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A New Weighted-Lindley Distribution: Properties, Classical and Bayesian Estimation with an Application

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Abstract

The choice of the most suitable statistical distribution for modeling data is very important. Generally, the new distributions are more flexible to model real data that present a high degree of skewness and kurtosis. In this paper, we define a new one-parameter lifetime distribution, so-called weighted-Lindley distribution. Some of its basic properties are investigated. Some classical and Bayesian methods of estimation have been used for estimating its parameter. The behavior of these estimators were investigated by a graphical simulation study. A real data set is analyzed to investigate the flexibility of the new weighted-Lindley distribution.

Key Words: Anderson-Darling estimator; Weighted distributions; Lindley distribution; Maximum likelihood; Bayesian estimation; Moments.

Mathematical Subject Classification: 60E05, 62E15.

1. Introduction

Although many distributions have been developed, there are always rooms for developing distributions which are either more flexible or for fitting specific real world scenarios. This has motivated researchers seeking and developing new and more flexible distributions.
Let $w(x)$ be a non-negative with finite non-zero expectation, the probability distribution function (pdf) of the weighted random variable X_w was introduced as

$$f_w(x) = \frac{w(x)\, f(x)}{E(w(X))}, \tag{1}$$

where $E(w(X))$ denotes the expectation of $w(X)$.
Ghitany et al. (2008) introduced the Lindley distribution. The cumulative distribution function (cdf) and pdf of the

Lindley distribution are given by

$$F_{Li}(x;\lambda) = 1 - (1 + \frac{\lambda\, x}{1+\lambda})\, \mathrm{e}^{-\lambda\, \mathrm{x}},\ \mathrm{x} > 0 \tag{2}$$

and

$$f_{Li}(x;\lambda) = \frac{\lambda^2}{1+\lambda}\,(1+x)\mathrm{e}^{-\lambda\, \mathrm{x}},\ \mathrm{x} > 0, \tag{3}$$

where $\lambda > 0$ is the shape parameter.

Many extensions of Lindley distributions have been introduced by several authors. For examples, the three-parameter Lindley distribution due to Zakerzadeh and Dolati (2009), generalized Lindley distribution by Nadarajah et al. (2011), generalized Poisson-Lindley distribution by Mahmoudi and Zakerzadeh (2010), power Lindley distribution by Ghitany et al. (2013), two parameter-Lindley distribution by Shanker and Mishra (2013), a new quasi Lindley distribution by Shanker et al. (2013), transmuted Lindley distribution by Merovci (2013), beta-Lindley distribution by Merovci and Sharma (2014), negative binomial-Lindley distribution Zamani and Ismail (2010) and gamma-Lindley distribution by Zeghdoudi and Nedjar (2016).

Our aim in this article, is to propose a new extension of the Lindley distribution, referred to as weighted-Lindley (WL) distribution and study its basic properties. The WL distribution can be considered as a suitable model for modeling skewed data encountered in different applied fields such public health, as biomedical studies, engineering, and survival and reliability analysis. The WL distribution outperforms some classical lifetime distributions with respect to a real data example.

Further, we study how different classical and Bayesian estimators of the WL parameter perform for several sample sizes. The studied classical and Bayesian estimation methods include the maximum likelihood, least-squares, Anderson-Darling, weighted least squares, right tail Anderson-Darling and Cramér–von Mises estimators. The behavior of these methods is explored by a graphical simulation study.

The random variable X is said to follow the WL distribution, if its pdf is

$$f_{WL}(x;\lambda) = \frac{2\lambda^2\,(1+x)\ \mathrm{e}^{-\lambda\, \mathrm{x}}}{(1+\lambda)\left[1 + (1+\frac{\lambda\, x}{1+\lambda})\, \mathrm{e}^{-\lambda\, \mathrm{x}}\right]^2}, \tag{4}$$

where $x > 0$ and $\lambda > 0$ is a shape parameter. The related cdf and hazard rate function (hrf) are given by

$$F_{WL}(x;\lambda) = \frac{1 - (1+\frac{\lambda\, x}{1+\lambda})\, \mathrm{e}^{-\lambda\, \mathrm{x}}}{1 + (1+\frac{\lambda\, x}{1+\lambda})\, \mathrm{e}^{-\lambda\, \mathrm{x}}}, \tag{5}$$

and

$$h_{WL}(x;\lambda) = \frac{\lambda^2\,(1+x)}{(1+\lambda+\lambda\, x)\left[1 + (1+\frac{\lambda\, x}{1+\lambda})\, \mathrm{e}^{-\lambda\, \mathrm{x}}\right]}. \tag{6}$$

It is clear that pdf (4) is a weighted lindley distribution with $w(x) = \frac{2}{\left[1+(1+\frac{\lambda\, x}{1+\lambda})\, \mathrm{e}^{-\lambda\, \mathrm{x}}\right]^2}$.

Theorem 1.1. *The hrf of WL(λ) is increasing.*
Proof:

$$\psi(x) = \frac{\partial \log(h_{WL}(x))}{\partial x} = \frac{1}{(1+x)(1+\lambda+\lambda\, x)} + \frac{\lambda^2\,(1+x)\mathrm{e}^{-\lambda\, \mathrm{x}}}{(1+\lambda)(1+(1+\frac{\lambda\, x}{1+\lambda})\, \mathrm{e}^{-\lambda\, \mathrm{x}})}. \tag{7}$$

Note that $\psi(x) > 0$ for any $x > 0$, then the proof is completed.

Here we draw the plots of density and hazard rate functions for some value of parameters in Figure 1. The density function of WL is ynimodal and right-skew. The hazard rate function of WL is increasing.

The rest of this paper is organized into 5 sections: Section 2 is devoted to main features of the WL distribution. In Section 3, we discuss some classical estimation methods and Bayesian methods to estimate the WL parameter.

