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Mixed Poisson Transmuted New Weighted Exponential Distribution with Applications on Skewed and Dispersed Count Data

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Abstract

This study proposes a new three-parameter mixed Poisson Cubic Rank Transmuted New Weighted Exponential Distribution. The new discrete distribution is obtained by mixing the Poisson distribution with a newly obtained Cubic Rank Transmuted New Weighted Exponential Distribution. Various shapes and mathematical properties of the mixing distribution and the new count distribution are examined. Special cases of the new proposition are also identified. The distribution, special cases, and other count distributions are assumed for skewed and dispersed count observations. The maximum likelihood estimation is used to estimate the parameters of all examined distributions. Results show that the new proposition, along with some of its special cases, provides a good fit for all the examined data.

Key Words: New weighted exponential distribution, Cubic rank transmutations, Dispersed count observations, Skewed count data, Maximum likelihood estimation.

Mathematical Subject Classification: 60E05, 62E15

1. Introduction

The classical Poisson distribution assumes equality for the mean and variance for count observation. However, such scenarios are rarely seen given count data. In most cases, count observations are often dispersed (mean ≠ variance). Hence, research in count distribution usually focuses on obtaining distributions that can be efficiently used to model dispersed observations. In most cases, dispersed data are skewed (Adcock *et al*., 2015; Omari *et al*., 2018) because of inherent variations in the mean and variance of observations.

In obtaining count distributions that can effectively model skewed and dispersed observations, the method of mixed Poisson pioneered by Greenwood and Yule (1920) and also recently explored further by Polito (2019) is well refereed. The process assumes a continuous distribution for the parameter of the Poisson distribution and obtains the marginal of the mixed distribution (mixture of the Poisson and the assumed mixing probability distributions). The mixed Poisson distribution has an inherent characteristic of always providing a higher variance than the mean, hence, they are more efficient to model dispersed observations when compared with the classical Poisson distribution. Many choices for mixing probability distributions have been made. (Greenwood & Yule, 1920) assumed gamma distribution; (Bhattacharya, 1966) assumed uniform distribution; (Sankaran, 1970) assumed the Lindley distribution; (Johnson et al., 1993) assumed exponential distribution; (Gómez-Déniz *et al*., 2012) assumed Lindley-Beta; while (Das *et al*., 2018) assumed a three-parameter Lindley distribution among many others.

When the count distribution is skewed and unimodal, the aim is to obtain a mixing distribution with similar properties. Following the earlier work of (Nasiru, 2015) on Weibull distribution, (Oguntunde *et al*., 2016) proposed a new weighted exponential distribution with a relatively simpler distribution function than the weighted exponential distribution of Gupta and Kundu (2009). To improve flexibility in handling unimodal, skewed and dispersed count data, this research utilizes the Cubic Rank Transmutation (CRT) map (Rahman *et al*., 2019) to compound the New

Weighted Exponential Distribution (Oguntunde *et al*., 2016) and obtain a new mixing probability distribution. The mixing distribution is then used to obtain a new three-parameter mixed Poisson distribution.

2. Mixing Distribution

The mixing distribution is obtained by using the one-parameter Cubic Rank Transmutation (CRT) map (Rahman et al., 2019) to compound the new weighted exponential distribution (Oguntunde et al., 2016). If the distribution function of the baseline distribution is given as G_x, and ρ is the introduced extra shape parameter, the distribution function for the CRT is defined as:

$$F_x = (1-\rho)G_x + 3\rho(G_x)^2 - 2\rho(G_x)^3. \tag{1}$$

A new weighted exponential distribution (Oguntunde et al., 2016) has the cumulative distribution function defined as:

$$G_x = 1 - e^{-(1+v)\theta x}. \tag{2}$$

Therefore, a new continuous mixing probability distribution called the Cubic Rank Transmuted New Weighted Exponential Distribution (CRTNWED) is obtained by inserting (2) into (1). Hence, the distribution function of the CRTNWED is obtained as:

$$F_x = 1 - e^{-(1+v)\theta x} + \rho e^{-(1+v)\theta x} + 2\rho e^{-3(1+v)\theta x} - 3\rho e^{-2(1+v)\theta x}. \tag{3}$$

The corresponding PDF is given as:

$$f_x = (1+v)\theta e^{-(1+v)\theta x}\left(1-\rho+6\rho e^{-(1+v)\theta x} - 6\rho e^{-2(1+v)\theta x}\right). \tag{4}$$

2.1 Special Cases:

1. If $v = 0$ in (4), the distribution becomes the CRT exponential distribution with PDF given as:
$$f_x = \theta e^{-\theta x}\left(1-\rho+6\rho e^{-\theta x} - 6\rho e^{-2\theta x}\right). \tag{5}$$
2. If $\rho = 0$ in (4), the distribution becomes the weighted exponential distribution (Oguntunde *et al*., 2016) with PDF given as:
$$f_x = (1+v)\theta e^{-(1+v)\theta x}. \tag{6}$$

2.2 Reliability Functions for CRTNWED

The survival function for the distribution is obtained by evaluating $(1-F_x)$. Hence, the survival function, denoted by S_x is given as:

$$S_x = e^{-(1+v)\theta x} + \rho e^{-(1+v)\theta x} + 2\rho e^{-3(1+v)\theta x} - 3\rho e^{-2(1+v)\theta x}. \tag{7}$$

