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Bayesian Estimation of Transmuted Lomax Mixture Model with an Application to Type-I Censored Windshield Data

Muntazir Mehdia[1*], Muhammad Aslam[1], Navid Feroze[2]

* Corresponding Author

1. Department of Mathematics and Statistics, Riphah International University, Islamabad, Pakistan, muntazir.mehdi22@gmail.com, m.aslam@riphah.edu.pk
2. Department of Statistics, University of Azad Jammu and Kashmir, Muzaffarabad, Pakistan, navid.feroze@ajku.edu.pk

Abstract

Transmuted distributions have been centered of focus for researchers recently due to their flexibility and applicability in statistics. However, the only few contributions have considered estimation for mixture of transmuted lifetime models especially under Bayesian methods has been explored more recently. We have considered the Bayesian estimation of transmuted Lomax mixture model (TLMM) for type-I censored samples. The Bayes estimates (BEs) and posterior risks (PRs) for informative and noninformative priors are evaluated using four different loss functions (LFs), two symmetric and two asymmetric, namely the squared error loss function (SELF), precautionary loss function (PLF), weighted balance loss function (WBLF), and general entropy loss function (GELF). Simulations are run using Lindley Approximation method to compare the BEs under various sample sizes and censoring rates. The estimates under informative prior and GELF were found superior to their counterparts. The applicability of the proposed estimates has been illustrated using the analysis of a real data regarding type-I censored failure times of windshields airplanes.

Key Words: Bayesian analysis; loss functions; posterior risk; Lindley approximation; Confidence Intervals.

1 Introduction

The skewed family of non-Gaussian distributions that are used in modeling and analyzing reliability data are called transmuted distributions. The QRTM technique due to Shaw and Buckley (2009) is used to build new families of non-Gaussian distributions. By adding a new parameter to an existing baseline probability distribution, the QRTM approach produces a flexible family of probability distributions. Currently, many fields such as lifetime analysis, reliability studies, insurance, medicine, economics, environmental sciences, and engineering are using transmuted distributions (AL-Kadim, 2018). Shaw and Buckley (2009) suggested that a random variable X is assumed to have a transmuted distribution with

$$F(x) = (1+\lambda)G(x) - \lambda\{G(x)\}^2 \quad (1)$$

$$f(x) = g(x)\{1 + \lambda - 2\lambda G(x)\} \quad (2)$$

f(x) and g(x) are probability density functions with the cumulative density functions *F(x)* and *G(x)*, respectively, whereas $x > 0$, and $|\lambda| \leq 1$ is the transmuted parameter. Several authors have worked on mixture models using classical and Bayesian methods. Based on a type-I censored sample, Al-Hussaini et al. (2001) used a mixture of 2-component Lomax model to generate Bayesian prediction boundaries for future observations. Sultan et al. (2007) looked at features of the inverse Weibull distribution's 2-component mixture. Saleem et al. (2010) used the SELF to evaluate BEs and PRs for the Bayesian analysis of the 2-component mixture of the power function distribution. They used uniform, Jefferys, and inverse chi-square priors. In order to simulate heterogeneous survival data, Erisoglu et al. (2011) evaluated exponential-gamma, exponential-Weibull, and gamma-Weibull as mixtures of two separate distributions. Various properties of the proposed model have also been determined, and the maximum likelihood estimation was considered using the expectation-maximization (EM) algorithm. A Dataset from the real world has also been analyzed. The Bayesian analysis of a 2-component mixture of the Maxwell distribution was performed by Kazmi et al. (2012). Under QLF, Majeed and Aslam (2012) performed a Bayesian study of a 2-component mixture of the inverted exponential distribution. Feroze et al. (2013) used the Bayesian technique to study 2-component mixtures of the Top-Leone distribution and evaluated several statistical characteristics. Sultan and Al-Moisheer (2013) used a Bayesian technique to model reliability function for the 2-component mixture of the inverse Weibull distribution under type-II censoring. Feroze and Aslam (2014) performed a Bayesian study of a 2-component mixture of the Weibull distribution with censored lifetime data. Sindhu et al. (2016) used doubly type-II censored samples for Bayesian analysis of a 2-component mixture of the inverse Weibull distribution. Rahman and Aslam (2017) performed Bayesian analysis of 2-component mixture of the inverse Lomax distribution. Reyad and Othman (2018) compared the E-Bayesian and Bayesian approaches for estimating the parameters of a 2-component mixture of the inverse Lomax distribution. Bayesian study of the 2-component mixture of the transmuted Weibull distribution was performed by Yousaf et al. (2019, b). Aslam et al. (2020, b) used several loss functions to develop a Bayesian analysis of a 2-component mixture of the transmuted Pareto distribution. Aslam et al. (2020, c) proposed Bayesian analysis of a 2-component mixture of the transmuted Fréchet distribution. Younis et al. (2021) considered Bayesian analysis of a 2-component mixture of the Lomax distribution (LD). Ashour and Eltehiwy (2013) proposed and studied generalization of the Lomax distribution (LD) known as transmuted Lomax Distribution (TLD). They took LD as base distribution and derived some structural properties i.e., moments, quantiles, mean deviation and order statistics. ML method was opted for parameter estimation.

From the above discussion it can be assessed that transmuted lifetime models have attracted many researchers to use these models in modeling lifetimes of different products. However, the only few contributions have considered estimation for mixture of transmuted lifetime models especially under Bayesian methods has been explored more recently. This paper proposes the Bayesian estimation of parameters from TLMM. The Bayesian estimation has been considered using informative and non-informative prior distributions. The Bayes estimates have been obtained using four different loss functions, namely, SELF, PLF, WBLF and GELF. The samples are assumed to be type-I right censored. The Lindley's approximation has been used to obtain the approximate Bayes estimates. The applicability of the proposed estimates has been illustrated using the analysis of a real data regarding type-I censored failure times of windshields airplanes.

The organization of this article is as follows: Section 2 discusses transmuted distributions, LD, TLD, likelihood function, derivation of the posterior distributions using NIP and IP, and estimation of model parameters using Lindley's approximation. The results and discussions have been given in Section 3. Section 4 covers the conclusion and some future suggestions.

2 Materials and Methods

In statistics, heterogeneous datasets can be dealt very efficiently by using finite mixture models in order to model and analyze them. Specifically, Pearson (1894) introduced the statistical modeling using finite mixtures of distributions by fitting a mixture of two normal distributions to model data provided by Weldon (1892, 1894). Due to the flexibility of the finite mixture distributions, researchers are inclined towards the analysis of mixture models. The models might have finite or infinite number of components. The analysis of mixture models has been discussed by many authors, for example, Bertoli et al. (2020), and Noor et al. (2021), Feroze et al. (2022). Engineering, physical sciences, chemical sciences, and biological sciences are just a few of the domains where mixture models are used (Aslam et al., 2015). Titterington et al. (1985) defined mixture distribution with m-component such that its pdf can be expressed as:

$$f(x)=\sum_{i=1}^{m} p_i f_i(x) \tag{3}$$

where the parameter p_i denotes the mixing proportion of i^{th} component and $f_i(x)$ denotes the density of i^{th} component. It must satisfy the following condition: $p_i > 0;\ i=1,2,...,m\ \ and\ \sum_{i=1}^{m} p_i = 1$. Each probability density function f_i can be same or different distributions, but in our research, we have considered f_i of the same distribution due to convenience. There are two types of mixture distributions: type-I mixture model and type-II mixture model. The type-I mixture model is concerned with probability density functions belonging to the same family whereas the type-II mixture model considers probability density functions from different families of distributions.

2.1 Transmuted Lomax Distribution

The probability density function of a random variable 'X' from a LD with parameters (α and β) > 0 is:

$$g\left(x;\alpha,\beta\right)=\alpha\beta\left(1+\beta x\right)^{-(\alpha+1)},\ x>0,\ \alpha,\beta>0 \tag{4}$$

with corresponding CDF is:

$$G(x;\alpha,\beta)=\left\{1-\left(1+\beta x\right)^{-\alpha}\right\},\ \ x>0,\ \alpha,\beta>0 \tag{5}$$

where α and β denote the shape and scale parameters, respectively.

The random variable ‘X’ is said to have the TLD with parameters α, β, and λ if its pdf is given by:

$$f\left(x;\alpha,\beta,\lambda\right)=\frac{\alpha\beta}{\left(1+\beta x\right)^{(\alpha+1)}}\left[1+\lambda-2\lambda\left\{1-\left(1+\beta x\right)^{-\alpha}\right\}\right],\ x>0,\ \alpha,\beta>0\ and\ |\lambda|\le 1 \tag{6}$$

Equation (6) has been obtained by substituting (4) and (5) in (2). Now the CDF can be obtained by substituting (5) in (1), then the CDF of the TLD with parameters α, β, and λ takes the form:

$$F\left(x;\alpha,\beta,\lambda\right)=\left(1+\lambda\right)\left\{1-\left(1+\beta x\right)^{-\alpha}\right\}-\lambda\left\{1-\left(1+\beta x\right)^{-\alpha}\right\}^{2},$$

After some algebraic simplification, we get:

$$F\left(x;\alpha,\beta,\lambda\right)=\left\{1-\left(1+\beta x\right)^{-\alpha}\right\}\left\{1+\lambda\left(1+\beta x\right)^{-\alpha}\right\},\ x>0,\ \alpha,\beta>0\ and\ |\lambda|\le 1 \tag{7}$$

If $\lambda=0$, then TLD reduces to ordinary LD and if $\beta=1$, then TLD reduces to the TPrD.

2.2 Transmuted Lomax Mixture Model

In this paper, a 2-component mixture of the TLD has been proposed. The Bayesian approach has been employed for estimating the model parameters. The Bayes estimates have been derived using NIP and IP with two symmetric and asymmetric LFs. Since the marginal posterior distributions are not in closed form, the Lindley’s approximation has been used to evaluate the BEs with corresponding PRs and 95% CIs. A density function for the mixture of two component densities with mixing weights $\left(p_1,1-p_1\right)$ is:

$$f\left(x\right)=p_1 f_1\left(x\right)+\left(1-p_1\right)f_2\left(x\right),\qquad 0<p_1<1 \tag{8}$$

with the CDF of the mixture model is:

$$F\left(x\right)=p_1F_1\left(x\right)+\left(1-p_1\right)F_2\left(x\right),\qquad 0<p_1<1 \tag{9}$$

where $f_1\left(x\right)$ and $f_2\left(x\right)$ represent the PDF of the first and second component of the TLD defined in (8), and $F_1\left(x\right)$ and $F_2\left(x\right)$ represent the CDF of the first and second component of the TLD defined in (9). Now using (8), the pdf for the mixture model of the TLD is:

$$\begin{aligned}f\left(x;\Omega\right)=&\,p_1\frac{\alpha_1\beta_1}{\left(1+\beta_1 x\right)^{(\alpha_1+1)}}\left[1+\lambda_1-2\lambda_1\left\{1-\left(1+\beta_1 x\right)^{-\alpha_1}\right\}\right]\\&+\left(1-p_1\right)\frac{\alpha_2\beta_2}{\left(1+\beta_2 x\right)^{(\alpha_2+1)}}\left[1+\lambda_2-2\lambda_2\left\{1-\left(1+\beta_2 x\right)^{-\alpha_2}\right\}\right],\end{aligned} \tag{10}$$

Using (9), the CDF for the mixture model of the TLD is:

$$F\left(x;\Omega\right)=p_1\left\{1-\left(1+\beta_1 x\right)^{-\alpha_1}\right\}\left\{1+\lambda_1\left(1+\beta_1 x\right)^{-\alpha_1}\right\}+\left(1-p_1\right)\left\{1-\left(1+\beta_2 x\right)^{-\alpha_2}\right\}\left\{1+\lambda_2\left(1+\beta_2 x\right)^{-\alpha_2}\right\}, \qquad (11)$$

where $\Omega=\left(\alpha_1,\alpha_2,\beta_1,\beta_2,\lambda_1,\lambda_2,p_1\right)$ represents the complete set of parameters of the TLD mixture model. Also $\alpha_1,\alpha_2,\beta_1,\beta_2>0$, $|\lambda_1,\lambda_2|\le 1$ and 'p_1' is the unknown weight.

