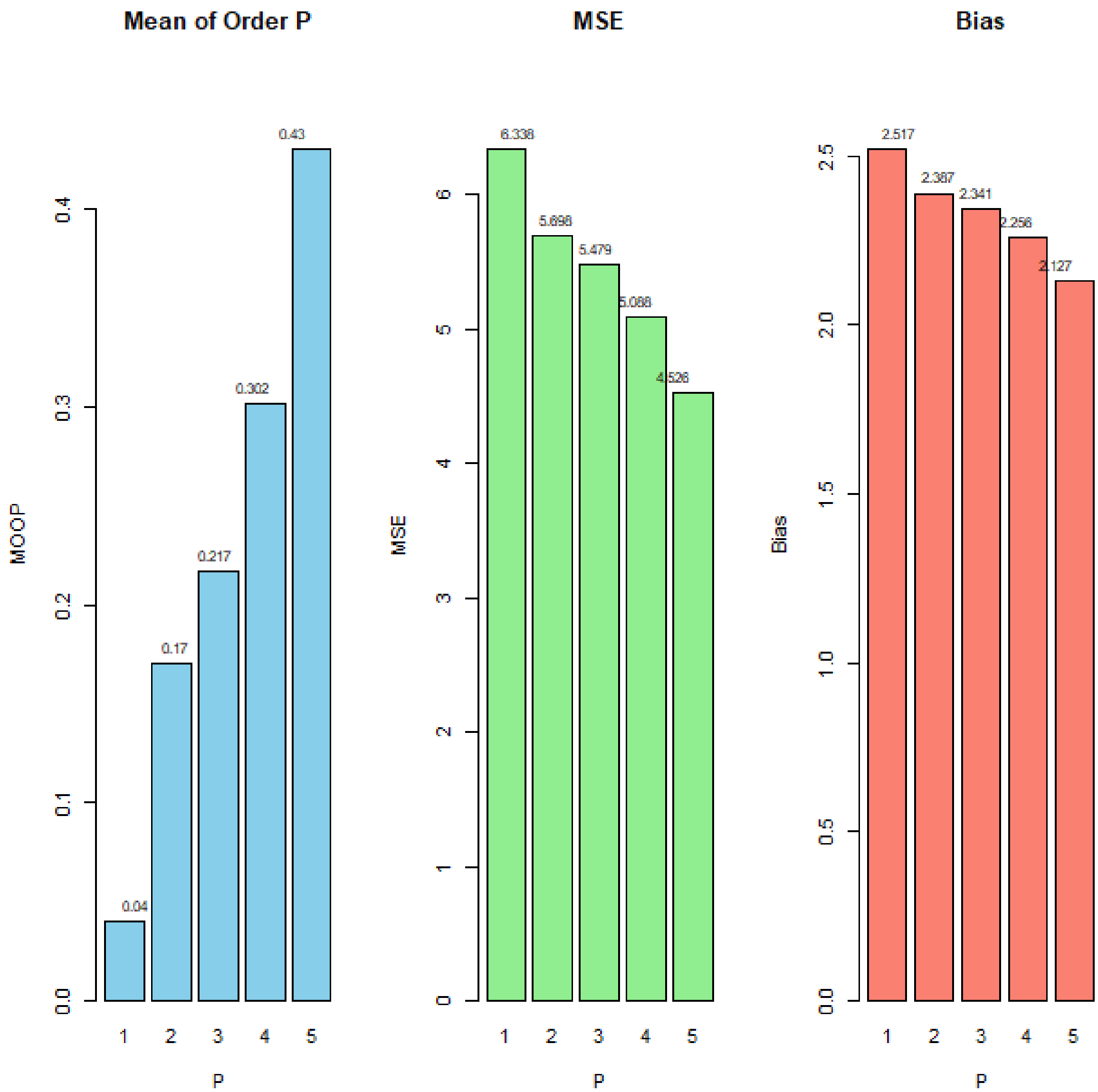


Figure 5: MOOP, MSE and Bias across the order of P for the failure times.

4.2 MOOP for the service times

Table 6 provides the MOOP analysis under the service times data for P=1,2,3,4 and 5. Both Tables list the results of the True mean (TRM), MOOP, the mean squared error (MSE) and the Bias for each data.

Table 6: MOOP analysis under the service times data.

