Pakistan Journal of Statistics and Operation Research

The Marshall-Olkin Pranav distribution: Theory and applications

Rehab Alsultan[1*]

* Corresponding Author

1. Department of Mathematics, Umm Al-Qura University, Makkah 24382, Saudi Arabia, rasultan@uqu.edu.sa

Abstract

The current paper presented new two-parameter life processes distribution, the Marshall-Olkin Pranav (MOEP) distribution. This study combines the Marshall-Olkin method with the Pranav distribution to produce a more accessible and flexible model for data survival techniques. Some of its critical statistical features are presented in this study. For instance, we mentioned its survival, hazard, reversed hazard, and cumulative hazard rate function. Then we discussed its Moment generating functions, The characteristic function, Incomplete moments, R`enyi and Entropies, and stochastic orderings. The research utilized maximization of chance in estimating parameters. These tests are done through simulations to achieve the desired results. With an assurance that the combined model has larger applications, such as in the strength of airplane glass and survival data set, the two real-world examples are given to explain the great possibility and validity of the extended distribution. These results demonstrate the usefulness of the proposed distribution and the need for more tilt parameters.

Key Words: Marshall-Olkin family of distributions; Pranav distribution; Stochastic ordering; Maximum likelihood, Quantile; Incomplete moments; Generating function.

1. Introduction

Developing classical distributions is as geriatric as statistics itself, and it has long been regarded as beneficial as many other valuable issues. These inferences began with the addition of new location, scale, or shape factors. This statistics case has concentrated in recent years, and several new generalized classes of distributions have been presented.

One of the main goals of offering and developing (models or classes) is to illustrate how the lifespan phenomenon arises in domains like statistics, probability, operation research, management science, medical, computer science, insurance, physics, engineering, biology, industry, communications, life-testing, Etc. The extended classes of distributions of modern distribution theory have been introduced using various methods. For instance, adding an extra parameter to a two-parameter Weibull distribution was proposed by Mudholkar and Srivastava (1993). Shaw and Buckley (2009) pioneered yet another well-known technique including a parameter in a family of distributions. Other authors have used it to extend notable distributions in recent years. Granzotto et al. (2017) introduced the Cubic Transmutation method as a new way of producing distributions. Rahman et al. (2018) have presented a general family of transmuted distributions. Kumaraswamy (1980) proposed the Kumaraswamy distribution, a two-parameter distribution on (0,1). Eugene et al. (2002) suggested the beta-generated technique, which develops, beta-produced distributions using the beta distribution with parameters α and λ as the generator. Mahdavi and Kundu (2017) proposed Alpha Power Transformation for presenting new statistical distributions. Ahmad (2020) recently introduced the Zubair-G family, a novel method for creating new distributions using the CDF. Marshall and Olkin (1997) pioneered a straightforward way of adding a single parameter to a family of distributions. If $Q(x)$ is the cumulative distribution function (CDF) and $\bar{Q}(x) = 1 - Q(x)$ is the survival rate function (SRT), then the SRF of the Marshall-Olkin (MO) family is as follows:

$$\bar{F}(x, \vartheta) = \frac{\lambda \bar{Q}(x;\vartheta)}{1-(1-\lambda)\bar{Q}(x;\vartheta)} \quad (1)$$

Where ϑ the parameters of the original distribution. An original distribution can be obtained with $\lambda = 1$. The parameter λ is generally referred to as the "tilt parameter". The probability density function of the family developed by Marshall Olkin is:

$$f(x, \vartheta) = \frac{\lambda q(x)}{[1-(1-\lambda)\bar{Q}(x)]^2}, \quad -\infty < x < \infty, \lambda > 0 \quad (2)$$

KK (2018) introduces a distribution with only one parameter, known as the Pranav distribution, based on its probability density function

$$q(x;\theta) = \frac{\theta^4}{\theta^4+6}(\theta + x^3)e^{-\theta x}; x > 0 , \theta > 0. \quad (3)$$

According to Shukla, the probability distribution function (PDF) is a blend of two distributions: the exponential distribution with a scale parameter and the gamma distribution with a scale parameter and shape parameter 4. The following is its cumulative distribution function:

$$Q(x;\theta) = 1 - \left[1 + \frac{\theta x(\theta^2 x^2 + 3\theta x + 6)}{\theta^4 + 6}\right] e^{-\theta x}; \; x > 0 , \theta > 0. \; (4)$$

where $x > 0$, $\theta > 0$. The corresponding the survival function (SRF) is given as

$$\bar{Q}(x;\theta) = \left[1 + \frac{\theta x(\theta^2 x^2 + 3\theta x + 6)}{\theta^4 + 6}\right] e^{-\theta x}; \; x > 0 , \theta > 0. \quad (5)$$

Several writers have adapted their strategy to expand many distributions in recent years. Cordeiro and Lemonte (2013) investigated the mathematical properties and applications of the Marshall-Olkin extended (MOE) Weibull distribution. Other instances include the Marshall-Olkin Marshall-Olkin Kappa distribution, which was introduced by Javed et al. (2019). The Marshall-Olkin Lindley-Log-logistic (MOLLLoG) distribution is a novel generalized distribution that was recently introduced by Moakofi et al. (2021). The Marshall-Olkin-odd power generalized Weibull (MO-OPGW-G) distribution is introduced by Chipepa et al. (2022) as a novel family of distributions.