2.3 The Likelihood Function and the Posterior Distributions

Let $x_1,x_2,...,x_n$ be a random sample of size 'n', taken from the TLD. Assuming that the true lifetimes for some of the objects were unavailable, type-I censoring scheme has been opted. This censoring scheme is associated with some pre-specified test termination time, see, (Kalbfleish and Prentice, 2011; Rabie and Li, 2020; and Elbatal et al., 2022). Suppose for the life testing experiment, out of 'n' elements which are taken from a 2-component mixture TLD, 'r' elements have failed until the test termination time 't' while remaining '$n-r$' objects still working after time 't'. According to Mendenhall and Hader (1958), subpopulation-I and subpopulation-II can contain 'r_1' and 'r_2' number of failed objects, respectively. It is known that $r=r_1+r_2$ which represents number of uncensored observations while the remaining $n-r$ observations represent the censored observations. Now, x_{ij} denotes the failure time of the j^{th} observation belonging to the i^{th} subpopulation, where $i=1,2$ and $j=1,2,...r_i$. The likelihood function for the mixture model under type-I censored samples (Mendenhall and Hader; 1958) is:

$$L\left(\mathbf{x};\Omega\right)=\left\{\prod_{j=1}^{r_1}p_1 f_1\left(x_{1j}\right)\right\}\left\{\prod_{j=1}^{r_1}\left(1-p_1\right)f_2\left(x_{2j}\right)\right\}\left\{1-F\left(x_r\right)\right\}^{n-r} \qquad (12)$$

where $\Omega=\left(\alpha_1,\alpha_2,\beta_1,\beta_2,\lambda_1,\lambda_2,p_1\right)$ and $\mathbf{x}=\left(x_{11},x_{12},...x_{1r_1},x_{21},x_{22},...x_{2r_2}\right)$

$$L\left(\mathbf{x};\Omega\right)=\left[\prod_{j=1}^{r_1}p_1\frac{\alpha_1\beta_1}{\left(1+\beta_1x_{1j}\right)^{(\alpha_1+1)}}\left\{1-\lambda_1+2\lambda_1\left(1+\beta_1x_{1j}\right)^{-\alpha_1}\right\}\right]\left[\prod_{j=1}^{r_2}\left(1-p_1\right)\frac{\alpha_2\beta_2}{\left(1+\beta_2x_{2j}\right)^{(\alpha_2+1)}}\left\{1-\lambda_2+2\lambda_2\left(1+\beta_2x_{2j}\right)^{-\alpha_2}\right\}\right]$$
$$\times\left[1-p_1\left\{1-\left(1+\beta_1x_r\right)^{-\alpha_1}\right\}\left\{1+\lambda_1\left(1+\beta_1x_r\right)^{-\alpha_1}\right\}+\left(1-p_1\right)\left\{1-\left(1+\beta_2x_r\right)^{-\alpha_2}\right\}\left\{1+\lambda_2\left(1+\beta_2x_r\right)^{-\alpha_2}\right\}\right]^{n-r}$$

After some algebraic simplifications, we get:

$$\begin{aligned}L\left(\mathbf{x};\Omega\right)\propto\ & p_1^{r_1}\left(1-p_1\right)^{r_2}\alpha_1^{r_1}e^{-(\alpha_1+1)\sum_{j=1}^{r_1}\log\left(1+\beta_1x_{1j}\right)}\alpha_2^{r_2}e^{-(\alpha_2+1)\sum_{j=1}^{r_2}\log\left(1+\beta_2x_{2j}\right)}\beta_1^{r_1}e^{\sum_{j=1}^{r_1}\log\left\{1-\lambda_1+2\lambda_1\left(1+\beta_1x_{1j}\right)^{-\alpha_1}\right\}}\\ & \times\beta_2^{r_2}e^{\sum_{j=1}^{r_2}\log\left\{1-\lambda_2+2\lambda_2\left(1+\beta_2x_{2j}\right)^{-\alpha_2}\right\}}\left[1-p_1\left\{1-\left(1+\beta_1x_r\right)^{-\alpha_1}\right\}\left\{1+\lambda_1\left(1+\beta_1x_r\right)^{-\alpha_1}\right\}\right.\\ & \left.+\left(1-p_1\right)\left\{1-\left(1+\beta_2x_r\right)^{-\alpha_2}\right\}\left\{1+\lambda_2\left(1+\beta_2x_r\right)^{-\alpha_2}\right\}\right]^{n-r}\end{aligned} \qquad (13)$$

The above expression can be presented as:

$$L\left(\mathbf{x};\Omega\right)\propto\left(\alpha_1\beta_1p_1\right)^{r_1}\left\{\alpha_2\beta_2\left(1-p_1\right)\right\}^{r_2}e^{-(\alpha_1+1)I_{11}}e^{-(\alpha_2+1)I_{12}}e^{I_{13}+I_{14}}I^{n-r}$$

Where $I_{11}=\sum_{j=1}^{r_1}\log\left(1+\beta_1x_{1j}\right)$, $I_{12}=\sum_{j=1}^{r_2}\log\left(1+\beta_2x_{2j}\right)$, $I_{13}=\sum_{j=1}^{r_1}\log\left\{1-\lambda_1+2\lambda_1\left(1+\beta_1x_{1j}\right)^{-\alpha_1}\right\}$,

$I_{14}=\sum_{j=1}^{r_2}\log\left\{1-\lambda_2+2\lambda_2\left(1+\beta_2x_{2j}\right)^{-\alpha_2}\right\}$, and

$$I=1-p_1\left\{1-\left(1+\beta_1x_r\right)^{-\alpha_1}\right\}\left\{1+\lambda_1\left(1+\beta_1x_r\right)^{-\alpha_1}\right\}+\left(1-p_1\right)\left\{1-\left(1+\beta_2x_r\right)^{-\alpha_2}\right\}\left\{1+\lambda_2\left(1+\beta_2x_r\right)^{-\alpha_2}\right\}$$

The log-likelihood function is:

$$l_3 = r_1 \log(\alpha_1\beta_1 p_1) + r_2 \log\{\alpha_2\beta_2(1-p_1)\} - (\alpha_1+1)\sum_{j=1}^{r_1}\log(1+\beta_1 x_{1j}) - (\alpha_2+1)\sum_{j=1}^{r_2}\log(1+\beta_2 x_{2j}) + \sum_{j=1}^{r_1}\log\left\{1-\lambda_1+2\lambda_1(1+\beta_1 x_{1j})^{-\alpha_1}\right\} + \sum_{j=1}^{r_2}\log\left\{1-\lambda_2+2\lambda_2(1+\beta_2 x_{2j})^{-\alpha_2}\right\} + (n-r)\log I \quad (14)$$

2.3.1 Posterior Distribution Assuming Noninformative Prior

Here, we take UP as NIP which has a little impact on the final result of the inference (Bernardo and Smith, 1994). It is used when prior information is not available. Now we take an assumption of independent NIP (uniform) for the parameters $\Omega = (\alpha_1,\alpha_2,\beta_1,\beta_2,\lambda_1,\lambda_2,p_1)$, i.e., $\alpha_1 \sim U(0,\infty)$, $\alpha_2 \sim U(0,\infty)$, $\beta_1 \sim U(0,\infty)$, $\beta_2 \sim U(0,\infty)$
For mixing proportion p_1, the UP over the interval $(0,1)$ is assumed, i.e., $p_1 \sim U(0,1)$. The priors for transmuted parameters λ_1 and λ_2 are $\lambda_1 \sim U(-1,1)$ and $\lambda_2 \sim U(-1,1)$. The joint prior distribution for the parameters $\Omega = (\alpha_1,\alpha_2,\beta_1,\beta_2,\lambda_1,\lambda_2,p_1)$ is:

$$\pi_{UP}(\Omega) \propto 1,\ \alpha_1,\alpha_2>0,\ \beta_1,\beta_2>0,\ |\lambda_1,\lambda_2|\le 1, \text{ and } 0<p_1<1 \quad (15)$$

Using the Bayes theorem, we can define the posterior distribution as:

$$g_{UP}(\Omega\,|\,\mathbf{x}) = \frac{L(\mathbf{x};\Omega)\,\pi_{UP}(\Omega)}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_{-1}^1\int_{-1}^1\int_0^1 L(\mathbf{x};\Omega)\,\pi_{UP}(\Omega)\,d\Omega}. \quad (16)$$

The joint posterior distribution of $\Omega = (\alpha_1,\alpha_2,\beta_1,\beta_2,\lambda_1,\lambda_2,p_1)$ given $\mathbf{x}$ under UP is:

$$g_{UP}(\Omega\,|\,\mathbf{x}) = \frac{(\alpha_1\beta_1 p_1)^{r_1}\{\alpha_2\beta_2(1-p_1)\}^{r_2} e^{-(\alpha_1+1)I_{11}} e^{-(\alpha_2+1)I_{12}} e^{I_{13}+I_{14}} I^{n-r}}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_{-1}^1\int_{-1}^1\int_0^1 (\alpha_1\beta_1 p_1)^{r_1}\{\alpha_2\beta_2(1-p_1)\}^{r_2} e^{-(\alpha_1+1)I_{11}} e^{-(\alpha_2+1)I_{12}} e^{I_{13}+I_{14}} I^{n-r}\,dp_1\,d\lambda_2\,d\lambda_1\,d\beta_2\,d\beta_1\,d\alpha_2\,d\alpha_1}, \quad (17)$$

$$\alpha_1,\alpha_2>0,\ \beta_1,\beta_2>0,\ |\lambda_1,\lambda_2|\le 1,\ and\ 0<p_1<1$$

Where $I_{11} = \sum_{j=1}^{r_1}\log(1+\beta_1 x_{1j})$, $I_{12} = \sum_{j=1}^{r_2}\log(1+\beta_2 x_{2j})$, $I_{13} = \sum_{j=1}^{r_1}\log\left\{1-\lambda_1+2\lambda_1(1+\beta_1 x_{1j})^{-\alpha_1}\right\}$,

$I_{14} = \sum_{j=1}^{r_2}\log\left\{1-\lambda_2+2\lambda_2(1+\beta_2 x_{2j})^{-\alpha_2}\right\}$, and

$$I = 1 - p_1\left\{1-(1+\beta_1 x_r)^{-\alpha_1}\right\}\left\{1+\lambda_1(1+\beta_1 x_r)^{-\alpha_1}\right\} + (1-p_1)\left\{1-(1+\beta_2 x_r)^{-\alpha_2}\right\}\left\{1+\lambda_2(1+\beta_2 x_r)^{-\alpha_2}\right\}$$

and it can be written in the form:

$$g_{UP}(\alpha,\beta,\lambda\,|\,\mathbf{x}) \propto (\alpha_1\beta_1 p_1)^{r_1}\{\alpha_2\beta_2(1-p_1)\}^{r_2} e^{-(\alpha_1+1)I_{11}} e^{-(\alpha_2+1)I_{12}} e^{I_{13}+I_{14}} I^{n-r}$$

2.3.2 Posterior Distribution Assuming Informative Prior

Following independent priors are assumed for the unknown parameters:
$\alpha_1 \sim gamma(a_1,b_1)$, $\alpha_2 \sim gamma(a_2,b_2)$, $\beta_1 \sim gamma(c_1,d_1)$, *and* $\beta_2 \sim gamma(c_2,d_2)$
While the UPs are assumed for the transmuted parameters, i.e., $\lambda_1 \sim U(-1,1)$ and $\lambda_2 \sim U(-1,1)$. Furthermore, the beta prior for the mixing proportion has been assumed as $p_1 \sim Beta(e_1,e_2)$. It is not the first time

that assumption of independence is being used, see, (Yousaf et al., 2019b; Aslam et al., 2020b; and Aslam et al., 2020c). The joint prior distribution for the parameters $\Omega=\left(\alpha_1,\alpha_2,\beta_1,\beta_2,\lambda_1,\lambda_2,p_1\right)$ is:

$$\pi_{GP}\left(\Omega\right)\propto\alpha_1^{a_1-1}e^{-b_1\alpha_1}\alpha_2^{a_2-1}e^{-b_2\alpha_2}\beta_1^{c_1-1}e^{-d_1\beta_1}\beta_2^{c_2-1}e^{-d_2\beta_2}p_1^{e_1-1}\left(1-p_1\right)^{e_2-1}, \tag{18}$$

