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Characterizations of Fourteen (2021-2022) Proposed Discrete Distributions

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Abstract

As we mentioned in our previous works, sometimes in real life cases, it is very difficult to obtain samples from a continuous distribution. The observed values are generally discrete due to the fact that they are not measured in continuum. In some cases, it may be possible to measure the observations via a continuous scale, however, they may be recorded in a manner in which a discrete model seems more suitable. Consequently, the discrete models are appearing quite frequently in applied fields and have attracted the attention of many researchers.
Characterizations of distributions are important to many researchers in the applied fields. An investigator will be vitally interested to know if their model fits the requirements of a particular distribution. To this end, one will depend on the characterizations of this distribution which provide conditions under which the underlying distribution is indeed that particular distribution. Here, we present certain characterizations of 14 recently introduced discrete distributions.

Key Words: Discrete Univaiate Distributions; Characterizations; Hazard Function; Reverse Hazard Function; Conditional Expectations.

1. Introduction

Characterizations of distributions are important to many researchers in the applied fields. An investigator will be vitally interested to know if their model fits the requirements of a particular distribution. To this end, one will depend on the characterizations of this distribution which provide conditions under which the underlying distribution is indeed that particular distribution. Here, we present certain characterizations of 14 recently introduced discrete distributions.
The cumulative distribution (cdf) F, the corresponding probability mass function (pmf) f, the reverse hazard r_F and hazard h_F functions of each of the 14 distributions are listed below in A)-N).

A) The cdf, pmf, reverse hazard and hazard functions of ”Discrete Inverted Topp-Leone” (DITL) distribution of (9), are given, respectively, by

$$F(x;\delta) = 1 - \frac{(3+2x)^{\delta}}{(2+x)^{2\delta}}, \quad x \in \mathbb{N}, \tag{1}$$

$$f\left(x;\delta\right)=\frac{\left(3+2(x-1)\right)^{\delta}}{\left(2+(x-1)\right)^{2\delta}}-\frac{\left(3+2x\right)^{\delta}}{\left(2+x\right)^{2\delta}},\quad x\in\mathbb{N}, \tag{2}$$

$$r_F\left(x\right)=\frac{\frac{\left(3+2(x-1)\right)^{\delta}}{\left(2+(x-1)\right)^{2\delta}}-\frac{\left(3+2x\right)^{\delta}}{\left(2+x\right)^{2\delta}}}{1-\frac{\left(3+2x\right)^{\delta}}{\left(2+x\right)^{2\delta}}},\quad x\in\mathbb{N}, \tag{3}$$

$$h_F\left(x\right)=\frac{\left(1+2x\right)^{\delta}\left(2+x\right)^{2\delta}}{\left(1+x\right)^{2\delta}\left(3+2x\right)^{\delta}}-1,\quad x\in\mathbb{N}, \tag{4}$$

where $\delta>0$ is a parameter and $\mathbb{N}$ is the set of all positive integers.

B) The cdf, pmf, reverse hazard and hazard functions of "Size-Biased Poisson-Gamma Lindley Discrete" (SBPGL) distribution of (14), are given, respectively, by

$$F\left(x;\theta,\beta\right)=C\left(\theta,\beta\right)\sum_{u=0}^{x}P\left(u\right)\left(1+\theta\right)^{-u},\quad x\in\mathbb{N}^*, \tag{5}$$

$$f\left(x;\theta,\beta\right)=C\left(\theta,\beta\right)P\left(x\right)\left(1+\theta\right)^{-x},\quad x\in\mathbb{N}^*, \tag{6}$$

$$r_F\left(x\right)=\frac{P\left(x\right)\left(1+\theta\right)^{-x}}{\sum_{u=0}^{x}P\left(u\right)\left(1+\theta\right)^{-u}},\quad x\in\mathbb{N}^*, \tag{7}$$

$$h_F\left(x\right)=\frac{C\left(\theta,\beta\right)P\left(x\right)\left(1+\theta\right)^{-x}}{1-C\left(\theta,\beta\right)\sum_{u=0}^{x}P\left(u\right)\left(1+\theta\right)^{-u}},\quad x\in\mathbb{N}^*, \tag{8}$$

where $\theta>0,\beta>\frac{\theta}{1+\theta}$ are parameters, $\mathbb{N}^*=\mathbb{N}\cup\{0\}\,, C\left(\theta\right)=\frac{\theta^3(1+\theta)^{-2}}{2\beta(1+\theta)-\theta}$ and $P\left(x\right)=x\left[\left(\beta+\beta\theta-\theta\right)x+\beta\left(1+\theta\right)+1\right]$.

