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Marginal and Conditional both Extreme Value Distributions: A Case of Stochastic Regression Model

Sulaxana Bharali[1*], Jiten Hazarika[2], Kuldeep Goswami[3]

* Corresponding Author

1. Department of Statistics, Dibrugarh University, Dibrugarh, Assam, India, sulaxana.bharali@gmail.com
2. Department of Statistics, Dibrugarh University, Dibrugarh, Assam, India, jitenhazarika@rediffmail.com
3. Department of Statistics, Dibrugarh University, Dibrugarh, Assam, India, kgkuldeepgoswami@gmail.com

Abstract

A mathematical model is a mathematical connection that describes some real-life scenario. To handle real-world problems securely and effectively, simulation modelling is required. In this article, the author investigates the stochastic regression model scenario in which the dependent and independent variables in a linear regression model follow a distribution. We assume that the dependent and independent variables both exhibit Type I Extreme Value Distribution. The estimators are then derived using the Modified Maximum Likelihood (MML) estimation method. In accordance with this, a hypothesis testing technique is developed.

Key Words: Econometrics; Regression Analysis; Modified Maximum Likelihood; Maximum Likelihood

1. Introduction

Main focus of univariate regression is to analyze the relationship between a dependent variable and one independent variable and formulates the linear relation equation between dependent and independent variable (Kinal and Lahiri,1983; Lai,1994; Magdalinos and kandilorou,2001; Narula, 1974). Regression models with one dependent variable and more than one independent variables are called multilinear regression. Simultaneously, the conditions that have to be met for its appropriate use and the situations in which regression analysis may result in disastrously inaccurate conclusions if these conditions don't seem to be met (Akkaya and Tiku,2001; Akkaya and Tiku,2005; Bharali and Hazarika,2019). Providing greater importance to the assumption that the independent variable X is historically called nonstochastic, it is realized that the assumption of a nonstochastic regressor is not necessarily plausible; the regressor can be stochastic in nature. It is true for experimental research, in which the experimenter has power over the independent variables and may evaluate the outcome of the dependent variable continuously for the same defined values or any of the independent variables assigned values (Reynolds,1982; Sazak et.al, 2006;Tiku,1980). In the social sciences, the independent variables in one equation are often generated as the outcome variables of other equations that are stochastic in nature (Bharali and Hazarika,2022; Hwang,1980; Judge et. all,1988). Thus they neither have the same fixed values in repeated samples nor do they have values that correspond to the investigator's desired experimental design (Bowden and Turkington,1981; Ehrenberg,1963; Hooper and Zellner,1961). Thus, the dependent variable is always under the influence of explanatory factors, which are stochastic in design, in a non-experimental or unregulated environment.

The term stochastic regressor refers that the regressors, i.e. the independent variables, becoming unpredictable with the time transition. In the case of Stochastic regressors, the basic assumption are: i) X, Y, ε random ii) (X,Y) obtained from i.i.d sampling iii) $E(\varepsilon \mid X)=0$ iv) X takes atleast two values v) $Var(\varepsilon \mid X)=\sigma^2$ vi) ε is normal. The Variables X is the independent variable, Y dependent variable and ε is the error term.

However, it has been acknowledged that in numerous applications X may be stochastic, and *e* the random error term may not be normal. This may give rise to three problems (a) X is non-stochastic, and *e* is non-normal. (b) X is stochastic, and *e* is normal (c) X is stochastic, and *e* is non-normal (Islam et al, 2001; Islam et all,2005; Islam et al, 2010, Kerridge,1967; Vaughan and Tiku,200; Tiku et al,2001; Tiku and Suresh, 1992).

2. Preliminaries

We consider a situation where the marginal distribution of X and the conditional distributions of (Y|X=x) are stochastic in nature. The distribution of variable *X* is assumed to be Extreme Value Distribution (Type I) with probability density function,

$$g(x) = \frac{1}{\sigma_1} e^{\left\{\left(\frac{x-\mu_1}{\sigma_1}\right) - e^{\left(\frac{x-\mu_1}{\sigma_1}\right)}\right\}} \tag{1}$$

and the conditional density function of (Y|X=x) is Extreme Value Distribution (Type I) -

$$h(y \mid x) = \frac{1}{\sigma_2 (1-\rho^2)^{\frac{1}{2}}} e^{\left(\frac{y-\mu_2-\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)}{\sigma_2\sqrt{1-\rho^2}}\right)} \exp[-e^{\left(\frac{y-\mu_2-\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)}{\sigma_2\sqrt{1-\rho^2}}\right)}] \tag{2}$$

where, $-\infty \le x \le \infty$; $-\infty < y < \infty$; $\mu_1, \mu_2 \in R$; $\sigma_1, \sigma_2 > 0$ and $-1 < \rho < 1$

3. Modified Maximum Likelihood Estimator (MMLE) and it's Properties:

While doing parameter estimations, sometimes the solutions may have no explicit solutions. Solving them by iteration can be a difficult task as is determining the properties of the resulting estimators, especially for small samples. However, there are some fundamental difficulties with iterative solutions. In such cases maximum likelihood doesn't work, and a modification was suggested over simple maximum likelihood method so that modified equation has a explicate solution. A modified method over Maximum Likelihood Estimation (MLE) Method called Modified Maximum Likelihood Estimation (MMLE) Method was introduced and developed by Tiku in 1967. Further, Mehrotra and Nanda 1974; Pearson and Rootzen 1977; Tiku and Suresh 1992; Rosaiah et al. 1993a; Rosaiah et al. 1993b; Kantam and Srinivasa Rao 1993; Kantam and Srinivasa Rao 2002; Kantam and Sriram 2003 and the references therein are some of the works in this direction. The MMLE method expresses likelihood equations in terms of order statistics and then linearizes the intractable terms. The resulting estimators are called Modified Maximum Likelihood (MML).

The following properties are known to have under the usual regularity conditions for the existence of maximum likelihood estimators (Vaugan and Tiku 2000).

a. asymptotically, MML estimators are efficient i.e. they are unbiased and their variances are equal to the Minimum Variance Bounds (MVB).
b. for small samples, the MML estimators are almost fully efficient i.e. they have no or unimportant bias and their variances are only marginally, if at all, greater than the MVB.
c. the MML estimators are explicit functions of sample interpretation and are easy to compute and their properties are uncomplicated to determine.

Moreover, the MML estimators are numerically very close to the ML estimates for all sample sizes. This method can be utilized to estimate the parameters in the bivariate distribution with density g(x,y).
g(x,y)= g(x)g(y|x)
where g(x) is the distribution of X and g(y|x) is conditional distribution of (y|x) respectively.

3.1. Estimation Procedure of MMLE:

Suppose, we consider the three parameter log normal distribution with parameter (λ, μ, σ^2), where λ is the location parameter, μ is the scale parameter and σ^2 is the shape parameter. Accordingly, pdf is given by-

$$f(x;\lambda,\mu,\sigma^2) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma^2}(x-\lambda)} \exp_{\frac{-1}{2\sigma^2}}[\ln(x-\lambda)-\mu]^2, \lambda < x < \infty, \sigma^2 > 0, -\infty < \mu < \infty \\ 0, \quad\quad\quad\quad elsewhere \end{cases}$$

The estimators for three unknown parameters $(\lambda,\ \mu,\ \sigma^2)$, for lognormal distribution can be obtained by using the following algorithm:

Step 1: Get the likelihood of the given distribution.

Step 2: Take the natural log of the likelihood function obtain in step 1.

Step 3: Find the partial derivatives of the log-likelihood with respect to the three unknown parameters.

Step 4: Equate the derivatives in step 3 with zero and solve for μ and σ^2.

Step 5: To obtain the MML estimators, which have all the properties, first order the values xi, $1\le i \le n$, (in ascending order) of magnitude. The equations reduce to a new form.

Step 6: Solving the new transform equations are difficult, so linearize the function by using the first two terms of Taylor Series expansion around $t_{(i)} = E(Z_{(i)})$.

Step 7: Substitute the terms of the tailor series expansion in the equations and estimated the parameters from the new set of equations.

MML estimators are asymptotically equivalent to the ML estimators, their asymptotic variances and the covariance are given by where is the Fisher information matrix.