In this paper, we develop the Pranav model using the (MO) approach. The primary reason for developing the new model is that it provides many additional features. Also, its pdf and HRF are simple, containing only two parameters. Furthermore, the splendor of the proposed model lies in its ability to fit a wide range of real data sets. Consequently, we introduce this model, hoping it will deliver a more accurate fit in specific applicable contexts than other Marshall-Olkin models.

Following is a summary of the paper's outline. In Section 2, the statistical functions are associated with the presented distribution. Section 3 examines the statistical properties. Section 4 offers a maximum likelihood estimation of the unknown parameters and a simulation approach. In Section 5, we discuss applications of the newly produced model. The final section of the paper discusses the summary.

2. The suggested model

This section proposes a new distribution, namely, the Marshal-Olkin Pranav distribution (MOEP) distribution. By using two Equations (1) and (5), the SRF of the MOEP model is given by

 DOI: http://dx.doi.org/10.18187/pjsor.v19i1.4058

$$SRF(x;\lambda,\theta)=\frac{\lambda e^{-\theta x}[(\theta^4+6)+\theta x(\theta^2x^2+3\theta x+6)]}{(\theta^4+6)-(1-\lambda)e^{-\theta x}[(\theta^4+6)+\theta x(\theta^2x^2+3\theta x+6)]} \quad (6)$$

Also, the cumulative (CDF) and density PDF function of the Marshall-Olkin Pranav distribution, respectively given as:

$$F(x;\lambda,\theta)=1-\mathrm{SRF}(x;\lambda,\theta)=1-\frac{\lambda\bar{Q}(x)}{1-(1-\lambda)\bar{Q}(x)}=\frac{1-\bar{Q}(x)+\lambda\bar{Q}(x)-\lambda\bar{Q}(x)}{1-(1-\lambda)\bar{Q}(x)}=\frac{1-\bar{Q}(x)}{1-(1-\lambda)\bar{Q}(x)}$$
$$=\frac{Q(x;\theta)}{1-(1-\lambda)\bar{Q}(x)}$$

By substituting (4) and (5) into $\frac{Q(x;\theta)}{1-(1-\lambda)\bar{Q}(x)}$, we obtained the CDF of the proposed MOEP model as

$$F(x;\lambda,\theta)=\frac{(\theta^4+6)-[(\theta^4+6)+\theta x(\theta^2x^2+3\theta x+6)]e^{-\theta x}}{(\theta^4+6)-(1-\lambda)[(\theta^4+6)+\theta x(\theta^2x^2+3\theta x+6)]e^{-\theta x}} \quad (7$$

The associated PDF of the suggested MOEP model is provided as

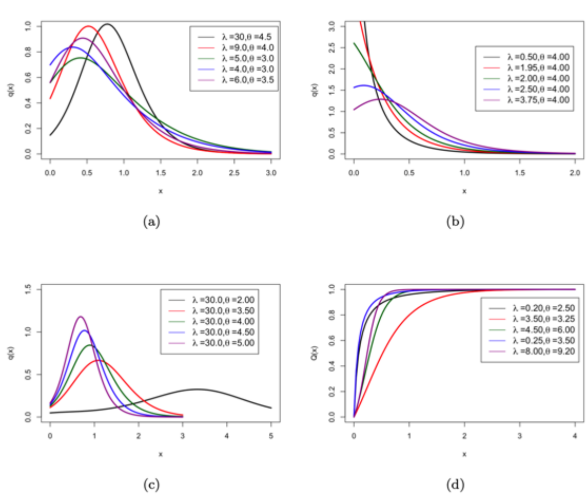
$$f(x;\lambda,\theta)=\frac{d}{dx}F(x;\lambda,\theta)=\frac{d}{dx}\left[\frac{Q(x;\theta)}{1-(1-\lambda)\bar{Q}(x)}\right]=\frac{\lambda q(x;\theta)}{[1-(1-\lambda)\bar{Q}(x)]^2}$$

By substituting (3) and (5) into $\frac{\lambda q(x;\theta)}{[1-(1-\lambda)\bar{Q}(x)]^2}$, we obtained the PDF of the proposed MOEP model

$$f(x;\lambda,\theta)=\frac{\lambda\theta^4(\theta^4+6)(\theta+x^3)e^{-\theta x}}{[(\theta^4+6)-(1-\lambda)[(\theta^4+6)+\theta x(\theta^2x^2+3\theta x+6)]e^{-\theta x}]^2} \quad (8)$$

Where $-\infty<x<\infty, \lambda>0, \theta>0$.

Fig 1. (d) represents cumulative function at different parameter values. Fig. 1(a), (b), and (c) depict the manners of the density function. It shows that the skewness of the density carries a smaller value as θ decreases, while the distribution exhibits unimodal, positively skewed behavior as λ increases.



Figur1: Density function (a), (b) and (c) and cumulative function (d) at different parameter values using R programming.