P	1	2	3	4	5
True mean			2.08527		
MOOP	0.046	0.093	0.112	0.146	0.1728
MSE	4.158621	3.969139	3.893794	3.760768	3.657541
Bias	2.03927	1.99227	1.97327	1.93927	1.91247

According to Table 6, the true mean service time of the aircraft windshields is approximately 2.08527 units. The initials for the Mean of Order P (MOOP), MSE, and Bias are provided for P=1 to P=5. MOOP represents the average of the smallest P ordered values of the service times. As P increases from 1 to 5, MOOP increases, indicating that the average of the smallest ordered values increases with more data points considered. MSE quantifies the average squared difference between the estimated mean (MOOP) and the true mean. As P increases from 1 to 5, MSE decreases, suggesting that the estimation of the mean becomes more accurate as more data points are included. Bias measures the deviation between the estimated mean (MOOP) and the true mean. Similar to MSE, Bias decreases as P increases, indicating that the estimation of the mean becomes less biased with more data points.

Recommendations for aircraft windshield companies about the problem of the service times:

I. Companies should acknowledge that the true mean service time of aircraft windshields is around 2.08527 units. This serves as a reference point for evaluating the reliability and longevity of their products.

II. Monitoring MOOP across different P values (from 1 to 5) provides insights into the average service times of aircraft windshields under various scenarios.

III. Companies can use MOOP to assess the average performance of their windshields in terms of service times and compare it against the true mean for accuracy.

IV. The decrease in MSE and Bias with increasing P indicates improved estimation of the true mean service time.

V. Companies should aim to gather sufficient data points (increasing P) to reduce variability in estimating the mean service time, thereby enhancing reliability assessments.

VI. Based on observed MOOP, MSE, and Bias trends, companies could enhance their design and maintenance strategies for aircraft windshields.

VII. This includes leveraging data-driven insights to optimize materials, manufacturing processes, and maintenance intervals to potentially extend service times beyond the current average.

VIII. Implementing continuous improvement initiatives based on statistical analyses of service times helps in identifying potential areas for enhancement.

IX. This proactive approach supports maintaining high levels of customer satisfaction, safety, and operational efficiency within the aviation sector.

By implementing these recommendations, aircraft windshield companies can strengthen their product development strategies, improve reliability assessments, and potentially increase the service life of their products. These actions contribute to enhancing overall safety and operational effectiveness in the aviation industry.