The method can solve particularly two problems of likelihood function:

When the value of any ordered sample $x_1, x_2,\ldots x_n$ will not tend to ∞ as (λ,μ,σ^2) approaches $(x, -\infty, \infty)$, this method can be utilized for any sample sizes. It avoids the problem of convergence in Newton-Raphson iteration method which produces no solutions for λ, when sample size is very small (n<10) (Colin Chen, 2005). However, this method also has some disadvantages. Some information about the sample is lost since the smallest value of x is not included in the formulation to calculate the estimate values of μ and σ^2.

4. Estimation of Parameters

The likelihood function based on the marginal distribution of X and the conditional distribution of (Y|X=x) as the Extreme Value Distribution of Type I is as follows,

$$L_x = \prod_{i=1}^{n} f(x;\mu_1,\sigma_1) \quad = \sigma_1^{-n} \exp\sum_{i=1}^{n}\left\{\left(\frac{x_i-\mu_1}{\sigma_1}\right) - \exp\left(\frac{x_i-\mu_1}{\sigma_1}\right)\right\}$$

and $L_{y|x} = \prod_{i=1}^{n} f(x,y;\ \mu_1,\mu_2,\sigma_1,\sigma_2,\rho)$

$$= \sigma_2^{-n}(1-\rho^2)^{-\frac{n}{2}} \exp\left\{\sum_{i=1}^{n}\left(\frac{y_i-\mu_2-\rho\frac{\sigma_2}{\sigma_1}(x_i-\mu_1)}{\sigma_2\sqrt{1-\rho^2}}\right)\right\} \exp\left\{-\sum_{i=1}^{n}\exp\left(\frac{y_i-\mu_2-\rho\frac{\sigma_2}{\sigma_1}(x_i-\mu_1)}{\sigma_2\sqrt{1-\rho^2}}\right)\right\}$$

$$L = L_x\ L_{(y|x)} = \sigma_1^{-n}\sigma_{2.1}^{-n}(1-\rho^2)^{-\frac{n}{2}} \exp\left[\sum_{i=1}^{n} z_i - \sum_{i=1}^{n}\exp z_i + \sum_{i=1}^{n}\left(\frac{e_i}{\sigma_{2.1}}\right) - \sum_{i=1}^{n}\exp\left(\frac{e_i}{\sigma_{2.1}}\right)\right]$$

$$L \propto \sigma_1^{-n}\sigma_{2.1}^{-n}(1-\rho^2)^{-\frac{n}{2}} \exp\left[\sum_{i=1}^{n} z_i - \sum_{i=1}^{n}\exp z_i + \sum_{i=1}^{n}\left(\frac{e_i}{\sigma_{2.1}}\right) - \sum_{i=1}^{n}\exp\left(\frac{e_i}{\sigma_{2.1}}\right)\right]$$

After taking log both sides,

$$\ln L = -n\ln\sigma_1 - n\ln\sigma_{2.1} + \sum_{i=1}^{n} z_i - \sum_{i=1}^{n}\exp z_i + \sum_{i=1}^{n}\left(\frac{e_i}{\sigma_{2.1}}\right) - \sum_{i=1}^{n}\exp\left(\frac{e_i}{\sigma_{2.1}}\right) \tag{3}$$

where $-\infty < z < \infty;\ -\infty < e < \infty;\ \mu_1,\ \mu_{2.1} \in \Re;\ \sigma_1, \sigma_{2.1} > 0$

and $z_i = \left(\frac{x_i-\mu_1}{\sigma_1}\right),\ \ e_i = (y_i - \theta x_i - \mu_{2.1}),\ \ \theta = \rho\frac{\sigma_2}{\sigma_1},\ \mu_{2.1} = \mu_2 - \theta\mu_1,\ \sigma_{2.1} = \sigma_2\sqrt{1-\rho^2}$

$$e_i = y_i - \mu_2 - \rho\tfrac{\sigma_1}{\sigma_2}(x_i-\mu_1) = y_i - \theta x_i - \mu_{2.1} \quad where\ \mu_{2.1} = \mu_2 - \theta\mu_1$$

Likelihood equations w.r.t the parameters $\mu_1,\ \sigma_1,\ \mu_{2.1}, \sigma_{2.1}\ and\, \theta\ are-$

$$\frac{\partial}{\partial\mu_1}\ln L = -\frac{n}{\sigma_1} + \frac{1}{\sigma_1}\sum_{i=1}^{n}\exp z_i = 0 \tag{4}$$

$$\frac{\partial}{\partial\sigma_1}\ln L = -\frac{n}{\sigma_1} - \frac{1}{\sigma_1}\sum_{i=1}^{n} z_i + \frac{1}{\sigma_1^{\ 2}}\sum_{i=1}^{n}\exp z_i = 0 \tag{5}$$

$$\frac{\partial}{\partial\mu_{2.1}}\ln L = -\frac{n}{\sigma_{2.1}} + \sum_{i=1}^{n}\exp\left(\frac{e_i}{\sigma_{2.1}}\right) = 0 \tag{6}$$

$$\frac{\partial}{\partial\sigma_{2.1}}\ln L = -\frac{n}{\sigma_{2.1}} - \frac{1}{\sigma_{2.1}^{\ 2}}\sum_{i=1}^{n} e_i + \frac{1}{\sigma_{2.1}^{\ 2}}\sum_{i=1}^{n}\exp\left(\frac{e_i}{\sigma_{2.1}}\right)(e_i) = 0 \tag{7}$$

$$\frac{\partial}{\partial\theta}\ln L = -\frac{\sigma_1}{\sigma_{2.1}}\sum_{i=1}^{n} z_i + \frac{\sigma_1}{\sigma_{2.1}}\sum_{i=1}^{n}\exp\left(\frac{e_i}{\sigma_{2.1}}\right)(z_i) = 0 \tag{8}$$

5. The Modified Maximum Likelihood (MML) Estimators

MMLE provides a framework to handle missing data more effectively by integrating over the missing values. This can lead to more accurate parameter estimates and better model fit. It allows for the incorporation of prior knowledge and this is particularly useful in Bayesian modeling, where prior information about the parameters can be combined with the observed data to obtain more informative estimates. MLE assumes that the observed data follow a specific probability distribution. While this assumption is often reasonable, it can limit the flexibility of the model. On the other hand, MMLE allows for more flexible modeling assumptions by considering a wider class of probability distributions or using nonparametric methods. This can be beneficial when the data do not adhere strictly to the assumed distribution. Though MMLE involves additional computational steps, it is helpful in the case of relatively complex situations. In this research work, because of the functions $e^{\left(\frac{e_i}{\sigma_{2.1}}\right)}e_i$ and $e^{\left(\frac{e_i}{\sigma_{2.1}}\right)}z_i$, in the above equations (4) to (8) are almost impossible to solve. Hence, we use Modified Maximum Likelihood (MML) Method to solve the above equations.

To find the MML estimators, we define, $w_i = y_i - \theta x_i$

We order the values x_i and $w_i,\ 1 \le i \le n$, in ascending order of magnitude as

$$x_{(1)} \le x_{(2)} \le\ x_{(3)} \le \ldots \le x_{(n)}$$
$$w_{(1)} \le w_{(2)} \le w_{(3)} \le \ldots \le w_{(n)} \tag{9}$$

Then, $e_{(i)} = w_{(i)} - \mu_{2.1} = (y_i - \theta x_1) - \mu_{2.1}$ has the same order with $w_{(i)}$ since $\mu_{2.1}$ is a constant and $z_{(i)} = \left(\frac{x_{(i)} - \mu_1}{\sigma_1}\right)$ has the same order with $x_{(i)}$ since μ_1 is a constant and σ_1 is positive.

We also define, $a_i = \frac{e_i}{\sigma_{2.1}}$ so that $a_{(i)} = \frac{e_{(i)}}{\sigma_{2.1}} = \frac{w_{(i)} - \mu_{2.1}}{\sigma_{2.1}}$.

Now, $w_{(i)} = y_{[i]} - \theta\, x_{[i]}$; $(x_{[i]}, y_{[i]})$ and $z_{[i]} = \left(\frac{x_{[i]} - \mu_1}{\sigma_1}\right)$ are the concomitants of $w_{(i)}$.