The corresponding hazard rate function of MOEP distribution is given as

$$hrf(x;\lambda,\theta) = \frac{f(x)}{\mathrm{SRF(x)}} = \frac{\lambda q(x)}{[1-(1-\lambda)\bar{Q}(x)]^2} \times \frac{1-(1-\lambda)\bar{Q}(x)}{\lambda \bar{Q}(x)} = \frac{q(x)}{\bar{Q}(x)[1-(1-\lambda)\bar{Q}(x)]}$$

By substituting (3) and (5) , we obtained the $hrf(x;\lambda,\theta)$ of the proposed MOEP model

$$= \frac{\theta^4(\theta+x^3)e^{-\theta x}(\theta^4+6)^2}{(\theta^4+6)\left[(\theta^4+6)-(1-\lambda)e^{-\theta x}[(\theta^4+6)+\theta x(\theta^2x^2+3\theta x+6)]\right][(\theta^4+6)+\theta x(\theta^2x^2+3\theta x+6)]e^{-\theta x}}$$

$$hrf(x;\lambda,\theta) = \frac{\theta^4(\theta^4+6)(\theta+x^3)}{\left[(\theta^4+6)-(1-\lambda)e^{-\theta x}[(\theta^4+6)+\theta x(\theta^2x^2+3\theta x+6)]\right][(\theta^4+6)+\theta x(\theta^2x^2+3\theta x+6)]} \quad (9)$$

The reversed rate hazard function of the MOEP distribution is given as

$$Rhr = \frac{f(x)}{F(x)} = \frac{\lambda\, q(x)}{Q(x)[1-(1-\lambda)\bar{Q}(x)]} = \frac{\lambda\,\theta^4(\theta^4+6)(\theta+x^3)e^{-\theta x}}{\left[(\theta^4+6)-(1-\lambda)e^{-\theta x}[(\theta^4+6)+\theta x(\theta^2x^2+3\theta x+6)]\right]\left[(\theta^4+6)-e^{-\theta x}[(\theta^4+6)+\theta x(\theta^2x^2+3\theta x+6)]\right]}$$
(10)

The cumulative hazard rate function of the MOEP is given as

$$chrf(x;\lambda,\theta) = -\ln\left[\frac{\lambda\,\bar{Q}(x)}{1-(1-\lambda)\bar{Q}(x)}\right] = -\ln\left[\frac{\lambda e^{-\theta x}[(\theta^4+6)+\theta x(\theta^2x^2+3\theta+6)]}{(\theta^4+6)-(1-\lambda)e^{\theta x}[(\theta^4+6)+\theta x(\theta^2x^2+3\theta x+6)]}\right] \quad (11)$$

Figure 2. (a) shows the survival rate function and (b) represents a reversed hazard rate function. Fig 2. (c) and (d) depict the behavior of the hazard rate function at different parameter values.

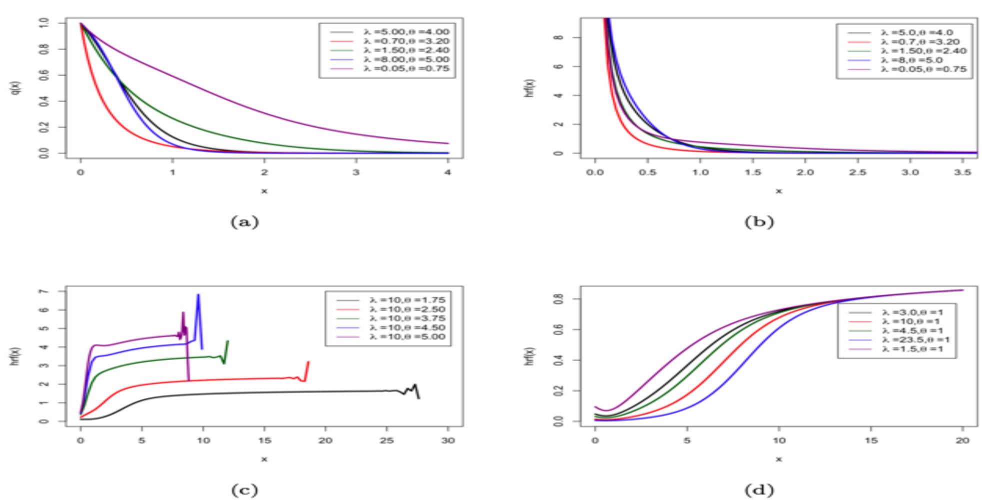


Figure 2: The survival rate plot (a), Plot of the reversed hazard rate function (b), hazard rate function (c) and (d) at different parameter values using R software**.**

2.1 The linear representation

To easily comprehend the possessions of the MOEP distribution, it is required to obtain the explicit term of the distribution. For this goal, we use some expansion functions that follow the generalized binomial theorem:

$$(b-z)^{-n} = \sum_{\delta=0}^{\infty} (-1)^{\delta} \binom{n+\delta-1}{\delta} z^{\delta} b^{-(n+\delta)}$$

$$(b+z)^{n} = \sum_{\delta=0}^{n} \binom{n}{\delta} b^{(n-\delta)} z^{\delta}, \quad n > 0$$

