Some Characterization Results Based on Conditional Expectation of Function of Dual Generalized Order Statistics

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Abstract

Two families of probability distributions are characterized through the conditional expectations of dual generalized order statistics ($dgos$) and spacing of two $dgos$ conditioned on a non-adjacent dual generalized order statistics. Further, some of its deductions are also discussed.

Keywords: Characterization, Conditional expectation, Dual generalized order statistics, Probability distributions.

1. Introduction

The concept of generalized order statistics (gos) has been introduced as a unified approach to a variety of models of ordered random variables with different interpretation (Kamps, 1995), such as ordinary order statistics, sequential order statistics, progressive type II censoring, record values and Pfeifer's records. Generalized order statistics serve as a common approach to a structural similarities and analogies. Several of these models can be effectively applied, e.g., in reliability theory. Using this concept of gos, Burkschat *et al*. (2003) introduced the concept of dual generalized order statistics ($dgos$) that enables a common approach to descending ordered random variables like reverse ordered order statistics, lower record values etc.

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed (iid) random variable with absolutely continuous distribution function (df) $F(x)$ and the probability density function (pdf) $f(x)$, $x \in (\alpha, \beta)$. Further, let $n \in N$, $n \geq 2$, $k \geq 1$, $\tilde{m} = (m_1, m_2, \cdots, m_{n-1}) \in \Re^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 1$, for all $r \in \{1, 2, \ldots, n-1\}$. Then, $X'(r, n, \tilde{m}, k)$, $r = 1, 2, \ldots, n$ are called $dgos$ if their joint pdf is given by

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[F(x_i)]^{m_i} f(x_i)\right)[F(x_n)]^{k-1} f(x_n) \qquad (1.1)$$

for $F^{-1}(1) \geq x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0)$.

Here we will assume two cases:

Case I: $m_1 = m_2 = \cdots = m_{n-1} = m$.

The *pdf* of the r^{th} *dgos* is given by (Burkschat *et al*., 2003)

$$f_{X'(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1} f(x)[g_m(F(x))]^{r-1} \tag{1.2}$$

The joint *pdf* of the r^{th} and s^{th} *dgos* is

$$f_{X'(r,n,m,k),X'(s,n,m,k)}(x,y) = \frac{c_{s-1}}{(r-1)!(s-r-1)!}[F(x)]^m f(x) g_m^{r-1}(F(x))$$
$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1} f(y), \qquad \alpha \le y < x \le \beta, \tag{1.3}$$

where

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1} x^{m+1} & , m \ne -1 \\ -\log x & , m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1), \quad x \in [0,1) .$$

The conditional *pdf* of $X'(s,n,m,k)$ given $X'(r,n,m,k) = x,\ 1 \le r < s \le n$

$$f_{s|r}(y \mid x) = \frac{c_{s-1}}{(s-r-1)! c_{r-1}}$$
$$\times \frac{[(F(x))^{m+1} - (F(y))^{m+1}]^{s-r-1}[F(y)]^{\gamma_s - 1}}{(m+1)^{s-r-1}[F(x)]^{\gamma_{r+1}}} f(y) \tag{1.4}$$

Case II: $\gamma_i \ne \gamma_j \; i \ne j, i, j = 1, \ldots, n-1$.

The *pdf* of the r^{th} *dgos* is given by (Burkschat *et al.*, 2003)

$$f_{X'(r,n,\tilde{m},k)}(x) = c_{r-1} \; f(x) \sum_{i=1}^{r} a_i(r) [F(x)]^{\gamma_i - 1} \tag{1.5}$$

and the joint *pdf* of the r^{th} and s^{th} *dgos* is

$$f_{X'(r,n,\tilde{m},k),X'(s,n,\tilde{m},k)}(x,y) = c_{s-1} \sum_{j=r+1}^{s} a_j^{(r)}(s) \left[\frac{F(y)}{F(x)}\right]^{\gamma_j}$$
$$\times \left(\sum_{i=1}^{r} a_i(r) [F(x)]^{\gamma_i} \right) \frac{f(x)}{F(x)} \frac{f(y)}{F(y)} \tag{1.6}$$