Since complete sums are invariant to ordering, the likelihood equations can be written as $g_1(z) = e^z$ and $g_2(a) = e^a$

$$\left.\begin{aligned}
&\frac{\partial}{\partial\mu_1}\ln L = -\frac{n}{\sigma_1} + \frac{1}{\sigma_1}\sum_{i=1}^{n} g_1(z_{(i)}) = 0 \qquad \frac{\partial}{\partial\sigma_1}\ln L = -\frac{n}{\sigma_1} - \frac{1}{\sigma_1}\sum_{i=1}^{n} z_{(i)} + \frac{1}{\sigma_1^{\ 2}}\sum_{i=1}^{n} g_1\left(z_{(i)}\right) = 0\\
&\frac{\partial}{\partial\mu_{2.1}}\ln L = -\frac{n}{\sigma_{2.1}} + \sum_{i=1}^{n} g_2\left(a_{(i)}\right) = 0 \qquad \frac{\partial}{\partial\sigma_{2.1}}\ln L = -\frac{n}{\sigma_{2.1}} - \frac{1}{\sigma_{2.1}^{\ 2}}\sum_{i=1}^{n} e_{(i)} + \frac{1}{\sigma_{2.1}^{\ 2}}\sum_{i=1}^{n} g_2\left(a_{(i)}\right)e_{(i)} = 0\\
&\frac{\partial}{\partial\theta}\ln L = -\frac{\sigma_1}{\sigma_{2.1}}\sum_{i=1}^{n} z_{(i)} + \frac{\sigma_1}{\sigma_{2.1}}\sum_{i=1}^{n} g_2\left(a_{(i)}\right)z_{(i)} = 0
\end{aligned}\right\} \tag{10}$$

To derive the MML estimators, we linearize the function $g_1(z_{(i)})$ and $g_2(a_{(i)})$ by using the first two terms of Taylor Series Expansion around $E(z_{(i)}) = t_{1(i)}$ and $E(z_{(i)}) = t_{2(i)}$

$$g_1(z_{(i)}) = g_1(t_{1(i)}) + [z_{(i)} - t_{1(i)}]\left(\tfrac{d}{dz} g_1\left(z_{(i)}\right)\right)_{z_{(i)}=t_{1(i)}} = e^{t_{1(i)}}(1 - t_{1(i)}) + z_{(i)} e^{t_{1(i)}} = \alpha_i + \beta_{1i}\, z_{(i)} \tag{11}$$

where $\alpha_i = e^{t_{1(i)}}(1 - t_{1(i)})$ *and* $\beta_{1i} = e^{t_{1(i)}}$

$$g_2(a_{(i)}) = g_2(t_{2(i)}) + [a_{(i)} - t_{2(i)}]\left(\tfrac{d}{da} g_2\left(a_{(i)}\right)\right)_{a_{(i)}=t_{2(i)}} = e^{t_{2(i)}}(1 - t_{2(i)}) + a_{(i)} e^{t_{2(i)}} = \alpha_i + \beta_{2i}\, a_{(i)} \tag{12}$$

where $\alpha_i = e^{t_{2(i)}}(1 - t_{2(i)})$, $\beta_{2i} = e^{t_{2(i)}}$

Substituting the values (11) and (12) to (10),

$$\left.\begin{aligned}
&\frac{\partial}{\partial \mu_1} \ln L^* = -\frac{n}{\sigma_1} + \frac{1}{\sigma_1}\sum_{i=1}^{n}(\alpha_i + \beta_{1i}\, z_{(i)}) = 0 \\
&\frac{\partial}{\partial \sigma_1} \ln L^* = -\frac{n}{\sigma_1} - \frac{1}{\sigma_1}\sum_{i=1}^{n} z_{(i)} + \frac{1}{\sigma_1^{\,2}}\sum_{i=1}^{n}(\alpha_i + \beta_{1i}\, z_{(i)}) = 0 \\
&\frac{\partial}{\partial \mu_{2.1}} \ln L^* = -\frac{n}{\sigma_{2.1}} + \frac{1}{\sigma_{2.1}}\sum_{i=1}^{n}(\alpha_i + \beta_{2i}\, z_{(i)}) = 0 \\
&\frac{\partial}{\partial \sigma_{2.1}} \ln L^* = -\frac{n}{\sigma_{2.1}} - \frac{1}{\sigma_{2.1}^{\,2}}\sum_{i=1}^{n} e_{(i)} + \frac{1}{\sigma_{2.1}^{\,2}}\sum_{i=1}^{n} e_i(\alpha_i + \beta_{2i}\, a_{(i)}) = 0 \\
&\frac{\partial}{\partial \theta} \ln L^* = -\frac{\sigma_1}{\sigma_{2.1}}\sum_{i=1}^{n} z_{(i)} + \frac{\sigma_1}{\sigma_{2.1}}\sum_{i=1}^{n} z_{(i)}(\alpha_i + \beta_{2i} a_{(i)}) = 0
\end{aligned}\right\} \tag{13}$$

The equations have explicit solutions which are the following MML estimators:

$$\hat{\mu}_1 = K_1 + \hat{\sigma}_1 D_1 \tag{14}$$

$$\hat{\sigma}_1 = \frac{B_1 + \sqrt{B_1^{\,2} + 4nC_1}}{2n} \tag{15}$$

$$\hat{\mu}_{2.1} = \overline{y_{[.]}} - \hat{\theta}\, \overline{x_{[.]}} - \frac{\Delta}{m_2}\hat{\sigma}_{2.1} \tag{16}$$

$$\hat{\sigma}_{2.1} = \frac{-B \pm \sqrt{B^2 + 4nc}}{2n} \tag{17}$$

$$\hat{\theta} = K_2 - D_2\, \hat{\sigma}_{2.1} \tag{18}$$

where $m_1 = \sum_{i=1}^{n}\beta_{1i}$, $m_2 = \sum_{i=1}^{n}\beta_{2i}$, $\Delta_i = \sum_{i=1}^{n}(1 - \alpha_i)$

$$K_1 = \frac{1}{m}\sum_{i=1}^{n}\beta_{1i} x_{(i)}\ , \qquad K_2 = \frac{\sum_{i=1}^{n}\beta_{2i}(x_i - \mu_1)(y_i - \mu_2)}{\sum_{i=1}^{n}\beta_{2i}(x_i - \mu_1)^2}$$

$$D_1 = \frac{1}{m_1}\left(\sum_{i=1}^{n}\alpha_i - n\right)\ , \qquad D_2 = \frac{\sum_{i=1}^{n}(1 - \alpha_i)(x_i - \mu_1)}{\sum_{i=1}^{n}\beta_{2i}(x_i - \mu_1)^2}$$

$$\overline{y_{[.]}} = \frac{\sum_{i=1}^{n}\beta_{2i} y_{(i)}}{\sum_{i=1}^{n}\beta_{2i}},\quad \overline{x_{[.]}} = \frac{\sum_{i=1}^{n}\beta_{2i} x_{(i)}}{\sum_{i=1}^{n}\beta_{2i}} \quad \textit{and} \quad B = \sum_{i=1}^{n}\{(1 - \alpha_i)(y_{[i]} - \theta\ x_{(i)} - \mu_{2.1})\}$$

6. Conditional and Marginal Likelihood function

All the estimators are very difficult than those based on a bivariate normal distribution. It is, therefore very important to recognize the true underlying distribution. In order to develop the estimation procedure for the parameters $\mu_1, \sigma_1, \mu_2, \sigma_2$ *and* ρ directly, we consider the likelihood function.

$$L = \sigma_1^{-n} \sigma_2^{-n} (1-\rho^2)^{-\frac{n}{2}} \exp\left[\sum_{i=1}^{n} z_i - \sum_{i=1}^{n} \exp z_i + \sum_{i=1}^{n}\left(\frac{e_i}{\sigma_{2.1}}\right) - \sum_{i=1}^{n} \exp\left(\frac{e_i}{\sigma_{2.1}}\right)\right]$$

$$\ln L = -n\ln\sigma_1 - n\ln\sigma_2 - \frac{n}{2}\ln(1-\rho^2) + \sum_{i=1}^{n} z_i - \sum_{i=1}^{n} \exp z_i + \sum_{i=1}^{n}\left(\frac{e_i}{\sigma_{2.1}}\right) - \sum_{i=1}^{n} \exp\left(\frac{e_i}{\sigma_{2.1}}\right) \tag{19}$$