Then,

$$\left[(\theta^4+6)-(1-\lambda)[(\theta^4+6)+\theta x(\theta^2x^2+3\theta x+6)]e^{-\theta x}\right]^{-2} = \sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{\delta=0}^{j} P_{kij\delta}\, x^{i+j+\delta} e^{-\theta x k}$$

Where $P_{kij\delta} = \binom{k+1}{k}\binom{k}{i}\binom{i}{j}\binom{j}{\delta}(-1)^k 6^{i-j} 3^{j-\delta}(1-\lambda)^k(\theta^4+6)^{-(i+2)}\theta^{i+j+\delta}$

Then, the pdf of the MOP density can be expressed as:

$$f(x;\lambda,x) = \sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{\delta=0}^{j} P_{kij\delta}\,\lambda\theta^4(\theta^4+6)\,(\theta+x^3)x^{i+j+\delta}\, e^{-\theta x(k+1)} \qquad (12)$$

Thus, Equation (12) is the linear expression of (8).

3. Statical Properties

This section contains unique phrases for some of the new distribution's most important attributes:

3.1 Moment generating functions (mgf)

The MOEP mgf for a random variable X is defined as

$$M_X(t) = \sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{\delta=0}^{j} P_{kij\delta}\,\lambda\theta^4(\theta^4+6)\int_0^{\infty}(\theta+x^3)x^{i+j+\delta}e^{-\theta x(k+1)+tx}dx$$

$$M_X(t) = C_0\left[\theta(\theta(k+1)+t)^{-(i+j+\delta+1)}\Gamma(i+j+\delta+1)+(\theta(k+1)+t)^{-(i+j+\delta+1)}\Gamma(i+j+\delta+4)\right]$$
(13)

Where $C_0 = \sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{\delta=0}^{j} P_{kij\delta}\,\lambda\theta^4(\theta^4+6)$

3.2 The characteristic function

$$\varphi_X = E(e^{ztx}) = \sum_{r=0}^{\infty} \frac{(zt)^r}{r!} E[x^r]$$

$$\varphi_X = C_1 \left[\theta\big(\theta(k+1)\big)^{-(i+j+\delta+r+1)}\Gamma(i+j+\delta+r+1) + \big(\theta(k+1)\big)^{-(i+j+\delta+r+4)}\Gamma(i+j+\delta+r+4)\right] \quad (14)$$

Where $C_1 = \sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{\delta=0}^{j}\sum_{r=0}^{\infty}\frac{zt^r}{r!}P_{kij\delta}\,\lambda\theta^4(\theta^4+6)$.

3.3 Incomplete moments

$$\psi = \int_0^x v^r f(v)dv = \sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{\delta=0}^{j} P_{kij\delta}\,\lambda\theta^4(\theta^4+6)\int_0^x(\theta+v^3)v^{i+j+\delta}e^{-\theta v(k+1)}\,dv$$

$$= \sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{\delta=0}^{j} P_{kij\delta}\lambda\theta^4(\theta^4+6)\big(\theta(k+1)\big)^{-(i+j+\delta+r+1)}\,\Gamma(i+j+\delta+r+1,\theta(k+1)x) \quad (15)$$

Where $\Gamma(a,x)$ is incomplete gamma functions.

3.4 R´enyi Entropies

A random variable's entropy is a measure of its uncertainty fluctuation. The MOEP distribution's R´enyi entropy is calculated as follows:

$$I_{RE}(X) = (1-a)^{-1}\log\{\int_0^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{\delta=0}^{j}\left[P_{kij\delta}\lambda\theta^4(\theta^4+6)\right]^a(\theta+x^3)^a x^{a(i+j+\delta)}e^{-a(\theta x(k+1))}\}$$

$$= (1-a)^{-1}\log\{\ C_2\left[a\big(\theta x(k+1)\big)\right]^{-(3\zeta+(i+j+\delta)a+1)}\ \Gamma(3\,\zeta+(i+j+\delta)a+1))\ \} \quad (16)$$

Where $C_2 = \sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{\delta=0}^{j}\left[P_{kij\delta}\lambda\theta^4(\theta^4+6)\right]^a\sum_{\zeta=0}^{a}\binom{a}{\zeta}\theta^{a-\zeta}$

3.5 Stochastic orderings

If the following ordering holds, X is said to be smaller than Y, if X and Y are independent random variables with CDFs F_X and F_Y, respectively. (see Shaked and Shanthikumar (2007)):

- if $F_X(x) \geq F_Y(x)$ for x, then $(X \leq_{st} Y)$ Stochastic order.
- if $f_X(x)/f_Y(x)$ is decreasing in x, then $(X \leq_{lr} Y)$ Likelihood ratio order.
- if $h_X(x) \geq h_Y(x)$ for all x, then $(X \leq_{hr} Y)$ Hazard rate order.
- if $m_X(x) \geq m_Y(x)$ for all x, then $(X \leq_{mrl} Y)$ Mean residual life order.

Theorem 1. Assume $X \sim$ MOP (λ_1,θ_1) and $Y \sim MOP(\lambda_2,\theta_2)$. If $\lambda_1 > \lambda_2$, $\theta_1 > \theta_2$, then $X \leq_{lr} Y, X \leq_{hr} Y, X \leq_{mrl} Y, and\ X \leq_{st} Y$.