where

$$a_i(r) = \prod_{\substack{j=1 \\ j \ne i}}^{r} \frac{1}{(\gamma_j - \gamma_i)} \quad , \qquad \gamma_j \ne \gamma_i, \quad 1 \le i \le r \le n \tag{1.7}$$

and

$$a_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j \ne i}}^{s} \frac{1}{(\gamma_j - \gamma_i)} \quad , \qquad \gamma_j \ne \gamma_i, \quad r+1 \le i \le s \le n \tag{1.8}$$

Thus, the conditional *pdf* of $X'(s,n,\tilde{m},k)$ given $X'(r,n,\tilde{m},k)=x$, $1 \le r < s \le n$ is

$$f_{s|r}(y \mid x)=\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{F(y)}{F(x)}\right)^{\gamma_i-1}\frac{f(y)}{F(x)}, \qquad x>y \tag{1.9}$$

If $m=0$, $k=1$, then $X'(r,n,m,k)$ reduces to $X_{n-r+1:n}$, the $(n-r+1)^{th}$ order statistics, from a sample of size n and when $m=-1$, then $X'(r,n,m,k)$ reduces to the r^{th} $k-$ lower record value (Pawlas and Szynal, 2001). A number of results on characterization of distributions of dual generalized order statistics are available in the literature. For a detailed survey one may refer to Ahsanullah (2004), Mbah and Ahsanullah (2007), Khan *et al.* (2009) and Khan *et al.* (2010) and references contained therein. In this paper, two general classes of distributions

$$F(x)=e^{-a\,h(x)},\ a \ne 0,\ x \in (\alpha,\beta) \tag{1.10}$$

and

$$F(x)=[a\,h(x)+b]^{c},\quad x \in (\alpha,\beta) \tag{1.11}$$

have been characterized through conditional expectation of function of *dgos* ,where a,b,c and $h(x)$ are so chosen that $F(x)$ in (1.10) and (1.11) are *df* .

It may be noted that the *df* $F(x)$ and the *pdf* $f(x)$ in (1.10) and (1.11) are related respectively as

$$F(x)=-\frac{f(x)}{a\,h'(x)} \tag{1.12}$$

and

$$\frac{f(x)}{F(x)}=\frac{ach'(x)}{[ah(x)+b]} \tag{1.13}$$

Khan *et al*. (2010) have characterized the general class of distributions through conditional expectation of *dgos* conditioned on non-adjacent *dgos* .We have extended the result of Khan *et al*. (2010) for the difference of the conditional expectations conditioned on non-adjacent *dgos* , its particular cases for order statistics, lower record statistics as obtained by Khan *et al.* (2011) and Faizan and Khan (2011) are discussed.

2. Characterizations of distributions when

$\gamma_i \ne \gamma_j\ i \ne j, i, j = 1,\ldots,n-1$.

Theorem 2.1: Let X be an absolutely continuous random variable with the *df* $F(x)$ and the *pdf* $f(x)$ in the interval (α,β), where α and β may be finite or infinite, then for $1 \le r < s < t \le n$,

$$E[h\{X'(s,n,\tilde{m},k)\}-h\{X'(t,n,\tilde{m},k)\} \mid X'(l,n,\tilde{m},k)=x]$$

$$=-\frac{1}{a}\sum_{j=t+1}^{s}\frac{1}{\gamma_j},\quad l=r,r+1 \tag{2.1}$$

if and only if

$$F(x) = e^{-a\,h(x)},\ a \neq 0 \tag{2.2}$$

where $h(x)$ is a monotonic and differentiable function of x such that $h(x) \to 0$ as $x \to \beta$ and $h(x)F(x) \to 0$ as $x \to \alpha$.