The likelihood equations for estimating the parameters $\mu_1, \sigma_1, \mu_2, \sigma_2$ *and* ρ are

$$\left.\begin{aligned}
&\frac{\partial}{\partial\mu_1}\ln L = -\frac{n}{\sigma_1} + \frac{1}{\sigma_1}\sum_{i=1}^{n} e^{z_i} + \frac{\rho}{\sigma_1\sqrt{1-\rho^2}}\left\{n - \sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}\right\} = 0 \\
&\frac{\partial}{\partial\sigma_1}\ln L = -\frac{n}{\sigma_1} - \frac{1}{\sigma_1}\sum_{i=1}^{n} z_i + \frac{1}{\sigma_1}\sum_{i=1}^{n} z_i\, e^{z_i} - \frac{\rho}{\sigma_1\sqrt{1-\rho^2}}\left\{\sum_{i=1}^{n} z_i + \sum_{i=1}^{n} z_i\, e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}\right\} = 0 \\
&\frac{\partial}{\partial\mu_2}\ln L = -\frac{1}{\sigma_2\sqrt{1-\rho^2}}\left\{n - \sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}\right\} = 0 \\
&\frac{\partial}{\partial\sigma_2}\ln L = -\frac{n}{\sigma_2} - \frac{1}{\sigma_2^{\,2}\sqrt{1-\rho^2}}\sum_{i=1}^{n} e_i + \frac{1}{\sigma_2^{\,2}\sqrt{1-\rho^2}}\sum_{i=1}^{n} e_i\, e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} + \frac{\rho}{\sigma_2\sqrt{1-\rho^2}}\left\{\sum_{i=1}^{n} z_i + \sum_{i=1}^{n} z_i\, e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}\right\} = 0 \\
&\frac{\partial}{\partial\rho}\ln L = \frac{n\rho}{(1-\rho^2)} + \frac{\rho}{\sigma_2(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n} e_i - \frac{\rho}{\sigma_2(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n} e_i e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} - \frac{1}{\sqrt{1-\rho^2}}\left\{\sum_{i=1}^{n} z_i + \sum_{i=1}^{n} z_i\, e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}\right\} = 0
\end{aligned}\right\} \tag{20}$$

Let,

$$n - \sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} = 0 \;\; and \;\; \sum_{i=1}^{n} z_i + \sum_{i=1}^{n} z_i\, e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} = 0$$

Thus, the likelihood equations reduces to-

$$\frac{\partial}{\partial\mu_1}\ln L = -\frac{n}{\sigma_1} + \frac{1}{\sigma_1}\sum_{i=1}^{n} e^{z_i} = 0 \tag{21}$$

$$\frac{\partial}{\partial\sigma_1}\ln L = -\frac{n}{\sigma_1} - \frac{1}{\sigma_1}\sum_{i=1}^{n} z_i + \frac{1}{\sigma_1}\sum_{i=1}^{n} z_i\, e^{z_i} = 0 \tag{22}$$

$$\frac{\partial}{\partial\mu_2}\ln L = -\frac{n}{\sigma_2\sqrt{1-\rho^2}} + \frac{\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}}{\sigma_2\sqrt{1-\rho^2}} = 0 \tag{23}$$

$$\frac{\partial}{\partial\sigma_2}\ln L = -\frac{n}{\sigma_2} - \frac{1}{\sigma_2^{\,2}\sqrt{1-\rho^2}}\sum_{i=1}^{n} e_i + \frac{1}{\sigma_2^{\,2}\sqrt{1-\rho^2}}\sum_{i=1}^{n} e_i\, e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} = 0 \tag{24}$$

$$\frac{\partial}{\partial\rho}\ln L=\frac{n\rho}{(1-\rho^2)}+\frac{\rho}{\sigma_2(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n}e_i-\frac{\rho}{\sigma_2(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n}e_i e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}=0 \tag{25}$$

Because of the intractable function $\sum_{i=1}^{n} e^{Z_i}$ and $\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}$ in equations (21) and (25) respectively cannot be solved. Hence, we use MML method to solve the equations (21) and (25).

To find the MML estimators, we define $w_i=(y_i-\rho\frac{\sigma_2}{\sigma_1}x_i)$

We order the values x_i and w_i (for a given θ), $1\le i\le n$, in ascending order of magnitude as-

$$x_{(1)}\le x_{(2)}\le\ldots\le x_{(n)}$$
$$w_{(1)}\le w_{(2)}\le\ldots\le w_{(n)} \tag{26}$$

Thus, $e_{(i)}=w_{(i)}-(\mu_2-\rho\frac{\sigma_2}{\sigma_1}\mu_1)$ has the same order as $w_{(i)}$ since $(\mu_2-\rho\frac{\sigma_2}{\sigma_1}\mu_1)$ is a constant and $z_{(i)}=\dfrac{(x_{(i)}-\mu_1)}{\sigma_1}$ has the same order as $x_{(i)}$ since μ_1 is a constant and σ_1 is positive.

We also write, $a_i=\dfrac{e_i}{\sigma_2\sqrt{1-\rho^2}}$

Thus we have,

$$a_{(i)}=\frac{e_{(i)}}{\sigma_2\sqrt{1-\rho^2}}=\frac{\left\{w_{(i)}-(\mu_2-\rho\frac{\sigma_2}{\sigma_1}\mu_1)\right\}}{\sigma_2\sqrt{1-\rho^2}}$$

Where,

$$w_{(i)}=y_{[i]}-\rho\frac{\sigma_2}{\sigma_1}x_{[i]}$$

$(x_{[i]},y_{[i]})$ *and* $z_{[i]}=\dfrac{(x_{[i]}-\mu_1)}{\sigma_1}$ are the concomitants of $w_{(i)}$.

Since complete sums are invariant to ordering, the likelihood equations can be written as-

$$\left.\begin{aligned}
&\frac{\partial}{\partial\mu_1}\ln L=-\frac{n}{\sigma_1}+\frac{1}{\sigma_1}\sum_{i=1}^{n}e^{Z_{(i)}}=0\\
&\frac{\partial}{\partial\sigma_1}\ln L=-\frac{n}{\sigma_1}-\frac{1}{\sigma_1}\sum_{i=1}^{n}z_{(i)}+\frac{1}{\sigma_1}\sum_{i=1}^{n}z_{(i)}e^{Z_i}=0\\
&\frac{\partial}{\partial\mu_2}\ln L=-\frac{n}{\sigma_2\sqrt{1-\rho^2}}+\frac{\sum_{i=1}^{n}e^{a_{(i)}}}{\sigma_2\sqrt{1-\rho^2}}=0\\
&\frac{\partial}{\partial\sigma_2}\ln L=-\frac{n}{\sigma_2}-\frac{1}{\sigma_2{}^2\sqrt{1-\rho^2}}\sum_{i=1}^{n}e_{(i)}+\frac{1}{\sigma_2{}^2\sqrt{1-\rho^2}}\sum_{i=1}^{n}e_{(i)}e^{a_{(i)}}=0\\
&\frac{\partial}{\partial\rho}\ln L=\frac{n\rho}{(1-\rho^2)}+\frac{\rho}{\sigma_2(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n}e_{(i)}-\frac{\rho}{\sigma_2(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n}e_{(i)}e^{a_{(i)}}=0
\end{aligned}\right\} \tag{27}$$

If we write,

$$g_1(z)=\sum_{i=1}^{n}e^{Z_{(i)}}\ \textit{and}\ g_2(z)=\sum_{i=1}^{n}e^{a_{(i)}}$$

And linearize them as before, the modified likelihood equations are

$$\frac{\partial}{\partial\mu_1}\ln L=-\frac{n}{\sigma_1}+\frac{1}{\sigma_1}g_1(z)=0$$

$$\begin{aligned}
&\frac{\partial}{\partial\sigma_1}\ln L = -\frac{n}{\sigma_1} - \frac{1}{\sigma_1}\sum_{i=1}^{n} z_{(i)} + \frac{1}{\sigma_1}\sum_{i=1}^{n} z_{(i)} g_1(z) = 0\\
&\frac{\partial}{\partial\mu_2}\ln L = -\frac{n}{\sigma_2\sqrt{1-\rho^2}} + \frac{1}{\sigma_2\sqrt{1-\rho^2}}\, g_2(a) = 0\\
&\frac{\partial}{\partial\sigma_2}\ln L = -\frac{n}{\sigma_2} - \frac{1}{\sigma_2^{\,2}\sqrt{1-\rho^2}}\sum_{i=1}^{n} e_{(i)} + \frac{1}{\sigma_2^{\,2}\sqrt{1-\rho^2}}\sum_{i=1}^{n} e_{(i)} g_2(a) = 0\\
&\frac{\partial}{\partial\rho}\ln L = \frac{n\rho}{(1-\rho^2)} + \frac{\rho}{\sigma_2(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n} e_{(i)} - \frac{\rho}{\sigma_2(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n} e_{(i)} g_2(a) = 0
\end{aligned} \tag{28}$$