The failure rate also called the hazard rate function for the distribution is obtained as:

$$h_x = \frac{(1+v)\theta\left(1-\rho+6\rho e^{-(1+v)\theta x} - 6\rho e^{-2(1+v)\theta x}\right)}{1-\rho-2\rho e^{-2(1+v)\theta x} + 3\rho e^{-(1+v)\theta x}}. \tag{8}$$

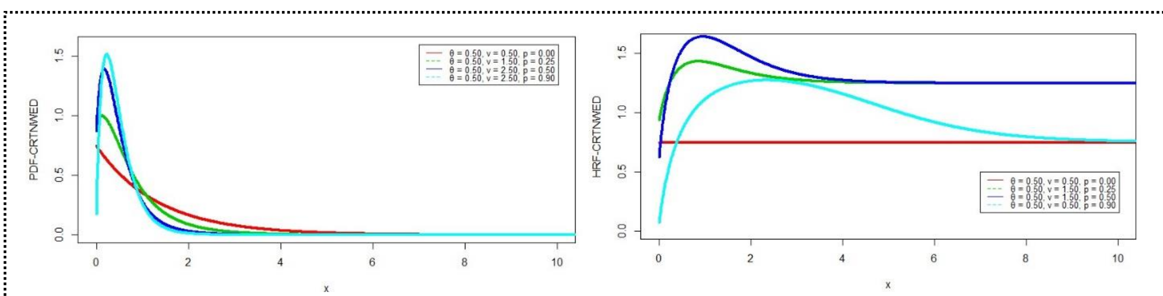

Figure 1: Shapes of the PDF and HRF of the CRTNWED

The PDF of the CRTNWED has a positive skew unimodal shape (Figure 1) for different parameters values except for when $\rho = 0$ when its shape resembles that of an exponential distribution. The hazard rate function has an increasing-decreasing-constant shape for different parameter combinations. When $\rho = 0$, the HRF has a constant hazard rate.

2.3 Moment

Proposition 1: The central moment of a random variable *X* with a CRTNWED, is given as:

$$E(X^r) = \left(1 - \rho + \frac{3\rho}{2^r} - \frac{2\rho}{3^r}\right)\frac{r!}{(1+v)^r\theta^r}. \tag{9}$$

Proof:

$$E(X^r) = \int_0^\infty x^r f_x \, dx$$

$$= \int_0^\infty x^r (1+v)\theta e^{-(1+v)\theta x}\left(1 - \rho + 6\rho e^{-(1+v)\theta x} - 6\rho e^{-2(1+v)\theta x}\right) dx$$

$$= (1+v)\theta \int_0^\infty x^r e^{-(1+v)\theta x} \, dx - \rho(1+v)\theta \int_0^\infty x^r e^{-(1+v)\theta x} \, dx + 6\rho(1+v)\theta \int_0^\infty x^r e^{-2(1+v)\theta x} \, dx$$

$$- 6\rho(1+v)\theta \int_0^\infty x^r e^{-3(1+v)\theta x} \, dx$$

$$= \frac{r!}{(1+v)^r\theta^r} - \frac{\rho r!}{(1+v)^r\theta^r} + \frac{3\rho r!}{(1+\rho)^r(2\theta)^r} - \frac{2\rho r!}{(1+\rho)^r(3\theta)^r}$$

$$= \left(1 - \rho + \frac{3\rho}{2^r} - \frac{2\rho}{3^r}\right)\frac{r!}{(1+v)^r\theta^r}$$

2.4 Moment Generating Function

Proposition 2: If a random variable X has a CRTNWED, the MGF is given as:

$$E(e^{tx}) = \frac{\theta(1+v)}{\theta + v\theta - t} - \frac{\rho\theta(1+v)}{\theta + v\theta - t} + \frac{6\rho\theta(1+v)}{2\theta + 2v\theta - t} - \frac{6\rho\theta(1+v)}{3\theta + 3v\theta - t}. \tag{10}$$

Proof:

$$E(e^{tx}) = \int_0^\infty e^{tx} f(x) \, dx$$

$$= \int_0^\infty e^{tx} (1+v)\theta e^{-(1+v)\theta x}\left(1 - \rho + 6\rho e^{-(1+v)\theta x} - 6\rho e^{-2(1+v)\theta x}\right) dx$$

$$= (1+v)\theta \int_0^\infty \left(e^{-(\theta+v\theta-t)x} - \rho e^{-(\theta+v\theta-t)x} + 6\rho e^{-(2\theta+2v\theta-t)x} - 6\rho e^{-(3\theta+3v\theta-t)x}\right) dx$$

$$= \frac{\theta(1+v)}{\theta + v\theta - t} - \frac{\rho\theta(1+v)}{\theta + v\theta - t} + \frac{6\rho\theta(1+v)}{2\theta + 2v\theta - t} - \frac{6\rho\theta(1+v)}{3\theta + 3v\theta - t}$$

From the equation 10, the mean and the variance for the distribution are respectively obtained as:

$$E(X) = \frac{6-\rho}{6(1+v)\theta}. \tag{11}$$

$$Var(X) = \frac{36 - 22\rho - \rho^2 - 34\rho v + 72v}{36(1+v)^2\theta^2}. \tag{12}$$

3. Mixed Poisson CRTNWED

Using the PDF for the CRTNWED in equation (4) as the mixing distribution, this section obtains a new three-parameter mixed Poisson distribution, called the mixed Poisson Cubic Rank Transmuted New Weighted Exponential Distribution (mPNWED).