C) The cdf, pmf, reverse hazard and hazard functions of "Discrete Kumaraswamy Marshall-Olkin Exponential" (DKMOE) distribution of (11), are given, respectively, by

$$F\left(x;\alpha,b,p\right)=1-\left\{1-\left[\frac{1-p^{x+1}}{1-\overline{\alpha}p^{x+1}}\right]\right\}^b,\quad x\in\mathbb{N}^*, \tag{9}$$

$$f\left(x;\alpha,b,p\right)=\left\{1-\left[\frac{1-p^{x}}{1-\overline{\alpha}p^{x}}\right]\right\}^b-\left\{1-\left[\frac{1-p^{x+1}}{1-\overline{\alpha}p^{x+1}}\right]\right\}^b,\quad x\in\mathbb{N}^*, \tag{10}$$

$$r_F\left(x\right)=\frac{\left\{1-\left[\frac{1-p^{x}}{1-\overline{\alpha}p^{x}}\right]\right\}^b-\left\{1-\left[\frac{1-p^{x+1}}{1-\overline{\alpha}p^{x+1}}\right]\right\}^b}{1-\left\{1-\left[\frac{1-p^{x+1}}{1-\overline{\alpha}p^{x+1}}\right]\right\}^b},\quad x\in\mathbb{N}^*, \tag{11}$$

$$h_F\left(x\right)=\frac{\left\{1-\left[\frac{1-p^{x}}{1-\overline{\alpha}p^{x}}\right]\right\}^b}{\left\{1-\left[\frac{1-p^{x+1}}{1-\overline{\alpha}p^{x+1}}\right]\right\}^b}-1,\quad x\in\mathbb{N}^*, \tag{12}$$

where $\alpha>0,b>0,p\in(0,1)$ are parameters and $\overline{\alpha}=1-\alpha$.

D) The cdf, pmf, reverse hazard and hazard functions of "Zero-Truncated Poisson Pseudo Lindley Discrete" (ZTPPsLD) distribution of (10), are given, respectively, by

$$F(x;\theta,\beta)=1-\frac{(\theta\beta+\theta+\beta+\theta x)}{(\theta\beta+\beta+\theta)(\theta+1)^{x}},\quad x\in\mathbb{N}, \tag{13}$$

$$f(x;\theta,\beta)=\frac{\theta(\theta\beta+\beta-1+\theta x)}{(\theta\beta+\beta+\theta)(\theta+1)^{x}},\quad x\in\mathbb{N}, \tag{14}$$

$$r_F(x)=\frac{\frac{\theta(\theta\beta+\beta-1+\theta x)}{(\theta\beta+\beta+\theta)(\theta+1)^{x}}}{1-\frac{(\theta\beta+\theta+\beta+\theta x)}{(\theta\beta+\beta+\theta)(\theta+1)^{x}}},\quad x\in\mathbb{N}, \tag{15}$$

$$h_F(x)=\frac{\theta(\theta\beta+\beta-1+\theta x)}{(\theta\beta+\theta+\beta+\theta x)},\quad x\in\mathbb{N}, \tag{16}$$

where $\theta>0,\beta>\frac{1}{\theta+1}$ are parameters.

E) The cdf, pmf, reverse hazard and hazard functions of "Size-Biased Polya-Aeppli Discrete" (SBPAD) distribution of (12), are given, respectively, by

$$F(x;\theta,p)=C(\theta,p)\left\{\sum_{u=1}^{x}P(u)\,p^{u}\right\},\quad x\in\mathbb{N}, \tag{17}$$

$$f(x;\theta,p)=C(\theta,p)\{P(x)\}\,p^{x},\quad x\in\mathbb{N}, \tag{18}$$

$$r_F(x)=\frac{\{P(x)\}\,p^{x}}{\sum_{u=1}^{x}P(u)\,p^{u}},\quad x\in\mathbb{N}, \tag{19}$$

$$h_F(x)=\frac{C(\theta,p)\{P(x)\}\,p^{x}}{1-C(\theta,p)\sum_{u=1}^{x}P(u)\,p^{u}},\quad x\in\mathbb{N}, \tag{20}$$

where $\theta>0, p\in(0,1)$ are parameters, $C(\theta,p)=\frac{pe^{-\theta}}{\theta^{2}}$ and $P(x)=\sum_{j=1}^{x}\binom{x}{j}\left[\left(\frac{\theta(1-p)^{j+1}}{p}\right)\left(\frac{1}{(j-1)!}\right)\right]$.

F) The cdf, pmf, reverse hazard and hazard functions of "Poisson XGamma Discrete" (PXGD) distribution of (4), are given, respectively, by

$$F(x;\theta)=1-\left(\frac{1}{2(1+\theta)^{4}}\right)Q(x)(1+\theta)^{-x},\quad x\in\mathbb{N}^{*}, \tag{21}$$

$$f(x;\theta)=\left(\frac{\theta^{2}}{2(1+\theta)^{4}}\right)P(x)(1+\theta)^{-x},\quad x\in\mathbb{N}^{*}, \tag{22}$$

$$r_F(x)=\frac{\left(\frac{\theta^{2}}{2(1+\theta)^{4}}\right)P(x)(1+\theta)^{-x}}{1-\left(\frac{1}{2(1+\theta)^{4}}\right)Q(x)(1+\theta)^{-x}},\quad x\in\mathbb{N}^{*}, \tag{23}$$

$$h_F(x)=\frac{\theta^{2}P(x)}{Q(x)},\quad \in\mathbb{N}^{*}, \tag{24}$$

where $\theta>0$ is a parameter, $P(x)=2(1+\theta)^{2}+\theta(x+2)(x+1)$ and $Q(x)=x^{2}\theta^{2}+5x\theta^{2}+2x\theta+2\theta^{3}+10\theta^{2}+8\theta+2$.