Proof

It is sufficient to show $\frac{f_X(x)}{f_Y(x)}$ is a decreasing function of x, the likelihood ratio

$$\frac{f_X(x)}{f_Y(x)} = \frac{\lambda_1\theta_1^4(\theta_1^4+6)(\theta_1+x^3)e^{-\theta_1 x}}{\left[(\theta_1^4+6)-(1-\lambda_1)[(\theta_1^4+6)+\theta_1 x(\theta_1^2x^2+3\theta_1 x+6)]e^{-\theta_1 x}\right]^2} \times \frac{\left[(\theta_2^4+6)-(1-\lambda_2)[(\theta_2^4+6)+\theta_2 x(\theta_2^2x^2+3\theta_2 x+6)]e^{-\theta_2 x}\right]^2}{\lambda_2\theta_2^4(\theta_2^4+6)(\theta_2+x^3)e^{-\theta_2 x}} \quad (17)$$

Therefore,

$$\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} = \frac{3x}{\theta_1+x^3} - \theta_1 - \frac{3x}{\theta_2+x^3} + \theta_2 - \frac{2(1-\lambda_1)\theta_1^4e^{-\theta_1 x}(\theta_1+x^3)}{(\theta_1^4+6)-(1-\lambda_1)[(\theta_1^4+6)+\theta_1 x(\theta_1^2x^2+3\theta_1 x+6)]e^{-\theta_1 x}} + \frac{2(1-\lambda_2)\theta_2^4e^{-\theta_2 x}(\theta_2+x^3)}{(\theta_2^4+6)-(1-\lambda_2)[(\theta_2^4+6)+\theta_2 x(\theta_2^2x^2+3\theta_2 x+6)]e^{-\theta_2 x}} < 0 \quad (18)$$

Thus, $\frac{f_X(x)}{f_Y(x)}$ is decreasing in x and hence $X \leq_{lr} Y$. In the same way, we can deduce that for $X \leq_{hr} Y$, $X \leq_{mrl} Y$, and $X \leq_{st} Y$.

4. Maximum likelihood estimation

Assume that random variable X belongs to the observed distribution and that the parameter vector $(\theta,\lambda)^T$ has size n. The sample likelihood function is calculated in the following way:

$$\prod_{i=0}^{n} f(x;\theta,\lambda) == \lambda^n\theta^{4n}(\theta^4+6)^n \prod_{i=0}^{n} \frac{(\theta+x^3)e^{-\theta x}}{\left[(\theta^4+6)-(1-\lambda)[(\theta^4+6)+\theta x(\theta^2x^2+3\theta x+6)]e^{-\theta x}\right]^2} \quad (19)$$

The log-likelihood function is

$$L = nlog(\lambda) + 4nlog(\theta) + nlog(\theta^4+6) + \sum log(\theta+x^3) - \theta x - 2\sum log\,\Lambda \quad (20)$$

Where $\Lambda = (\theta^4+6) - (1-\lambda)e^{-\theta x}[(\theta^4+6) + \theta x(\theta^2x^2+3\theta x+6)]$.

To obtain the ML estimates of the unknown parameters of the Marshall-Olkin-Pranav distribution, we must maximize the log-likelihood function given in Eq. (19). We do this by taking the first derivative of the log-likelihood equation concerning the parameters and setting them to zero.

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} - \frac{2e^{-tx}[(\theta^4+6)+\theta x(\theta^2x^2+3\theta x+6)]}{\Lambda} \quad (21)$$

$$\frac{\partial L}{\partial \theta} = \frac{4n}{\theta} + \frac{4n\theta^3}{\theta^4+6} - \theta + \sum\frac{1}{\theta+x^3} - 2\sum\frac{\theta^3\left(4+(1-\lambda)e^{-\theta x}(\theta x+x^4-4)\right)}{\Lambda} \quad (22)$$

The accurate solution of the generated ML estimator of equations (20)-(21) for unknown parameters is indeed impossible. As a result, it's more practical to employ non-linear optimization algorithms like the Newton-Raphson algorithm to maximize the likelihood function numerically. To obtain $\hat{\lambda}$ and $\hat{\theta}$, we used R's optimum function.

5. Application

5.1 Simulation

We used the Monte Carlos simulation approach with 10,000 repeats to test the performance of the Marshall-Olkin-Pranav distribution based on the bias and mean square error of the predicted parameters of the maximum likelihood estimation method. The simulation is carried out in the following manner: G(x) = u/u generates data, where u is uniformly distributed (0, 1). The actual parameter values are assumed to be (λ =5, θ=20), (λ =20, θ=10). The simulation is run for the values n = 20, 150, and 200. The results of the Monte Carlos simulation study are shown in

Table 1. We assess biases, mean square errors (MSE), and the mean of the predicted values. These results are based on the anticipated first-order asymptotic theory, which signifies that bias and MSEs will drop near zero as the sample size increases. Table 1 shows that as sample size increases, the MSE of the ML estimators of λ and θ reduces, and their biases decay, approaching 0. The MSE of estimated parameters rises as shape parameters increase.

Table 1: MOEP parameters estimations.