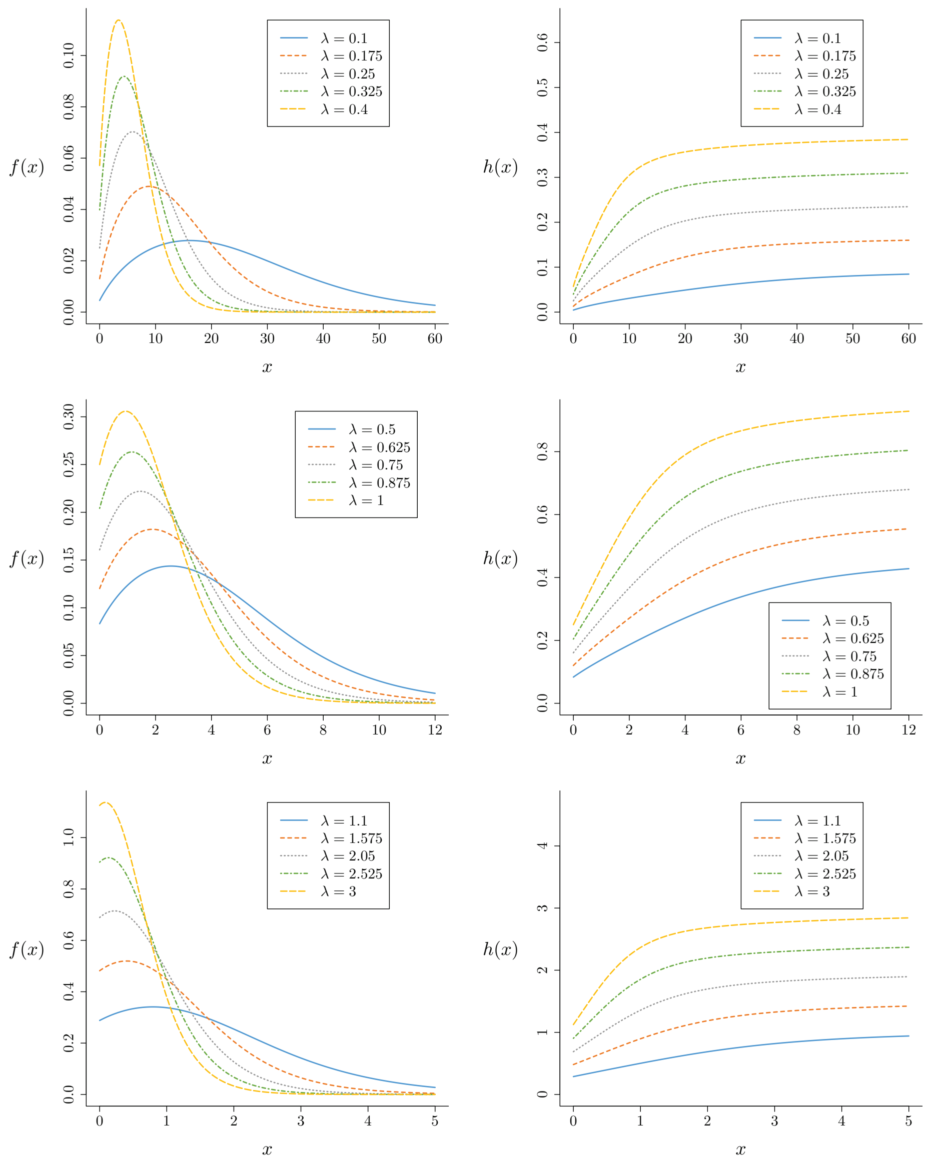

Figure 1: Plots of density and hazard functions of the WL for selected parameters value.

Section 4 provides a graphical simulation study to explore the performance of these estimation methods. In Section 5, the flexibility of the proposed WL distribution is studied by a real-life data set. Finally, the paper is concluded in Section 6.

2. Main properties

2.1. Asymptotic properties

The asymptotic of the cdf, pdf and hrf of the WL distribution as $x \to 0$ are, respectively, given by

$$\begin{aligned} F_{WL}(x) &\sim 0 \quad as \quad x \to 0, \\ f_{WL}(x) &\sim \frac{\lambda^2}{2(1+\lambda)} \quad as \quad x \to 0, \\ h_{WL}(x) &\sim \frac{\lambda^2}{2(1+\lambda)} \quad as \quad x \to 0, \end{aligned}$$

The respective asymptotic of the cdf, pdf and hrf of the WL distribution as $x \to \infty$ are

$$\begin{aligned} 1 - F_{WL}(x) &\sim \ 0 \quad as \quad x \to \infty, \\ f_{WL}(x) &\sim \frac{2\,\lambda^2}{1+\lambda} \quad as \quad x \to \infty, \\ h_{WL}(x) &\sim \lambda \quad as \quad x \to \infty. \end{aligned}$$

These equations show the effect of parameters on the tails of the WL distribution.

2.2. Extreme value

One may be interested in the asymptotic of the extreme values $M_n = \max(X_1, ..., X_n)$ and $m_n = \min(X_1, ..., X_n)$. Let $\tau(x) = \frac{1}{\lambda}$, we obtain the following equations for the cdf in 5 as

$$\lim_{t\to 0} \frac{F_{WL}(t\,x)}{F_{WL}(t)} = \lim_{t\to 0} \frac{1-(1+\frac{\lambda\,t\,x}{1+\lambda})\,\mathrm{e}^{-\lambda\,\mathrm{t\,x}}}{1+(1+\frac{\lambda\,t\,x}{1+\lambda})\,\mathrm{e}^{-\lambda\,\mathrm{x}}} \left[\frac{1-(1+\frac{\lambda\,t}{1+\lambda})\,\mathrm{e}^{-\lambda\,\mathrm{t}}}{1+(1+\frac{\lambda\,t}{1+\lambda})\,\mathrm{e}^{-\lambda\,\mathrm{x}}}\right]^{-1} = x, \tag{8}$$

and

$$\lim_{t\to\infty} \frac{1-F_{WL}(t+x\,\tau(t))}{1-F_{WL}(t)} = \lim_{t\to\infty} \frac{2(1+\frac{\lambda\,(t+\frac{x}{\lambda})}{1+\lambda})\,\mathrm{e}^{-\lambda\,(\mathrm{t}+\frac{\mathrm{x}}{\lambda})}}{1+(1+\frac{\lambda\,(t+\frac{x}{\lambda})}{1+\lambda})\,\mathrm{e}^{-\lambda\,\mathrm{x}}} \left[\frac{2(1+\frac{\lambda\,t}{1+\lambda})\,\mathrm{e}^{-\lambda\,\mathrm{t}}}{1+(1+\frac{\lambda\,t}{1+\lambda})\,\mathrm{e}^{-\lambda\,\mathrm{x}}}\right]^{-1} = \mathrm{e}^{-\mathrm{x}}. \tag{9}$$

Thus, from Leadbetter et al. (2012), there must be norming constants $a_n > 0, \quad b_n, \quad c_n > 0$ and d_n such that

$$Pr\left[a_n(M_n - b_n) \le x\right] \to \mathrm{e}^{-\mathrm{e}^{-\mathrm{x}}},$$

and

$$Pr\left[c_n(m_n - d_n) \le x\right] \to 1 - \mathrm{e}^{-\mathrm{x}},$$

as $n \to \infty$. The forms of norming constants can also be determined. For instance, using Corollary 1.6.3 in Leadbetter et al. (2012), one can see that $b_n = F^{-1}(1 - \frac{1}{n})$ and $a_n = \lambda$, where $F^{-1}(\cdot)$ denotes the inverse function of $F(\cdot)$.