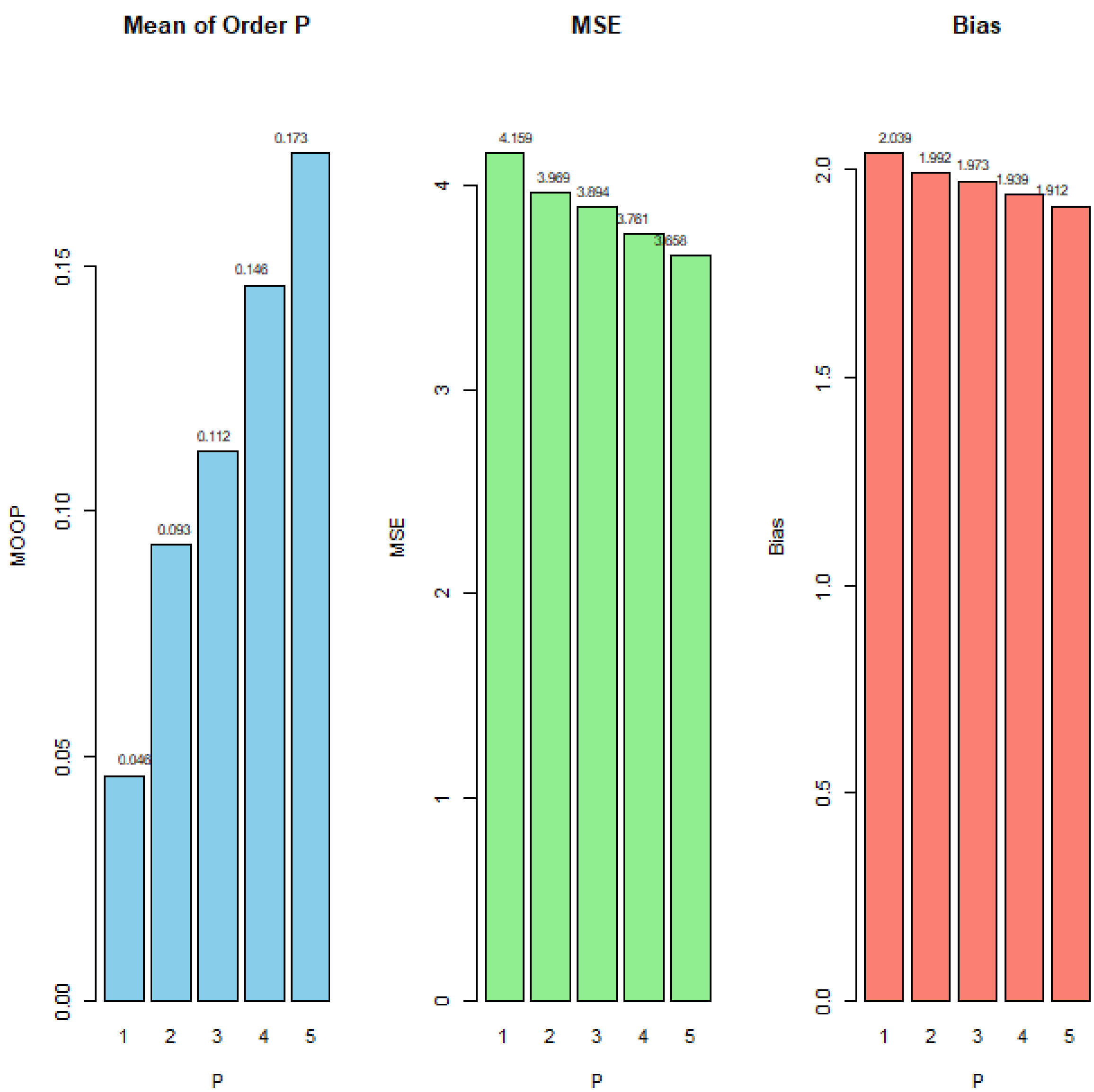


Figure 4: MOOP, MSE and Bias across the order of P for the services times.

5. PORT VaR analysis

The PORT analysis, particularly in the context of Value-at-Risk (VaR), holds significant importance for aircraft windshield companies when analyzing the service and failure times of their products. It identifies and quantifies extreme events in service and failure times that exceed a predetermined threshold. For aircraft windshields, these extreme events could indicate critical failures or unusually long service times that may impact operational safety and maintenance schedules. Also, PORT VaR analysis provides insights into tail risks associated with service and failure times of aircraft windshields. It helps in understanding the potential financial and operational impacts of rare but severe events, such as unexpected failures during flight operations. By identifying peaks in failure times through PORT analysis, companies can implement proactive maintenance strategies. This includes scheduling inspections, replacements, or repairs based on identified risk thresholds, thereby minimizing the likelihood of catastrophic failures. Aircraft windshields must adhere to stringent safety regulations. PORT analysis aids in ensuring compliance by quantifying and managing risks associated with potential failures, thereby contributing to overall aviation safety.

Understanding the likelihood and severity of extreme events through PORT VaR analysis enables companies to allocate resources effectively. This includes budgeting for maintenance, insurance coverage, and contingency planning to mitigate financial impacts from unexpected failures. Table 7 presents the PORT VaR analysis including the Min., 1st Qu., Median, Mean, 3rd Qu., Max. under the failure times data. Table 8 gives the PORT VaR analysis under the service times data.

Table 7: PORT VaR analysis under the failure times data.

CL	N. Peaks	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
50%	42	2.385	2.910	3.409	3.475	4.155	4.663
70%	59	1.912	2.264	2.962	3.085	3.739	4.663
90%	74	1.303	2.017	2.639	2.793	3.475	4.663
99%	83	0.301	1.871	2.385	2.588	3.409	4.663