True parameters		Sample size	parameters	Mean	bias	MSE
λ	θ					
5	20	20	λ	9.189	4.189	11.116
			θ	21.167	1.167	5.709
		150	λ	5.548	0.548	1.748
			θ	20.301	0.301	2.117
		200	λ	5.266	0.266	1.349
			θ	20.047	0.047	1.810
20	10	20	λ	75.671	55.671	137.015
			θ	12.788	2.799	2.990
		150	λ	24.604	4.604	9.331
			θ	10.619	0.619	0.891
		200	λ	24.417	4.417	7.961
			θ	10.525	0.525	0.764

5.2 Goodness of fit

In this part, real-world examples are used to demonstrate the new model's capabilities and flexibility compared to some other existing lifespan models. Bjerkedal et al. (1960) reported the survival periods in days of 72 guinea pigs infected with virulent tubercle bacilli in the first data set. The second data set contains the data from Fuller Jr et al. (1994) on the strength of airplane glass.

Table 2: Data set for 72 guinea pigs afflicted with virulent tubercle bacilli (survival in days).

10, 33, 44, 56, 59, 72, 74, 77, 92, 93, 96, 100, 100, 102, 105, 107,107, 108, 108, 108, 109, 112, 113, 115, 116, 120, 121, 122, 122, 124, 130, 134, 136, 139, 144, 146, 153, 159, 160, 163, 163, 168, 171, 172, 176, 183, 195, 196, 197, 202, 213, 215, 216, 222, 230, 231,240, 245, 251, 253, 254, 254, 278, 293, 327, 342, 347, 361, 402, 432, 458, 555

Table 3: data set of the strength of airplane glass.

18.83, 20.8, 21.657, 23.03, 23.23, 24.05, 24.321, 25.5, 27.67, 29.9, 31.11, 33.2, 33.73, 33.76, 33.89, 34.76, 35.75, 44.045, 45.29, 45.381,25.52,25.8, 26.69,26.77, 26.78,27.05, 35.91, 36.98, 37.08,37.09, 39.59

Table 4: Descriptive statistics for First data set.

Min	1st Qu.	Median	Mean	3rd Qu	Max	Skewness	Kurtosis
10.0	108.0	149.5	176.8	224.0	555.0	1.342	4.991

Table 5: Descriptive statistics for Second data set.

Min	1st Qu.	Median	Mean	3rd Qu	Max	Skewness	Kurtosis
18.83	25.51	29.90	30.81	35.83	45.38	0.405	2.286

The descriptive statistics for all the data sets are shown in Tables 4, and 5. The MOEP distribution and the Pranav, quasi-Lindley, and new weighted Lindley distributions were fitted to the given datasets. The other existing models' PDF and CDF functions are provided below.

Quasi Lindley distribution by Shanker and Mishra (2013)

$$f(x,\theta,\beta) = \frac{\theta(\beta + x\theta)}{\beta + 1} e^{-\theta x}, \quad \theta, \beta > 0$$

$$F(x) = 1 - \frac{1 + \beta + \theta x}{\beta + 1} e^{-\theta x}, \quad \theta, \beta > 0$$

A new weighted Lindley distribution by Asgharzadeh et al. (2016)

$$f(x) = \frac{\theta^2(1+\beta)^2(1+x)}{\beta\theta(1+\beta) + \beta(2+\beta)} \left(1 - e^{-\theta\beta x}\right) e^{-\theta x}, \qquad \theta, \beta > 0$$

$$F(x) = 1 - \frac{e^{-\theta x}\left[(1+\beta)^2(1+\theta+\theta x) - [\theta(1+\beta)(1+x)+1]e^{-\beta\theta x}\right]}{\beta\theta(1+\beta) + \beta(2+\beta)}, \qquad \theta, \beta > 0$$

We use the log-likelihood function (2L), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC), Hanna-Quinn Information Criterion (HQIC), and Kolmogorov-Smirnov (K-S) quality of fit measurements to compare our model to other existing models.

The estimated parameters and goodness of fit of two real data sets are shown in Tables 6 - 9. The AIC, BIC, CAIC, HQIC, K-S, and p-value of the newly constructed MOEP distribution are lower than those of the Pranav, QLD, and NWL distributions and are thus regarded the best-fit model, as displayed in the tables 6-9. Figures 3, and 4 depict the data histogram (right side) with the calculated pdf curves and represent the estimated and empirical CDF curves (left side). The MOEP distribution, when compared to other existing models, shows a more suitable fit, as seen in the figures and tables. All findings are conducted using the R language.

 DOI: http://dx.doi.org/10.18187/pjsor.v19i1.4058

Table 6: The parameters estimate, K-S, and p-values of the fitted model using First data.

Model	Parameters Estimates (Std.Error)		K-S	P-value
	$\hat{\theta}$	$\hat{\beta}$		
MOP($\boldsymbol{\theta}, \boldsymbol{\beta}$)	0.016348(0.004094)	0.32051(0.25264)	0.069875	0.8735
Pranav($\boldsymbol{\theta}, \boldsymbol{\beta}$)	0.02264(0.001331)	-	0.10706	0.3812
QLD($\boldsymbol{\theta}, \boldsymbol{\beta}$)	0.01133814(0.00126)	0.00001(0.19144)	0.16899	0.03274
NWL($\boldsymbol{\theta}, \boldsymbol{\beta}$)	0.016865537(NaN)	0.008430701(NaN)	0.096919	0.5082

Table 7: Statistics for (First Data set).