Again, to derive the MML estimators, we linearize the function $g_1(z_{(i)})$ and $g_2(a_{(i)})$ by using the first two terms of Taylor Series Expansion around $E(z_{(i)}) = t_{1(i)}$ and $E(z_{(i)}) = t_{2(i)}$

$$g_1(Z_{(i)}) = g_1(t_{1(i)}) + [z_{(i)} - t_{1(i)}]\left(\tfrac{d}{dz} g_1\left(z_{(i)}\right)\right)_{z_{(i)}=t_{1(i)}} = e^{t_{1(i)}}(1-t_{1(i)}) + z_{(i)}e^{t_{1(i)}} = \alpha_i + \beta_{1i}\, z_{(i)} \tag{29}$$

$$where\ \ \alpha_i = e^{t_{1(i)}}(1-t_{1(i)})\ \ and\ \ \beta_{1i} = e^{t_{1(i)}}$$

$$g_2(a_{(i)}) = g_2(t_{2(i)}) + [a_{(i)} - t_{2(i)}]\left(\tfrac{d}{da} g_2\left(a_{(i)}\right)\right)_{a_{(i)}=t_{2(i)}} = e^{t_{2(i)}}(1-t_{2(i)}) + a_{(i)}e^{t_{2(i)}} = \alpha_i + \beta_{2i}\, a_{(i)} \tag{30}$$

$$where\ \alpha_i = e^{t_{2(i)}}(1-t_{2(i)})\ and\ \beta_{2i} = e^{t_{2(i)}}$$

Substituting these values we get the following equations,

$$\left.\begin{aligned}
&\frac{\partial}{\partial\mu_1}\ln L^* = -\frac{n}{\sigma_1} + \frac{1}{\sigma_1}\sum_{i=1}^{n}(\alpha_i + \beta_{1i} z_{(i)}) = 0\\
&\frac{\partial}{\partial\sigma_1}\ln L^* = -\frac{n}{\sigma_1} - \frac{1}{\sigma_1}\sum_{i=1}^{n} z_{(i)} + \frac{1}{\sigma_1}\sum_{i=1}^{n} z_{(i)}(\alpha_i + \beta_{1i} z_{(i)}) = 0\\
&\frac{\partial}{\partial\mu_2}\ln L^* = -\frac{n}{\sigma_2\sqrt{1-\rho^2}} + \frac{1}{\sigma_2\sqrt{1-\rho^2}}\sum_{i=1}^{n}(\alpha_i + \beta_{2i} a_{(i)}) = 0\\
&\frac{\partial}{\partial\sigma_2}\ln L^* = -\frac{n}{\sigma_2} - \frac{1}{\sigma_2^{\,2}\sqrt{1-\rho^2}}\sum_{i=1}^{n} e_{(i)} + \frac{1}{\sigma_2^{\,2}\sqrt{1-\rho^2}}\sum_{i=1}^{n} e_{(i)}\ (\alpha_i + \beta_{2i} a_{(i)}) = 0\\
&\frac{\partial}{\partial\rho}\ln L^* = \frac{n\rho}{(1-\rho^2)} + \frac{\rho}{\sigma_2(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n} e_{(i)}\ (\alpha_i + \beta_{2i} a_{(i)}) = 0
\end{aligned}\right\} \tag{31}$$

To solve (20) we use the functional relations $\rho = \theta\dfrac{\sigma_1}{\sigma_2}$ *and* $\sigma_{2.1} = \sigma_2\sqrt{1-\rho^2}$ we obtain the following MMLE

$$\hat{\mu}_1 = K_1 - D_1\hat{\sigma}_1 \tag{32}$$

$$\hat{\sigma}_1 = \frac{-B \pm \sqrt{B^2 + 4nC}}{2n} \tag{33}$$

$$\hat{\mu}_2 = \overline{y_{[.]}} - \hat{\theta}(\overline{x_{[.]}} - \hat{\mu}_1) - \frac{\Delta}{m_2}\hat{\sigma}_2\sqrt{1-\hat{\rho}^{\,2}} \tag{34}$$

$$\hat{\sigma}_2 = \sqrt{\hat{\sigma}_{2.1}^{\,2} + \hat{\theta}_1^{\,2}\,\hat{\sigma}_1^{\,2}} \tag{35}$$

$$\hat{\rho} = \hat{\theta}\frac{\hat{\sigma}_1}{\hat{\sigma}_2} \tag{36}$$

Since the estimators can be obtained from the equations by simple substitution, it follows that, like the ML estimators, the MML estimators have the invariance property even in this complex situation. This is a very interesting result indeed.

7. Asymptotic Covariance Matrix

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Case 1: The asymptotic covariance matrix of the estimators $\hat{\mu}_1, \hat{\sigma}_1, \hat{\mu}_2, \hat{\sigma}_2, and\ \hat{\rho}$ is given by $I^{-1}(\mu_1, \sigma_1, \mu_2, \sigma_2, \rho)$, where I is the Fisher information matrix. The Fisher information matrix is given by (i, j=1, 2, 3,4, 5)

$$I = [I_{ij}] = \left[-E\left(\frac{\partial^2 \ln L}{\partial \tau_i \partial \tau_j}\right)\right] \ ,\tau_1 = \mu_1,\ \tau_2 = \sigma_1,\ \tau_3 = \mu_2,\ \tau_4 = \sigma_2,\ \tau_5 = \rho$$

If we let $I = n\,A$, the elements of the matrix A are

$$A_{\mu_1\mu_1} = -\frac{1}{\sigma_1^{\,2}}\sum_{i=1}^{n} e^{z_i} + \frac{n\theta_1\rho}{\sigma_1^{\,2}(1-\rho^2)}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}$$

$$A_{\sigma_1\mu_1} = \frac{n}{\sigma_1^{\,2}} - \frac{1}{\sigma_1^{\,2}}\sum_{i=1}^{n} e^{z_i} - \frac{1}{\sigma_1^{\,2}}\sum_{i=1}^{n} e^{z_i} z_i - \frac{n\rho}{\sigma_1^{\,2}\sqrt{1-\rho^2}} - \frac{\rho^2}{\sigma_1^{\,2}(1-\rho^2)}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} z_i + \frac{\rho}{\sigma_1^{\,2}(1-\rho^2)}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}$$

$$A_{\mu_1\mu_2} = \frac{\rho}{\sigma_1\sigma_2(1-\rho^2)}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}$$

$$A_{\mu_1\sigma_2} = \frac{\rho^2}{\sigma_1\sigma_2(1-\rho^2)}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} z_i + \frac{\rho}{\sigma_1\sigma_2^{\,2}(1-\rho^2)}\sum_{i=1}^{n} e_i\, e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}$$

$$A_{\mu_1\rho} = \frac{n}{\sigma_1\sqrt{1-\rho^2}} + \frac{n\rho^2}{\sigma_1(1-\rho^2)^{\frac{3}{2}}} - \frac{\rho}{\sigma_1(1-\rho^2)}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} z_i + \frac{1}{\sigma_1\sqrt{1-\rho^2}}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} - \frac{\rho^2}{\sigma_1\sigma_2(1-\rho^2)^2}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} e_i - \frac{\rho^2}{\sigma_1(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}$$

$$A_{\sigma_1\sigma_1} = \frac{n}{\sigma_1^{\,2}} + \frac{2}{\sigma_1^{\,2}}\sum_{i=1}^{n} z_i - \frac{1}{\sigma_1^{\,2}}\sum_{i=1}^{n} e^{z_i} z_i^{\,2} - \frac{2}{\sigma_1^{\,2}}\sum_{i=1}^{n} e^{z_i} z_i + \frac{2\rho}{\sigma_1^{\,2}\sqrt{1-\rho^2}}\sum_{i=1}^{n} z_i - \frac{\rho^2}{\sigma_1^{\,2}(1-\rho^2)}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} z_i^{\,2}$$

$$A_{\sigma_1\mu_2} = \frac{\rho}{\sigma_1\sigma_2(1-\rho^2)}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} z_i$$

$$A_{\sigma_1\sigma_2} = \frac{\rho^2}{\sigma_1\sigma_2(1-\rho^2)^2}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} z_i^{\,2} + \frac{\rho}{\sigma_1\sigma_2(1-\rho^2)}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} e_i\ z_i$$