2.3. Quantile function

Let $X \sim \text{WL}(\lambda)$ random variable, then the quantile function (qf) denoted by $Q(p)$, is defined by $F[Q(p)] = p$, where

$$\frac{1 - [1 + \frac{\lambda}{1+\lambda}\, Q(p)]\, \mathrm{e}^{-\lambda\, Q(p)}}{1 + [1 + \frac{\lambda}{1+\lambda}\, Q(p)]\, \mathrm{e}^{-\lambda\, Q(p)}} = p,$$

it implies

$$[1 + \lambda + \lambda\, Q(p)]\, \mathrm{e}^{-\lambda\, Q(p)} = (1+\lambda)\frac{1-p}{1+p},\ 0 < p < 1. \tag{10}$$

Substituting $Z(p) = -1 - \lambda - \lambda\, Q(p)$, one can write (10) as

$$Z(p)\, \mathrm{e}^{Z(p)} = -(1+\lambda)(\frac{1-p}{1+p})\mathrm{e}^{-1-\lambda}. \tag{11}$$

Hence, the solution $Z(p)$ is

$$Z(p) = W_{-1}\left\{-(1+\lambda)\mathrm{e}^{-1-\lambda}(\frac{1-\mathrm{p}}{1+\mathrm{p}})\right\}, \tag{12}$$

where $W_{-1}[.]$ is the negative branch of Lambert function by Corless et al. (1996). Inserting (12), we obtain

$$Q(p) = -1 - \frac{1}{\lambda} - \frac{1}{\lambda}\, W_{-1}\left\{-(1+\lambda)\mathrm{e}^{-1-\lambda}(\frac{1-\mathrm{p}}{1+\mathrm{p}})\right\}. \tag{13}$$

Here, we also propose two different algorithms for generating random data from the WL distribution.

(a) The first algorithm is based on generating random data from the Lindley distribution using the inverse cdf.

Algorithm 1 (Mixture form of the Lindley distribution)

- Generate $U_i \sim \text{Uniform}(0,1), \quad i = 1, \ldots, n;$
- Generate $V_i \sim \text{Exponential}(\lambda), \quad i = 1, \ldots, n;$
- Generate $W_i \sim \text{Gamma}(2, \lambda), \quad i = 1, \ldots, n;$
- If $\frac{2\,U_i}{1+U_i} \le \frac{\lambda}{1+\lambda}$ set $X_i = V_i$, otherwise, set $X_i = W_i, \quad i = 1, \ldots, n.$

(b) The second algorithm is based on generating random data from the inverse cdf (5) of the WL distribution.

Algorithm 2 (Inverse cdf)

- Generate $U_i \sim$ Uniform(0,1), $\quad i = 1, \ldots, n;$
- Set

$$X_i = -1 - \frac{1}{\lambda} - \frac{1}{\lambda}\, W_{-1}\left\{-(1+\lambda)\mathrm{e}^{-1-\lambda}(\frac{1-\mathrm{U_i}}{1+\mathrm{U_i}})\right\}, \quad i = 1, \ldots, n.$$

2.4. Mixture representations

The cdf and pdf can be written as mixture representations and such forms of cdf and pdf can be used to derive some mathematical properties, e.g., moments, moments of residual life and incomplete moments. Using the geometric expansion, we can write

$$\begin{aligned} F_{WL}(x) &= \frac{2\left[1-(1+\frac{\lambda}{1+\lambda}x)e^{-\lambda x}\right]}{2-\left[1-(1+\frac{\lambda}{1+\lambda}x)e^{-\lambda x}\right]} = \sum_{k=0}^{\infty}(\frac{1}{2})^{k+1}\left[1-(1+\frac{\lambda}{1+\lambda}x)e^{-\lambda x}\right]^{k+1} \\ &= \sum_{k=1}^{\infty} z_k \left[1-(1+\frac{\lambda}{1+\lambda}x)e^{-\lambda x}\right]^{k}, \end{aligned} \tag{14}$$

where $z_k = (\frac{1}{2})^k$ and for $k \geq 1$.
Equation (14) can be interpreted as a linear combination of generalized Lindley distribution by Nadarajah et al. (2011). Using this equation, the mixture representation of pdf is given by

$$f_{WL}(x) = \sum_{k=1}^{\infty} z_k\, g(x;\lambda) G(x;\lambda)^{k-1}. \tag{15}$$

2.5. Some moments

Here, we derive the nth moment, kth central moment and moment generating function (mgf) of the WL distribution. In addition, we provide the nth incomplete moment, and mean deviations. First of all, assume that $X \sim WL(\lambda)$. Now, using (15), we define

$$A(a_1,a_2,a_3,a_4;\lambda) = \int_0^{\infty} x^{a_1}\,(1+x)^{a_2}\,\mathrm{e}^{-a_3\,x}\left[1-(1+\frac{\lambda}{1+\lambda}\,x)\mathrm{e}^{-\lambda x}\right]^{a_4}\mathrm{d}x.$$

By using generalized binomial expansion, it can be shown that

$$A(a_1,a_2,a_3,a_4;\lambda) = \sum_{l,r=0}^{\infty}\sum_{k=0}^{l}(-1)^l\binom{a_4}{l}\binom{l}{k}\binom{a_2}{r}(\frac{\lambda}{1+\lambda})^l \times \frac{\Gamma(a_1+1+k+r)}{(\lambda\, l+a_3)^{a_1+1+k+r}}. \tag{16}$$