G) The cdf, pmf, reverse hazard and hazard functions of "Discrete Generalized Lindley" (DGLi) distribution of (7), are given, respectively, by

$$F(x;\alpha,\eta) = 1 - \frac{1}{\alpha(1-\ln\eta)} P(x+1)\,\eta^{x+1}, \quad x \in \mathbb{N}^*, \tag{25}$$

$$f(x;\alpha,\eta) = \frac{1}{(1-\ln\eta)} \left[P(x) - \eta P(x+1)\right]\eta^{x}, \quad x \in \mathbb{N}^*, \tag{26}$$

$$r_F(x) = \frac{(1-\ln\eta)^{-1}\left[P(x) - \eta P(x+1)\right]\eta^{x}}{1 - (\alpha(1-\ln\eta))^{-1} P(x+1)\,\eta^{x+1}}, \quad x \in \mathbb{N}^*, \tag{27}$$

$$h_F(x) = \frac{\alpha\left[P(x) - \eta P(x+1)\right]}{P(x+1)\,\eta}, \quad x \in \mathbb{N}^*, \tag{28}$$

where $\alpha > 0, \eta \in (0,1)$ are parameters and $P(x+1) = \left(1 - \ln\eta^{x+1}\right)(\alpha - \alpha\ln\eta + \ln\eta) - \ln\eta$.

H) The cdf, pmf, reverse hazard and hazard functions of " Binomial-Natural Discrete Lindley " (BNDL) distribution of (15), are given, respectively, by

$$F(x;p) = 1 - \frac{(1-p)^{x+1}\left(3 + x + p - p^2\right)}{(p+1)(2-p)^{x+2}}, \quad x \in \mathbb{N}^*, \tag{29}$$

$$f(x;p) = \frac{(1-p)^{x}\left(1 + x + 2p - p^2\right)}{(p+1)(2-p)^{x+2}}, \quad x \in \mathbb{N}^*, \tag{30}$$

$$r_F(x) = \frac{(1-p)^{x}\left(1 + x + 2p - p^2\right)}{(p+1)(2-p)^{x+2} - (1-p)^{x+1}(3 + x + p - p^2)}, \quad x \in \mathbb{N}^*, \tag{31}$$

$$h_F(x) = \frac{\left(1 + x + 2p - p^2\right)}{(1-p)(3 + x + p - p^2)}, \quad x \in \mathbb{N}^*, \tag{32}$$

where $p \in (0,1)$ is a parameter.

I) The cdf, pmf, reverse hazard and hazard functions of " Transmuted Record Type Geometric " (TRTG) distribution of (3), are given, respectively, by

$$F(x;\theta,p) = 1 - p^{x+1}\left[1 + \theta\log\left(p^{x+1}\right)\right], \quad x \in \mathbb{N}^*, \tag{33}$$

$$f(x;\theta,p) = p^{x}\left[p(\theta\log\left(p^{x+1}\right) - 1) - \theta\log(p^{x}) + 1\right], \quad x \in \mathbb{N}^*, \tag{34}$$

$$r_F(x) = \frac{p^{x}\left[p(\theta\log\left(p^{x+1}\right) - 1) - \theta\log(p^{x}) + 1\right]}{1 - p^{x+1}\left[1 + \theta\log(p^{x+1})\right]}, \quad x \in \mathbb{N}^*, \tag{35}$$

$$h_F(x) = 1 - \frac{2p + \theta\log(p^{x}) - 1}{p\left[1 + \theta\log(p^{x+1})\right]}, \quad x \in \mathbb{N}^*, \tag{36}$$

where $\theta > 0, p \in (0,1)$ are parameters. We will use $P(x) = [p(\theta\log\left(p^{x+1}\right) - 1) - \theta\log(p^{x}) + 1]$ in the next Section.

J) The cdf, pmf, reverse hazard and hazard functions of " Quasi-Binomial " (QB) distribution of (16), are given, respectively, by

$$F\left(x;\phi,p,m\right)=\sum_{u=0}^{x}p^{u}P\left(u\right),\quad x\in I=\left\{0,1,...,m\right\}, \tag{37}$$

$$f\left(x;\phi,p,m\right)=p^{x}P\left(x\right),\quad x\in I, \tag{38}$$

$$r_{F}\left(x\right)=\frac{p^{x}P\left(x\right)}{\sum_{u=0}^{x}p^{u}P\left(u\right)},\quad x\in I, \tag{39}$$

$$h_{F}\left(x\right)=\frac{p^{x}P\left(x\right)}{1-\sum_{u=0}^{x}p^{u}P\left(u\right)},\quad x\in I, \tag{40}$$

where $p\in\left(0,1\right),m\in\mathbb{N},-p/m<\phi<\left(1-p\right)/m$ are parameters and $P\left(x\right)=\binom{m}{x}\left(1+\frac{\phi x}{p}\right)^{x-1}\left(1-p-x\phi\right)^{m-x}$.