3.1 The PMF of the mPNWED

Proposition 3: If a random variable $N \sim Poisson(X)$ and $X \sim CRTNWED\ (\theta, v, \rho)$, a discrete random variable N has a mPNWED if its PMF is given as:

$$f(n) = \frac{(1-\rho)\theta(1+v)}{(1+\theta+v\theta)^{n+1}} + \frac{6\rho\theta(1+v)}{(1+2\theta+2v\theta)^{n+1}} - \frac{6\rho\theta(1+v)}{(1+3\theta+3v\theta)^{n+1}}. \tag{13}$$

Proof:

$$f(n) = \int_0^\infty f(n|x) \times f_x \, dx$$

Therefore,

$$f(n) = \int_0^\infty \frac{x^n e^{-x}}{n!} (1+v)\theta e^{-(1+v)\theta x}\left(1-\rho+6\rho e^{-(1+v)\theta x}-6\rho e^{-2(1+v)\theta x}\right) dx$$

$$= \frac{\theta(1+v)}{n!}\int_0^\infty x^n \left((1-\rho)e^{-(1+\theta+v\theta)x}+6\rho e^{-(1+2\theta+2v\theta)x}-6\rho e^{-(1+3\theta+3v\theta)x}\right) dx$$

$$= \frac{\theta(1+v)}{n!}\int_0^\infty \left(\frac{(1-\rho)u^n e^{-u}}{(1+\theta+v\theta)^{n+1}}+\frac{6\rho u^n e^{-u}}{(1+2\theta+2v\theta)^{n+1}}-\frac{6\rho u^n e^{-u}}{(1+3\theta+3v\theta)^{n+1}}\right) du$$

$$= \frac{(1-\rho)\theta(1+v)}{(1+\theta+v\theta)^{n+1}}+\frac{6\rho\theta(1+v)}{(1+2\theta+2v\theta)^{n+1}}-\frac{6\rho\theta(1+v)}{(1+3\theta+3v\theta)^{n+1}}$$

3.2 Special Cases:

1. If $v = 0$ in (13), the distribution becomes the mixed Poisson CRT exponential distribution (mPED) with PMF given as:
$$f(n) = \frac{(1-\rho)\theta}{(1+\theta)^{n+1}}+\frac{6\rho\theta}{(1+2\theta)^{n+1}}-\frac{6\rho\theta}{(1+3\theta)^{n+1}}. \tag{14}$$
2. If $\rho = 0$ in (13), the distribution becomes a new two-parameter discrete distribution (2PDD) with PMF given as:
$$f(n) = \frac{\theta(1+v)}{(1+\theta+v\theta)^{n+1}}. \tag{15}$$
3. If $\rho = 0$ and $v = 0$ in (13), the distribution becomes the geometric distribution (Mixed Poisson Exponential, MPE) with PMF given as:
$$f(n) = \frac{\theta}{(1+\theta)^{n+1}}. \tag{16}$$

Shapes of the PMF of the mPNWED as shown in the Figure 2 show that for different values of the parameters of the distribution, its PMF is unimodal and can effectively model highly skewed observations and observations with many zeros.

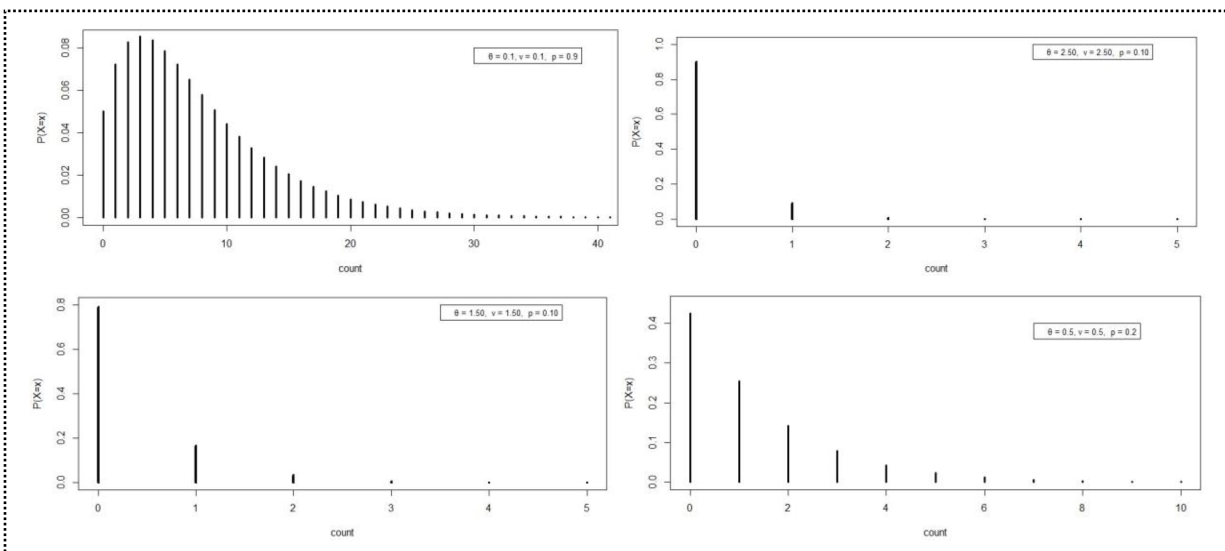

Figure 2: Shapes of the PMF of the mPNWED

3.3 The Proof of Legitimacy of the mPNWED

To prove that the mPNWED is a legitimate PMF, we show that it sums up to 1.