So, the nth moment of the WL distribution reduces to

$$E\left[X^n\right] = \frac{\lambda^2}{1+\lambda}\sum_{k=1}^{\infty} z_k\; A(n,1,\lambda,k-1;\lambda). \tag{17}$$

The central moments $\mu_k = E(X-\mu)^k$ of the WL distribution follows from (15) as

$$\mu_k = E(X-\mu)^k = \sum_{r=0}^{k}\binom{k}{r}\mu'_r(-\mu)^{k-r}, \tag{18}$$

where $\mu'_k = E(X^k)$, $\mu = \mu'_1 = E(X)$ and k is an integer value.
The mean and variance of X can be particularly obtained using equations (17) and (18). In additional, these equations are used to derive the skewness as

$$S = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1}{(\mu'_2 - \mu'^2_1)^{3/2}},$$

and the kurtosis as

$$K = \frac{\mu_4}{\mu_2^2} = \frac{\mu'_4 - 4\mu'_1\mu'_3 + 6\mu'^2_1\mu'_2 - 3\mu'^4_1}{\mu'_2 - \mu'^2_1}.$$

It is to highlighted that the equation (17) can be easily computed numerically using mathematical or statistical softwares. For this purpose, one can compute this equation for a large natural number, say N, instead of infinity in the sums. Therefore, several quantities of X such as moments, skewness and kurtosis can be computed numerically using (17). Figure 2 shows plots of the mean, variance, skewness and kurtosis of the WL distribution for different values of λ. These plots illustrate that the mean and variance are decreasing and the skewness and kurtosis are increasing function of λ.

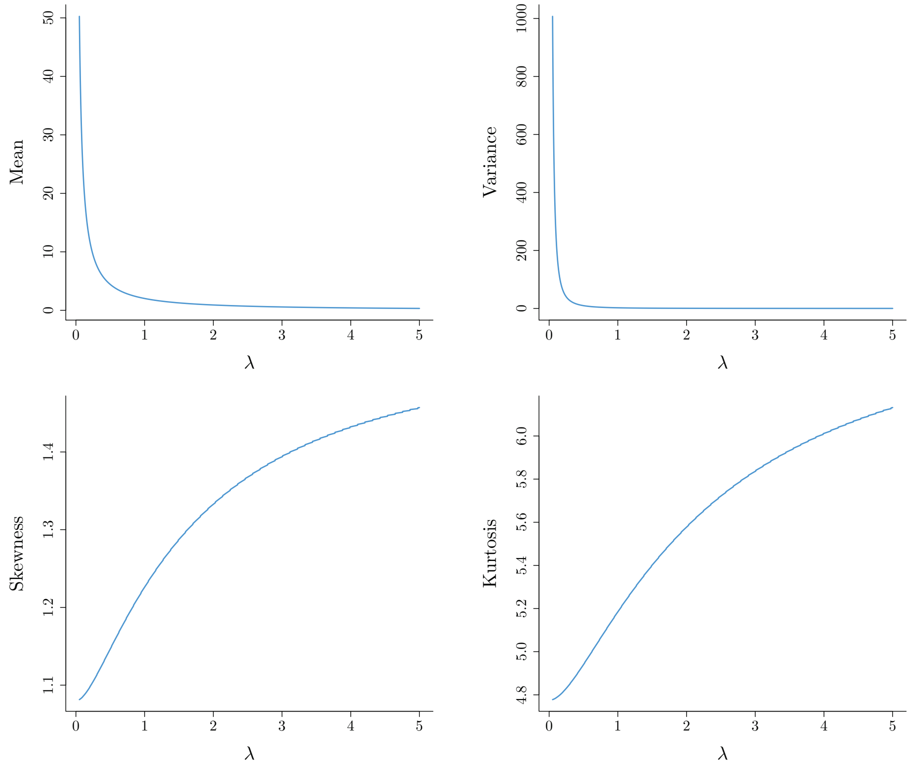

Figure 2: Plots of mean, variance, skewness and kurtosis of the WL distribution as a function of λ.

Moreover, it is easy to verify that the moment generating function of the WL distribution is

$$M_X(t) = E\left[\mathrm{e}^{t\,X}\right] = \frac{\lambda^2}{1+\lambda}\sum_{k=1}^{\infty} z_k\, A(0,1,\lambda-t,k-1;\lambda).$$

To obtain the nth incomplete moment of the WL distribution, we define

$$B(a_1,a_2,a_3,a_4;y,\lambda) \quad = \int_0^y x^{a_1}\,(1+x)^{a_2}\,\mathrm{e}^{-a_3\,x}\left[1-(1+\tfrac{\lambda}{1+\lambda}\,x)\mathrm{e}^{-\lambda x}\right]^{a_4}\mathrm{d}x, \tag{19}$$

After some simple algebraic manipulation, we obtain

$$B(a_1,a_2,a_3,a_4;y,\lambda) \quad = \sum_{l,r=0}^{\infty}\sum_{k=0}^{l}(-1)^l\binom{a_4}{l}\binom{l}{k}\binom{a_2}{r}\,(\tfrac{\lambda}{1+\lambda})^l \times \frac{\gamma(a_1+1+k+r,\frac{y}{\lambda\,l+a_3})}{(\lambda\,l+a_3)^{a_1+1+k+r}}, \tag{20}$$

where $\gamma(\lambda,z)=\int_0^z t^{\lambda-1}\,\mathrm{e}^{-t}\,dt$ stands for the incomplete gamma function. Note that the second equality of (20) is obtained by generalized binomial expansion. Hence, using (20) the nth incomplete moment of the WL distribution takes the form

$$m_n(y) = E\left[X^n\,|X<y\right] = \frac{\lambda^2}{(1+\lambda)F_{WL}(y)}\sum_{k=0}^{\infty} z_k\;B(n,1,\lambda,k-1;y,\lambda). \tag{21}$$

Now, we provide two measures of deviation called, mean deviation about the mean (δ_1) and the mean deviation about

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the median (δ_2). These measures are given by

$$\delta_1\,(X) = 2\mu\, F(\mu) - 2\int_0^{\mu} x\, f(x)\,\mathrm{d}x,$$

and

$$\delta_2\,(X) = \mu - 2\int_0^{M} x\, f(x)\,\mathrm{d}x,$$

where M denotes the median of X. Therefore, using (20), we obtain

$$\delta_1\,(X) = 2\mu\, F_{WL}(\mu) - \frac{\lambda^2}{1+\lambda}\sum_{k=1}^{\infty} z_k\, A(1,\lambda,k-1;\lambda),$$

and

$$\delta_2\,(X) = \mu - \frac{\lambda^2}{1+\lambda}\sum_{k=1}^{\infty} z_k\, B(1,\lambda,k-1;M,\lambda).$$