K) The cdf, pmf, reverse hazard and hazard functions of " Binomial-Poisson Lindley Discrete " (Bin-PLD) distribution of (5), are given, respectively, by

$$F\left(x;p,\theta\right)=1-\frac{Q\left(x\right)p^{x+1}}{\left(1+\theta\right)},\quad x\in\mathbb{N}^{*}, \tag{41}$$

$$f\left(x;p,\theta\right)=\frac{\theta^{2}P\left(x\right)p^{x}}{\left(1+\theta\right)},\quad x\in\mathbb{N}^{*}, \tag{42}$$

$$r_{F}\left(x\right)=\frac{\theta^{2}P\left(x\right)p^{x}}{\left(1+\theta\right)-Q\left(x\right)p^{x+1}},\quad x\in\mathbb{N}^{*}, \tag{43}$$

$$h_{F}\left(x\right)=\frac{\theta^{2}P\left(x\right)}{Q\left(x\right)p},\quad x\in\mathbb{N}^{*}, \tag{44}$$

where $p\in\left(0,1\right),\theta>0$ are parameters, $P\left(x\right)=\frac{x+\theta+p+1}{\left(p+\theta\right)^{x+2}}$ and $Q\left(x\right)=\frac{\theta^{2}+p\theta+2\theta+x\theta+p}{\left(p+\theta\right)^{x+2}}$.

L) The cdf, pmf, reverse hazard and hazard functions of " Discrete Odd Weibull-G " (DOW-G) distribution of (8), are given, respectively, by

$$F\left(x;p,\beta,\mathbf{\Phi}\right)=1-p^{\left(\frac{G\left(x+1;\mathbf{\Phi}\right)}{1-G\left(x+1;\mathbf{\Phi}\right)}\right)^{\beta}},\quad x\in\mathbb{N}^{*}, \tag{45}$$

$$f\left(x;p,\beta,\mathbf{\Phi}\right)=p^{\left(\frac{G\left(x;\mathbf{\Phi}\right)}{1-G\left(x;\mathbf{\Phi}\right)}\right)^{\beta}}-p^{\left(\frac{G\left(x+1;\mathbf{\Phi}\right)}{1-G\left(x+1;\mathbf{\Phi}\right)}\right)^{\beta}},\quad x\in\mathbb{N}^{*}, \tag{46}$$

$$r_{F}\left(x\right)=\frac{p^{\left(\frac{G\left(x;\mathbf{\Phi}\right)}{1-G\left(x;\mathbf{\Phi}\right)}\right)^{\beta}}-p^{\left(\frac{G\left(x+1;\mathbf{\Phi}\right)}{1-G\left(x+1;\mathbf{\Phi}\right)}\right)^{\beta}}}{1-p^{\left(\frac{G\left(x+1;\mathbf{\Phi}\right)}{1-G\left(x+1;\mathbf{\Phi}\right)}\right)^{\beta}}},x\in\mathbb{N}^{*}, \tag{47}$$

$$h_{F}\left(x\right)=\frac{p^{\left(\frac{G\left(x;\mathbf{\Phi}\right)}{1-G\left(x;\mathbf{\Phi}\right)}\right)^{\beta}}}{p^{\left(\frac{G\left(x+1;\mathbf{\Phi}\right)}{1-G\left(x+1;\mathbf{\Phi}\right)}\right)^{\beta}}}-1,\quad x\in\mathbb{N}^{*}, \tag{48}$$

where $p\in\left(0,1\right),\beta>0$ are parameters and $G\left(x;\mathbf{\Phi}\right)$ is a baseline cdf which depends on the parameter vector $\mathbf{\Phi}$.

M) The cdf, pmf, reverse hazard and hazard functions of " Discrete Exponentiated Burr-Hatke " (DEBuH) distribution of (6), are given, respectively, by

$$F(x;p,\beta)=\left(1-\frac{p^{x+1}}{x+2}\right)^{\beta},\quad x\in\mathbb{N}^{*}, \tag{49}$$

$$f(x;p,\beta)=\left(1-\frac{p^{x+1}}{x+2}\right)^{\beta}-\left(1-\frac{p^{x}}{x+1}\right)^{\beta},\quad x\in\mathbb{N}^{*}, \tag{50}$$

$$r_F(x)=1-\frac{\left(1-\frac{p^{x}}{x+1}\right)^{\beta}}{\left(1-\frac{p^{x+1}}{x+2}\right)^{\beta}},\quad x\in\mathbb{N}^{*}, \tag{51}$$

$$h_F(x)=\frac{\left(1-\frac{p^{x+1}}{x+2}\right)^{\beta}-\left(1-\frac{p^{x}}{x+1}\right)^{\beta}}{1-\left(1-\frac{p^{x+1}}{x+2}\right)^{\beta}},\quad x\in\mathbb{N}^{*}, \tag{52}$$

where $p\in(0,1),\beta>0$ are parameters.

N) The cdf, pmf, reverse hazard and hazard functions of " Binomial New Poisson-Weighted Exponential " (BN-PWE) distribution of (1), are given, respectively, by

$$F(x;\alpha,\beta,\theta)=1-\left(\frac{\beta}{\alpha+\beta+\alpha\theta}\right)^{x+1},\quad x\in\mathbb{N}^{*}, \tag{53}$$

$$f(x;\alpha,\beta,\theta)=\frac{\alpha(1+\theta)\beta^{x}}{(\alpha+\beta+\alpha\theta)^{x+1}},\quad x\in\mathbb{N}^{*}, \tag{54}$$

$$r_F(x)=\frac{\frac{\alpha(1+\theta)\beta^{x}}{(\alpha+\beta+\alpha\theta)^{x+1}}}{1-\left(\frac{\beta}{\alpha+\beta+\alpha\theta}\right)^{x+1}},\quad x\in\mathbb{N}^{*}, \tag{55}$$

$$h_F(x)=\alpha(1+\theta)\beta^{-1},\quad x\in\mathbb{N}^{*}, \tag{56}$$

where $\alpha>0,\theta>0,\beta\in(0,1)$ are parameters.