$$A_{\sigma_1\rho} = \frac{1}{\sigma_1\sqrt{1-\rho^2}}\sum_{i=1}^{n} z_i + \frac{\rho^2}{\sigma_1(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n} z_i - \frac{\rho}{\sigma_1(1-\rho^2)}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} z_i^{\,2} + \frac{1}{\sigma_1\sqrt{1-\rho^2}}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} z_i$$
$$-\frac{\rho^2}{\sigma_1\sigma_2(1-\rho^2)^2}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} z_i . e_i - \frac{\rho^2}{\sigma_1(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} z_i$$

$$A_{\mu_2\mu_2} = -\frac{1}{\sigma_2^{\,2}(1-\rho^2)}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}$$

$$A_{\mu_2\sigma_2} = \frac{n}{\sigma_2^{\,2}\sqrt{1-\rho^2}} - \frac{\rho}{\sigma_2^{\,2}(1-\rho^2)}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} z_i - \frac{1}{\sigma_2^{\,3}(1-\rho^2)}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} e_i - \frac{1}{\sigma_2^{\,2}\sqrt{1-\rho^2}}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}$$

$$A_{\mu_2\rho} = -\frac{n\rho}{\sigma_2(1-\rho^2)} + \frac{1}{\sigma_2(1-\rho^2)}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} z_i + \frac{\rho}{\sigma_2{}^2(1-\rho^2)^2}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} e_i + \frac{\rho}{\sigma_2(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}$$

$$A_{\sigma_2\sigma_2} = \frac{n}{\sigma_2{}^2} + \frac{2\rho}{\sigma_2{}^2\sqrt{1-\rho^2}}\sum_{i=1}^{n} z_i + \frac{2}{\sigma_2{}^3\sqrt{1-\rho^2}}\sum_{i=1}^{n} e_i + \frac{\rho^2}{\sigma_2{}^2(1-\rho^2)}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} z_i{}^2 - \frac{2\rho}{\sigma_2{}^3(1-\rho^2)}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} e_i .\, z_i$$
$$-\frac{\rho}{\sigma_2{}^2\sqrt{1-\rho^2}}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} z_i - \frac{2}{\sigma_2{}^3\sqrt{1-\rho^2}}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} e_i - \frac{1}{\sigma_2{}^4(1-\rho^2)}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} e_i{}^2$$

$$A_{\sigma_2\rho} = \frac{1}{\sigma_2\sqrt{1-\rho^2}}\sum_{i=1}^{n} z_i + \frac{\rho^2}{\sigma_2(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n} z_i - \frac{n}{\sigma_2\sqrt{1-\rho^2}}\sum_{i=1}^{n} z_i + \frac{\rho}{\sigma_2{}^2(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n} e_i + \frac{\rho}{\sigma_2(1-\rho^2)}\sum_{i=1}^{n} e^{\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}} z_i{}^2$$
$$-\frac{\rho^2}{\sigma_2{}^2(1-\rho^2)^2}\sum_{i=1}^{n} e^{\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}} z_i e_i - \frac{\rho^2}{\sigma_2(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n} e^{\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}} z_i + \frac{1}{\sigma_2{}^2(1-\rho^2)}\sum_{i=1}^{n} e^{\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}} e_i z_i - \frac{\rho}{\sigma_2(1-\rho^2)^2}\sum_{i=1}^{n} e^{\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}} e_i{}^2$$

$$A_{\rho\rho} = \frac{n}{(1-\rho^2)} + \frac{2np}{(1-\rho^2)^2} - \frac{2\rho}{(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n} z_i + \frac{1}{\sigma_2(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n} e_i + \frac{3\rho^2}{\sigma_2(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n} e_i + \frac{1}{(1-\rho^2)}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} z_i{}^2$$
$$+\frac{\rho}{\sigma_2(1-\rho^2)^2}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} z_i\; e_i + \frac{\rho}{(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} z_i + \frac{\rho}{\sigma_2{}^2(1-\rho^2)^2}\sum_{i=1}^{n} e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)} e_i$$

Case 2: Define Fisher information matrix $I(\mu_1,\sigma_1,\mu_{2.1},\sigma_{2.1},\theta)$ for estimating $\mu_1,\sigma_1,\mu_{2.1},\sigma_{2.1}$ *and* θ. As usual, the asymptotic covariance matrix of the estimators $\hat{\mu}_1,\hat{\sigma}_1,\hat{\mu}_{2.1},\hat{\sigma}_{2.1}$ *and* $\hat{\theta}$ is given by $\sum \equiv I^{-1}(\mu_1,\sigma_1,\mu_{2.1},\sigma_{2.1},\theta)$. If we write I= nA, the elements of the matrix A are

$$A_{\mu_1\mu_1} = \frac{\partial}{\partial\mu_1\partial\mu_1} = -\frac{1}{\sigma_1{}^2}\sum_{i=1}^{n} z_i$$

$$A_{\mu_1\sigma_1} = \frac{\partial}{\partial\mu_1\partial\sigma_1} = \frac{1}{\sigma_1{}^2}\sum_{i=1}^{n} z_i - \frac{1}{\sigma_1{}^3}\sum_{i=1}^{n}\exp(z_i)$$

$$A_{\mu_1\mu_{2.1}} = \frac{\partial}{\partial\mu_1\partial\mu_{2.1}} = 0, \quad A_{\mu_1\sigma_{2.1}} = \frac{\partial}{\partial\mu_1\partial\sigma_{2.1}} = 0$$

$$A_{\mu_1\theta} = \frac{\partial}{\partial\mu_1\partial\theta} = \frac{1}{\sigma_{2.1}} + \frac{\sigma_1}{\sigma_{2.1}}\sum_{i=1}^{n}\exp\left(\frac{e_i}{\sigma_{2.1}}\right)e_i$$

$$A_{\sigma_1\sigma_1} = \frac{\partial}{\partial\sigma_1\partial\sigma_1} = \frac{1}{\sigma_1{}^2}\sum_{i=1}^{n} z_i - \frac{1}{\sigma_1{}^3}\sum_{i=1}^{n}\exp(z_i) + \frac{1}{\sigma_{2.1}}\sum_{i=1}^{n}\left(\frac{e_i}{\sigma_{2.1}}\right)e_i$$

$$A_{\sigma_1\mu_{2.1}} = \frac{\partial}{\partial\sigma_1\partial\mu_{2.1}} = 0, \quad A_{\sigma_1\sigma_{2.1}} = \frac{\partial}{\partial\sigma_1\partial\sigma_{2.1}} = 0$$

$$A_{\sigma_1\theta} = \frac{\partial}{\partial\sigma_1\partial\theta} = -\frac{1}{\sigma_{2.1}} + \frac{1}{\sigma_{2.1}}\sum_{i=1}^{n}\left(\frac{e_i}{\sigma_{2.1}}\right)e_i$$

$$A_{\mu_{2.1}\mu_{2.1}} = \frac{\partial}{\partial\mu_{2.1}\partial\mu_{2.1}} = \frac{1}{\sigma_{2.1}{}^2} + \frac{\sigma_1}{\sigma_{2.1}}\sum_{i=1}^{n}\left(\frac{e_i}{\sigma_{2.1}}\right)(z_i)$$

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$$A_{\mu_{2.1}\sigma_{2.1}} = \frac{\partial}{\partial\mu_{2.1}\partial\sigma_{2.1}} = \frac{1}{\sigma_{2.1}{}^2} + \frac{1}{\sigma_{2.1}{}^2}\sum_{i=1}^{n}\left(\frac{e_i}{\sigma_{2.1}}\right)e_i$$

$$A_{\mu_{2.1}\theta} = \frac{\partial}{\partial\mu_{2.1}\partial\theta} = \frac{\sigma_1}{\sigma_{2.1}}\sum_{i=1}^{n}\left(\frac{e_i}{\sigma_{2.1}}\right)(z_i)$$

$$A_{\sigma_{2.1}\sigma_{2.1}} = \frac{\partial}{\partial\sigma_{2.1}\partial\sigma_{2.1}} = \frac{n}{\sigma_{2.1}{}^2} + \frac{2}{\sigma_{2.1}{}^3}\sum_{i=1}^{n}e_i - \frac{2}{\sigma_{2.1}{}^3}\sum_{i=1}^{n}\left(\frac{e_i}{\sigma_{2.1}}\right)(e_i)$$

$$A_{45} = \frac{\partial}{\partial\sigma_{2.1}\partial\theta} = \frac{\sigma_1}{\sigma_{2.1}{}^2}\sum_{i=1}^{n}z_i - \frac{\sigma_1}{\sigma_{2.1}{}^2}\sum_{i=1}^{n}\exp\left(\frac{e_i}{\sigma_{2.1}}\right)(z_i)$$

$$A_{\theta\theta} = \frac{\partial}{\partial\theta\partial\theta} = \frac{n}{\sigma_{2.1}{}^2} + \frac{2}{\sigma_{2.1}{}^3}\sum_{i=1}^{n}e_i - \frac{2}{\sigma_{2.1}{}^3}\sum_{i=1}^{n}\left(\frac{e_i}{\sigma_{2.1}}\right)(e_i)$$