2.6. Bonferroni and Lorenz curves

Bonferroni and Lorenz curves were first presented by Bonferroni (1961) to measure the inequality of the distribution for a random variable, respectively. These curves are widely used in reliability, economics, insurance, etc. The Bonferroni and Lorenz indexes are defined as

$$B(p) = \frac{1}{p\mu}\int_0^{q} x f(x)\mathrm{d}x,$$

and

$$L(p) = \frac{1}{\mu}\int_0^{q} x f(x)\mathrm{d}x,$$

respectively, where $q = F^{-1}(p)$ is the qf. If $X \sim WL(\lambda)$, then it can be shown that the Bonferroni curve of the WL distribution has the form

$$B(p) = \frac{1}{\mu\, p} \times \frac{\lambda^2}{1+\lambda}\sum_{k=1}^{\infty} z_k\, B(1,\lambda,k+-1;q,\lambda).$$

The Lorenz curve of the WL distribution follows as

$$L(p) = \frac{1}{\mu} \times \frac{\lambda^2}{1+\lambda}\sum_{k=1}^{\infty} z_k\, B(1,\lambda,k-1;q,\lambda).$$

In order to use the Lorenz curve as a measure of inequality of income, one should investigate the area between the Lorenz curve and the line $L(p) = p$ that is called the area of concentration and such area is important in economics, reliability, insurance and medicine.

3. Estimation methods

In this section, we investigate the estimation of the WL parameter using classical and Bayesian estimators. The classical estimators include the maximum likelihood estimator (MLE), least squares estimator (LSE), weighted least squares estimator (WLSE), Cramér–von–Mises estimator (CME), Anderson-Darling estimator (ADE), and right-tailed Anderson-Darling estimator (RTADE). The statistical literature contains comprehensive comparisons of different estimation methods for many distributions, such as the quasi xgamma-geometric by Afify et al. (2019), Weibull Marshall–Olkin Lindley by Afify et al. (2020) and odd exponentiated half-logistic exponential by ldahlan et al.(2020) distributions.

3.1. Maximum likelihood estimator

In this section, we estimate the WL parameter using the maximum likelihood method. The MLE is one of the most common point estimators, and it is very applicable in confidence intervals and hypothesis testing. Various statistics are built based on the MLE for assessing the goodness-of-fit in a model, such as: the maximum log-likelihood ($\hat{\ell}_{max}$), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Anderson-Darling (A^*) and Cramér–von Mises (W^*), described by Chen and Balakrishnan (1995). These statistics are used in section 5.

Let $x_1, x_2, ..., x_n$ be the observations from the pdf 4 of the WL model. In this case, the log-likelihood function reduces to

$$\begin{aligned}\ell_n(\lambda) &= n(\ln 2 + 2\ln\lambda - \ln(1+\lambda)) + \sum_{i=0}^{n}\ln(1+x_i) - \lambda\sum_{i=0}^{n}x_i \\ &\quad -2\sum_{i=0}^{n}\ln[1+(1+\frac{\lambda x_i}{1+\lambda})\exp^{\lambda x_i}] \qquad (22)\end{aligned}$$

By numerically solving the following equations, the MLE can be obtained.

3.1.1. Least squares and weighted least squares estimators

The LSE and WLSE are introduced by Swain et al. (1988). Let $\{t_i; i = 1, 2, ..., n\}$ be the associated order statistics of a random sample from the WL distribution. The LSE and WLSE of the WL parameter are obtained by minimizing the following functions:

$$S_{\text{LSE}}(\lambda) = \sum_{i=1}^{n}\left(F_{WL}(t_i;\lambda) - \frac{i}{n+1}\right)^2$$

and

$$S_{\text{WLSE}}(\lambda) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(F_{WL}(t_i;\lambda) - \frac{i}{n+1}\right)^2.$$

3.1.2. Cramer- von- Mises estimator

The CME is introduced by Choi and Bulgern (1968). The CME of the parameter λ is obtained by minimizing the following function

$$S_{CME}(\lambda) = \frac{1}{12n} + \sum_{i=1}^{n}\left(F_{WL}(t_i;\lambda) - \frac{2i-1}{2n}\right)^2.$$

3.1.3. Anderson - Darling and right-tailed Anderson- Darling

The ADE and RTADE are introduced by Anderson and Darling (1952). The ADE and RTADE the parameter λ are obtained by minimizing the following two functions

$$S_{\text{ADE}}(\lambda) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\{\log F_{WL}(t_i;\lambda) + \log\overline{F}_{WL}(t_{n+1-i};\lambda)\}$$

and

$$S_{\text{RTADE}}(\lambda) = \frac{n}{2} - 2\sum_{i=1}^{n}F_{WL}(t_i;\lambda) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\overline{F}_{WL}(t_{n+1-i};\lambda),$$

where $\overline{F}_{WL}(\cdot) = 1 - F_{WL}(\cdot)$.

3.2. Bayesian estimators: posterior mean and maximum-a-posteriori

The Bayesian estimators have significant applications in a wide variety of statistical studies. They provide good estimation for parameters by combining past experience with current observations. In this section, posterior mean (PM) and maximum-a-posteriori (MAP) estimator are examined to the estimation of the WL parameter. In Bayesian analysis after specifying the prior distribution of parameter, posterior distribution can be obtained as follows:

$$\lambda \sim \pi(\lambda)$$

$$f_{WL}(\boldsymbol{x}|\lambda) = \prod_{i=1}^{n} f_{WL}(x_i|\lambda)$$

$$\pi(\lambda|\boldsymbol{x}) = c\pi(\lambda) f_{WL}(\boldsymbol{x}|\lambda);\ c = \left(\int_{\Lambda} \pi(\lambda) f_{WL}(\boldsymbol{x}|\lambda) d\lambda\right)^{-1}$$

Now if one consider mean squared error loss function and posterior mean (PM) will be the bayes estimator. Also, the mode of posterior, MAP, is other estimator that is equivalent to MLE in Bayesian statistics; It means:

$$\mathrm{PM} = \mathrm{E}(\lambda|\boldsymbol{x})$$

$$\mathrm{MAP} = \arg\max_{\lambda} \pi(\lambda|\boldsymbol{x})$$

4. Simulation study

In order to explore the behaviour of the previously introduced estimators, we conduct a simulation study of those estimators for different samples. In order to do that, two pdf of Figure 3 were selected in which two deferent cases of WL distribution are visible: $\lambda = 0.2$ and $\lambda = 5$.