2. Characterizations Results

In this Section, we present our characterizations of the 14 distributions listed in the Introduction in three subsections: (i) in terms of the truncated moments of certain functions of the random variables (the choice of each function depends on the form of the pmf), (ii) based on the reverse hazard function and (iii) based on the hazard function. Most of the proofs follow the same scheme, we will give all of them for the sake of completeness.

2.1. Characterizations Based on Conditional Expectation

(A) DITL Distribution

Proposition 2.1. *Let $X:\Omega\to\mathbb{N}$ be a random variable. The pmf of X is* (2) *if and only if*

$$E\left\{\left[\frac{(3+2(X-1))^{\delta}}{(2+(X-1))^{2\delta}}+\frac{(3+2X)^{\delta}}{(2+X)^{2\delta}}\right]\mid X>k\right\}=\frac{(3+2k)^{\delta}}{(2+k)^{2\delta}} \tag{57}$$

Proof. If X has pmf (2), then for $k\in\mathbb{N}$, the left-hand side of (57), using telescoping sum formula, will be

$$
\begin{aligned}
&(1-F(k))^{-1}\sum_{x=k+1}^{\infty}\left\{\left(\frac{(3+2(x-1))^{\delta}}{(2+(x-1))^{2\delta}}\right)^{2}-\left(\frac{(3+2x)^{\delta}}{(2+x)^{2\delta}}\right)^{2}\right\}\\
&=\frac{(2+k)^{2\delta}}{(3+2k)^{\delta}}\sum_{x=k+1}^{\infty}\left\{\left(\frac{(3+2(x-1))^{\delta}}{(2+(x-1))^{2\delta}}\right)^{2}-\left(\frac{(3+2x)^{\delta}}{(2+x)^{2\delta}}\right)^{2}\right\}\\
&=\frac{(2+k)^{2\delta}}{(3+2k)^{\delta}}\left(\frac{(3+2k)^{\delta}}{(2+k)^{2\delta}}\right)^{2}=\frac{(3+2k)^{\delta}}{(2+k)^{2\delta}}.
\end{aligned}
$$

Conversely, if (57) holds, then

$$
\begin{aligned}
&\sum_{x=k+1}^{\infty}\left\{\left[\frac{(3+2(x-1))^{\delta}}{(2+(x-1))^{2\delta}}+\frac{(3+2x)^{\delta}}{(2+x)^{2\delta}}\right]f(x)\right\}\\
&=(1-F(k))\frac{(3+2k)^{\delta}}{(2+k)^{2\delta}}\\
&=(1-F(k+1)+f(k+1))\frac{(3+2k)^{\delta}}{(2+k)^{2\delta}}. \qquad (58)
\end{aligned}
$$

From (58), we also have

$$
\sum_{x=k+2}^{\infty}\left\{\left[\frac{(3+2(x-1))^{\delta}}{(2+(x-1))^{2\delta}}+\frac{(3+2x)^{\delta}}{(2+x)^{2\delta}}\right]f(x)\right\}=(1-F(k+1))\frac{(3+2(k+1))^{\delta}}{(2+(k+1))^{2\delta}}. \qquad (59)
$$

Now, subtracting (59) from (58), yields

$$
\begin{aligned}
&\left[\frac{(3+2k)^{\delta}}{(2+k)^{2\delta}}+\frac{(3+2(k+1))^{\delta}}{(2+(k+1))^{2\delta}}\right]f(k+1)\\
&=(1-F(k+1))\left\{\frac{(3+2k)^{\delta}}{(2+k)^{2\delta}}-\frac{(3+2(k+1))^{\delta}}{(2+(k+1))^{2\delta}}\right\}\\
&+f(k+1)\left[\frac{(3+2k)^{\delta}}{(2+k)^{2\delta}}\right].
\end{aligned}
$$

From the above equality, we have

$$
\frac{f(k+1)}{1-F(k+1)}=\frac{\frac{(3+2k)^{\delta}}{(2+k)^{2\delta}}-\frac{(3+2(k+1))^{\delta}}{(2+(k+1))^{2\delta}}}{\frac{(3+2(k+1))^{\delta}}{(2+(k+1))^{2\delta}}}=\frac{(3+2k)^{\delta}(3+k)^{2\delta}}{(2+k)^{2\delta}(5+2k)^{\delta}}-1,
$$

which is the hazard function, (4) corresponding to the pmf (2), so X has pmf (2). □

(B) SBPGLD Distribution

Proposition 2.2. *Let* $X:\Omega\rightarrow\mathbb{N}^{*}$ *be a random variable. The pmf of* X *is* (6) *if and only if*

$$
E\left\{[P(x)]^{-1}\mid X\leq k\right\}=\frac{(1+\theta)}{\theta}\left(\frac{1-(1+\theta)^{-(k+1)}}{\sum_{x=0}^{k}P(x)(1+\theta)^{-x}}\right). \qquad (60)
$$