Using (18), the joint posterior distribution of parameters Ω given $\mathbf{x}$ under GP is:

$$g_{GP}\left(\Omega\mid\mathbf{x}\right)=\frac{\alpha_1^{r_1+a_1-1}\alpha_2^{r_2+a_2-1}\beta_1^{r_1+c_1-1}\beta_2^{r_2+c_2-1}e^{-\alpha_1I_{21}-\alpha_2I_{22}}e^{-d_1\beta_1-d_2\beta_2}e^{I_{23}+I_{24}}K^{n-r}p_1^{r_1+e_1-1}\left(1-p_1\right)^{r_2+e_2-1}}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_{-1}^1\int_{-1}^1\int_0^1\alpha_1^{r_1+a_1-1}\alpha_2^{r_2+a_2-1}\beta_1^{r_1+c_1-1}\beta_2^{r_2+c_2-1}e^{-\alpha_1I_{21}-\alpha_2I_{22}}e^{-d_1\beta_1-d_2\beta_2}e^{I_{23}+I_{24}}K^{n-r}\times p_1^{r_1+e_1-1}\left(1-p_1\right)^{r_2+e_2-1}dp\,d\lambda_2\,d\lambda_1\,d\beta_2\,d\beta_1\,d\alpha_2\,d\alpha_1}, \tag{19}$$

$$\alpha_1,\alpha_2>0,\ \beta_1,\beta_2>0,\ |\lambda_1,\lambda_2|\le1\ and\ 0<p_1<1$$

Where $I_{21}=\sum_{j=1}^{r_1}\log\left(1+\beta_1x_{1j}\right)+b_1$, $I_{22}=\sum_{j=1}^{r_2}\log\left(1+\beta_2x_{2j}\right)+b_2$, $I_{23}=\sum_{j=1}^{r_1}\log\left\{1-\lambda_1+2\lambda_1\left(1+\beta_1x_{1j}\right)^{-\alpha_1}\right\}-\sum_{j=1}^{r_1}\log\left(1+\beta_1x_{1j}\right)$,

$$I_{24}=\sum_{j=1}^{r_2}\log\left\{1-\lambda_2+2\lambda_2\left(1+\beta_2x_{2j}\right)^{-\alpha_2}\right\}-\sum_{j=1}^{r_2}\log\left(1+\beta_2x_{2j}\right)\ and$$

$$K=1-p_1\left\{1-\left(1+\beta_1x_r\right)^{-\alpha_1}\right\}\left\{1+\lambda_1\left(1+\beta_1x_r\right)^{-\alpha_1}\right\}+\left(1-p_1\right)\left\{1-\left(1+\beta_2x_r\right)^{-\alpha_2}\right\}\left\{1+\lambda_2\left(1+\beta_2x_r\right)^{-\alpha_2}\right\}$$

and it can be written in the form:

$$g_{GP}\left(\Omega\mid\mathbf{x}\right)\propto\alpha_1^{r_1+a_1-1}\alpha_2^{r_2+a_2-1}\beta_1^{r_1+c_1-1}\beta_2^{r_2+c_2-1}e^{-\alpha_1I_{21}-\alpha_2I_{22}}e^{-d_1\beta_1-d_2\beta_2}e^{I_{23}+I_{24}}K^{n-r}p_1^{r_1+e_1-1}\left(1-p_1\right)^{r_2+e_2-1}$$

2.3.3 Bayes Estimators and Posterior Risks under Different Loss Functions

This section explains the derivations of the BEs under different LFs and their respective PRs. The BEs are evaluated under the SELF, PLF, WBLF and GELF. The following Table 1 shows the BEs and their PRs under above mentioned LFs.

Table 1 Bayes Estimators and Posterior Risks under different LFs

Loss function	Expression	BE	PR
SELF	$\left(\theta-d\right)^2$	$E\left(\theta\mid\mathbf{x}\right)$	$Var\left(\theta\mid\mathbf{x}\right)$
PLF	$\frac{\left(\theta-d\right)^2}{d}$	$\sqrt{E\left(\theta^2\mid\mathbf{x}\right)}$	$2\left\{\sqrt{E\left(\theta^2\mid\mathbf{x}\right)}-E\left(\theta\mid\mathbf{x}\right)\right\}$
WBLF	$\left(\frac{\theta-d}{d}\right)^2$	$\frac{E\left(\theta^2\mid\mathbf{x}\right)}{E\left(\theta\mid\mathbf{x}\right)}$	$1-\frac{\left\{E\left(\theta\mid\mathbf{x}\right)\right\}^2}{E\left(\theta^2\mid\mathbf{x}\right)}$
GELF	$\left(\frac{d}{\theta}\right)^p-p\ln\left(\frac{d}{\theta}\right)-1,\ p\ne0$	$\left\{E\left(\theta^{-p}\mid\mathbf{x}\right)\right\}^{-\frac{1}{p}}$	$\ln\left\{E\left(\theta^{-p}\mid\mathbf{x}\right)\right\}+pE\left(\ln\theta\mid\mathbf{x}\right)$

2.4 Lindley's Approximation

To derive the Bayes estimators for the parameters $\Omega=(\alpha_1,\alpha_2,\beta_1,\beta_2,\lambda_1,\lambda_2,p_1)$, we apply Lindley (1980) approximation by introducing the function $I(x)$, such that:

$$I(x)=E\left[h(\Omega)\right]=\frac{\iiint h(\Omega)e^{L(\Omega)+G(\Omega)}d(\Omega)}{\iiint e^{L(\Omega)+G(\Omega)}d(\Omega)} \tag{20}$$

whereas $h(\Omega)$: function of Ω only, $L(\Omega)$: Log-likelihood function and $G(\Omega)$: Log joint prior of Ω,

For Lindley approximation method, the availability of the ML estimates of the parameters and sample size 'n' is sufficiently large enough are the requirements. The ratio of the integral presented in (20) can be approximated as:

$$I(x)=h(\Omega)+h_1k_1+h_2k_2+h_3k_3+h_4k_4+h_5k_5+h_6k_6+h_7k_7+k_8+k_9+\frac{1}{2}(D_1E_1+D_2E_2+D_3E_3+D_4E_4+D_5E_5+D_6E_6+D_7E_7), \tag{21}$$

where Ω: MLE of Ω,

$$E_i=h_1S_{i1}+h_2S_{i2}+h_3S_{i3}+h_4S_{i4}+h_5S_{i5}+h_6S_{i6}+h_7S_{i7},$$

$$\begin{aligned}D_i=&S_{11}L_{11i}+S_{22}L_{22i}+S_{33}L_{33i}+S_{44}L_{44i}+S_{55}L_{55i}+S_{66}L_{66i}+S_{77}L_{77i}+2S_{12}L_{12i}+2S_{13}L_{13i}+2S_{14}L_{14i}+2S_{15}L_{15i}\\&+2S_{16}L_{16i}+2S_{17}L_{17i}+2S_{23}L_{23i}+2S_{24}L_{24i}+2S_{25}L_{25i}+2S_{26}L_{26i}+2S_{27}L_{27i}+2S_{34}L_{34i}\\&+2S_{35}L_{35i}+2S_{36}L_{36i}+2S_{37}L_{37i}+2S_{45}L_{45i}+2S_{46}L_{46i}+2S_{47}L_{47i}+2S_{56}L_{56i}+2S_{57}L_{57i}+2S_{67}L_{67i},\end{aligned}$$

$$k_i=\phi_1S_{i1}+\phi_2S_{i2}+\phi_3S_{i3}+\phi_4S_{i4}+\phi_5S_{i5}+\phi_6S_{i6}+\phi_7S_{i7},\ i=1,2,3,4,5,6,7$$

$$\begin{aligned}k_8=&h_{12}S_{12}+h_{13}S_{13}+h_{14}S_{14}+h_{15}S_{16}+h_{17}S_{17}+h_{23}S_{23}+h_{24}S_{24}+h_{25}S_{25}+h_{26}S_{26}+h_{27}S_{27}\\&+h_{34}S_{34}+h_{35}S_{35}+h_{36}S_{36}+h_{37}S_{37}+h_{45}S_{45}+h_{46}S_{46}+h_{47}S_{47}+h_{56}S_{56}+h_{57}S_{57}+h_{67}S_{67},\end{aligned}$$

$$k_9=\frac{1}{2}(h_{11}S_{11}+h_{22}S_{22}+h_{33}S_{33}+h_{44}S_{44}+h_{55}S_{55}+h_{66}S_{66}+h_{77}S_{77})$$

and on the right-hand side, subscripts like 1, 2, 3, 4, 5, 6, 7 refers to as parameters $\alpha_1,\alpha_2,\beta_1,\beta_2,\lambda_1,\lambda_2,\ and\ p_1$, respectively. Now, let

$$\Omega_1=\alpha_1,\ \Omega_2=\alpha_2,\ \Omega_3=\beta_1,\ \Omega_4=\beta_2,\Omega_5=\lambda_1,$$

$$\Omega_6=\lambda_2,\ \Omega_7=p_1,\ \phi_i=\frac{\partial G(\Omega)}{\partial\Omega_i},\ h_i=\frac{\partial h(\Omega)}{\partial\Omega_i},h_{ij}=\frac{\partial^2h(\Omega)}{\partial\Omega_i\partial\Omega_j},\ L_{ij}=\frac{\partial^2L(\Omega)}{\partial\Omega_i\partial\Omega_j},\ L_{ijk}=\frac{\partial^3L(\Omega)}{\partial\Omega_i\partial\Omega_j\partial\Omega_k},$$

$$i,j,k=1,2,3,4,5,6,7,\ \Omega=(\alpha_1,\alpha_2,\beta_1,\beta_2,\lambda_1,\lambda_2,p_1)$$

and S_{ij}: $(i,j)^{th}$ element of the inverse of the matrix $\{-L_{ij}\}$ which is represented as:

$$\begin{pmatrix}-L_{11}&-L_{12}&-L_{13}&-L_{14}&-L_{15}&-L_{16}&-L_{17}\\-L_{21}&-L_{22}&-L_{23}&-L_{24}&-L_{25}&-L_{26}&-L_{27}\\-L_{31}&-L_{32}&-L_{33}&-L_{34}&-L_{35}&-L_{36}&-L_{37}\\-L_{41}&-L_{42}&-L_{43}&-L_{44}&-L_{45}&-L_{46}&-L_{47}\\-L_{51}&-L_{52}&-L_{53}&-L_{54}&-L_{55}&-L_{56}&-L_{57}\\-L_{61}&-L_{62}&-L_{63}&-L_{64}&-L_{65}&-L_{66}&-L_{67}\\-L_{71}&-L_{72}&-L_{73}&-L_{74}&-L_{75}&-L_{76}&-L_{77}\end{pmatrix}^{-1}=\begin{pmatrix}S_{11}&S_{12}&S_{13}&S_{14}&S_{15}&S_{16}&S_{17}\\S_{21}&S_{22}&S_{23}&S_{24}&S_{25}&S_{26}&S_{27}\\S_{31}&S_{32}&S_{33}&S_{34}&S_{35}&S_{36}&S_{37}\\S_{41}&S_{42}&S_{43}&S_{44}&S_{45}&S_{46}&S_{47}\\S_{51}&S_{52}&S_{53}&S_{54}&S_{55}&S_{56}&S_{57}\\S_{61}&S_{62}&S_{63}&S_{64}&S_{65}&S_{66}&S_{67}\\S_{71}&S_{72}&S_{73}&S_{74}&S_{75}&S_{76}&S_{77}\end{pmatrix},$$