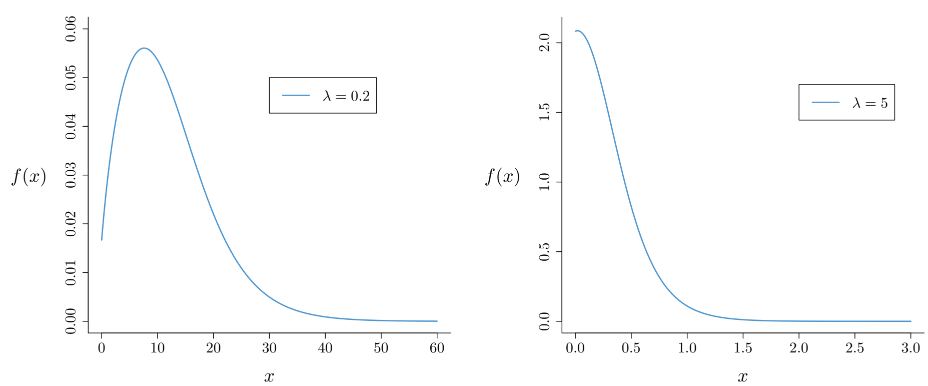

Figure 3: Two density functions for simulation study.

To verify the validity of the estimators, bias and mean square error (MSE) of the estimate have been adopted. For example, as described in Section 2.3, for $\lambda = 0.2$, samples of $n = 20, 40, 70, 100, 140, 200, 270, 350, 450, 600$ of $\mathrm{WL}(0.2)$ are generated for $N = 10000$ times. For PM and MAP, the noninformative $\lambda \sim \Gamma(0.01, 0.01)$ is used. This prior has a large standard deviation and there is no information on the positive parameters. According to Section 3.2 and using these priors, MAP and MLE produce similar results approximately.

To obtain the posterior distribution, we have the following relationships:

$$\pi(\lambda) = \frac{0.01^{0.01}}{\Gamma(0.01)} \lambda^{-0.99} e^{-0.01\lambda},$$

$$f(x_i|\lambda) = \frac{2\lambda^2 (1 + x_i) \mathrm{e}^{-\lambda \mathrm{x_i}}}{(1+\lambda)\left[1 + (1 + \frac{\lambda x_i}{1+\lambda})\, \mathrm{e}^{-\lambda \mathrm{x_i}}\right]^2},\ \ i = 1, 2, ...n$$

and

$$\pi(\lambda|\boldsymbol{x}) \propto \pi(\lambda) \prod_{i=1}^{n} f(x_i|\lambda),$$

where $\pi(\lambda|\boldsymbol{x})$ is not a known distribution and the value of PM is obtained using Hamiltonian Monte Carlo (HMC). The HMC is a method of the MCMC simulation, applying Stan software by Carpenter et al. (2016), which is becoming more popular. In this method the gradient of the log posterior is utilized. Fortunately, the definition of new distribution in Stan software is very straightforward. $rstan$ package needs to run Stan in R.

In order to obtain the MAP estimator, Stan has a function using numerical methods ($optimizing$ function in $rstan$). For any simulation by n volume and $i = 1, 2, ..., N$, the estimations are obtained as $\widehat{\lambda}_i$. The standard deviation of estimators is obtained through the standard deviation of posterior distribution, because noninformative priors is used. The estimation of standard deviation is shown by $s_{\widehat{\lambda}}$. In this case, the $\widehat{\lambda}$, Bias and MSE are calculated by the following formulas:

$$\widehat{\lambda}(n) = \frac{1}{N} \sum_{i=1}^{N} \widehat{\lambda}_i,$$

$$Bias_{\widehat{\lambda}}(n) = \frac{1}{N} \sum_{i=1}^{N} (\widehat{\lambda}_i - \lambda_i)$$

and

$$MSE_{\widehat{\lambda}}(n) = \frac{1}{N} \sum_{i=1}^{N} (\widehat{\lambda}_i - \lambda_i)^2.$$

Figures 4 and 5 represent the plots of the biases and MSE of the $\lambda = 0.2$ and $\lambda = 5$, respectively. As expected, in all methods, the bias and MSE of estimated parameter converge to zero while n is growing. The plots of parameters vector $\lambda = 5$ have the same position as one can see in Figures 5.

It is observed that, the behavior of the estimates of the WL parameter obtained using all estimation methods are quite reliable, showing small bias and creditable MSE in the two studied cases. Further, the bias approaches to zero as n increases, hence these estimates are asymptotically unbiased estimators. These estimators are consistent for the WL parameter because the MSE decreases as n increases. We conclude that all estimators provide adequate estimates of the parameter λ in terms of bias and MSE.

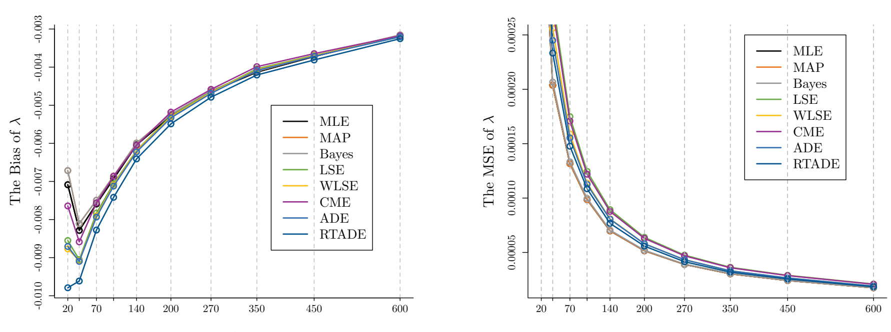

Figure 4: The bias and MSE of $\hat{\lambda}$ versus n for $\lambda = 0.2$.