Proof. If X has pmf (6), then for $k \in \mathbb{N}^*$, the left-hand side of (60), using finite geometric sum formula, will be

$$\begin{aligned} C(\theta,\beta)(F(k))^{-1}\sum_{x=0}^{k}(1+\theta)^{-x} &= \left(\frac{1}{\sum_{x=0}^{k}P(x)(1+\theta)^{-x}}\right)\sum_{x=0}^{k}(1+\theta)^{-x} \\ &= \frac{(1+\theta)}{\theta}\left(\frac{1-(1+\theta)^{-(k+1)}}{\sum_{x=0}^{k}P(x)(1+\theta)^{-x}}\right). \end{aligned}$$

Conversely, if (60) holds, then

$$\sum_{x=0}^{k}\left\{[P(x)]^{-1}\ f(x)\right\} = F(k)\left\{\frac{(1+\theta)}{\theta}\left(\frac{1-(1+\theta)^{-(k+1)}}{\sum_{x=0}^{k}P(x)(1+\theta)^{-x}}\right)\right\}. \tag{61}$$

From (61), we also have

$$\begin{aligned} \sum_{x=0}^{k-1}\left\{[P(x)]^{-1}\ f(x)\right\} &= F(k-1)\left\{\frac{(1+\theta)}{\theta}\left(\frac{1-(1+\theta)^{-(k)}}{\sum_{x=0}^{k-1}P(x)(1+\theta)^{-x}}\right)\right\} \\ &= (F(k)-f(k))\left\{\frac{(1+\theta)}{\theta}\left(\frac{1-(1+\theta)^{-(k)}}{\sum_{x=0}^{k-1}P(x)(1+\theta)^{-x}}\right)\right\}. \end{aligned} \tag{62}$$

Now, subtracting (62) from (61), yields

$$\begin{aligned} \left[[P(k)]^{-1}\right]f(k) =& F(k)\frac{(1+\theta)}{\theta}\left\{\left(\frac{1-(1+\theta)^{-(k+1)}}{\sum_{x=0}^{k}P(x)(1+\theta)^{-x}}\right)-\left(\frac{1-(1+\theta)^{-(k)}}{\sum_{x=0}^{k-1}P(x)(1+\theta)^{-x}}\right)\right\} \\ &+ f(k)\left[\frac{(1+\theta)}{\theta}\left(\frac{1-(1+\theta)^{-(k)}}{\sum_{x=0}^{k-1}P(x)(1+\theta)^{-x}}\right)\right], \end{aligned}$$

or

$$\begin{aligned} & f(k)\left[[P(k)]^{-1}-\frac{(1+\theta)}{\theta}\left(\frac{1-(1+\theta)^{-k}}{\sum_{x=0}^{k-1}P(x)(1+\theta)^{-x}}\right)\right] \\ &= F(k)\frac{(1+\theta)}{\theta}\left\{\left(\frac{1-(1+\theta)^{-(k+1)}}{\sum_{x=0}^{k}P(x)(1+\theta)^{-x}}\right)-\left(\frac{1-(1+\theta)^{-k}}{\sum_{x=0}^{k-1}P(x)(1+\theta)^{-x}}\right)\right\}. \end{aligned}$$

From the above equality, we have

$$\frac{f(k)}{F(k)} = \frac{\frac{(1+\theta)}{\theta}\left\{\left(\frac{1-(1+\theta)^{-(k+1)}}{\sum_{x=0}^{k}P(x)(1+\theta)^{-x}}\right)-\left(\frac{1-(1+\theta)^{-k}}{\sum_{x=0}^{k-1}P(x)(1+\theta)^{-x}}\right)\right\}}{\left[[P(k)]^{-1}-\frac{(1+\theta)}{\theta}\left(\frac{1-(1+\theta)^{-k}}{\sum_{x=0}^{k-1}P(x)(1+\theta)^{-x}}\right)\right]},$$

which is the reverse hazard function, (7) corresponding to the pmf (6), so X has pmf (6). □

(C) DKMOE Distribution

Proposition 2.3. *Let* $X : \Omega \to \mathbb{N}^*$ *be a random variable. The pmf of* X *is* (10) *if and only if*

$$E\left\{\left[\left\{1-\left[\frac{1-p^{X}}{1-\overline{\alpha}p^{X}}\right]\right\}^{b}+\left\{1-\left[\frac{1-p^{X+1}}{1-\overline{\alpha}p^{X+1}}\right]\right\}^{b}\right] \mid X>k\right\}$$
$$=\left\{1-\left[\frac{1-p^{k+1}}{1-\overline{\alpha}p^{k+1}}\right]\right\}^{b}. \tag{63}$$

Proof of Proposition 2.3 . If X has pmf (10), then for $k\in\mathbb{N}^{*}$, the left-hand side of (63), using telescoping sum formula, will be

$$\left\{1-\left[\frac{1-p^{k+1}}{1-\overline{\alpha}p^{k+1}}\right]\right\}^{-b}\sum_{x=k+1}^{\infty}\left\{\begin{array}{c}\left\{1-\left[\frac{1-p^{x}}{1-\overline{\alpha}p^{x}}\right]\right\}^{2b}\\ -\left\{1-\left[\frac{1-p^{x+1}}{1-\overline{\alpha}p^{x+1}}\right]\right\}^{2b}\end{array}\right\}$$
$$=\left\{1-\left[\frac{1-p^{k+1}}{1-\overline{\alpha}p^{k+1}}\right]\right\}^{-b}\left\{1-\left[\frac{1-p^{k+1}}{1-\overline{\alpha}p^{k+1}}\right]\right\}^{2b}$$
$$=\left\{1-\left[\frac{1-p^{k+1}}{1-\overline{\alpha}p^{k+1}}\right]\right\}^{b}.$$