Proof: First we shall prove that (2.2) implies (2.1). It can be seen (Athar *et al.*, 2008) that

$$E[h\{X'(t,n,\tilde{m},k)\} - h\{X'(t-1,n,\tilde{m},k)\} \mid X'(r,n,\tilde{m},k) = x]$$
$$= \frac{c_{t-2}}{c_{r-1}} \sum_{i=r+1}^{t} a_i^{(r)}(t) \int_\alpha^x h'(y) \left(\frac{F(y)}{F(x)}\right)^{\gamma_i} dy$$

Therefore, for $1 \le r < s < t \le n$

$$E[h\{X'(s,n,\tilde{m},k)\} - h\{X'(t,n,\tilde{m},k)\} \mid X'(r,n,\tilde{m},k) = x]$$
$$= \sum_{i=0}^{s-t-1} E[h\{X'(s-i,n,\tilde{m},k)\} - h\{X'(s-i-1,n,\tilde{m},k)\} \mid X'(r,n,\tilde{m},k) = x]$$
$$= \sum_{j=t+1}^{s} E[h\{X'(j,n,m,k)\} - h\{X'(j-1,n,m,k)\} \mid X'(r,n,m,k) = x]$$
$$= \sum_{j=t+1}^{s} \frac{c_{j-2}}{c_{r-1}} \sum_{i=r+1}^{j} a_i^{(r)}(j) \int_\alpha^x h'(y) \left(\frac{F(y)}{F(x)}\right)^{\gamma_i} dy$$
$$= -\frac{1}{a} \sum_{j=t+1}^{s} \frac{1}{\gamma_j}, \text{ in view of (1.9) and (1.12)}$$

This proves the necessary part.

For the sufficiency part, we have at $c = \frac{1}{a} \sum_{j=t+1}^{s} \frac{1}{\gamma_j}$

$$E[h\{X'(t,n,\tilde{m},k)\} - h\{X'(s,n,\tilde{m},k)\} \mid X'(r,n,\tilde{m},k) = x] = c$$

$$\frac{c_{t-1}}{c_{r-1}} \sum_{i=r+1}^{t} a_i^{(r)}(t) \int_\alpha^x h(y) \left(\frac{F(y)}{F(x)}\right)^{\gamma_i} \frac{f(y)}{F(y)}\, dy - \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \int_\alpha^x h(y) \left(\frac{F(y)}{F(x)}\right)^{\gamma_i} \frac{f(y)}{F(y)}\, dy = c \tag{2.3}$$

Differentiating (2.3) $w.r.t.x$ we have

$$\frac{f(x)}{F(x)} \frac{c_{t-1}}{c_{r-1}} \sum_{i=r+1}^{t} a_i^{(r)}(t)\, h(x) - \frac{f(x)}{F(x)} \frac{c_{t-1}}{c_{r-1}} \sum_{i=r+1}^{t} \gamma_i a_i^{(r)}(t) \times \int_\alpha^x \frac{h(y)[F(y)]^{\gamma_i - 1} f(y)}{[F(x)]^{\gamma_i}}\, dy - \frac{f(x)}{F(x)} \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s)\, h(x)$$
$$+ \frac{f(x)}{F(x)} \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} \gamma_i a_i^{(r)}(s) \int_\alpha^x \frac{h(y)[F(y)]^{\gamma_i - 1} f(y)}{[F(x)]^{\gamma_i}}\, dy = 0 \tag{2.4}$$