All evaluated at the MLEs of the parameters which can be determined using the nonlinear system of equations. Using (4.7), we maximize l_3 with respect to $\alpha_1,\alpha_2,\beta_1,\beta_2,\lambda_1,\lambda_2$, and p_1. We have the following system of non-linear equations:

$$\frac{\partial l_3}{\partial \alpha_1}=\frac{r_1}{\alpha_1}-\sum_{j=1}^{r_1}\ln\left(\beta_1 x_{1j}+1\right)-\sum_{j=1}^{r_1}\left[\frac{2\lambda_1\left(\beta_1 x_{1j}+1\right)^{-\alpha_1}\ln\left(\beta_1 x_{1j}+1\right)}{1+\lambda_1-2\lambda_1\left\{1-\left(\beta_1 x_{1j}+1\right)^{-\alpha_1}\right\}}\right]$$
$$+\frac{(n-r)\left[-p_1\left(\beta_1 x_r+1\right)^{-\alpha_1}\left\{1+\lambda_1\left(\beta_1 x_r+1\right)^{-\alpha_1}\right\}\ln\left(\beta_1 x_r+1\right)+p_1\left\{1-\left(\beta_1 x_r+1\right)^{-\alpha_1}\right\}\lambda_1\left(\beta_1 x_r+1\right)^{-\alpha_1}\ln\left(\beta_1 x_r+1\right)\right]}{1-p_1\left\{1-\left(\beta_1 x_r+1\right)^{-\alpha_1}\right\}\left\{1+\lambda_1\left(\beta_1 x_r+1\right)^{-\alpha_1}\right\}-(1-p_1)\left\{1-\left(\beta_2 x_r+1\right)^{-\alpha_2}\right\}\left\{1+\lambda_2\left(\beta_2 x_r+1\right)^{-\alpha_2}\right\}}=0$$

$$\frac{\partial l_3}{\partial \alpha_2}=\frac{r_2}{\alpha_2}-\sum_{j=1}^{r_2}\ln\left(\beta_2 x_{2j}+1\right)-\sum_{j=1}^{r_2}\left[\frac{2\lambda_2\left(\beta_2 x_{2j}+1\right)^{-\alpha_2}\ln\left(\beta_2 x_{2j}+1\right)}{1+\lambda_2-2\lambda_2\left\{1-\left(\beta_2 x_{2j}+1\right)^{-\alpha_2}\right\}}\right]$$
$$+\frac{(n-r)\left[-(1-p_1)\left(\beta_2 x_r+1\right)^{-\alpha_2}\left\{1+\lambda_2\left(\beta_2 x_r+1\right)^{-\alpha_2}\right\}\ln\left(\beta_2 x_r+1\right)+(1-p_1)\left\{1-\left(\beta_2 x_r+1\right)^{-\alpha_2}\right\}\lambda_2\left(\beta_2 x_r+1\right)^{-\alpha_2}\ln\left(\beta_2 x_r+1\right)\right]}{1-p_1\left\{1-\left(\beta_1 x_r+1\right)^{-\alpha_1}\right\}\left\{1+\lambda_1\left(\beta_1 x_r+1\right)^{-\alpha_1}\right\}-(1-p_1)\left\{1-\left(\beta_2 x_r+1\right)^{-\alpha_2}\right\}\left\{1+\lambda_2\left(\beta_2 x_r+1\right)^{-\alpha_2}\right\}}=0$$

$$\frac{\partial l_3}{\partial \beta_1}=\frac{r_1}{\beta_1}-(\alpha_1+1)\sum_{j=1}^{r_1}\frac{x_{1j}}{\beta_1 x_{1j}+1}-\sum_{j=1}^{r_1}\left\{\frac{2\lambda_1\left(\beta_1 x_{1j}+1\right)^{-\alpha_1-1}\alpha_1 x_{1j}}{2\lambda_1\left(\beta_1 x_{1j}+1\right)^{-\alpha_1}-\lambda_1+1}\right\}$$
$$+\frac{p_1\left\{2\lambda_1\left(\beta_1 x_r+1\right)^{-\alpha_1}-\lambda_1+1\right\}\alpha_1(n-r)x_r\left(\beta_1 x_r+1\right)^{-\alpha_1-1}}{\left[\left(\beta_1 x_r+1\right)^{-2\alpha_1}\lambda_1 p_1-p_1(\lambda_1-1)\left(\beta_1 x_r+1\right)^{-\alpha_1}-\left(\beta_2 x_r+1\right)^{-\alpha_2}\left\{\lambda_2\left(\beta_2 x_r+1\right)^{-\alpha_2}-\lambda_2+1\right\}(-1+p_1)\right]}=0$$

$$\frac{\partial l_3}{\partial \beta_2}=\frac{r_2}{\beta_2}-(\alpha_2+1)\sum_{j=1}^{r_2}\frac{x_{2j}}{\beta_2 x_{2j}+1}-\sum_{j=1}^{r_2}\left\{\frac{2\lambda_2\left(\beta_2 x_{2j}+1\right)^{-\alpha_2-1}\alpha_2 x_{2j}}{2\lambda_2\left(\beta_2 x_{2j}+1\right)^{-\alpha_2}-\lambda_2+1}\right\}$$
$$+\frac{(n-r)(-1+p_1)\left(\beta_2 x_r+1\right)^{-\alpha_2-1}\alpha_2 x_r\left\{2\lambda_2\left(\beta_2 x_r+1\right)^{-\alpha_2}-\lambda_2+1\right\}}{\left[-\lambda_2(-1+p_1)\left(\beta_2 x_r+1\right)^{-2\alpha_2}+(-1+p_1)(\lambda_2-1)\left(\beta_2 x_r+1\right)^{-\alpha_2}+\left(\beta_1 x_r+1\right)^{-\alpha_1}p_1\left\{\lambda_1\left(\beta_1 x_r+1\right)^{-\alpha_1}-\lambda_1+1\right\}\right]}=0$$

$$\frac{\partial l_3}{\partial \lambda_1}=\sum_{j=1}^{r_1}\frac{-1+2\left(\beta_1 x_{1j}+1\right)^{-\alpha_1}}{1+\lambda_1-2\lambda_1\left\{1-\left(\beta_1 x_{1j}+1\right)^{-\alpha_1}\right\}}$$
$$-\frac{(n-r)p_1\left\{1-\left(\beta_1 x_r+1\right)^{-\alpha_1}\right\}\left(\beta_1 x_r+1\right)^{-\alpha_1}}{1-p_1\left\{1-\left(\beta_1 x_r+1\right)^{-\alpha_1}\right\}\left\{1+\lambda_1\left(\beta_1 x_r+1\right)^{-\alpha_1}\right\}-(1-p_1)\left\{1-\left(\beta_2 x_r+1\right)^{-\alpha_2}\right\}\left\{1+\lambda_2\left(\beta_2 x_r+1\right)^{-\alpha_2}\right\}}=0$$

$$\frac{\partial l_3}{\partial \lambda_2}=\sum_{j=1}^{r_2}\frac{-1+2\left(\beta_2 x_{2j}+1\right)^{-\alpha_2}}{1+\lambda_2-2\lambda_2\left\{1-\left(\beta_2 x_{2j}+1\right)^{-\alpha_2}\right\}}$$
$$-\frac{(n-r)(1-p_1)\left\{1-\left(\beta_2 x_r+1\right)^{-\alpha_2}\right\}\left(\beta_2 x_r+1\right)^{-\alpha_2}}{1-p_1\left\{1-\left(\beta_1 x_r+1\right)^{-\alpha_1}\right\}\left\{1+\lambda_1\left(\beta_1 x_r+1\right)^{-\alpha_1}\right\}-(1-p_1)\left\{1-\left(\beta_2 x_r+1\right)^{-\alpha_2}\right\}\left\{1+\lambda_2\left(\beta_2 x_r+1\right)^{-\alpha_2}\right\}}=0$$

$$\frac{\partial l_3}{\partial p_1}=\frac{r_1}{p_1}-\frac{r_2}{1-p_1}$$
$$-\frac{(n-r)\left[-\left\{1-\left(\beta_1 x_r+1\right)^{-\alpha_1}\right\}\left\{1+\lambda_1\left(\beta_1 x_r+1\right)^{-\alpha_1}\right\}+\left\{1-\left(\beta_2 x_r+1\right)^{-\alpha_2}\right\}\left\{1+\lambda_2\left(\beta_2 x_r+1\right)^{-\alpha_2}\right\}\right]}{1-p_1\left\{1-\left(\beta_1 x_r+1\right)^{-\alpha_1}\right\}\left\{1+\lambda_1\left(\beta_1 x_r+1\right)^{-\alpha_1}\right\}-(1-p_1)\left\{1-\left(\beta_2 x_r+1\right)^{-\alpha_2}\right\}\left\{1+\lambda_2\left(\beta_2 x_r+1\right)^{-\alpha_2}\right\}}=0$$

This system of non-linear equations can be solved numerically by any software to obtain the ML estimates. Also, the asymptotic CIs can be obtained for different LFs. Hence, the approximated $100\left(1-\frac{\gamma}{2}\right)\%$ CIs for the parameters Ω are as follows:

$$\alpha_i \pm Z_{\frac{\gamma}{2}}\sqrt{Var\left(\alpha_i\right)},\ \beta_i \pm Z_{\frac{\gamma}{2}}\sqrt{Var\left(\beta_i\right)},\ \lambda_i \pm Z_{\frac{\gamma}{2}}\sqrt{Var\left(\lambda_i\right)},\ and\ \ p_1 \pm Z_{\frac{\gamma}{2}}\sqrt{Var\left(p_1\right)},\quad i=1,2$$

γ denotes the level of significance. If $\gamma = 0.05$, it means that 95% chance that true parameter falls in the interval e.g., $(\alpha_{LL}, \alpha_{UL})$, where LL=Lower Limit, UL= Upper Limit.

3 Results and Discussions

A simulation study has been used to compare the performance of different BEs. Sample of sizes n = 20, 60, 100 and 500 have been generated by the inverse transformation method from the 2-component mixture of the TLD with the following formula

$$u_i = \left\{1-\left(1+\beta x_i\right)^{-\alpha}\right\}\left\{1+\lambda\left(1+\beta x_i\right)^{-\alpha}\right\},$$

After simplification we get, $x_i = \frac{1}{\beta_i}\left[\left\{\frac{\lambda_i - 1 + \sqrt{(1+\lambda_i)^2 - 4\lambda_i u_i}}{2\lambda_i}\right\}^{-\frac{1}{\alpha_i}} - 1\right]$, where $u_i \sim U(0,1)$ and $i = 1, 2.$

In order to generate the censored data using the above inverse transformation methods, we need to fix the censoring time. The parametric sets assumed for this study are $(\alpha_1, \beta_1, \lambda_1, p_1, \alpha_2, \beta_2, \lambda_2) \in \{(0.7, 0.9, 0.8, 0.5, 1.2, 1.3, 0.6), (1.0, 1.2, 0.6, 0.7, 0.8, 0.7, 0.7), (1.3, 1.4, 0.7, 0.3, 0.6, 0.9, 0.8)\}$. After generating the desired samples, the BEs, the PRs, and associated 95% asymptotic CIs are computed assuming the UP and the IP under the SELF, PLF, WBLF and GELF. It is important to mention that type-I censoring is implemented so that the impact of censoring rate on the BEs can be evaluated using different censoring rates. The censoring rate in the resultant sample is assumed to be between 10% and 20%. Because one iteration does not satisfy the aim of performance assessment of the estimator, we repeat the samples N = 10,000 times to compute the average BEs along the appropriate PRs and the 95 % asymptotic CIs using the Mathematica software. Iteration size is fixed after convergence of parameter has been carefully observed making the estimated value getting closer and closer to the parametric value. In Bayesian analysis under IP (gamma), the values of the hyperparameters are selected such that the prior mean becomes the expected value of the pertinent parameter. The numerical results of simulation study are presented in Tables 2-25, whereas the PRs are tabulated in the column next to BEs. The simulation study's findings revealed some intriguing BEs characteristics for the TLMM which are as follows in general:

- As the sample size increases, the estimated values of the parameters converge to the true values of the parameters, and PRs diminish. This trend is not limited to any particular LF.
- These findings show that BEs perform better under the IP than those under UP for all of the LFs evaluated in this study. As a result, the estimates based on the IP are preferable in terms of PRs.
- It is observed that censoring rate is directly proportional to the PRs, i.e., the increased censoring rate leads to increased PRs.
- From the results of CIs given in the Tables 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23 and 25. and for the aforementioned LFs, it is noticed that all CIs contain the true value of the respective unknown parameters.
- The pattern of these intervals demonstrates that the width of intervals becomes narrow as the sample size is increased.
- By comparing the results of censored data based on 10% and 20% censoring rates, the PRs for 10% censoring are found smaller than 20% censoring rate. In general, the PRs (complete data) < PRs (10% censored data) < PRs (20% censored data).
- The pattern of these findings can also be seen in the figures 1-8. Furthermore, the CIs for 10% censoring rate are narrower than the 20% censoring rate due to a less loss of information when a small censoring rate is used.
- For the purpose of brevity, we have only reported the results under UP. However, remaining tables and figures can be seen in the supplementary material.