Conversely, if (63) holds, then

$$\sum_{x=k+1}^{\infty}\left\{\left[\left\{1-\left[\frac{1-p^{x}}{1-\overline{\alpha}p^{x}}\right]\right\}^{b}+\left\{1-\left[\frac{1-p^{x+1}}{1-\overline{\alpha}p^{x+1}}\right]\right\}^{b}\right] f(x)\right\}$$
$$=(1-F(k))\left\{1-\left[\frac{1-p^{k+1}}{1-\overline{\alpha}p^{k+1}}\right]\right\}^{b}$$
$$=(1-F(k+1)+f(k+1))\left\{1-\left[\frac{1-p^{k+1}}{1-\overline{\alpha}p^{k+1}}\right]\right\}^{b}. \tag{64}$$

From (64), we also have

$$\sum_{x=k+2}^{\infty}\left\{\left[\left\{1-\left[\frac{1-p^{x}}{1-\overline{\alpha}p^{x}}\right]\right\}^{b}+\left\{1-\left[\frac{1-p^{x+1}}{1-\overline{\alpha}p^{x+1}}\right]\right\}^{b}\right] f(x)\right\}$$
$$=(1-F(k+1))\left\{1-\left[\frac{1-p^{k+2}}{1-\overline{\alpha}p^{k+2}}\right]\right\}^{b}. \tag{65}$$

Now, subtracting (65) from (64), yields

$$\left[\left\{1-\left[\frac{1-p^{k+1}}{1-\overline{\alpha}p^{k+1}}\right]\right\}^{b}+\left\{1-\left[\frac{1-p^{k+2}}{1-\overline{\alpha}p^{k+2}}\right]\right\}^{b}\right] f(k+1)$$
$$=(1-F(k+1))\left\{\left[\left\{1-\left[\frac{1-p^{k+1}}{1-\overline{\alpha}p^{k+1}}\right]\right\}^{b}\right]-\left[\left\{1-\left[\frac{1-p^{k+2}}{1-\overline{\alpha}p^{k+2}}\right]\right\}^{b}\right]\right\}$$
$$+f(k+1)\left[\left\{1-\left[\frac{1-p^{k+1}}{1-\overline{\alpha}p^{k+1}}\right]\right\}^{b}\right],$$

or

$$f\left(k+1\right)\left[\left\{1-\left[\frac{1-p^{k+2}}{1-\overline{\alpha}p^{k+2}}\right]\right\}^{b}\right]$$
$$=\left(1-F\left(k+1\right)\right)\left\{\left[\left\{1-\left[\frac{1-p^{k+1}}{1-\overline{\alpha}p^{k+1}}\right]\right\}^{b}\right]-\left[\left\{1-\left[\frac{1-p^{k+2}}{1-\overline{\alpha}p^{k+2}}\right]\right\}^{b}\right]\right\}.$$

From the above equality, we have

$$\frac{f\left(k+1\right)}{1-F\left(k+1\right)}=\frac{\left\{1-\left[\frac{1-p^{k+1}}{1-\overline{\alpha}p^{k+1}}\right]\right\}^{b}-\left\{1-\left[\frac{1-p^{k+2}}{1-\overline{\alpha}p^{k+2}}\right]\right\}^{b}}{\left\{1-\left[\frac{1-p^{k+2}}{1-\overline{\alpha}p^{k+2}}\right]\right\}^{b}}$$
$$=\frac{\left\{1-\left[\frac{1-p^{k+1}}{1-\overline{\alpha}p^{k+1}}\right]\right\}^{b}}{\left\{1-\left[\frac{1-p^{k+2}}{1-\overline{\alpha}p^{k+2}}\right]\right\}^{b}}-1,$$

which is the hazard function, (12) corresponding to the pmf (10), so X has pmf (10). □

(D) ZTPPsLD Distribution

Proposition 2.4. *Let* $X:\Omega\rightarrow\mathbb{N}$ *be a random variable. The pmf of* X *is* (14) *if and only if*

$$E\left\{\left[\theta\beta+\beta-1+\theta X\right]^{-1}\mid X>k\right\}=\left[\theta\beta+\theta+\beta+\theta k\right]^{-1}. \tag{66}$$

Proof. If X has pmf (14), then for $k\in\mathbb{N}$, the left-hand side of (66), using infinite geometric sum formula, will be

$$\frac{\left(\theta\beta+\beta+\theta\right)\left(\theta+1\right)^{k}}{\left(\theta\beta+\theta+\beta+\theta k\right)}\sum_{x=k+1}^{\infty}\left\{\left(\frac{\theta}{\theta\beta+\beta+\theta}\right)\left(\theta+1\right)^{-x}\right\}$$
$$=\frac{\left(\theta\beta+\beta+\theta\right)\left(\theta+1\right)^{k}}{\left(\theta\beta+\theta+\beta+\theta k\right)}\left(\frac{\theta}{\theta\beta+\beta+\theta}\right)\left[\frac{1}{\theta}\left(\theta+1\right)^{-k}\right]$$
$$=\left[\left(\theta\beta+\theta+\beta+\theta k\right)\right]^{-1}.$$