5. Real data application

In this section, the flexibility of the new WL distribution is investigated based on a real data set as compared with some other competing distributions such as the Weibull, gamma, Rayleigh, Lindley, exponential (Exp), Log Normal (Lnorm), and HLog distributions.

The data consist of service times of 63 aircraft windshield as reported in Murthy et al. (2004). The data set is:
0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915,

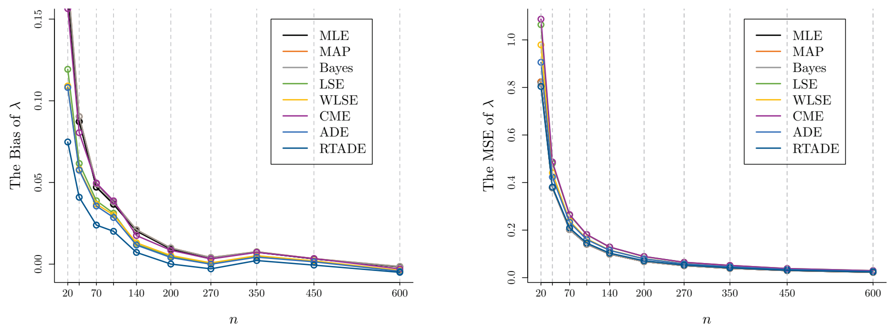

Figure 5: The bias and MSE of $\hat{\lambda}$ versus n for $\lambda = 5$.

2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140

We used AIC, BIC, W^* and A^* statistics for comparision. These measures are computed with the following equations.

$$AIC = -2\,\hat{l}(.) + 2\,p, \qquad BIC = -2\,\hat{l}(.) + p\log(n),$$

$$A^* = (1 + \frac{0.5}{n})\left[\sum_{i=1}^{n}(\hat{F}(x_{(i)}) - \frac{i-0.5}{n})^2 + \frac{1}{12\,n}\right]$$

$$W^* = (1 + \frac{0.75}{n} + \frac{2.25}{n^2})\left[\sum_{i=1}^{n}\frac{2\,i-1}{n}(\ln(\hat{F}(x_{(i)})) - \ln(\hat{F}(x_{(n+1-i)})))^2\right]$$

where n denote the number of observation, p denote the number of parameters, $\hat{l}(.)$ denote the log-likelihood function evalutaed at the MLEs, $\hat{F}(.)$ denote the estimated of cdf $F(.)$ and $x_{(i)}$ denote the i-th of observed order statistics.

Table 1 illustrates the values of AIC, BIC, W^* and A^* for the fitted models. The estimated pdf, cdf, sf and PP plots of the WL model for service times of aircraft windshield data are illustrated in Figure 6. Furthermore, the previous methods of estimation are adopted to estimate the WL parameter from service times of aircraft windshield data and the results are reported in Table 3. The PP plots of these methods are displayed in Figure 7. These tables and graphs show that the WL distribution provide better fit than other competitive models.

Table 1: Goodness-of-fit measures of the fitted models for service times of aircraft windshield data.

Model	AIC	BIC	W^*	A^*
WL	203.99	206.13	0.09	0.53
Weibull	204.64	208.92	0.10	0.63
Rayleigh	206.98	209.13	0.08	0.46
Gamma	209.67	213.95	0.18	1.11
HLog	209.70	211.84	0.12	0.73
Lindley	211.16	213.30	0.14	0.84
Exp	220.60	222.74	21.44	126.44
Lnorm	229.74	234.03	0.49	2.84

Table 2: MLEs for service times of aircraft windshield data.

Model	Estimates
WL	$\widehat{\lambda} = 0.99$ $s_{\widehat{\lambda}} = 0.079$
Weibull	$(\widehat{\lambda}, \widehat{k}) = (1.629, 2.31)$ $(s_{\widehat{\lambda}}, s_{\widehat{k}}$ = (0.168 , 0.187)
Rayleigh	$\widehat{\sigma} = 1.714$ $s_{\widehat{\sigma}} = 0.108$
Gamma	$(\widehat{\alpha}, \widehat{\beta}) = (1.908, 0.915)$ $(s_{\widehat{\alpha}}, s_{\widehat{\beta}}$ =(0.315 , 0.173)
HLog	$\widehat{k} = 0.687$ $s_{\widehat{k}} = 0.07$
Lindley	$\widehat{\theta} = 0.753$ $s_{\widehat{\theta}} = 0.07$
Exp	$\widehat{\lambda} = 0.48$ $s_{\widehat{\lambda}} = 0.06$
Lnorm	$(\widehat{\mu}, \widehat{\sigma}) = (0.451, 0.925)$ $(s_{\widehat{\mu}}, s_{\widehat{\sigma}}$ =(0.117 , 0.082)

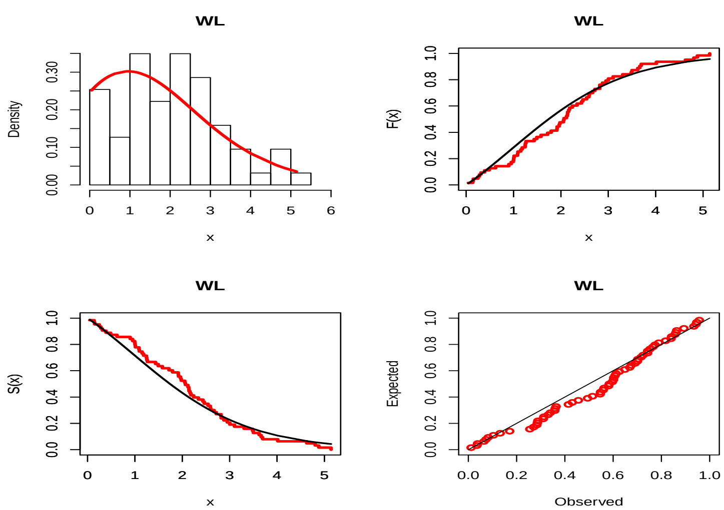

Figure 6: The estimated pdf, cdf, sf and PP plots of the WL model for service times of aircraft windshield data.

Table 3: The estimates of λ using several estimation methods and goodness-of-fit measures for service times of aircraft windshield data.