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Table 2 BEs, and PRs of TLMM under UP with 10% Censoring

Parameter	Size n	SELF		PLF		WBLF		GELF	
$\alpha_1 = 0.7$	20	0.66835	*(0.00995)*	0.67575	*(0.01479)*	0.68323	*(0.02176)*	0.65682	*(0.00582)*
$\beta_1 = 0.9$		0.84232	*(0.04025)*	0.86587	*(0.04709)*	0.89007	*(0.05361)*	0.80806	*(0.01339)*
$\lambda_1 = 0.8$		0.83051	*(0.02996)*	0.84846	*(0.03590)*	0.86685	*(0.04201)*	0.79624	*(0.01523)*
$p_1 = 0.5$		0.49711	*(0.00694)*	0.49009	*(0.01405)*	0.48316	*(0.02888)*	0.50975	*(0.00899)*
$\alpha_2 = 1.2$		1.08865	*(0.03643)*	1.10517	*(0.03305)*	1.12196	*(0.02955)*	1.06693	*(0.00621)*
$\beta_2 = 1.3$		1.18711	*(0.08008)*	1.22029	*(0.06635)*	1.25440	*(0.05352)*	1.14219	*(0.01203)*
$\lambda_2 = 0.6$		0.68455	*(0.03192)*	0.70788	*(0.04664)*	0.73215	*(0.06560)*	0.62797	*(0.03364)*
$\alpha_1 = 0.7$	60	0.68997	*(0.00352)*	0.69251	*(0.00509)*	0.69507	*(0.00733)*	0.68559	*(0.00222)*
$\beta_1 = 0.9$		0.88120	*(0.01418)*	0.88921	*(0.01602)*	0.89729	*(0.01793)*	0.86774	*(0.00531)*
$\lambda_1 = 0.8$		0.80976	*(0.01050)*	0.81622	*(0.01292)*	0.82273	*(0.01577)*	0.79787	*(0.00528)*
$p_1 = 0.5$		0.49914	*(0.00231)*	0.49682	*(0.00464)*	0.49451	*(0.00936)*	0.50331	*(0.00297)*
$\alpha_2 = 1.2$		1.16575	*(0.01483)*	1.17209	*(0.01269)*	1.17847	*(0.01080)*	1.15530	*(0.00309)*
$\beta_2 = 1.3$		1.26430	*(0.02964)*	1.27597	*(0.02333)*	1.28774	*(0.01820)*	1.24507	*(0.00525)*
$\lambda_2 = 0.6$		0.62698	*(0.01261)*	0.63696	*(0.01997)*	0.64711	*(0.03114)*	0.60708	*(0.01179)*
$\alpha_1 = 0.7$	100	0.69404	*(0.00212)*	0.69556	*(0.00305)*	0.69709	*(0.00438)*	0.69136	*(0.00136)*
$\beta_1 = 0.9$		0.88877	*(0.00857)*	0.89358	*(0.00962)*	0.89842	*(0.01074)*	0.88046	*(0.00328)*
$\lambda_1 = 0.8$		0.80579	*(0.00627)*	0.80968	*(0.00777)*	0.81358	*(0.00957)*	0.79870	*(0.00315)*
$p_1 = 0.5$		0.49949	*(0.00139)*	0.49810	*(0.00278)*	0.49672	*(0.00559)*	0.50199	*(0.00178)*
$\alpha_2 = 1.2$		1.17961	*(0.00913)*	1.18347	*(0.00772)*	1.18734	*(0.00652)*	1.17301	*(0.00195)*
$\beta_2 = 1.3$		1.27870	*(0.01804)*	1.28574	*(0.01407)*	1.29281	*(0.01091)*	1.26669	*(0.00328)*
$\lambda_2 = 0.6$		0.61615	*(0.00770)*	0.62237	*(0.01243)*	0.62865	*(0.01988)*	0.60425	*(0.00705)*
$\alpha_1 = 0.7$	500	0.69881	*(0.00043)*	0.69912	*(0.00061)*	0.69942	*(0.00088)*	0.69826	*(0.00028)*
$\beta_1 = 0.9$		0.89775	*(0.00173)*	0.89872	*(0.00193)*	0.89968	*(0.00215)*	0.89603	*(0.00068)*
$\lambda_1 = 0.8$		0.80116	*(0.00126)*	0.80194	*(0.00157)*	0.80272	*(0.00195)*	0.79974	*(0.00063)*
$p_1 = 0.5$		0.49990	*(0.00028)*	0.49962	*(0.00055)*	0.49935	*(0.00111)*	0.50040	*(0.00036)*
$\alpha_2 = 1.2$		1.19594	*(0.00188)*	1.19673	*(0.00157)*	1.19751	*(0.00131)*	1.19454	*(0.00041)*
$\beta_2 = 1.3$		1.29575	*(0.00367)*	1.29717	*(0.00283)*	1.29859	*(0.00218)*	1.29323	*(0.00069)*
$\lambda_2 = 0.6$		0.60324	*(0.00158)*	0.60454	*(0.00261)*	0.60585	*(0.00431)*	0.60086	*(0.00141)*

Some findings which can be seen in the results are as under:

- If ($\alpha_1 < \alpha_2$), it is observed that for first parametric set of values the estimates of the shape parameter for second component are underestimated for both 10% and 20% censored data under all the LFs and both priors. While for first component, the estimates of the shape parameter are underestimated for 10% censored data under all the cases. The said estimates overestimated the true parametric values for 20% censored data under PLF and WBLF using the UP, while underestimated under all the LFs except the WBLF using the GP.

- If ($\alpha_1 > \alpha_2$) it is observed that, using second parametric set of values, the shape parameters for first and second components are underestimated for both 10% and 20% censored data under all the LFs using the UP and the GP. It is observed that for third parameter set of values, the shape parameter for first component are underestimated for both 10% and 20% censored data under all the cases, while for second component, the shape parameters are underestimated for 10% censored data under all the cases.

- If ($\beta_1 < \beta_2$), it is observed that for first parametric set of values the estimates for scale parameter for second component are underestimated for majority of the cases.

- If ($\beta_1 > \beta_2$), it is observed that for second and third parametric sets the scale parameters for first and second components are underestimated throughout. On the other hand, the transmuted parameter is overestimated using all LFs except GELF.

- In the case of the first and third sets of true parametric values, estimates for mixing parameter under SELF were observed to be superior to those under other three LFs. While for the second parametric set of values, the GELF provides better estimates.

- It is observed that for third set of true parametric values, the estimates of the mixing component parameter are overestimated for both 10% and 20% censored data under all the LFs using the UP and the GP.

- It is observed that for first set of values, the estimates of mixing component parameter are underestimated for both 10% and 20% censored data under all the LFs except the GELF using the UP. Using GP, the estimates of the mixing component parameter are underestimated for 10% censored data under all the LFs except the GELF.

- It is observed that for second parameter set of values, the estimates of the mixing component parameter are underestimated for both 10% and 20% censored data under all the LFs using the UP. Using GP, the estimates of the mixing component parameter are overestimated for 10% censored data under all the LFs except the WBLF.

- In short, the estimates under the GELF are having the best convergence among all the LFs, because the PRs are smaller for the GELF as compared to the PRs for other LFs involved in the study. Also, IP (gamma) is better because the PRs under the GP are smaller as compared to the PRs under the UP.

Table 3 BEs, and PRs of TLMM under UP with 20% Censoring

Parameter	Size n	SELF		PLF		WBLF		GELF	
$\alpha_1 = 0.7$	20	0.69360	*(0.01245)*	0.70252	*(0.01783)*	0.71155	*(0.02521)*	0.67816	*(0.00784)*
$\beta_1 = 0.9$		0.86193	*(0.04416)*	0.88716	*(0.05047)*	0.91314	*(0.05605)*	0.82278	*(0.01539)*
$\lambda_1 = 0.8$		0.82720	*(0.05313)*	0.85876	*(0.06310)*	0.89156	*(0.07212)*	0.76810	*(0.02626)*
$p_1 = 0.5$		0.48576	*(0.00889)*	0.47652	*(0.01849)*	0.46745	*(0.03918)*	0.50141	*(0.01113)*
$\alpha_2 = 1.2$		1.12560	*(0.04753)*	1.14649	*(0.04178)*	1.16777	*(0.03606)*	1.09479	*(0.00904)*
$\beta_2 = 1.3$		1.19730	*(0.09280)*	1.23541	*(0.07621)*	1.27473	*(0.06068)*	1.14453	*(0.01421)*
$\lambda_2 = 0.6$		0.61699	*(0.05864)*	0.66299	*(0.09201)*	0.71264	*(0.13409)*	0.53640	*(0.04750)*
$\alpha_1 = 0.7$	60	0.69775	*(0.00414)*	0.70071	*(0.00592)*	0.70368	*(0.00843)*	0.69249	*(0.00267)*
$\beta_1 = 0.9$		0.88713	*(0.01509)*	0.89559	*(0.01693)*	0.90414	*(0.01881)*	0.87264	*(0.00572)*
$\lambda_1 = 0.8$		0.80831	*(0.01810)*	0.81943	*(0.02224)*	0.83070	*(0.02695)*	0.78804	*(0.00901)*
$p_1 = 0.5$		0.49558	*(0.00291)*	0.49264	*(0.00588)*	0.48972	*(0.01198)*	0.50078	*(0.00370)*
$\alpha_2 = 1.2$		1.17581	*(0.01694)*	1.18299	*(0.01436)*	1.19021	*(0.01210)*	1.16368	*(0.00359)*
$\beta_2 = 1.3$		1.26659	*(0.03349)*	1.27974	*(0.02630)*	1.29302	*(0.02044)*	1.24482	*(0.00594)*
$\lambda_2 = 0.6$		0.60649	*(0.01962)*	0.62246	*(0.03194)*	0.63886	*(0.05066)*	0.57781	*(0.01699)*
$\alpha_1 = 0.7$	100	0.69862	*(0.00248)*	0.70040	*(0.00354)*	0.70217	*(0.00505)*	0.69546	*(0.00161)*
$\beta_1 = 0.9$		0.89226	*(0.00909)*	0.89734	*(0.01016)*	0.90245	*(0.01129)*	0.88339	*(0.00350)*
$\lambda_1 = 0.8$		0.80488	*(0.01087)*	0.81161	*(0.01345)*	0.81839	*(0.01650)*	0.79269	*(0.00542)*
$p_1 = 0.5$		0.49740	*(0.00174)*	0.49565	*(0.00350)*	0.49391	*(0.00707)*	0.50052	*(0.00222)*
$\alpha_2 = 1.2$		1.18558	*(0.01027)*	1.18990	*(0.00865)*	1.19424	*(0.00725)*	1.17809	*(0.00222)*
$\beta_2 = 1.3$		1.28010	*(0.02036)*	1.28803	*(0.01586)*	1.29601	*(0.01227)*	1.26652	*(0.00371)*
$\lambda_2 = 0.6$		0.60402	*(0.01173)*	0.61366	*(0.01927)*	0.62345	*(0.03116)*	0.58668	*(0.01028)*
$\alpha_1 = 0.7$	500	0.69972	*(0.00050)*	0.70008	*(0.00071)*	0.70043	*(0.00101)*	0.69909	*(0.00032)*
$\beta_1 = 0.9$		0.89844	*(0.00183)*	0.89946	*(0.00204)*	0.90049	*(0.00227)*	0.89662	*(0.00072)*
$\lambda_1 = 0.8$		0.80095	*(0.00220)*	0.80232	*(0.00274)*	0.80369	*(0.00341)*	0.79849	*(0.00110)*
$p_1 = 0.5$		0.49949	*(0.00035)*	0.49914	*(0.00069)*	0.49879	*(0.00139)*	0.50011	*(0.00044)*
$\alpha_2 = 1.2$		1.19713	*(0.00209)*	1.19800	*(0.00174)*	1.19888	*(0.00145)*	1.19558	*(0.00046)*
$\beta_2 = 1.3$		1.29603	*(0.00415)*	1.29763	*(0.00320)*	1.29924	*(0.00246)*	1.29318	*(0.00078)*
$\lambda_2 = 0.6$		0.60082	*(0.00236)*	0.60278	*(0.00392)*	0.60475	*(0.00650)*	0.59729	*(0.00209)*