8. Hypothesis Testing

Case 1: Our major interest is testing the null hypothesis *H_0: $\rho=0$*.
Since the MML estimators are asymptotically equivalent to the ML estimators, the likelihood function L is maximized (asymptotically) by the MML estimators. Thus, the likelihood ratio is (asymptotically)

$$\hat{\lambda} = \frac{\max(L\,|\,H_0)}{\max(L\,|\,H_1)}$$

$$= \frac{\prod_{i=1}^{n}\left[\frac{1}{\hat{\sigma}_3}e^{\left(\frac{y_i-\hat{\mu}_3}{\hat{\sigma}_3}\right)}\exp\left(-e^{\left(\frac{y_i-\hat{\mu}_3}{\hat{\sigma}_3}\right)}\right)\right]}{\prod_{i=1}^{n}\left[\frac{1}{\hat{\sigma}_2(1-\hat{\rho}^2)^{\frac{1}{2}}}e^{\left(\frac{\hat{y}_i-\hat{\mu}_2-\hat{\theta}(x_i-\hat{\mu}_1)}{\hat{\sigma}_{2.1}}\right)}\exp\left(-e^{\left(\frac{y_i-\hat{\mu}_2-\hat{\theta}(x_i-\hat{\mu}_1)}{\hat{\sigma}_{2.1}}\right)}\right)\right]}$$

It can be shown that for large n, $\hat{\lambda}$ is a monotonic function of $\hat{\rho}^2$. Thus, to test *H_0: $\rho>0$,* we propose the statistic

$$W = \hat{\rho}\left[\begin{array}{l}\frac{n}{(1-\rho^2)} + \frac{2np}{(1-\rho^2)^2} - \frac{2\rho}{(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n}z_i + \frac{1}{\sigma_2(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n}e_i + \frac{3\rho^2}{\sigma_2(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n}e_i + \frac{1}{(1-\rho^2)}\sum_{i=1}^{n}e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}z_i{}^2 \\ \quad + \frac{\rho}{\sigma_2(1-\rho^2)^2}\sum_{i=1}^{n}e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}z_i\,e_i + \frac{\rho}{(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n}e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}z_i + \frac{\rho}{\sigma_2{}^2(1-\rho^2)^2}\sum_{i=1}^{n}e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}e_i\end{array}\right]_{\rho=0}^{-\frac{1}{2}}$$

Where,

$$\left[\begin{array}{l}\frac{n}{(1-\rho^2)} + \frac{2np}{(1-\rho^2)^2} - \frac{2\rho}{(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n}z_i + \frac{1}{\sigma_2(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n}e_i + \frac{3\rho^2}{\sigma_2(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n}e_i + \frac{1}{(1-\rho^2)}\sum_{i=1}^{n}e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}z_i{}^2 \\ \quad + \frac{\rho}{\sigma_2(1-\rho^2)^2}\sum_{i=1}^{n}e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}z_i\,e_i + \frac{\rho}{(1-\rho^2)^{\frac{3}{2}}}\sum_{i=1}^{n}e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}z_i + \frac{\rho}{\sigma_2{}^2(1-\rho^2)^2}\sum_{i=1}^{n}e^{\left(\frac{e_i}{\sigma_2\sqrt{1-\rho^2}}\right)}e_i\end{array}\right]$$

is the asymptotic variance of $\hat{\rho}$ under H_0, obtained from $\{I^{-1}(\mu_1,\,\sigma_1,\,\mu_2,\,\sigma_2,\,\rho)\}_{\rho=0}$

The null distribution of *W* is referred to normal N(0,1). Large values of *W* lead to rejection of H_0 against H_1: $\rho>0$.
Testing the null hypothesis

$$H_0:\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

is also of great practical importance.

Since, $\hat{\mu}_1$ and $\hat{\mu}_2$ are asymptotically equivalent to the ML estimators, the distribution of the random vector $\sqrt{n}(\hat{\mu}_1,\ \hat{\mu}_2)$ is bivariate normal with zero mean vector and (estimated) covariance matrix.

$$\hat{\Omega}=n\begin{bmatrix}\hat{\sigma}_{11} & \hat{\sigma}_{13}\\ \hat{\sigma}_{13} & \hat{\sigma}_{33}\end{bmatrix}$$

where, $\hat{\sigma}_{11}$, $\hat{\sigma}_{13}$ and $\hat{\sigma}_{33}$ are the estimated elements of the asymptotic covariance matrix $\sum$, more specifically, $\sigma_{ij}=\Sigma_{ij}=I_{ij}{}^{-1}(\mu_1,\sigma_1,\mu_2,\sigma_2,\rho)$. Since in these elements $\hat{\sigma}_1$ and $\hat{\sigma}_2$ converge to σ_1 and σ_2, respectively, the asymptotic null distribution of

$$\hat{T}^2=n(\hat{\mu}_1,\hat{\mu}_2)\hat{\Omega}^{-1}\begin{pmatrix}\hat{\mu}_1\\ \hat{\mu}_2\end{pmatrix}$$

is chi-square with 2 d.f. We reject H_0 at the 5% level of significance if the value of $\hat{T}^2$ is greater than $\chi^2{}_{0.05}(2)$. The non null distribution (asymptotic) of $\hat{T}^2$ is noncentral chi-square with 2 d.f. and non centrality parameter.

$$\lambda^2=n(\mu_1,\ \mu_2)\Omega^{-1}\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}$$

For small *n*, the null distribution of $\frac{(n-2)}{2(n-1)}\hat{T}^2$ is approximately central- F with (2, n-2) d.f. and the non-null distribution is approximately noncentral –F with (2,n-2) d.f. and non-centrality parameter λ^2.

Case 2: We are interested in the null hypothesis $(\mu_1,\ \mu_{2.1})=(0,0)$ which is same as

$$H_o:\begin{pmatrix}\mu_1\\ \mu_{2.1}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

Since $\hat{\mu}_1$ and $\hat{\mu}_{2.1}$ are asymptotically equivalent to the ML estimators, the distribution of the random vector $\sqrt{n}(\hat{\mu}_1,\ \hat{\mu}_{2.1})$ is bivariate normal with zero mean vector and (estimated) covariance matrix

$$\hat{\Omega}=n\begin{bmatrix}\hat{\sigma}_{11} & 0\\ 0 & \hat{\sigma}_{33}\end{bmatrix}$$

where $\hat{\sigma}_{11}$ and $\hat{\sigma}_{33}$ are the estimated elements of the asymptotic covariance matrix $\sum$, more specifically , $\sigma_{ij}=\Sigma_{ij}=I_{ij}{}^{-1}(\mu_1,\sigma_1,\mu_{2.1},\sigma_{2.1},\theta_1)$. The covariance between $\hat{\mu}_1$ and $\hat{\mu}_{2.1}$ is zero since they are orthogonal components, so there is no need to estimate it. Since in these elements $\hat{\sigma}_1$ and $\hat{\sigma}_{2.1}$ converge to σ_1 and $\sigma_{2.1}$, respectively, the asymptotic null distribution of

$$\hat{T}_1^{\,2} = n(\hat{\mu}_1, \hat{\mu}_{2.1})\hat{\Omega}^{-1}\begin{pmatrix}\hat{\mu}_1 \\ \hat{\mu}_{2.1}\end{pmatrix}$$

$$= \frac{\hat{\mu}_1^{\,2}}{\hat{\sigma}_{11}} + \frac{\hat{\mu}_{2.1}^{\,2}}{\hat{\sigma}_{33}}$$

is chi-square with 2 d.f. We reject H_o at the 5% level of significance if the value of $\hat{T}_1^{\,2}$ is greater than $\chi^2_{0.05}(2)$. The nonnull distribution (asymptotic) of is $\hat{T}_1^{\,2}$ non central chi-square with 2 d.f. and non-centrality parameter

$$\lambda^2 = n(\mu_1, \mu_{2.1})\Omega^{-1}\begin{pmatrix}\mu_1 \\ \mu_{2.1}\end{pmatrix}$$

$$= \frac{\mu_1^{\,2}}{\sigma_{11}} + \frac{\mu_{2.1}^{\,2}}{\sigma_{33}}$$

For small n, the null distribution of $\frac{(n-2)}{2(n-1)}\hat{T}^2$ is approximately central-F with (2, n-2) d.f. and the nonnull distribution is approximately noncentral-F with (2, n-2) d.f and non-central parameter λ^2.