Rearranging the terms in (2.4) and noting that $\sum_{i=r+1}^{s} a_i^{(r)}(s) = 0$, $c_r = \gamma_{r+1} c_{r-1}$ and $a_i^{(r+1)}(t) = (\gamma_{r+1} - \gamma_i) a_i^{(r)}(t)$, we get

$$[g_{t|r}(x) - g_{t|r+1}(x)] - [g_{s|r}(x) - g_{s|r+1}(x)] = 0$$

where

$$g_{s|r}(x) = E[h\{X'(s,n,\tilde{m},k)\} \mid X'(r,n,\tilde{m},k) = x]$$

or,

$$g_{t|r}(x) - g_{s|r}(x) = g_{t|r+1}(x) - g_{s|r+1}(x) = \cdots = g_{t|s}(x) - g_{s|s}(x) = c \tag{2.5}$$

Noting that $g_{s|s}(x) = h(x)$, we have

$$g_{t|s}(x) = h(x) + c$$

i.e. $$E[h\{X'(t,n,\tilde{m},k)\} \mid X'(s,n,\tilde{m},k) = x] = h(x) + \frac{1}{a} \sum_{j=s+1}^{t} \frac{1}{\gamma_j} \tag{2.6}$$

Using the result (Khan *et al.*, 2006)

$$E[h\{X'(t,n,\tilde{m},k)\} \mid X'(s,n,\tilde{m},k) = x] = g_{t|s}(x),$$

implies

$$F(x) = e^{-\int_x^{\beta} A(u)\,du} \tag{2.7}$$

where

$$A(u) = \frac{g'_{t|s}(u)}{\gamma_{s+1}[g_{t|s+1}(u) - g_{t|s}(u)]} = -a\,h'(u) \tag{2.8}$$

we get,

$$F(x) = e^{-a\,h(x)}, a \neq 0$$

and hence the Theorem.

Remark 2.1: At $s = r$, $E[h\{X'(t,n,\tilde{m},k)\} \mid X'(s,n,\tilde{m},k) = x] = \frac{1}{a} \sum_{j=s+1}^{t} \frac{1}{\gamma_j} + h(x)$ as obtained by Khan *et al*. (2010).

Remark 2.2: At $m = 0, k = 1$, we will get the following result for order statistics for $1 \le r < s < t \le n$

$$E[h\{(X'_{s:n})\} - h\{(X'_{t:n})\} \mid X'_{r:n} = x] = -\frac{1}{a} \sum_{j=t+1}^{s} \frac{1}{n-j+1}$$

$$E[h\{(X_{s:n})\} - h\{(X_{r:n})\} \mid X_{t:n} = y] = -\frac{1}{a} \sum_{j=r}^{s-1} \frac{1}{j}$$

as obtained by Khan *et al.* (2011).

Remark 2.3: At $m = -1$, $k = 1$ and $c = \frac{1}{a}$ it will give result for lower record statistics as obtained by Faizan and Khan (2011).

Corollary 2.1: Under the conditions as given in Theorem 2.1 and for $1 \le r < s < t \le n$

$$E[h\{X'(s,n,\tilde{m},k)\} - h\{X'(r,n,\tilde{m},k)\}] + h(x) = E[h\{X'(s,n,\tilde{m},k)\} \mid X'(r,n,\tilde{m},k) = x] \quad (2.9)$$

if and only if

$$F(x) = e^{-a\,h(x)},\ a \ne 0 \quad (2.10)$$

Proof: The proof from Theorem 2.1 and Remark 2.1.

Further, putting $m = 0, k = 1$ in equation (2.9), we will get the result for order statistics as follows,

$$E[h\{(X_{r:n})\} - h\{(X_{s:n})\}] + h(x) = E[h\{(X_{r:n})\} \mid X_{s:n} = x]$$

Theorem 2.2: Let X be an absolutely continuous random variable with the *df* $F(x)$ and the *pdf* $f(x)$ in the interval (α, β), where α and β may be finite or infinite, then for $1 \le r < s < t \le n$,

$$E[h\{X'(t,n,\tilde{m},k)\} \mid X'(l,n,\tilde{m},k) = x] = a^*_{t|s}\, E[h\{X'(s,n,\tilde{m},k)\} \mid X'(l,n,\tilde{m},k) = x] + b^*_{t|s} \quad ,l = r,\ r+1 \quad (2.11)$$

if and only if

$$F(x) = [a\,h(x) + b]^c \quad (2.12)$$

where

$$a^*_{t|s} = \prod_{j=s+1}^{t} \frac{c\gamma_j}{1 + c\,\gamma_j} \text{ and } b^*_{t|s} = -\frac{b}{a}(1 - a^*_{t|s})$$