Method	$\hat{\lambda}$	W^*	A^*	KS	$KS - p\ value$
MLE	0.9897201	0.08751962	0.5341839	0.13359881	0.19262998
LSE	0.9165538	0.08917642	0.5439644	0.09178829	0.62986785
WLSE	0.9424206	0.08850525	0.5400078	0.10547292	0.45419878
CME	0.9182760	0.08912865	0.5436830	0.09089235	0.64189102
ADE	0.9339710	0.08871374	0.5412374	0.10035390	0.51738436
RTADE	0.9603694	0.08809578	0.5375909	0.11625328	0.33586452
MAP	0.9897307	0.08751943	0.5341828	0.1336051	0.1925886
Bayes	0.9896075	0.08752162	0.5341957	0.133533	0.1930669

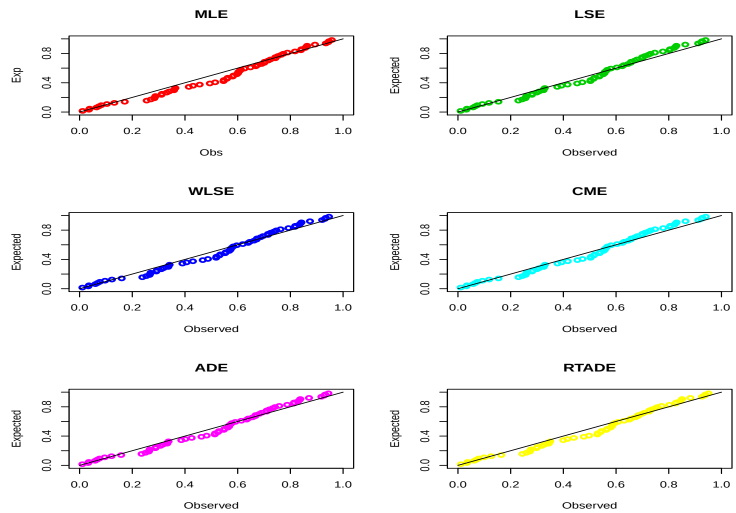

Figure 7: PP plots for the estimation methods for service times of aircraft windshield data.

6. Concluding remarks

In this paper, we introduce a new one-parameter lifetime distribution called weighted-Lindley (WL) distribution. Some of its mathematical properties are derived. We estimate the parameter of the WL model using some estimation methods including the maximum likelihood, least squares, weighted least squares, Cramér–von–Mises, Anderson-Darling, right-tailed Anderson-Darling, and Bayesian estimators. These methods are examined using a graphical simulation in terms of their mean square errors and biases to verify the validity of the these estimation methods. The flexibility of this distribution is assessed by applying it to a real data set as compared with other distributions.

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7. The R codes of WL

The program is developed in R to obtain the value of density (dWL), distribution (pWL), hazard (hWL) and random generation (rWL) for the WL distribution.

```
dWL = function(x, par)
    {
    ex = exp(-par*x)
    lex = (1+par*x/(1+par))*ex
    d = 2 * par^2 * (1+x) * ex / ( (1+par) * (1+lex)^2)
    d[x < 0] = 0
    return(d)
    } # end of pWL

pWL = function(x, par)
    {
    ex = exp(-par*x)
    lex = (1+par*x/(1+par))*ex
    d = (1-lex)/(1+lex)
    d[x < 0] = 0
    return(d)
    } # end of pWL

hWL = function(x, par)
    {
    pdf = dWL(x=x, par=par)
    cdf = pWL(x=x, par=par)
    hrf = pdf/(1 - cdf)
    hrf[!is.finite(hrf)] = NA
    hrf
    } # end of hWL

rWL = function(n, par)
    {
    library(lamW)
    u = runif(n)
    a = -(1+par)*exp(-1-par)*(1-u) / (1+u)
    d = -1-1/par-lambertWm1(a)/par


    return(d)
    } # end of qWL
```

The program is developed in R of calculation for one-dimensional integral based on observations and the trapezoidal rule integration:

```
intob = function(x, y) 0.5*sum(diff(x)*
                        (y[1:length(x)-1]+y[2:length(x)]))
```

The program is developed in R of calculation for the value of moment, skewness and kurtosis:

```
moment = function(par, order)
    {
    x = seq(par[3], par[4], le=10000)
    y = dWL(x = x, par = par)
```

```
        return(intob(x, x^order * y))
        } # end of moment

skew = function(par)
        {
        x = seq(0, 1, le=10000)
        y = dWL(x = x, par = par)
        m1 = intob(x, x*y)
        m2 = intob(x, (x-m1)^2*y)
        return(intob(x, ((x-m1)^3*y))/sqrt(m2)^3)
        } # end of skew

kurt = function(par)
        {
        x = seq(0, 1, le=10000)
        y = dWL(x = x, par = par)
        m1 = intob(x, x*y)
        m2 = intob(x, (x-m1)^2*y)
        return(intob(x, (x-m1)^4*y)/sqrt(m2)^4)
        } # end of kurt
```

Stan model and codes for WL distribution using optimization for simulations and application.

```
data
    {
    int<lower=1> n;
    real<lower=0> x[n];
    }
parameters
    {
    real<lower=0> lambda;
    }
model
    {
    real lae;
    real retall;
    real ret[2];
    ret[1] = 0;
    ret[2] = 0;
    lambda ~ gamma(0.01, 0.01);
    for(i in 1:n)
         {
         lae = 1 + (1 + lambda * x[i]/(1+lambda))*exp(-lambda*x[i]);          ret[1] = ret[1] +
log(1+x[i]);
         ret[2] = ret[2] + log(lae);
    retall = ret[1] - 2*ret[2] - lambda * sum(x)
            + rows(x)*(log(2)+2*log(lambda)-log(1+lambda));
    target += retall;
    }
```

After saving the above code in ”WLModel.stan” file, one can use following *rstan* codes for simulation and optimization of posterior.

```
mystanWL = stan_model("WLModel.stan")
```

 DOI: http://dx.doi.org/10.18187/pjsor.v18i4.4106

```
optimizing(mystanWL, data = list(n=length(x), x=x)
           hessian=TRUE)
sampling(mystanWL, data = list(n=length(x), x=x),
         iter=10000)
```

Program developed in R of optimization for LSE method in section 4. Other methods are the same.

```
t = sort(x)
LSE = function(para)
        sum((pWL(t, para) - 1:n/(n+1) )^2)
optim(par=c(1,1), fn=LSE, lower=0.005, upper=Inf,
      method="L-BFGS-B", hessian=TRUE)
```