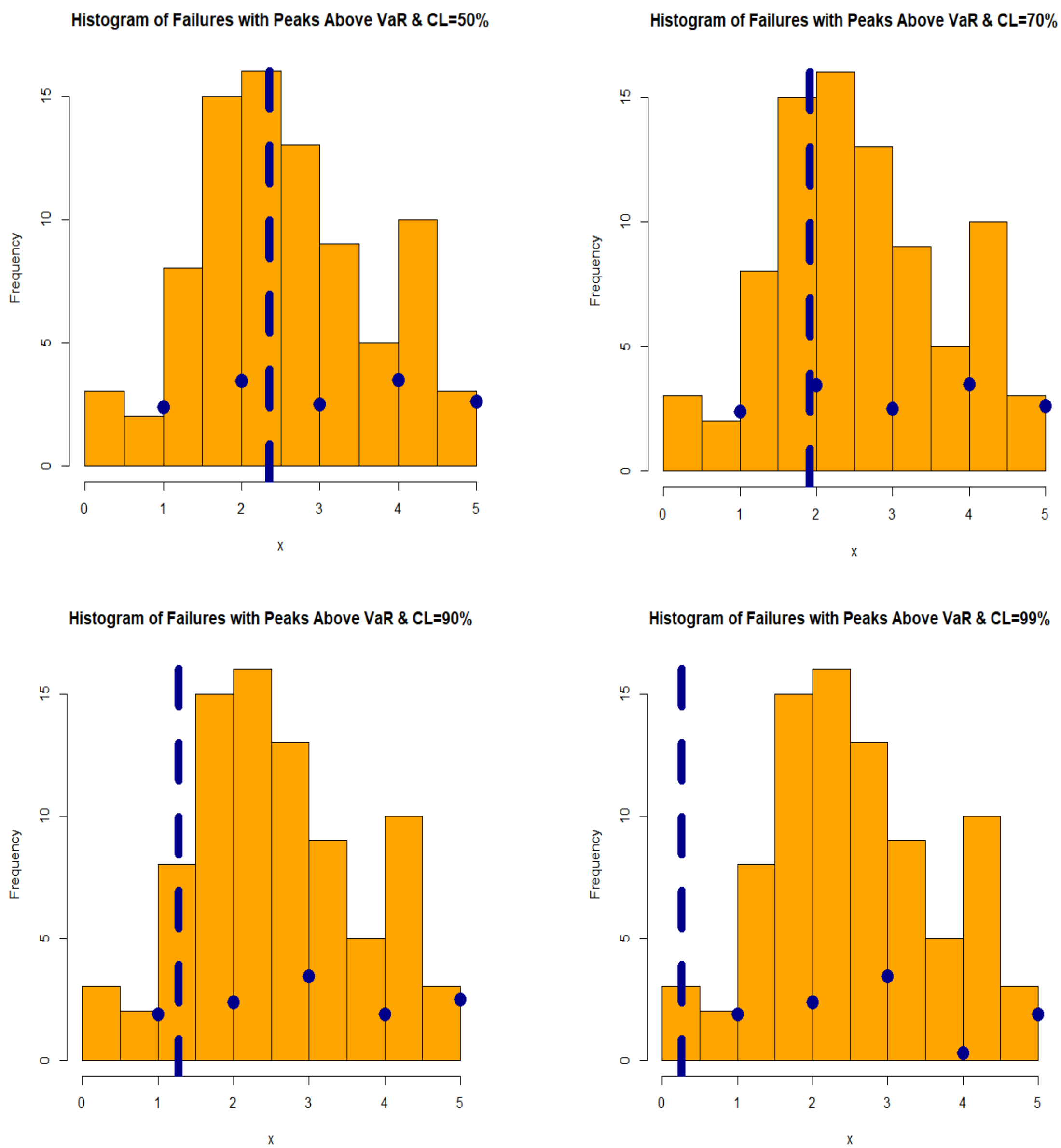
Table 8: PORT analysis under the service times data.

CL	N. Peaks	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
50%	31	2.117	2.450	2.820	3.086	3.561	5.140
70%	44	1.249	2.034	2.503	2.684	3.152	5.140
90%	56	0.487	1.393	2.152	2.318	2.896	5.140
99%	62	0.140	1.160	2.091	2.118	2.820	5.140