9. Simulation Study

The ratio of the variance of the MMLE to the corresponding LSE multiplied by 100 is used in this section to calculate the simulated relative efficiency of Least Square Estimators (LSE). Results have been provided for multiple n (sample size) values. We present the findings for fixed values of ρ = 0.5 and various values of n, which are 20, 40, 80, and 100. Then, 10,000 Monte Carlo runs were used to generate the results. Without loss of generality, $\mu_1, \sigma_1, \mu_2, \sigma_2$ are considered to be 0, 1, 0, 1. The other parameters take values from the relations $\theta = \rho\frac{\sigma_2}{\sigma_1}$, $\mu_{2.1} = \mu_2 - \theta\mu_1$, $\sigma_{2.1} = \sigma_2\sqrt{1-\rho^2}$. The computer program to do simulations is written in R studio. The simulated estimated value for the marginal distribution of X and the conditional distributions of Y given $X=x$ are both Extreme Value Distribution of Type I for fixed value of ρ and different values of n are presented in the Table: 8.1 through Table: 8.4.

Table 8.1: Simulated values for n =20, $\rho =$ 0.5

		μ_1	σ_1	μ_2	σ_2	$\mu_{2.1}$	$\sigma_{2.1}$	θ	ρ
MMLE	mean	0.0525	1.0275	-0.0032	1.062	-0.0456	0.8962	0.5123	0.5012
	n*bias2	0.0457	0.0135	0.0085	0.0456	0.0312	0.062	0.0012	0.0042
	n*variance	5.4612	0.8445	7.8921	0.7351	5.859	0.6563	0.9015	0.615
	n*mse	5.5069	0.858	7.9006	0.7807	5.8902	0.7183	0.9027	0.6192
LSE	mean	-0.0327	0.978	-0.1452	0.9803	-0.0697	0.8459	0.4995	0.5156
	n*bias2	0.0026	0.006	0.1385	0.0162	0.0676	0.0015	0.0012	0.0025
	n*variance	5.5747	0.982	7.9678	0.8013	6.3586	0.739	1.0253	0.6969
	n*mse	5.5773	0.988	8.1063	0.8175	6.4262	0.7405	1.0265	0.6994
	effvar	97.9640	85.9979	99.0499	91.7384	92.1429	88.8092	87.9254	88.2479
	effmse	98.7377	86.8421	97.4624	95.4984	91.6591	97.0020	87.9396	88.5330

Table 8.2: Simulated Values for n = 40, $\rho =$ 0.5

		μ_1	σ_1	μ_2	σ_2	$\mu_{2.1}$	$\sigma_{2.1}$	θ	ρ
MMLE	mean	0.0423	1.0137	0.0076	1.045	0.0111	0.8856	0.5162	0.5015
	n*bias2	0.0437	0.007	0.0051	0.0375	0.0123	0.0051	0.0011	0.0035

	n*variance	5.3552	0.8142	7.2156	0.7521	5.4563	0.6468	0.845	0.6038
	n*mse	5.3989	0.8212	7.2207	0.7896	5.4686	0.6519	0.8461	0.6073
LSE	mean	0.0311	0.985	-0.1321	0.9758	-0.0458	0.8156	0.4986	0.5745
	n*bias2	0.0025	0.004	0.1221	0.0151	0.0013	0.0011	0.0011	0.0021
	n*variance	5.5088	1.0121	7.5678	0.825	6.1236	0.7563	1.0001	0.6912
	n*mse	5.5113	1.0161	7.6899	0.8401	6.1249	0.7574	1.0012	0.6933
	effvar	97.2117	80.4466	95.3460	91.1636	89.1028	85.5216	84.4915	87.3553
	effmse	97.9605	80.8188	93.8984	93.9888	89.2847	86.0707	84.5085	87.5955

Table 8.3: Simulated Values for n = 80, $\rho = 0.5$

		μ_1	σ_1	μ_2	σ_2	$\mu_{2.1}$	$\sigma_{2.1}$	θ	ρ
MMLE	mean	0.0343	1.004	0.0052	1.031	0.0112	0.8695	0.5369	0.5125
	n*bias2	0.0345	0.001	0.0015	0.0025	0.0113	0.0045	0.0009	0.0031
	n*variance	5.2039	0.7707	6.6596	0.6521	5.1569	0.6213	0.812	0.5986
	n*mse	5.2384	0.7717	6.6611	0.6546	5.1682	0.6258	0.8129	0.6017
LSE	mean	-0.0269	0.992	-0.1289	0.9786	-0.0125	0.8145	0.4978	0.5645
	n*bias2	0.0019	0.002	0.1645	0.0021	0.0009	0.001	0.0005	0.0015
	n*variance	5.4501	1.0215	6.9878	0.8368	5.9869	0.7769	0.9865	0.6915
	n*mse	5.452	1.0235	7.1523	0.8389	5.9878	0.7779	0.987	0.693
	effvar	95.4826	75.4478	95.3032	77.9278	86.1364	79.9716	82.3112	86.5654
	effmse	96.0821	75.3981	93.1322	78.0307	86.3121	80.4473	82.3606	86.8254

Table 8.4: Simulated Values for n =100, ρ=0.5

		μ_1	σ_1	μ_2	σ_2	$\mu_{2.1}$	$\sigma_{2.1}$	θ	ρ
MMLE	mean	0.0261	1.003	0.0012	1.011	0.0071	0.8611	0.5372	0.5369
	n*bias2	0.0272	0.001	0.0011	0.0015	0.0073	0.0036	0.0007	0.0025
	n*variance	5.0011	0.7617	6.2457	0.6198	4.9863	0.5991	0.7526	0.5875
	n*mse	5.0283	0.7627	6.2468	0.6213	4.9936	0.6027	0.7533	0.59
LSE	mean	-0.0218	0.992	-0.1289	0.9869	-0.0078	0.8012	0.4965	0.5398
	n*bias2	0.0011	0.002	0.1056	0.0011	0.0008	0.001	0.0005	0.0004
	n*variance	5.3828	1.002	6.7589	0.8563	5.8963	0.7781	0.9521	0.69
	n*mse	5.3839	1.004	6.8645	0.8574	5.8971	0.7791	0.9526	0.6904
	effvar	92.9088	76.0179	92.4070	72.3811	84.5665	76.9952	79.0463	85.144
	effmse	93.3951	75.9661	91.0015	72.4632	84.6789	77.3584	79.0783	85.457

10. Conclusions

In the paper Tiku (1967), the author briefly mention a new family of estimators that can be derived from Tiku's MMLE estimator. The new family of estimators could be useful for a variety of applications, such as medical research, quality control, and financial forecasting. Also, MMLE get the better fitting in censored sample. The censoring can be either upper or lower, and the method is applicable to both symmetrical and asymmetrical censoring. Utilizing this, a method for testing hypothesis has been derived for the case where X's marginal distribution is the Extreme Value Distribution of Type I and Y's conditional distributions is also the Extreme Value Distribution of Type I given that X=x. According to a simulation research, the MML estimators are more effective than the comparable LS estimators for all sample sizes (n=20, 40, 80, and 100) and all parameters. Due to the fact that MML estimators are asymptotically MVB estimators, their relative efficiencies grow along with sample size, which is another benefit of using MML estimators.

The value of θ and ρ is the primary focus of regression analysis. It is evident from Table 8.1 to 8.4 that the effectiveness of LS estimators slowly declines as sample size increases and stays close to 80%. The simulated mean, variance, and MSE for MML estimators and LS estimators are shown in this study along with their respective efficiency. According to the analysis, MML estimators are more effective than the analogous LS estimators, which suggest that MMLE efficiency is directly proportional to sample size. Additionally, this outcome is consistent with the published theoretical findings.

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