Table 4 95% CIs of TLMM under UP with 10% Censoring

Parameter	Size	Loss Functions							
	n	SELF		PLF		WBLF		GELF	
$\alpha_1 = 0.7$	20	*0.47286*	*0.86384*	*0.43737*	*0.91413*	*0.39410*	*0.97235*	*0.50729*	*0.80635*
$\beta_1 = 0.9$		*0.44910*	*1.23554*	*0.44054*	*1.29119*	*0.43625*	*1.34390*	*0.58122*	*1.03490*
$\lambda_1 = 0.8$		*0.49126*	*1.16975*	*0.47708*	*1.21983*	*0.46512*	*1.26858*	*0.55437*	*1.03811*
$p_1 = 0.5$		*0.33389*	*0.66034*	*0.25776*	*0.72242*	*0.15010*	*0.81622*	*0.32396*	*0.69554*
$\alpha_2 = 1.2$		*0.71458*	*1.46272*	*0.74887*	*1.46148*	*0.78505*	*1.45886*	*0.91242*	*1.22145*
$\beta_2 = 1.3$		*0.63247*	*1.74176*	*0.71540*	*1.72517*	*0.80095*	*1.70785*	*0.92718*	*1.35721*
$\lambda_2 = 0.6$		*0.33438*	*1.03473*	*0.28457*	*1.13118*	*0.23015*	*1.23416*	*0.26846*	*0.98748*
$\alpha_1 = 0.7$	60	*0.57375*	*0.80619*	*0.55273*	*0.83230*	*0.52724*	*0.86289*	*0.59319*	*0.77799*
$\beta_1 = 0.9$		*0.64780*	*1.11460*	*0.64115*	*1.13728*	*0.63483*	*1.15976*	*0.72490*	*1.01059*
$\lambda_1 = 0.8$		*0.60896*	*1.01056*	*0.59344*	*1.03899*	*0.57659*	*1.06887*	*0.65538*	*0.94035*
$p_1 = 0.5$		*0.40492*	*0.59336*	*0.36330*	*0.63033*	*0.30486*	*0.68415*	*0.39648*	*0.61015*
$\alpha_2 = 1.2$		*0.92707*	*1.40443*	*0.95132*	*1.39286*	*0.97482*	*1.38212*	*1.04637*	*1.26423*
$\beta_2 = 1.3$		*0.92686*	*1.60174*	*0.97658*	*1.57536*	*1.02332*	*1.55216*	*1.10304*	*1.38710*
$\lambda_2 = 0.6$		*0.40689*	*0.84706*	*0.35996*	*0.91397*	*0.30124*	*0.99298*	*0.39424*	*0.81992*
$\alpha_1 = 0.7$	100	*0.60376*	*0.78431*	*0.58726*	*0.80387*	*0.56731*	*0.82688*	*0.61911*	*0.76362*
$\beta_1 = 0.9$		*0.70730*	*1.07025*	*0.70135*	*1.08582*	*0.69533*	*1.10150*	*0.76818*	*0.99274*
$\lambda_1 = 0.8$		*0.65056*	*0.96103*	*0.63694*	*0.98241*	*0.62183*	*1.00533*	*0.68869*	*0.90872*
$p_1 = 0.5$		*0.42651*	*0.57247*	*0.39477*	*0.60143*	*0.35021*	*0.64323*	*0.41932*	*0.58467*
$\alpha_2 = 1.2$		*0.99237*	*1.36684*	*1.01121*	*1.35573*	*1.02913*	*1.34556*	*1.08638*	*1.25964*
$\beta_2 = 1.3$		*1.01546*	*1.54195*	*1.05327*	*1.51821*	*1.08808*	*1.49754*	*1.15436*	*1.37902*
$\lambda_2 = 0.6$		*0.44422*	*0.78809*	*0.40385*	*0.84089*	*0.35233*	*0.90497*	*0.43965*	*0.76886*
$\alpha_1 = 0.7$	500	*0.65825*	*0.73937*	*0.65061*	*0.74763*	*0.64141*	*0.75743*	*0.66555*	*0.73098*
$\beta_1 = 0.9$		*0.81616*	*0.97934*	*0.81263*	*0.98480*	*0.80890*	*0.99047*	*0.84491*	*0.94715*
$\lambda_1 = 0.8$		*0.73168*	*0.87063*	*0.72434*	*0.87954*	*0.71609*	*0.88935*	*0.75059*	*0.84889*
$p_1 = 0.5$		*0.46727*	*0.53254*	*0.45346*	*0.54579*	*0.43403*	*0.56467*	*0.46347*	*0.53733*
$\alpha_2 = 1.2$		*1.11097*	*1.28092*	*1.11904*	*1.27442*	*1.12651*	*1.26852*	*1.15463*	*1.23446*
$\beta_2 = 1.3$		*1.17699*	*1.41452*	*1.19286*	*1.40148*	*1.20703*	*1.39014*	*1.24175*	*1.34471*
$\lambda_2 = 0.6$		*0.52544*	*0.68103*	*0.50444*	*0.70465*	*0.47717*	*0.73453*	*0.52729*	*0.67443*

Table 5 95% CIs of TLMM under UP with 20% Censoring

Parameter	Size n	SELF		PLF		WBLF		GELF	
$\alpha_1 = 0.7$	20	*0.47488*	*0.91232*	*0.44080*	*0.96424*	*0.40037*	*1.02272*	*0.50460*	*0.85171*
$\beta_1 = 0.9$		*0.45004*	*1.27381*	*0.44684*	*1.32748*	*0.44912*	*1.37716*	*0.57962*	*1.06593*
$\lambda_1 = 0.8$		*0.37543*	*1.27898*	*0.36639*	*1.35112*	*0.36518*	*1.41794*	*0.45051*	*1.08568*
$p_1 = 0.5$		*0.30091*	*0.67062*	*0.21002*	*0.74302*	*0.07951*	*0.85539*	*0.29465*	*0.70817*
$\alpha_2 = 1.2$		*0.69830*	*1.55291*	*0.74587*	*1.54711*	*0.79559*	*1.53995*	*0.90840*	*1.28119*
$\beta_2 = 1.3$		*0.60022*	*1.79437*	*0.69432*	*1.77649*	*0.79193*	*1.75753*	*0.91087*	*1.37818*
$\lambda_2 = 0.6$		*0.14238*	*1.09160*	*0.06845*	*1.25753*	*0.00508*	*1.43035*	*0.10920*	*0.96359*
$\alpha_1 = 0.7$	60	*0.57164*	*0.82386*	*0.54990*	*0.85152*	*0.52372*	*0.88364*	*0.59122*	*0.79376*
$\beta_1 = 0.9$		*0.64634*	*1.12792*	*0.64056*	*1.15063*	*0.63530*	*1.17298*	*0.72439*	*1.02089*
$\lambda_1 = 0.8$		*0.54464*	*1.07198*	*0.52714*	*1.11172*	*0.50891*	*1.15249*	*0.60205*	*0.97404*
$p_1 = 0.5$		*0.38991*	*0.60126*	*0.34230*	*0.64298*	*0.27520*	*0.70423*	*0.38161*	*0.61996*
$\alpha_2 = 1.2$		*0.92071*	*1.43091*	*0.94811*	*1.41787*	*0.97460*	*1.40583*	*1.04624*	*1.28112*
$\beta_2 = 1.3$		*0.90792*	*1.62525*	*0.96188*	*1.59759*	*1.01278*	*1.57326*	*1.09372*	*1.39593*
$\lambda_2 = 0.6$		*0.33197*	*0.88101*	*0.27217*	*0.97275*	*0.19770*	*1.08002*	*0.32234*	*0.83328*
$\alpha_1 = 0.7$	100	*0.60105*	*0.79620*	*0.58373*	*0.81706*	*0.56286*	*0.84148*	*0.61691*	*0.77401*
$\beta_1 = 0.9$		*0.70536*	*1.07916*	*0.69976*	*1.09492*	*0.69417*	*1.11073*	*0.76736*	*0.99942*
$\lambda_1 = 0.8$		*0.60052*	*1.00925*	*0.58429*	*1.03893*	*0.56659*	*1.07019*	*0.64840*	*0.93698*
$p_1 = 0.5$		*0.41570*	*0.57911*	*0.37970*	*0.61160*	*0.32907*	*0.65875*	*0.40825*	*0.59278*
$\alpha_2 = 1.2$		*0.98696*	*1.38420*	*1.00765*	*1.37215*	*1.02732*	*1.36116*	*1.08575*	*1.27042*
$\beta_2 = 1.3$		*1.00040*	*1.55980*	*1.04120*	*1.53486*	*1.07886*	*1.51316*	*1.14711*	*1.38593*
$\lambda_2 = 0.6$		*0.39171*	*0.81632*	*0.34154*	*0.88577*	*0.27744*	*0.96945*	*0.38800*	*0.78536*
$\alpha_1 = 0.7$	500	*0.65603*	*0.74341*	*0.64785*	*0.75230*	*0.63802*	*0.76284*	*0.66380*	*0.73437*
$\beta_1 = 0.9$		*0.81451*	*0.98238*	*0.81094*	*0.98799*	*0.80717*	*0.99380*	*0.84399*	*0.94924*
$\lambda_1 = 0.8$		*0.70911*	*0.89279*	*0.69975*	*0.90490*	*0.68923*	*0.91816*	*0.73357*	*0.86341*
$p_1 = 0.5$		*0.47488*	*0.91232*	*0.44080*	*0.96424*	*0.40037*	*1.02272*	*0.50460*	*0.85171*
$\alpha_2 = 1.2$		*0.45004*	*1.27381*	*0.44684*	*1.32748*	*0.44912*	*1.37716*	*0.57962*	*1.06593*
$\beta_2 = 1.3$		*0.37543*	*1.27898*	*0.36639*	*1.35112*	*0.36518*	*1.41794*	*0.45051*	*1.08568*
$\lambda_2 = 0.6$		*0.30091*	*0.67062*	*0.21002*	*0.74302*	*0.07951*	*0.85539*	*0.29465*	*0.70817*

(Loss Functions: SELF, PLF, WBLF, GELF)

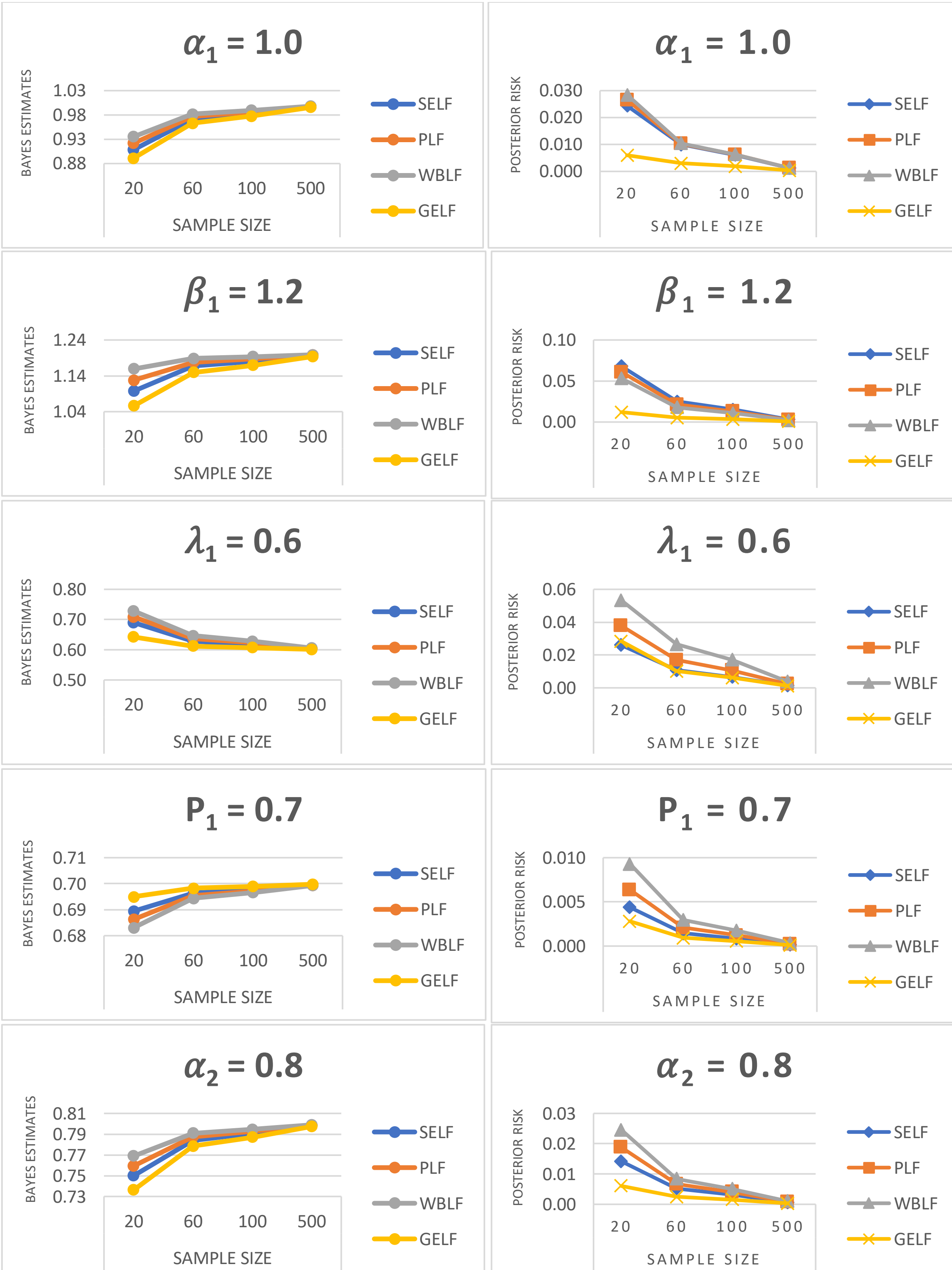

Figure 1 (Continued)