Proof: First we shall prove that (2.12) implies (2.11). In view of Khan *et al.* (2010), we have

$$E[h\{X'(t,n,\tilde{m},k)\} \mid X'(r,n,\tilde{m},k) = x] = a^*_{t|r}\ h(x) + b^*_{t|r} = a^*_{t|r}\left(h(x) + \frac{b}{a}\right) - \frac{b}{a} \quad (2.13)$$

where

$$a^*_{t|r} = \prod_{j=r+1}^{t} \frac{c\gamma_j}{1 + c\,\gamma_j} = a^*_{s|r}\ a^*_{t|s}$$

and

$$b^*_{t|r} = -\frac{b}{a}(1 - a^*_{t|r})$$

$$E[h\{X'(t,n,\tilde{m},k)\} \mid X'(r,n,\tilde{m},k) = x] = a^*_{t|s} a^*_{s|r}\left(h(x) + \frac{b}{a}\right) - \frac{b}{a}$$

$$= a^*_{t|s} a^*_{s|r}\left[\left(h(x) + \frac{b}{a}\right) - \frac{b}{a}\right] + \frac{b}{a} a^*_{t|s} - \frac{b}{a}$$

$$= a^*_{t|s} E[h\{X'(s,n,\tilde{m},k)\} \mid X'(r,n,\tilde{m},k) = x] + b^*_{t|s}$$

This proves the necessary part.

For the sufficiency part, we have

$$\frac{c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t} a_i^{(r)}(t)\int_{\alpha}^{x} h(y)\left[\frac{F(y)}{F(x)}\right]^{\gamma_i}\frac{f(y)}{F(y)}\,dy$$
$$= a_{t|s}^{*}\,\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s} a_i^{(r)}(s)\int_{\alpha}^{x} h(y)\left[\frac{F(y)}{F(x)}\right]^{\gamma_i}\frac{f(y)}{F(y)}\,dy + b_{t|s}^{*} \qquad (2.14)$$

Differentiating (2.14) $w.r.t.\ x$ and rearranging, we get

$$\left[\frac{c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t} a_i^{(r)}(t)\,h(x) - \frac{c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t}\gamma_i a_i^{(r)}(t)\int_{\alpha}^{x}\frac{h(y)[F(y)]^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}\,dy\right]$$
$$= a_{t|s}^{*}\left[\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s} a_i^{(r)}(s)\,h(x) - \frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}\gamma_i a_i^{(r)}(s)\right.$$
$$\left.\times\int_{\alpha}^{x}\frac{h(y)[F(y)]^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}\,dy\right]$$

After noting that $\sum_{i=r+1}^{s} a_i^{(r)}(s)=0$, $c_r=\gamma_{r+1}\,c_{r-1}$ and $a_i^{(r+1)}(t)=(\gamma_{r+1}-\gamma_i)\,a_i^{(r)}(t)$, we get

$$\frac{\gamma_{r+1}\,c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t} a_i^{(r)}(t)\int_{\alpha}^{x}\frac{h(y)[F(y)]^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}\,dy$$
$$-\ \frac{\gamma_{r+1}\,c_{t-1}}{c_r}\sum_{i=r+2}^{t} a_i^{(r+1)}(t)\int_{\alpha}^{x}\frac{h(y)[F(y)]^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}\,dy$$
$$= a_{t|s}^{*}\left[\frac{\gamma_{r+1}\,c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s} a_i^{(r)}(s)\int_{\alpha}^{x}\frac{h(y)[F(y)]^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}\,dy\right.$$
$$\left.-\ \frac{\gamma_{r+1}\,c_{s-1}}{c_r}\sum_{i=r+2}^{s} a_i^{(r+1)}(s)\int_{\alpha}^{x}\frac{h(y)[F(y)]^{\gamma_i-1}f(y)}{[F(x)]^{\gamma_i}}\,dy\right]$$