$$\sum_{n=0}^{\infty} f(n) = 1$$

Therefore,

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\frac{(1-\rho)\theta(1+v)}{(1+\theta+v\theta)^{n+1}}+\frac{6\rho\theta(1+v)}{(1+2\theta+2v\theta)^{n+1}}-\frac{6\rho\theta(1+v)}{(1+3\theta+3v\theta)^{n+1}}\right)\\
&\quad=\sum_{n=0}^{\infty}\frac{(1-\rho)\theta(1+v)}{(1+\theta+v\theta)^{n+1}}+\sum_{n=0}^{\infty}\frac{6\rho\theta(1+v)}{(1+2\theta+2v\theta)^{n+1}}-\sum_{n=0}^{\infty}\frac{6\rho\theta(1+v)}{(1+3\theta+3v\theta)^{n+1}}\\
&\quad=(1-\rho)\theta(1+v)\left(\tfrac{1}{(1+\theta+v\theta)^{1}}+\cdots\right)+6\rho\theta(1+v)\left(\tfrac{1}{(1+2\theta+2v\theta)^{1}}+\cdots\right)-6\rho\theta(1+v)\left(\tfrac{1}{(1+3\theta+3v\theta)^{1}}+\cdots\right)\\
&\quad=\theta(1-\rho)(1+v)\left(\frac{1}{\theta(1+v)}\right)+6\rho\theta(1+v)\left(\frac{1}{2\theta(1+v)}\right)-6\rho\theta(1+v)\left(\frac{1}{3\theta(1+v)}\right)\\
&\quad=1-\rho+3\rho-2\rho=1
\end{aligned}$$

3.4 The CDF of the mPNWED

Proposition 4: If a random variable *N* has the mPNWED, the CDF is defined as:

$$F(n)=1-\left(\frac{(1-\rho)}{(1+\theta+v\theta)^{n+1}}+\frac{3\rho}{(1+2\theta+2v\theta)^{n+1}}-\frac{2\rho}{(1+3\theta+3v\theta)^{n+1}}\right). \tag{17}$$

Proof:

$$\begin{aligned}
F(n)&=P(N\le n)\\
&=1-P(N>n)\\
&=1-\sum_{k=n+1}^{\infty} f(k)\\
&=1-\sum_{k=n+1}^{\infty}\left(\frac{(1-\rho)\theta(1+v)}{(1+\theta+v\theta)^{k+1}}+\frac{6\rho\theta(1+v)}{(1+2\theta+2v\theta)^{k+1}}-\frac{6\rho\theta(1+v)}{(1+3\theta+3v\theta)^{k+1}}\right)\\
&=1-\left(\sum_{k=n+1}^{\infty}\frac{(1-\rho)\theta(1+v)}{(1+\theta+v\theta)^{k+1}}+\sum_{k=n+1}^{\infty}\frac{6\rho\theta(1+v)}{(1+2\theta+2v\theta)^{k+1}}-\sum_{k=n+1}^{\infty}\frac{6\rho\theta(1+v)}{(1+3\theta+3v\theta)^{k+1}}\right)\\
&=1-\left(\frac{(1-\rho)}{(1+\theta+v\theta)^{n+1}}+\frac{3\rho}{(1+2\theta+2v\theta)^{n+1}}-\frac{2\rho}{(1+3\theta+3v\theta)^{n+1}}\right)
\end{aligned}$$

3.5 Reliability Measures for the mPNWED

The survival function is:

$$S(n)=\frac{(1-\rho)}{(1+\theta+v\theta)^{n+1}}+\frac{3\rho}{(1+2\theta+2v\theta)^{n+1}}-\frac{2\rho}{(1+3\theta+3v\theta)^{n+1}}. \tag{18}$$

The hazard rate function is:

$$h(n)=\frac{(1-\rho)\theta(1+v)(1+2\theta+2v\theta)^{n+1}(1+3\theta+3v\theta)^{n+1}+6\rho\theta(1+v)(1+\theta+v\theta)^{n+1}(1+3\theta+3v\theta)^{n+1}-6\rho\theta(1+v)(1+\theta+v\theta)^{n+1}(1+2\theta+2v\theta)^{n+1}}{(1-\rho)(1+2\theta+2v\theta)^{n+1}(1+3\theta+3v\theta)^{n+1}+3\rho(1+\theta+v\theta)^{n+1}(1+3\theta+3v\theta)^{n+1}-2\rho(1+\theta+v\theta)^{n+1}(1+2\theta+2v\theta)^{n+1}}. \tag{19}$$

4. Mathematical Properties of the mPNWED

4.1 Probability Generating Function

The PGF of the mPNWED is obtained as:

$$P_n(z)=\int_0^{\infty} e^{x(z-1)}\, f(x)\ dx$$

$$= \int_0^\infty e^{x(z-1)}(1+v)\theta e^{-(1+v)\theta x}\left(1-\rho+6\rho e^{-(1+v)\theta x}-6\rho e^{-2(1+v)\theta x}\right)dx$$

$$= \theta(1+v)\int_0^\infty \left((1-\rho)e^{-(1+\theta+v\theta-z)x}+6\rho e^{-(1+2\theta+2v\theta-z)x}-6\rho e^{-(1+3\theta+3v\theta-z)x}\right)dx$$

$$= \frac{(1-\rho)\theta(1+v)}{(1+\theta+v\theta-z)}+\frac{6\rho\theta(1+v)}{(1+2\theta+2v\theta-z)}-\frac{6\rho\theta(1+v)}{(1+3\theta+3v\theta-z)}$$

Therefore, the PGF is

$$P_n(z) = \frac{(1-\rho)\theta(1+v)}{(1+\theta+v\theta-z)}+\frac{6\rho\theta(1+v)}{(1+2\theta+2v\theta-z)}-\frac{6\rho\theta(1+v)}{(1+3\theta+3v\theta-z)}. \tag{20}$$