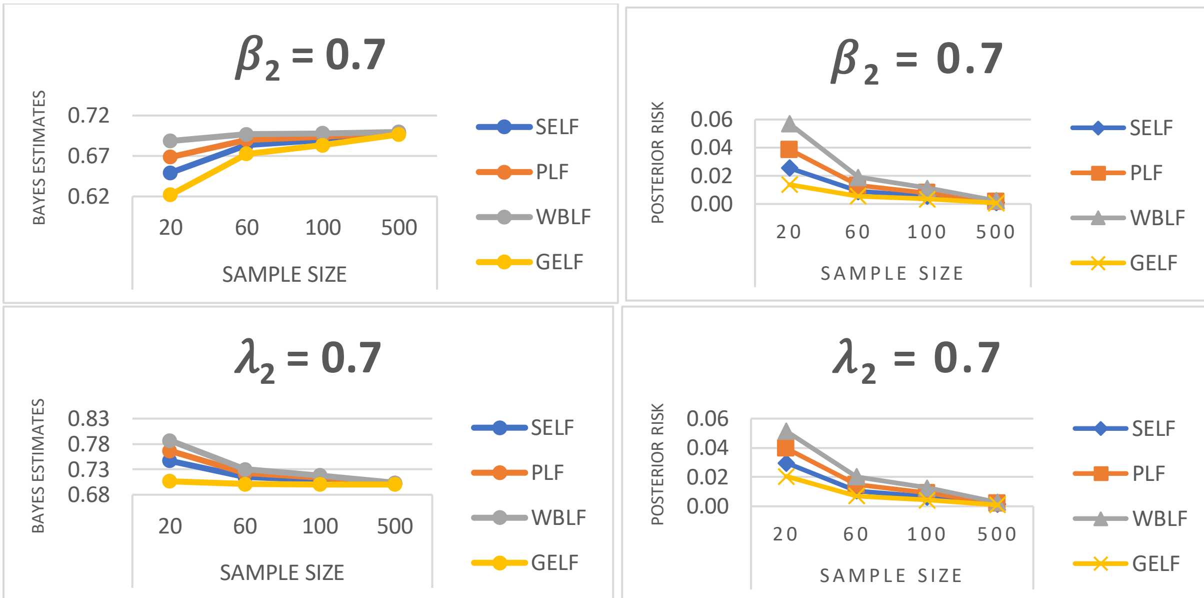

Figure 1. 10% censoring simulations for (α_1, β_1, λ_1, P_1, α_2, β_2, λ_2) using UP

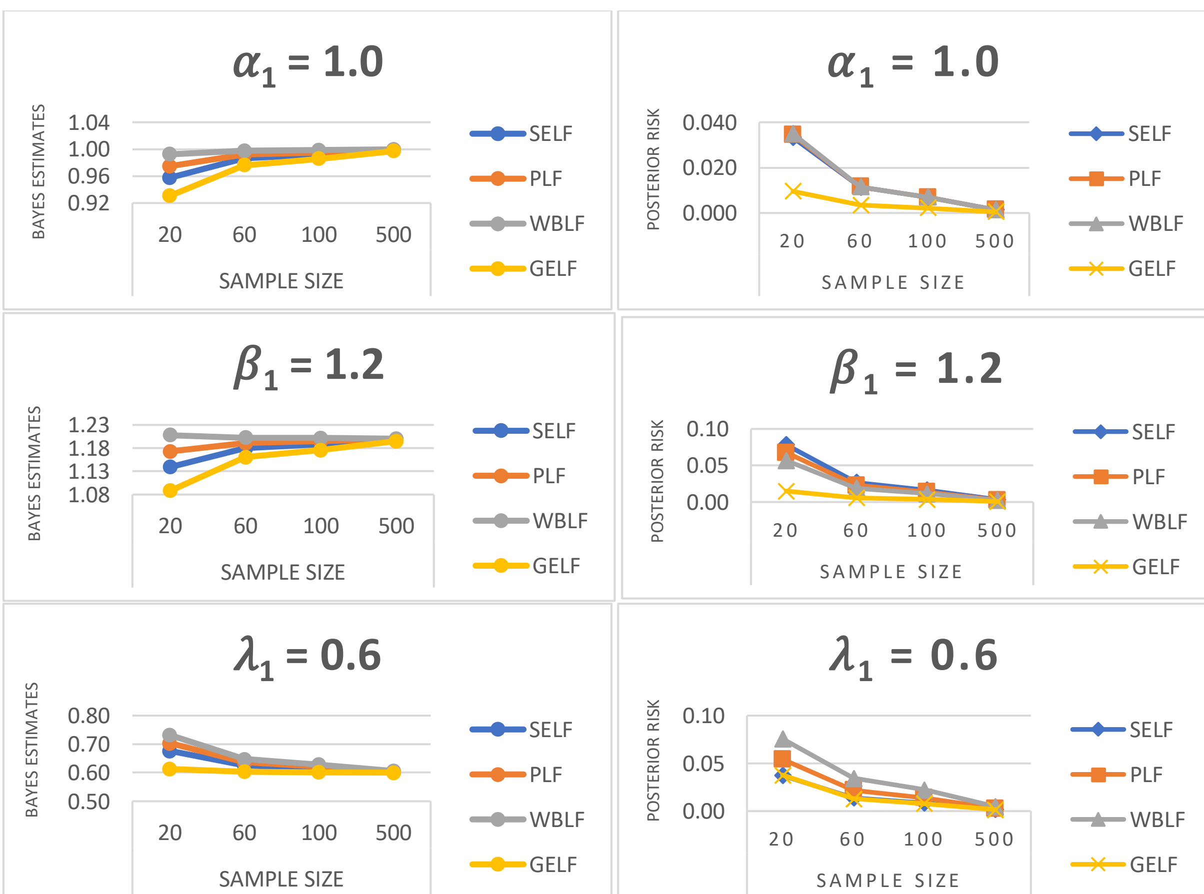

Figure 2 (Continued)

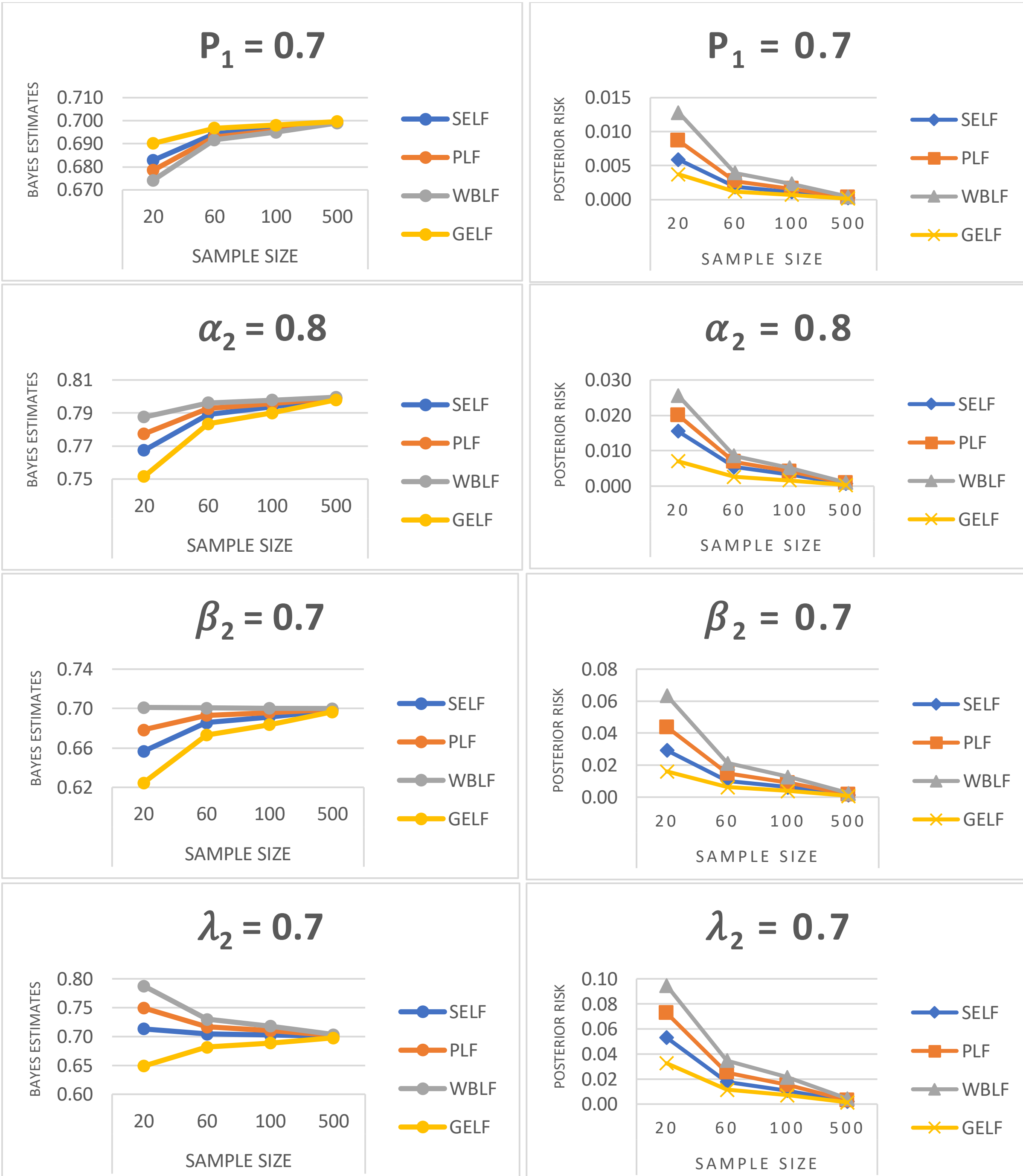

Figure 2. 20% censoring simulations for (α_1, β_1, λ_1, P_1, α_2, β_2, λ_2) using UP

3.1 A Real-life Application of the 2-component Transmuted Lomax Mixture Model

A large airplane's windshield is a complicated piece of equipment with a thick upper layer and a heated layer at the bottom with numerous layers of material. The data regarding windshield of airplane was reported by Murthy et al. (2004). Windshield failure statistics and censored times for a certain type of Windshield are included in dataset. There are 153 observations in the windshield data, with 65 service times of windshields that had not failed at the time of observation and the remaining 88 as failing windshields. Based on the cause of failure, it is now assumed that the item belongs to subpopulation-I or subpopulation-II. It is assumed that the windshield is expected to fail in the given circumstance due to either a heating system failure or damage to the non-structural outer layer. As a result, the data is randomly constructed into two subpopulations utilizing probabilistic mixing $p_1 = 0.5$ in order to run the Bayesian analysis assuming the 2-component combination of TLD. The following is a summary of the information acquired from the above data that is required for our suggested model:

For 10% Censoring

$x_r = 0.18, n_1 = 77, r_1 = 70, \sum_{j=1}^{r_1} x_{1j} = 146.88, n_2 = 76, r_2 = 69, \sum_{j=1}^{r_2} x_{2j} = 151.33$

For 20% Censoring

$x_r = 0.18, n_1 = 77, r_1 = 62, \sum_{j=1}^{r_1} x_{1j} = 116.51, n_2 = 76, r_2 = 61, \sum_{j=1}^{r_2} x_{2j} = 121.10$

For the Windshield Failure Data, we first conduct a goodness of fit test to see if the data set under consideration followed the TLMM. The K-S test resulted in a value of 0.08339 with a p-value of 0.953, indicating that the aforementioned real-life data set fits the TLMM reasonably well.