That is,

$$\gamma_{r+1}[g_{t|r}(x)-g_{t|r+1}(x)] \qquad = a_{t|s}^{*}\,\gamma_{r+1}[g_{s|r}(x)-g_{s|r+1}(x)]$$

where

$$g_{s|r}(x)=E[h\{X'(s,n,\tilde{m},k)\}\mid X'(r,n,\tilde{m},k)=x]$$

or,

$$g_{t|r}(x)-a_{t|s}^{*}\,g_{s|r}(x)=g_{t|r+1}(x)-a_{t|s}^{*}\,g_{s|r+1}(x)$$
$$=\cdots=g_{t|s}(x)-a_{t|s}^{*}\,g_{s|s}(x)=b_{t|s}^{*} \qquad (2.15)$$

Noting that $g_{s|s}(x)=h(x)$, we have

$$g_{t|s}(x)=a_{t|s}^{*}\,h(x)+b_{t|s}^{*}$$

i.e.

$$E[h\{X'(t,n,\tilde{m},k)\}\mid X'(s,n,\tilde{m},k)=x]=a_{t|s}^{*}\,h(x)+b_{t|s}^{*} \qquad (2.16)$$

Using the result (Khan *et al.*, 2006)

$$E[h\{X'(t,n,\tilde{m},k)\} \mid X'(s,n,\tilde{m},k)=x]=g_{t|s}(x),$$

implies

$$F(x)=e^{-\int_x^{\beta} A(u)\,du} \tag{2.17}$$

We get

$$F(x)=[a\,h(x)+b]^c$$

and hence the Theorem.

Remark 2.4: It may be seen that when $\gamma_i \neq \gamma_j$ but $m_i = m_j = m$, then

$$a_i^{(r)}(t)=\frac{1}{(m+1)^{t-r-1}}(-1)^{t-i}\frac{1}{(i-r-1)!(t-i)!}$$

$$a_i(r)=\frac{1}{(m+1)^{r-1}}(-1)^{r-i}\frac{1}{(i-1)!\,(r-i)!}$$

and consequently (1.5) will reduce to (1.2), (1.6) to (1.3).

Remark 2.5: At $m=0,\ k=1$, we will get result for order statistics as follows,

$$E[h\{(X'_{t:n})\} \mid (X'_{r:n})=x]$$
$$=a^*_{t|s}\,E[h\{(X'_{s:n})\} \mid (X'_{r:n})=x]+b^*_{t|s}$$

or

$$E[h\{(X_{r:n})\} \mid (X_{t:n})=y]=a^*_{r|s}\,E[h\{(X_{s:n})\} \mid (X_{t:n})=y]+b^*_{r|s}$$

where

$$a^*_{r|s}=\prod_{j=r}^{s-1}\frac{c\,j}{1+c\,j} \quad \text{and} \quad b^*_{r|s}=-\frac{b}{a}(1-a^*_{r|s})$$

as obtained by Khan *et al.*(2011).

Remark 2.6: At $s=r$, it reduces to as obtained by Khan *et al.* (2010).

Remark 2.7: At $a=-\frac{a}{c}$, $b=1$ and $c\to\infty$ then $F(x)=[a\,h(x)+b]^c \to e^{-a\,h(x)}$ as obtained in Theorem 2.1.