4.2 Moment Generating Function

The MGF for the distribution is obtained by replacing z with e^t in (20). Hence, the MGF is defined as:

$$M_N(t) = \frac{(1-\rho)\theta(1+v)}{(1+\theta+v\theta-e^t)}+\frac{6\rho\theta(1+v)}{(1+2\theta+2v\theta-e^t)}-\frac{6\rho\theta(1+v)}{(1+3\theta+3v\theta-e^t)}. \tag{21}$$

4.3 The Dispersion Index (DI)

DI measures dispersion in an observations. There is overdispersion when DI > 1, underdispersion when DI < 1 and there is equi-dispersion when DI = 1. For the mPNWED, DI is obtained as:

$$DI(N) = \frac{36-22\rho-\rho^2-6\theta(\rho-6)(1+v)+v(72-34\rho)}{6\theta(1+v)(6-\rho)}. \tag{22}$$

4.4 Skewness and Kurtosis

Using (21), the first four moments of the mPNWED are obtained from the relation $E(N^k) = \frac{d^k M_N(t)}{dt^k}|_{t=0}$ for *k = 1, 2, 3, 4.*

$$E(N) = \frac{6-\rho}{6\theta(1+v)}. \tag{23}$$

$$E(N^2) = \frac{36-17\rho-3\theta(1+\alpha)(\rho-6)}{18\theta^2(1+\alpha)^2}. \tag{24}$$

$$E(N^3) = \frac{216-151\rho-6\theta^2(1+v)^2(\rho-6)-6\theta(1+v)(17\rho-36)}{36\theta^3(1+v)^3}. \tag{25}$$

$$E(N^4) = \frac{1296-1085\rho-9\theta^3(1+v)^3(\rho-6)-21\theta^2(1+v)^2(17\rho-36)-9\theta(1+v)(151\rho-216)}{54\theta^4(1+v)^4}. \tag{26}$$

Using equations (23) – (26), the skewness and kurtosis for the distribution are obtained (De Jong & Heller, 2008) as:

$$S_k(N) = \frac{E(N^3)-3E(N^2)E(N)+2\left(E(N)\right)^3}{\left(Var(N)\right)^{\frac{3}{2}}}$$

$$Kurt(N) = \frac{E(N^4)-4E(N^3)E(N)+6E(N^2)\left(E(N)\right)^2-3\left(E(N)\right)^4}{\left(Var(N)\right)^2}$$

Skewness, Kurtosis, and the Dispersion Index of simulated values for different parameter combinations for the distribution are presented in tables 1 – 3.

Table 1: Skewness for selected parameter for mPNWED

	$v = 0.5$			$v = 2.5$			$v = 7.5$		
	$\theta = 0.1$	$\theta = 2.0$	$\theta = 10$	$\theta = 0.1$	$\theta = 2.0$	$\theta = 10$	$\theta = 0.1$	$\theta = 2.0$	$\theta = 10$
$\rho = -0.9$	1.668	2.439	4.061	1.764	3.084	5.822	1.945	4.270	8.796
$\rho = -0.5$	1.820	2.478	4.146	1.887	3.134	5.970	2.026	4.363	9.044
$\rho = 0.0$	2.005	2.500	4.250	2.023	3.182	6.167	2.095	4.478	9.382

$\rho = 0.5$	2.098	2.460	4.346	2.050	3.200	6.377	2.059	4.590	9.757
$\rho = 0.9$	1.771	2.339	4.411	1.739	3.176	6.555	1.804	4.670	10.089

Table 2: Kurtosis for selected parameter for mPNWED

	$v = 0.5$			$v = 2.5$			$v = 7.5$		
	$\theta = 0.1$	$\theta = 2.0$	$\theta = 10$	$\theta = 0.1$	$\theta = 2.0$	$\theta = 10$	$\theta = 0.1$	$\theta = 2.0$	$\theta = 10$
$\rho = -0.9$	6.604	10.608	21.614	7.057	14.460	39.118	7.962	23.375	82.649
$\rho = -0.5$	7.557	10.981	22.299	7.905	14.834	40.805	8.623	24.154	86.979
$\rho = 0.0$	9.020	11.250	23.063	9.091	15.125	43.028	9.391	25.056	93.012
$\rho = 0.5$	10.404	10.974	23.592	9.931	15.011	45.333	9.594	25.758	99.843
$\rho = 0.9$	8.719	9.822	23.675	8.173	14.341	47.165	7.962	26.020	105.971

Table 3: Dispersion Index for selected parameter for mPNWED

	$v = 0.5$			$v = 2.5$			$v = 7.5$		
	$\theta = 0.1$	$\theta = 2.0$	$\theta = 10$	$\theta = 0.1$	$\theta = 2.0$	$\theta = 10$	$\theta = 0.1$	$\theta = 2.0$	$\theta = 10$
$\rho = -0.9$	9.855	1.443	1.089	4.795	1.190	1.038	2.563	1.078	1.016
$\rho = -0.5$	8.992	1.400	1.080	4.425	1.171	1.034	2.410	1.071	1.014
$\rho = 0.0$	7.667	1.333	1.067	3.857	1.143	1.029	2.177	1.059	1.012
$\rho = 0.5$	6.000	1.250	1.050	3.143	1.107	1.021	1.882	1.044	1.009
$\rho = 0.9$	4.353	1.168	1.034	2.437	1.072	1.014	1.592	1.030	1.006

Remarks:
For fixed ρ and v, both skewness and kurtosis increase while the dispersion index decreases as θ increases.
When ρ and θ are fixed, skewness and kurtosis increase while the dispersion index decreases as v increases.
For fixed v and θ, skewness, and kurtosis are symmetrical in most cases while the dispersion index decreases as ρ increases.