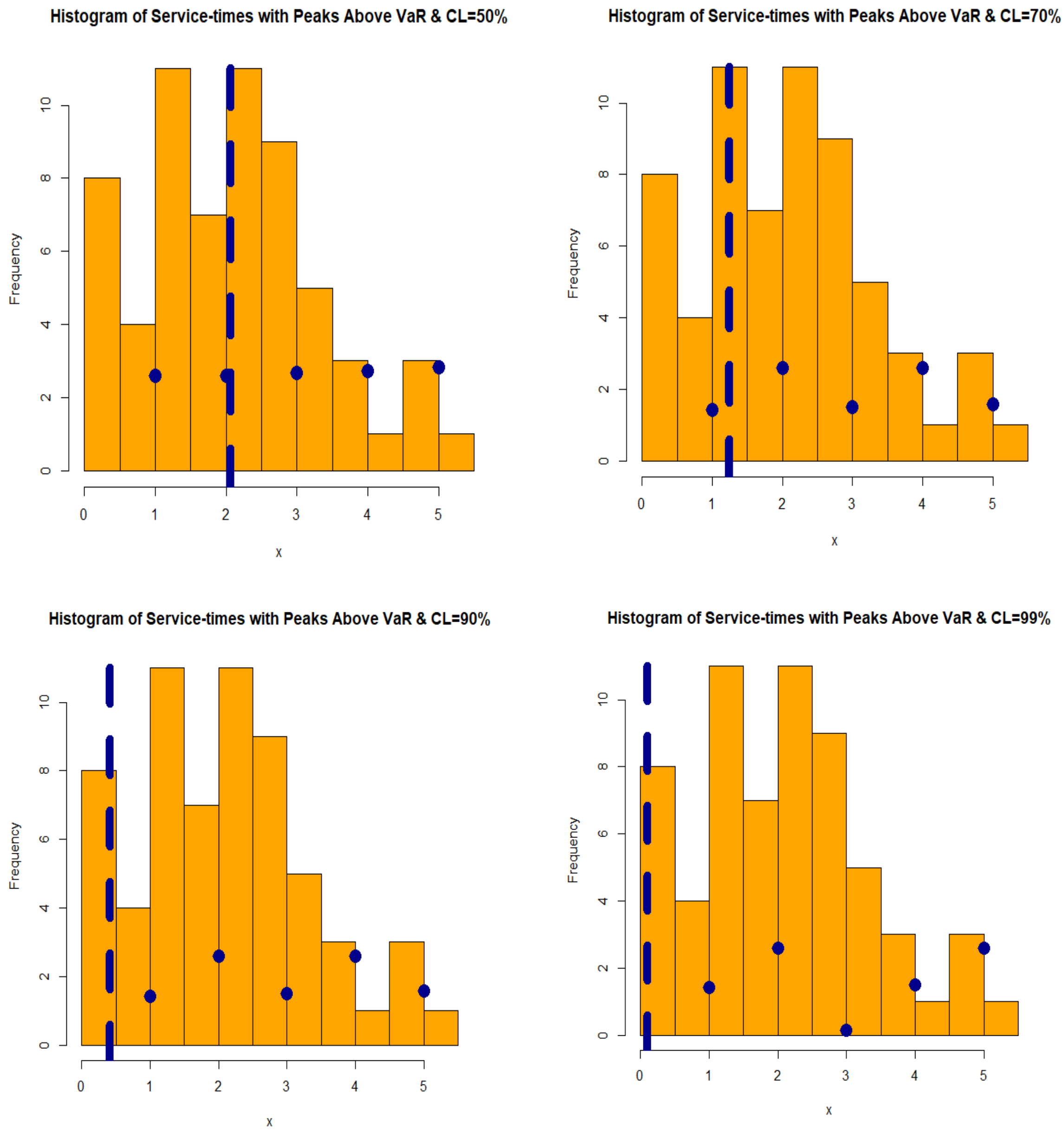
The analysis in Table 7 and Table 8 provides results for different confidence levels (CL): 50%, 70%, 90%, and 99%. These levels indicate the probability of exceeding a certain threshold value. As the confidence level increases (from 50% to 99%), the threshold value decreases, indicating higher confidence in capturing extreme events (peaks). The number of peaks (extreme events) identified increases as the confidence level (CL) increases. This suggests that at higher confidence levels, more extreme failure events (peaks) are observed in the failure times of aircraft windshields.