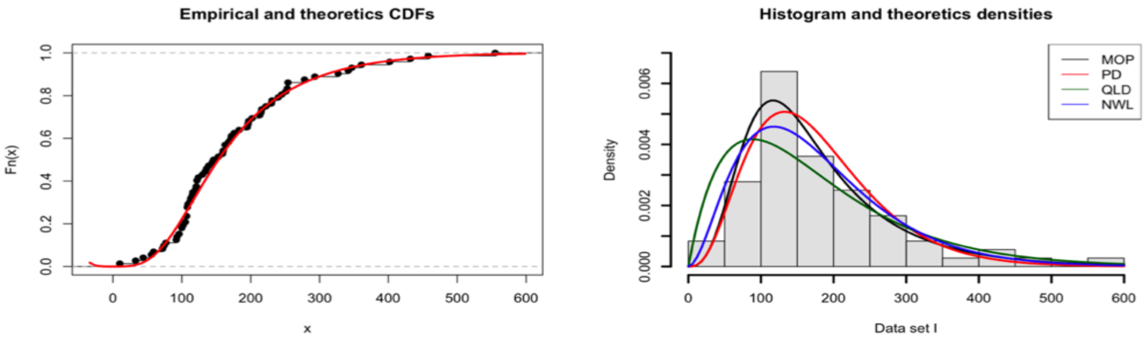
Model	-2L	AIC	BIC	CAIC	HQIC
MOP($\boldsymbol{\theta}, \boldsymbol{\beta}$)	851.4876	855.4876	860.0409	855.6615	857.3003
Pranav($\boldsymbol{\theta}, \boldsymbol{\beta}$)	854.5282	856.5282	858.8049	856.5853	857.4345
QLD($\boldsymbol{\theta}, \boldsymbol{\beta}$)	858.194	862.194	866.7473	862.3679	864.0067
NWL($\boldsymbol{\theta}, \boldsymbol{\beta}$)	851.5944	855.5944	860.1477	855.7683	857.4071

Table 8: The parameters estimate, K-S, and p-values of the fitted model of the second data set

Model	Parameters Estimates (Std.Error)		K-S	P-value
	$\hat{\theta}$	$\hat{\beta}$		
MOP($\boldsymbol{\theta}, \boldsymbol{\beta}$)	0.31537(0.0394)	73.52025(68.116)	0.14351	0.5007
Pranav($\boldsymbol{\theta}, \boldsymbol{\beta}$)	0.12982(0.01166)	-	0.25372	0.03021
QLD($\boldsymbol{\theta}, \boldsymbol{\beta}$)	0.06491528(0.007534971)	0.00001000(NaN)	0.35861	0.0004476
NWL($\boldsymbol{\theta}, \boldsymbol{\beta}$)	0.09588328(0.02122129)	0.00001000(0.39111114)	0.30193	0.005285

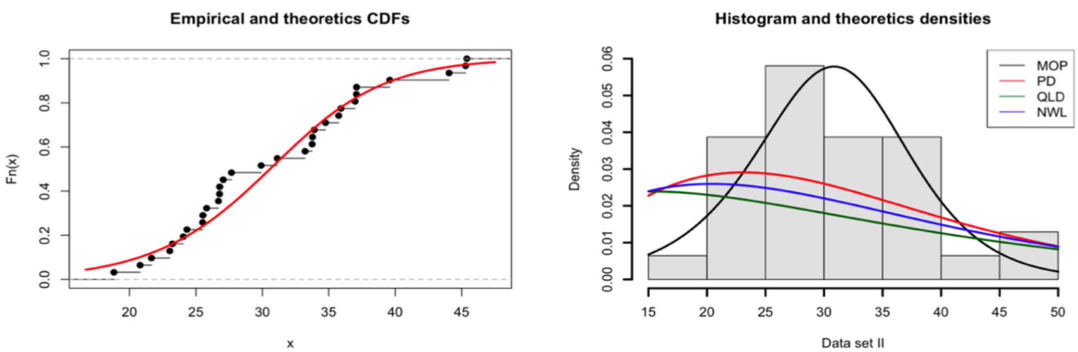
Table 9 : Statistics for (Second data set).

Model	-2L	AIC	BIC	CAIC	HQIC
MOP($\boldsymbol{\theta}, \boldsymbol{\beta}$)	212.1008	216.1008	218.9688	216.5294	217.0357
Pranav($\boldsymbol{\theta}, \boldsymbol{\beta}$)	232.7762	234.7762	236.2102	234.9141	235.2436
QLD($\boldsymbol{\theta}, \boldsymbol{\beta}$)	252.2316	256.2316	259.0996	256.6602	257.1665
NWL($\boldsymbol{\theta}, \boldsymbol{\beta}$)	241.3072	245.3072	248.17522	245.7358	246.2421



(a) (b)

Figure 3: The fitted densities (a) and estimated CDF (b) for the first data using R software.



(a) (b)

Figure 4 : The fitted densities (a) and estimated CDF(b) for the second data using R software .