5. Parameter Estimation for the mPNWED

Assuming $n_1, n_2, \dots, n_k$ are random samples of size k drawn from the mPNWED with (θ, v, ρ), the log-likelihood function for the distribution is obtained below.

$$\mathcal{L} = \prod_{i=1}^{k} f(n_i) = \prod_{i=1}^{k} \left(\frac{(1-\rho)\theta(1+v)}{(1+\theta+v\theta)^{n_i+1}} + \frac{6\rho\theta(1+v)}{(1+2\theta+2v\theta)^{n_i+1}} - \frac{6\rho\theta(1+v)}{(1+3\theta+3v\theta)^{n_i+1}} \right)$$

$$\ell = log\,\mathcal{L} = \sum_{i=1}^{k} log\left(\frac{(1-\rho)\theta(1+v)}{(1+\theta+v\theta)^{n_i+1}} + \frac{6\rho\theta(1+v)}{(1+2\theta+2v\theta)^{n_i+1}} - \frac{6\rho\theta(1+v)}{(1+3\theta+3v\theta)^{n_i+1}} \right). \quad (27)$$

The ML estimators for (θ, v, ρ) denoted with $(\hat{\theta}, \hat{v}, \hat{\rho})$ are the solutions using the **MaxLik** function in the R language.

6. Applications

6.1 Data

Five referred datasets are used to assess the performance of the new distribution. The first dataset is the frequency of third-party claims for Australian vehicle owners (De Jong & Heller, 2008). Dataset II is the claim frequency of insurance policyholders in a Turkish Insurance Firm. Several authors (Meytrianti *et al*., 2019; Sarul & Sahin, 2015) have previously used the data. The third and fourth datasets represent yeast cell counts per square metre and the number of European red mites on apple leaves found in Shanker and Hagos (2015). Dataset V is the distribution of mistakes in copying groups of random digits previously used in many discrete distribution propositions (Kemp & Kemp, 1965; Sah & Mishra, 2019; Samutwachirawong, 2021; Sankaran, 1970; Shanker & Mishra, 2014).

Table 4 shows that with 93.19%, dataset I has the highest percentage of 0, while dataset IV has the least with 47.50%. The highest skewness (4.067) is obtained for the first dataset, while dataset II has the highest kurtosis (7.312). With 14.027, dataset IV is the most dispersed, while dataset I (1.055) has the least.

Table 4: Summary of count datasets

	Dataset I	Dataset II	Dataset III	Dataset IV	Dataset V
Mean	0.073	0.265	0.460	0.150	0.783
Variance	0.077	0.334	0.605	2.104	1.257
Skewness	4.067	2.558	1.751	1.265	1.307
Kurtosis	18.504	7.312	2.799	0.961	0.722
Dispersion Index	1.055	1.260	1.315	14.027	1.605
% of zero	93.19%	79.01%	68.45%	47.50%	58.33%

6.2 Competing Distributions

The new proposition is assessed on the count distributions using its three other (mPED, 2PDD, and MPE) special cases and other count distributions.

i. When $v = 0$, the mPNWED becomes the mPED with the PMF given in equation (14) as:
$$f(n) = \frac{(1-\rho)\theta}{(1+\theta)^{n+1}} + \frac{6\rho\theta}{(1+2\theta)^{n+1}} - \frac{6\rho\theta}{(1+3\theta)^{n+1}}$$

ii. When $\rho = 0$, the mPNWED becomes a new two-parameter discrete distribution (2PDD) with PMF given as:
$$f(n) = \frac{\theta(1+v)}{(1+\theta+v\theta)^{n+1}}$$

iii. If $\rho = 0$ and $v = 0$ in, the mPNWED becomes the geometric (Mixed Poisson Exponential, MPE) distribution with PMF given as:
$$f(n) = \frac{\theta}{(1+\theta)^{n+1}}$$

iv. Poisson Distribution: A discrete random variable *N* has Poisson distribution if its PMF is defined as:
$$f(n) = \frac{e^{-\theta}\theta^n}{n!}$$

v. Zero Inflated Poisson Distribution: A discrete random variable *N* has a Zero-Inflated Poisson (ZIP) distribution with ρ zero inflation parameter if its PMF is given as:
$$f(n) = \begin{cases} \rho + (1-\rho)e^{-\theta}, & n = 0 \\ (1-\rho)\dfrac{e^{-\theta}\theta^n}{n!}, & n = 1,2,3,\dots \end{cases}$$

vi. Negative Binomial Distribution: A discrete random variable *N* has a Negative Binomial distribution if its PMF denoted by P_n is given has:
$$f(n) = \binom{n+v-1}{n}\left(\frac{\theta}{1+\theta}\right)^n\left(\frac{1}{1+\theta}\right)^v, \qquad n = 0,1,2,\dots, \qquad \theta, v > 0$$

vii. Zero-Inflated Negative Binomial Distribution: A discrete random variable *N* has a Zero-Inflated Negative Binomial (ZINB) distribution with $\wp$ zero inflation parameter if its PMF is given as:
$$f(n) = \begin{cases} \rho + (1-\rho)\left(\dfrac{1}{1+\theta}\right)^{\alpha}, & n = 0 \\ (1-\rho)\left[\dbinom{n+v-1}{n}\left(\dfrac{\theta}{1+\theta}\right)^n\left(\dfrac{1}{1+\theta}\right)^v\right], & n = 1,2,3,\dots \end{cases}$$

Remark: The distribution with the lowest –*LL*, AIC, BIC and/or the lowest Chi-Square value is considered the best fit for a given dataset.