Recommendations for aircraft windshield companies about the failure and the service times:

- **I.** Understand the Peaks Over Random Threshold Value-at-risk results at different confidence levels (50%, 70%, 90%, 99%) to assess the likelihood of extreme failure events in aircraft windshields. Companies should prioritize risk management strategies based on these insights to mitigate potential failures and their impacts.
- **II.** Consider adjusting the threshold values based on the desired confidence level and risk tolerance. Higher confidence levels (e.g., 99%) provide a more conservative estimate of extreme events but may require more stringent mitigation strategies.
- **III.** Implement monitoring systems that track failure times in real-time or through periodic inspections. Early detection of potential peaks can help in proactive maintenance and replacement of aircraft windshields, thereby minimizing operational disruptions and safety risks.
- **IV.** Use the Peaks Over Random Threshold Value-at-risk analysis iteratively to refine design, manufacturing processes, and materials selection for aircraft windshields. Incorporate lessons learned from extreme events to enhance product durability and reliability over time.
- **V.** Ensure compliance with aviation safety regulations regarding failure prediction and prevention. Maintain transparent reporting practices on failure times and risk assessment outcomes to stakeholders, including regulatory bodies and customers.
- **VI.** By applying these recommendations, aircraft windshield companies can enhance their understanding of failure risks, improve operational resilience, and uphold high standards of safety and reliability in the aviation industry. This proactive approach contributes to maintaining trust among stakeholders and ensuring long-term success in the market.
- **VII.** Finally, by implementing these recommendations, aircraft windshield companies can effectively manage risks associated with service times, enhance operational resilience, and uphold safety standards in the aviation industry. This proactive approach not only mitigates potential failures but also contributes to maintaining customer trust and satisfaction.



Histogram of failures with peaks above VaR & CL=50%, 70%, 90% and 99%.



Histogram of service-times with peaks above VaR & CL=50%, 70%, 90% and 99%.

7. Conclusions

The paper introduces a novel probability distribution called the Burr Pareto type-II (BUPII) probability distribution, and explores its applications in engineering contexts, focusing specifically on the failure and service times of aircraft windshields. Through rigorous analysis, the study demonstrates the applicability and effectiveness of the BUPII probability distribution in both scenarios. Firstly, the BUPII probability distribution is applied to analyze the failure times of aircraft windshields. By leveraging its theoretical foundation and computational framework, the probability distribution provides insights into the probabilistic behavior of failures, offering engineers valuable tools for predicting and managing risks associated with windshield reliability. Secondly, the paper extends the application of the BUPII probability distribution to analyze the service times of aircraft windshields. This application showcases the versatility of the probability distribution in different engineering domains, where understanding the distribution and characteristics of service times is crucial for optimizing maintenance schedules and operational efficiency. Moreover, the paper includes a comprehensive reliability due to the mean-of-order P (MOOP) analysis under both failure and service time datasets. This analysis not only validates the BUPII probability distribution's performance but also highlights its robustness in assessing reliability metrics essential for engineering decision-making. Lastly, the paper incorporates a PORT (Peaks Over Random Threshold) VaR (Value-at-Risk) analysis, underscoring the BUPII probability distribution's utility in risk management within the context of aircraft windshield operations. By quantifying the potential financial impacts associated with extreme events, this analysis further underscores the probability distribution's practical relevance in real-world applications. The BUPII probability distribution presented in this paper emerges as a valuable addition to the toolkit of engineers and researchers involved in reliability and risk analysis. Its demonstrated efficacy in analyzing failure and service times of aircraft windshields, coupled with robust MOOP and PORT VaR analyses, positions it as a promising framework for advancing the understanding and management of engineering systems' reliability and risk. Future research avenues could explore broader applications across other engineering disciplines and further refine its methodologies to enhance predictive capabilities and decision support tools.

In future research, we aim to enhance the validation of right-censored distributions using advanced goodness-of-fit tests tailored for the Burr PII probability distribution. These tests include the Nikulin-Rao-Robson goodness-of-fit test statistic, modified Nikulin-Rao-Robson goodness-of-fit statistic test, Bagdonavicius-Nikulin goodness-of-fit statistic test, and modified Bagdonavicius-Nikulin goodness-of-fit statistic test. These methods have been successfully applied by researchers such as Alizadeh et al. (2023, 2018a,b), Ibrahim et al. (2019), Goual et al. (2019, 2020), Salah et al. (2020), Mansour et al. (2020a, d), Ibrahim et al. (2020), Yadav et al. (2020), Goual and Yousof (2020, 2021b), Aidi et al. (2021), Shehata and Shehata (2021 and 2022), El-Morshedy et al. (2021), Shehata et al. (2021 and 2022), Elgohari, and Yousof (2020 and 2021a,b) and Elgohari et al. (2021) among others. These efforts will contribute to a comprehensive evaluation of the Burr PII probability distribution's performance and its applicability in diverse practical scenarios.

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