6. Conclusions

The Marshall-Olkin technique is used in this research to create adaptable distributions for two parameters Marshall-Olkin Pranav. It has been possible to deduce some of its helpful statistical characteristics. The suggested MOEP distribution's flexibility behavior was investigated using simulation studies. The maximum likelihood technique was used to estimate the suggested MOEP distribution parameters, and its numerical applications were examined using two real data sets. The proposed MOEP distribution has superior goodness of fit for the two datasets studied than the Pranav, QLD, and NWL distributions. As a result, in addition to Pranav, QLD, and NWL, the MOP distribution might be utilized to represent real-life circumstances. Additional studies can examine other statistical properties of the suggested model that were not addressed in this study.

References

1. Ahmad, Z. (2020). The zubair-G family of distributions: properties and applications. Annals of Data Science, 7(2):195–208. https://doi.org/10.1007/s40745-018-0169-9
2. Asgharzadeh, A., Bakouch, H. S., Nadarajah, S., & Sharafi, F. (2016). A new weighted lindley distribution with application. Brazilian Journal of Probabil- ity and Statistics, 30(1):1–27. DOI: 10.1214/14-BJPS253
3. Bjerkedal, T. et al. (1960). Acquisition of resistance in guinea pies infected with different doses of virulent tubercle bacilli. American Journal of Hygiene, 72(1):130–48. DOI: 10.1093/oxfordjournals.aje.a120129
4. Cordeiro, G.M.,& Lemonte, A.J.,(2013). On the Marshall-Olkin extended Weibull distribution. *Statistical papers*. 54, 333-353. DOIhttps://doi.org/10.1007/s00362-012-0431-8
5. Chipepa, F., Moakofi, T., & Oluyede, B. (2022). The Marshall-Olkin-Odd Power Generalized Weibull-G Family of Distributions with Applications of COVID-19 Data. *Journal of Probability and Statistical Science*, *20*(1), 1-20. DOI: https://doi.org/10.37119/jpss2022.v20i1.509
6. Eugene, N., Lee, C., & Famoye, F. (2002). Beta-normal distribution and its applications. Communications in Statistics-Theory and methods, 31(4):497– 512. https://doi.org/10.1081/STA-120003130
7. Fuller Jr, E. R., Freiman, S. W., Quinn, J. B., Quinn, G. D., & Carter, W. C. (1994). Fracture mechanics approach to the design of glass aircraft windows: A case study. In Window and dome technologies and materials IV, volume 2286, pages 419–430. International Society for Optics and Photonics. https://doi.org/10.1117/12.187363
8. Granzotto, D., Louzada, F., & Balakrishnan, N. (2017). Cubic rank trans- muted distributions: inferential issues and applications. Journal of statistical Computation and Simulation, 87(14):2760–2778. https://doi.org/10.1080/00949655.2017.1344239
9. Javed, M., Nawaz, T., & Irfan, M. (2019). The Marshall-Olkin kappa distribution: properties and applications. *Journal of King Saud University-Science*, *31*(4), 684-691. https://doi.org/10.1016/j.jksus.2018.01.001
10. KK, S. (2018). Pranav distribution with properties and its applications. Biom Biostat Int J, 7(3):244–254. DOI: 10.15406/bbij.2018.07.00215
11. Kumaraswamy, P. (1980). A generalized probability density function for double- bounded random processes. Journal of hydrology, 46(1-2):79–88. https://doi.org/10.1016/0022-1694(80)90036-0
12. Mahdavi, A. & Kundu, D. (2017). A new method for generating distributions with an application to exponential distribution. Communications in Statistics-Theory and Methods, 46(13):6543–6557. https://doi.org/10.1080/03610926.2015.1130839
13. Marshall, A. W. & Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and weibull families. Biometrika, 84(3):641–652. https://doi.org/10.1093/biomet/84.3.641
14. Moakofi, T., Oluyede, B., & Makubate, B. (2021). Marshall-Olkin Lindley-Log-logistic distribution: Model, properties and applications. *Mathematica Slovaca*, *71*(5), 1269-1290. https://doi.org/10.1515/ms-2021-0052
15. Mudholkar, G. S. & Srivastava, D. K. (1993). Exponentiated weibull family for analyzing bathtub failure-rate data. IEEE transactions on reliability, 42(2):299–302. **DOI:** 10.1109/24.229504
16. Rahman, M. M., Al-Zahrani, B., & Shahbaz, M. Q. (2018). A general trans- muted family of distributions. Pakistan Journal of Statistics and Operation Research, pages 451–469. https://doi.org/10.18187/pjsor.v14i2.2334
17. Shaked, M. & Shanthikumar, J. G. (2007). Stochastic orders. New York, Ny: Springer New York. https://doi.org/10.1007/978-0-387-34675-5_1
18. Shanker, R. and Mishra, A. (2013). A quasi Lindley distribution. African Journal of Mathematics and Computer Science Research, 6(4), 64-71. DOI 10.5897/AJMCSR 12.067
19. Shaw, W. T. & Buckley, I. R. (2009). The alchemy of probability distributions: beyond gram-charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. arXiv preprint arXiv:0901.0434. https://doi.org/10.48550/arXiv.0901.0434