3. Conclusion

In this paper, conditional expectation of the difference of two *dgos* conditioned on non-adjacent *dgos* are considered to characterize the *df* $F(x)=e^{-ah(x)}$ whose particular cases are given in the Table 2.1 with proper choice of $a, b,\ c$ and $h(x)$. Also $F(x)=[ah(x)+b]^c$ is characterized through the conditional expectation of *dgos* conditioned on non-adjacent *dgos* as given in Table 2.2. Further, its various deductions are discussed.

Table 2.1: Examples based on the distribution function $F(x)=e^{-a\,h(x)}$

Distribution	$F(x)$	a	$h(x)$
Inverse Weibull	$e^{-\theta x^{-p}}$ $0<x<\infty$	θ	x^{-p}
Power function	$\left(\frac{x}{a}\right)^{p}$ $0<x<a$	$-p$	$\log(x/a)$
Logistic	$(1+e^{-x})^{-1}$ $-\infty<x<\infty$	1	$\log(1+e^{-x})$
Burr Type II	$(1+e^{-x})^{-k}$ $-\infty<x<\infty$	k	$\log(1+e^{-x})$
Burr Type III	$(1+x^{-c})^{-k}$ $0<x<\infty$	k	$\log(1+x^{-c})$
Burr Type IV	$\left[1+\left(\frac{c-x}{x}\right)^{1/c}\right]^{-k}$ $0<x<c$	k	$\log\left[1+\left(\frac{c-x}{x}\right)^{1/c}\right]$
Burr Type V	$(1+ce^{-\tan x})^{-k}$ $-\frac{\pi}{2}<x<\frac{\pi}{2}$	k	$\log(1+ce^{-\tan x})$
Burr Type VI	$(1+ce^{-k\sinh x})^{-k}$ $-\infty<x<\infty$	k	$\log(1+ce^{-k\sinh x})$
Burr Type VII	$\left(\frac{1+\tanh x}{2}\right)^{k}$ $-\infty<x<\infty$	$-k$	$\log\left(\frac{1+\tanh x}{2}\right)$
Burr Type VIII	$\left(\frac{2}{\pi}\tan^{-1}e^{x}\right)^{k}$ $-\infty<x<\infty$	$-k$	$\log\left(\frac{2}{\pi}\tan^{-1}e^{x}\right)$
Burr Type X	$(1-e^{-x^{2}})^{k}$ $0<x<\infty$	$-k$	$\log(1-e^{-x^{2}})$
Burr Type XI	$(x-\frac{1}{2\pi}\sin 2\pi x)^{k}$ $0<x<1$	$-k$	$\log(x-\frac{1}{2\pi}\sin 2\pi x)$
Gumbel	$\exp[-e^{-x}]$ $-\infty<x<\infty$	1	e^{-x}
Extreme value II	$e^{-\left(\frac{\theta}{x}\right)^{p}}$ $0<x<\infty$	θ^{p}	x^{-p}

Table 2.2: Examples based on the distribution function $F(x) = [a\,h(x) + b]^c$

Distribution	$F(x)$	a	b	c	$h(x)$
Power function	$a^{-p}x^{p}$ $0 < x \le a$	a^{-q}	0	p/q	x^q, $q \ne 0$
Pareto	$1 - a^p x^{-p}$ $a \le x < \infty$	a^p $-a^p$	$1 - a^p$ 1	1 1	$1 - x^{-p}$ x^{-p}
Inverse Weibull	$e^{-\theta x^{-p}}$ $0 \le x < \infty$	$-\theta / c$ 1	1 0	∞ θ / q	x^{-p} $e^{-q x^{-p}}$, $q \ne 0$
Burr type III	$(1 + x^{-c})^{-k}$ $0 \le x < \infty$	1 1	1 0	$-k$ $-k/q$	x^{-c} $(1 + x^{-c})^q$, $q \ne 0$
Cauchy	$\frac{1}{2} + \frac{1}{\pi}\tan^{-1} x$ $-\infty < x < \infty$	$\frac{1}{\pi}$	$\frac{1}{2}$	1	$\tan^{-1} x$

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