6.3 Results

Table 5: Performance of distributions on the Australian Claims (Dataset I)

Obs.	Freq.	mPNWED	mPED	2PDD	MPE	Poisson	ZIP	Neg. Bin	ZINB
0	63232	63231.84	63232.78	63253.86	63252.72	63091.61	63230.49	63230.60	63317.89
1	4333	4331.49	4332.29	4290.02	4291.00	4593.07	4325.83	4330.57	4252.49
2	271	273.85	272.34	290.96	291.10	167.19	286.59	276.48	261.98
3	18	17.56	17.36	19.73	19.75	4.06	12.66	17.22	21.45
4	2	1.17	1.15	1.34	1.34	0.07	0.42	1.06	1.97
Estimates		$\hat{\theta}$=7.4619 $\hat{\upsilon}$=0.4005 $\hat{\rho}$=0.7190	$\hat{\theta}$=12.7789 $\hat{\rho}$=0.4248	$\hat{\theta}$=1.8439 $\hat{\upsilon}$=6.4540	$\hat{\theta}$=13.7408	$\hat{\theta}$=0.0728	$\hat{\theta}$=0.1325 $\hat{\rho}$=0.4507	$\hat{\theta}$=1.1569 $\hat{\upsilon}$=0.9408	$\hat{\theta}$=0.0067 $\hat{\upsilon}$=0.8776 $\hat{\rho}$=-76.06
– LL		18049.56	18049.57	18050.45	18050.45	18101.5	18052.2	18049.68	18105.58
AIC		36105.12	36103.14	36104.90	36104.90	36209.00	36108.40	36103.36	36215.16
BIC		36132.50	36121.39	36123.15	36123.15	36236.38	36126.65	36121.61	36233.41
Chi-Square		0.64	0.66	2.29	2.29	177.66	9.07	0.98	2.51

Table 6: Performance of distributions on the Turkish claim frequency (Dataset II)

Obs.	Freq.	mPNWED	mPED	2PDD	MPE	Poisson	ZIP	Neg. Bin	ZINB
0	8544	8547.32	8558.09	8544.49	8545.30	8292.42	8544.19	8543.47	8561.78
1	1796	1789.64	1787.22	1793.21	1792.74	2201.64	1759.23	1795.62	1807.66
2	370	376.74	371.23	376.34	376.11	292.27	430.75	375.71	331.89
3	81	79.25	77.16	78.98	78.90	25.87	70.31	78.50	81.03
4	23	16.64	16.07	16.58	16.55	1.72	8.61	16.39	22.23
Estimates		$\hat{\theta}$=33.5931 $\hat{\rho}$=-0.0247 $\hat{\upsilon}$=-0.8874	$\hat{\theta}$=3.7820 $\hat{\rho}$=-0.0244	$\hat{\theta}$=1.4747 $\hat{\upsilon}$=1.5530	$\hat{\theta}$=3.7666	$\hat{\theta}$=0.2655	$\hat{\theta}$=0.4897 $\hat{\rho}$=0.4579	$\hat{\theta}$=1.0090 $\hat{\upsilon}$=0.7917	$\hat{\theta}$=0.0055 $\hat{\upsilon}$=0.6348 $\hat{\rho}$=-82.430
– LL		7029.72	7029.72	7029.72	7029.72	7153.16	7038.91	7029.71	7057.06
AIC		14065.44	14063.44	14063.44	14063.44	14312.32	14081.82	14063.42	14118.12
BIC		14087.31	14078.02	14078.02	14078.02	14334.19	14096.40	14078.00	14132.70
Chi-Square		2.61	3.26	2.65	2.67	484.41	35.02	2.84	4.51

Table 7: Performance of distributions on counts of yeast cell per square metre (Dataset III)

Obs.	Freq.	mPNWED	mPED	2PDD	MPE	Poisson	ZIP	Neg. Bin	ZINB
0	128	127.37	127.37	128.09	128.09	118.06	128.00	126.73	180.68
1	37	38.75	38.75	40.35	40.35	54.30	38.35	42.08	4.51
2	18	16.03	16.04	12.71	12.71	12.49	15.49	12.84	1.15
3	3	3.91	3.91	4.00	4.00	1.91	4.17	3.80	0.39
4	1	0.77	0.77	1.26	1.26	0.22	0.84	1.11	0.15
Estimates		$\hat{\theta}$=55.4673 $\hat{\rho}$=-8.5781 $\hat{\upsilon}$=-0.9053	$\hat{\theta}$=5.252 $\hat{\rho}$=-8.578	$\hat{\theta}$=0.9777 $\hat{\upsilon}$=1.2240	$\hat{\theta}$=2.1744	$\hat{\theta}$=0.4599	$\hat{\theta}$=0.8077 $\hat{\rho}$=0.4306	$\hat{\theta}$=1.1950 $\hat{\upsilon}$=0.7221	$\hat{\theta}$=0.0037 $\hat{\upsilon}$=0.4902 $\hat{\rho}$=-11.72
– LL		168.70	168.70	170.09	170.09	173.83	168.80	170.02	172.05
AIC		343.40	341.40	344.18	344.18	353.66	341.60	344.04	348.10
BIC		353.09	347.86	350.64	350.64	363.35	348.06	350.50	354.56
Chi-Square		0.60	0.60	2.78	2.78	12.16	0.81	2.88	517.31