Table 26 Windshield Failure Data

Subpopulation-I	Subpopulation-II
0.040, 2.154, 3.595, 1.183, 3.003, 0.309, 2.194, 3.779, 2.223, 3.924, 1.262, 4.035, 1.360, 3.500, 1.436, 3.622, 1.124, 3.665, 1.248, 2.324, 1.281, 2.349, 4.255, 1.281, 2.385, 4.278, 1.303, 2.481, 4.305, 1.432, 2.610, 4.376, 1.480, 2.625, 4.449, 1.505, 2.632, 4.485, 1.568, 2.661, 4.602, 2.065, 1.619, 2.823, 4.694, 2.137, 1.652, 2.890, 0.046, 2.141, 1.757, 2.934, 0.150, 2.183, 1.866, 2.964, 0.280, 2.341, 1.876, 3.000, 0.313, 2.435, 1.899, 3.103, 0.389, 2.464, 1.912, 3.117, 0.622, 2.560, 1.981, 3.344, 0.952, 2.600, 2.089, 3.467, 1.085	0.301, 2.190, 3.699, 1.244, 3.102, 1.249, 3.304, 0.557, 3.483, 0.943, 2.224, 1.070, 2.229, 4.121, 2.300, 4.167, 1.492, 4.240, 1.580, 3.695, 1.719, 4.015, 1.794, 4.628, 1.915, 4.806, 1.920, 4.881, 1.963, 5.140, 1.978, 1.506, 2.640, 4.570, 2.053, 1.615, 2.688, 4.663, 2.117, 1.652, 2.902, 0.140, 2.163, 1.795, 2.962, 0.248, 2.240, 1.911, 3.114, 0.487, 2.543, 1.914, 3.166, 0.900, 2.592, 2.010, 3.376, 0.996, 2.670, 2.038, 3.385, 1.003, 2.717, 2.085, 3.443, 1.010, 2.819, 2.820, 2.097, 3.478, 1.092, 2.878, 2.135, 3.578, 1.152, 2.950

Findings are as follows:

- By comparing the performance of BEs, in terms of LFs under the UP and IP, Tables 27-30 shows that the PRs in majority of cases under GELF are smaller for the unknown parameters as compared to those under SELF, PLF and WBLF.

- We considered 10% and 20% censoring rates and by comparing the findings of 10% and 20% censored data, one can see that more precise results have been obtained in case of 10% censored data as compared to 20% censored data.

- The PRs for 10% censored data are smaller than the PRs for 20% censored data. Also, the trend of the intervals presented in Tables 27-30 shows that the width of intervals under the GP is narrower as compared to the width of the intervals under the UP which reveals that BEs under the GP are more likely to be precise as compared to the BEs under the UP.

Table 27 BEs and PRs of TLMM under UP

Parameters	Loss Functions							
	SELF		PLF		WBLF		GELF	
10% Censoring								
α_1	1.47422	*(0.01298)*	1.47862	*(0.00879)*	1.48303	*(0.00594)*	1.46672	*(0.00178)*
β_1	0.88409	*(0.00760)*	0.88838	*(0.00858)*	0.89269	*(0.00963)*	0.87679	*(0.00288)*
λ_1	0.80726	*(0.00326)*	0.80928	*(0.00404)*	0.81130	*(0.00498)*	0.80354	*(0.00165)*
p_1	0.50000	*(0.00182)*	0.49818	*(0.00364)*	0.49637	*(0.00732)*	0.50329	*(0.00234)*
α_2	1.16329	*(0.01058)*	1.16783	*(0.00907)*	1.17238	*(0.00775)*	1.15586	*(0.00220)*
β_2	1.26587	*(0.02009)*	1.27378	*(0.01582)*	1.28174	*(0.01238)*	1.25276	*(0.00358)*
λ_2	0.62686	*(0.00639)*	0.63193	*(0.01015)*	0.63705	*(0.01600)*	0.61659	*(0.00609)*
20% Censoring								
α_1	1.47412	*(0.01406)*	1.47888	*(0.00952)*	1.48365	*(0.00643)*	1.46599	*(0.00193)*
β_1	0.88331	*(0.00843)*	0.88807	*(0.00952)*	0.89286	*(0.01069)*	0.87524	*(0.00319)*
λ_1	0.80811	*(0.00442)*	0.81084	*(0.00546)*	0.81357	*(0.00672)*	0.80307	*(0.00224)*
p_1	0.50005	*(0.00205)*	0.49799	*(0.00411)*	0.49595	*(0.00826)*	0.50376	*(0.00264)*
α_2	1.16604	*(0.01109)*	1.17079	*(0.00949)*	1.17555	*(0.00809)*	1.15821	*(0.00231)*
β_2	1.26643	*(0.02203)*	1.27510	*(0.01734)*	1.28383	*(0.01355)*	1.25205	*(0.00393)*
λ_2	0.62634	*(0.00751)*	0.63231	*(0.01194)*	0.63834	*(0.01879)*	0.61434	*(0.00712)*

Table 28 95% CIs of TLMM under UP

Parameters	Loss Functions							
	SELF		PLF		WBLF		GELF	
10% Censoring								
α_1	*1.25094*	*1.69751*	*1.29485*	*1.66238*	*1.33202*	*1.63404*	*1.38407*	*1.54937*
β_1	*0.71321*	*1.05497*	*0.70686*	*1.06989*	*0.70034*	*1.08503*	*0.77155*	*0.98202*
λ_1	*0.69531*	*0.91920*	*0.68476*	*0.93379*	*0.67297*	*0.94963*	*0.72390*	*0.88319*
p_1	*0.41647*	*0.58354*	*0.37994*	*0.61642*	*0.32870*	*0.66405*	*0.40851*	*0.59808*
α_2	*0.96173*	*1.36485*	*0.98113*	*1.35452*	*0.99979*	*1.34497*	*1.06402*	*1.24770*
β_2	*0.98809*	*1.54366*	*1.02727*	*1.52029*	*1.06366*	*1.49982*	*1.13547*	*1.37005*
λ_2	*0.47017*	*0.78354*	*0.43443*	*0.82943*	*0.38910*	*0.88500*	*0.46369*	*0.76949*
20% Censoring								
α_1	*1.24172*	*1.70651*	*1.28762*	*1.67013*	*1.32651*	*1.64080*	*1.37999*	*1.55199*
β_1	*0.70336*	*1.06327*	*0.69686*	*1.07929*	*0.69022*	*1.09550*	*0.76455*	*0.98593*
λ_1	*0.67781*	*0.93840*	*0.66601*	*0.95566*	*0.65288*	*0.97427*	*0.71035*	*0.89579*
p_1	*0.41133*	*0.58876*	*0.37240*	*0.62358*	*0.31780*	*0.67410*	*0.40304*	*0.60447*
α_2	*0.95963*	*1.37245*	*0.97983*	*1.36174*	*0.99925*	*1.35185*	*1.06391*	*1.25252*
β_2	*0.97549*	*1.55737*	*1.01701*	*1.53319*	*1.05566*	*1.51200*	*1.12919*	*1.37490*
λ_2	*0.45647*	*0.79622*	*0.41817*	*0.84645*	*0.36968*	*0.90699*	*0.44899*	*0.77968*

Table 29 BEs and PRs of TLMM under GP with $(a_1, b_1, a_2, b_2, c_1, d_1, c_2, d_2, e_1, e_2)$ = (0.5, 0.5, 0.8, 1.0, 0.4, 0.3, 1.5, 2.0, 2.0, 2.5)

Parameters	Loss Functions							
	SELF		PLF		WBLF		GELF	
10% Censoring								
α_1	1.46613	*(0.01250)*	1.47038	*(0.00851)*	1.47465	*(0.00578)*	1.45898	*(0.00169)*
β_1	0.87934	*(0.00743)*	0.88355	*(0.00843)*	0.88778	*(0.00951)*	0.87228	*(0.00279)*
λ_1	0.81886	*(0.00296)*	0.82066	*(0.00361)*	0.82247	*(0.00439)*	0.81539	*(0.00154)*
p_1	0.50178	*(0.00182)*	0.49996	*(0.00363)*	0.49815	*(0.00728)*	0.50509	*(0.00236)*
α_2	1.15599	*(0.00999)*	1.16030	*(0.00862)*	1.16463	*(0.00742)*	1.14907	*(0.00204)*
β_2	1.23623	*(0.01719)*	1.24317	*(0.01386)*	1.25014	*(0.01112)*	1.22557	*(0.00289)*
λ_2	0.65630	*(0.00394)*	0.65930	*(0.00599)*	0.66231	*(0.00907)*	0.64898	*(0.00439)*
20% Censoring								
α_1	1.46630	*(0.01359)*	1.47093	*(0.00926)*	1.47557	*(0.00628)*	1.45852	*(0.00184)*
β_1	0.87882	*(0.00826)*	0.88351	*(0.00937)*	0.88822	*(0.01058)*	0.87098	*(0.00309)*
λ_1	0.82155	*(0.00402)*	0.82400	*(0.00489)*	0.82645	*(0.00592)*	0.81682	*(0.00210)*
p_1	0.50203	*(0.00205)*	0.49999	*(0.00410)*	0.49794	*(0.00821)*	0.50578	*(0.00266)*
α_2	1.16128	*(0.01074)*	1.16589	*(0.00923)*	1.17053	*(0.00790)*	1.15376	*(0.00222)*
β_2	1.23571	*(0.01903)*	1.24339	*(0.01535)*	1.25111	*(0.01231)*	1.22392	*(0.00320)*
λ_2	0.65852	*(0.00478)*	0.66214	*(0.00724)*	0.66578	*(0.01090)*	0.64965	*(0.00531)*

Table 30 95% CIs of TLMM under GP with $(a_1, b_1, a_2, b_2, c_1, d_1, c_2, d_2, e_1, e_2)$ = (0.5, 0.5, 0.8, 1.0, 0.4, 0.3, 1.5, 2.0, 2.0, 2.5)

Parameters	Loss Functions							
	SELF		PLF		WBLF		GELF	
10% Censoring								
α_1	*1.24703*	*1.68522*	*1.28957*	*1.65119*	*1.32564*	*1.62365*	*1.37834*	*1.53961*
β_1	*0.71043*	*1.04825*	*0.70364*	*1.06347*	*0.69661*	*1.07896*	*0.76881*	*0.97575*
λ_1	*0.71224*	*0.92548*	*0.70290*	*0.93842*	*0.69255*	*0.95239*	*0.73843*	*0.89235*
p_1	*0.41817*	*0.58538*	*0.38183*	*0.61810*	*0.33093*	*0.66538*	*0.40992*	*0.60026*
α_2	*0.96013*	*1.35185*	*0.97830*	*1.34230*	*0.99582*	*1.33343*	*1.06053*	*1.23761*
β_2	*0.97929*	*1.49317*	*1.01240*	*1.47393*	*1.04345*	*1.45682*	*1.12013*	*1.33101*
λ_2	*0.53324*	*0.77936*	*0.50756*	*0.81103*	*0.47565*	*0.84897*	*0.51908*	*0.77887*
20% Censoring								
α_1	*1.23779*	*1.69481*	*1.28236*	*1.65949*	*1.32022*	*1.63092*	*1.37442*	*1.54262*
β_1	*0.70069*	*1.05695*	*0.69374*	*1.07327*	*0.68660*	*1.08984*	*0.76196*	*0.97999*
λ_1	*0.69727*	*0.94583*	*0.68698*	*0.96101*	*0.67562*	*0.97727*	*0.72690*	*0.90674*
p_1	*0.41323*	*0.59084*	*0.37452*	*0.62545*	*0.32032*	*0.67557*	*0.40460*	*0.60696*
α_2	*0.95812*	*1.36444*	*0.97756*	*1.35423*	*0.99628*	*1.34478*	*1.06138*	*1.24614*
β_2	*0.96535*	*1.50608*	*1.00055*	*1.48623*	*1.03367*	*1.46855*	*1.11300*	*1.33484*
λ_2	*0.52300*	*0.79404*	*0.49537*	*0.82891*	*0.46111*	*0.87045*	*0.50677*	*0.79252*

4 Conclusion

We considered the Transmuted Lomax Mixture Model Bayesian Analysis using IP and NIP under various LFs. Based on the characteristics of the posterior distribution, we draw the conclusion that IP performs roughly equally to NIP and has lower PR. In terms of the selection of LF, it is clear from the information supplied that the GELF is preferable to other symmetrical LFs (based on PR). One trend emerged: PRs decreased as sample size increased. Be aware that PRs earned using the SELF are likewise smaller, placing them second. Future extensions of this work to 3-component analysis are possible, and a single component analysis of the Transmuted Lomax distribution using the Metropolis hasting algorithm within a Gibbs sampler is an option. We also used Lindley's approximation to evaluate BEs and their associated PRs, although it would be interesting to compare this method with MCMC and the quadrature method.

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