Table 8: Performance of distributions on number European red mites on apple leaves (Dataset IV)

Obs.	Freq.	mPNWED	mPED	2PDD	MPE	Poisson	ZIP	Neg. Bin	ZINB
0	38	41.22	41.22	37.21	37.21	25.33	38.00	36.56	38.11
1	17	16.27	16.27	19.90	19.90	29.13	14.55	20.36	22.48
2	10	9.57	9.57	10.65	10.65	16.75	13.42	10.92	8.50
3	9	5.84	5.84	5.69	5.69	6.42	8.25	5.78	4.28
4	3	3.36	3.36	3.05	3.05	1.85	3.80	3.04	2.42
5	2	1.83	1.83	1.63	1.63	0.42	1.40	1.59	1.46
6	1	0.96	0.96	0.87	0.87	0.08	0.43	0.83	0.92
Estimates		$\hat{\theta}$=14.4910 $\hat{\rho}$=-1.1895 $\hat{\upsilon}$=-0.9256	$\hat{\theta}$=1.0785 $\hat{\rho}$=-1.1889	$\hat{\theta}$=0.5386 $\hat{\upsilon}$=0.6146	$\hat{\theta}$=0.8696	$\hat{\theta}$=1.1500	$\hat{\theta}$=1.8440 $\hat{\rho}$=0.3763	$\hat{\theta}$=1.0801 $\hat{\upsilon}$=0.4843	$\hat{\theta}$=0.00287 $\hat{\upsilon}$=0.2461 $\hat{\rho}$=-129.4
– LL		118.57	118.57	118.80	118.80	127.89	118.02	118.78	121.66
AIC		243.14	241.14	241.60	241.60	261.78	240.04	241.56	247.32
BIC		250.29	245.90	246.36	246.36	268.93	244.80	246.32	252.08
Chi-Square		2.07	2.08	2.50	2.50	32.08	2.53	2.62	7.16

Table 9: Performance of distributions on mistakes in copying groups of random digits (Dataset V)

Obs.	Freq.	mPNWED	mPED	2PDD	MPE	Poisson	ZIP	Neg. Bin	ZINB
0	35	34.96	34.97	33.64	33.64	27.41	35.00	33.95	35.08
1	11	10.95	10.95	14.78	14.78	21.47	11.25	14.49	14.75
2	8	8.61	8.60	6.49	6.49	8.41	8.04	6.39	5.07
3	4	3.70	3.70	2.85	2.85	2.20	3.83	2.85	2.32
4	2	1.25	1.25	1.25	1.25	0.43	1.37	1.28	1.19
Estimates		$\hat{\theta}$=1.4191 $\hat{\rho}$=-7.2467 $\hat{\upsilon}$=0.9815	$\hat{\theta}$=2.8106 $\hat{\rho}$=-7.2368	$\hat{\theta}$=0.595 $\hat{\upsilon}$=1.145	$\hat{\theta}$=1.2766	$\hat{\theta}$=0.7833	$\hat{\theta}$=1.4302 $\hat{\rho}$=0.4523	$\hat{\theta}$=0.9381 $\hat{\upsilon}$=0.5449	$\hat{\theta}$=0.00279 $\hat{\upsilon}$=0.3151 $\hat{\rho}$=-128.1
– LL		72.03	72.03	73.38	73.38	77.55	71.92	73.37	74.96
AIC		150.06	148.06	150.76	150.76	161.10	147.84	150.74	153.92
BIC		156.34	152.25	154.95	154.95	167.38	152.03	154.93	158.11
Chi-Square		0.51	0.51	2.28	2.28	14.44	0.30	2.15	4.43

7. Discussion

Tables 5 – 9 show results of analysing five count data using the new proposition and three of its special cases. Both mPED and 2PDD have two parameters while the MPE has a single parameter.

The new proposition (mPNWED) has the least chi-square values, indicating the best fit for most datasets. For the first dataset as shown in Table 5, the two-parameter mPED provides the overall best fit based on the utilized selection criteria. Although the negative binomial distribution gives a relatively better fit to the second dataset, the new proposition and its special forms also provide appreciable alternatives. Table 7 shows that the new propositions (mPNWED) and mPED give the best fit to the third dataset while the ZIP also provides competing fitness values.

Results for assuming different distribution for dataset IV as presented in Table 8 reveals that the new proposition (mPNWED) and its special cases has the best fit for the data. For the fifth dataset, the new proposition competes favourably with the ZIP distribution.

Results also show that both two-parameter special forms of the new proposition (mPED and 2PDD) give same statistics in all cases despite having different PMFs. Also, the MPE (one parameter form of the distribution) also provide better fit than the classical Poisson distribution.

8. Conclusions

A new mixed discrete distribution using the mixed Poisson architecture is proposed in this study. The one-parameter cubic rank transmutation map is used to extend the new weighted exponential distribution to obtain a new mixing continuous distribution assumed for the parameter of the classical Poisson distribution in the mixed Poisson paradigm.

Various moment-based mathematical properties of the new propositions are obtained. Also, the shape of the new distribution resembles the shape of the continuous mixing distribution.
By assuming some fixed values for some of the parameters of the new proposition, special forms of the new propositions are also obtained and are compared with some classical distributions. Real life applications are assessed on five count datasets using the maximum likelihood estimation technique to estimate parameters. Using four selection criteria, results show that the new proposition performs creditably well and can be used as an alternative for the competing distributions.

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