Conversely, if (66) holds, then

$$\sum_{x=k+1}^{\infty}\left\{\left[\theta\beta+\beta-1+\theta x\right]^{-1}f\left(x\right)\right\}=\left(1-F(k)\right)\left[\left(\theta\beta+\theta+\beta+\theta k\right)\right]^{-1}$$
$$=\left(1-F\left(k+1\right)+f\left(k+1\right)\right)\left[\left(\theta\beta+\theta+\beta+\theta k\right)\right]^{-1}. \tag{67}$$

From (67), we also have

$$\sum_{x=k+2}^{\infty}\left\{\left[\theta\beta+\beta-1+\theta x\right]^{-1}f\left(x\right)\right\}=\left(1-F\left(k+1\right)\right)\left[\left(\theta\beta+\theta+\beta+\theta(k+1)\right)\right]^{-1}. \tag{68}$$

Now, subtracting (68) from (67) , yields

$$\begin{aligned}
&\left[\left[(\theta\beta+\beta-1+\theta(k+1))\right]^{-1}\right] f\left(k+1\right) \\
&=\left(1-F\left(k+1\right)\right)\left\{\left[(\theta\beta+\theta+\beta+\theta k)\right]^{-1}-\left[(\theta\beta+\theta+\beta+\theta(k+1))\right]^{-1}\right\} \\
&+f\left(k+1\right)\left[(\theta\beta+\theta+\beta+\theta k)\right]^{-1},
\end{aligned}$$

or

$$\begin{aligned}
&f\left(k+1\right)\left[\left[(\theta\beta+\beta-1+\theta(k+1))\right]^{-1}-\left[(\theta\beta+\theta+\beta+\theta k)\right]^{-1}\right] \\
&=\left(1-F\left(k+1\right)\right)\left\{\left[(\theta\beta+\theta+\beta+\theta k)\right]^{-1}-\left[(\theta\beta+\theta+\beta+\theta(k+1))\right]^{-1}\right\}.
\end{aligned}$$

From the above equality, we have

$$\begin{aligned}
\frac{f\left(k+1\right)}{1-F\left(k+1\right)} &= \frac{\left[(\theta\beta+\theta+\beta+\theta k)\right]^{-1}-\left[(\theta\beta+\theta+\beta+\theta(k+1))\right]^{-1}}{\left[(\theta\beta+\beta-1+\theta(k+1))\right]^{-1}-\left[(\theta\beta+\theta+\beta+\theta k)\right]^{-1}} \\
&= \frac{\theta\left(\theta\beta+\beta-1+\theta(k+1)\right)}{\left(\theta\beta+\theta+\beta+\theta(k+1)\right)},
\end{aligned}$$

which is the hazard function, (16) corresponding to the pmf (14), so X has pmf (14). □

(E) SBPAD Distribution

Proposition 2.5. *Let* $X:\Omega\rightarrow\mathbb{N}$ *be a random variable. The pmf of* X *is* (18) *if and only if*

$$E\left\{\left[P\left(X\right)\right]^{-1} \mid X\leq k\right\}=\left(\frac{p}{1-p}\right)\left(\frac{1-p^{k}}{\sum_{x=1}^{k}P\left(x\right)p^{x}}\right). \tag{69}$$

Proof. If X has pmf (18), then for $k\in\mathbb{N}$, the left-hand side of (69), using finite geometric sum formula, will be

$$\begin{aligned}
\frac{1}{C\left(\theta,p\right)\sum_{x=1}^{k}P\left(x\right)p^{x}}\sum_{x=1}^{k}\left\{C\left(\theta,p\right)p^{x}\right\} &= \frac{1}{\sum_{x=1}^{k}P\left(x\right)p^{x}}\sum_{x=1}^{k}p^{x} \\
&= \left(\frac{p}{1-p}\right)\left(\frac{1-p^{k}}{\sum_{x=1}^{k}P\left(x\right)p^{x}}\right).
\end{aligned}$$

Conversely, if (69) holds, then

$$\sum_{x=1}^{k}\left\{\left[P\left(x\right)\right]^{-1}\ f\left(x\right)\right\}=\left(F(k)\right)\left(\frac{p}{1-p}\right)\left(\frac{1-p^{k}}{\sum_{x=1}^{k}P\left(x\right)p^{x}}\right). \tag{70}$$

From (70), we also have

$$\sum_{x=1}^{k-1}\left\{\left[P\left(x\right)\right]^{-1}\ f\left(x\right)\right\}=\left(F\left(k\right)-f\left(k\right)\right)\left(\frac{p}{1-p}\right)\left(\frac{1-p^{k-1}}{\sum_{x=1}^{k-1}P\left(x\right)p^{x}}\right). \tag{71}$$

Now, subtracting (71) from (70), yields

$$[P(k)]^{-1} f(k) = F(k)\left(\frac{p}{1-p}\right)\left\{\left(\frac{1-p^k}{\sum_{x=1}^{k} P(x)p^x}\right) - \left(\frac{1-p^{k-1}}{\sum_{x=1}^{k-1} P(x)p^x}\right)\right\}$$
$$+ f(k)\left(\frac{p}{1-p}\right)\left(\frac{1-p^{k-1}}{\sum_{x=1}^{k-1} P(x)p^x}\right),$$

or

$$f(k)\left[[P(k)]^{-1} - \left(\frac{p}{1-p}\right)\left(\frac{1-p^{k-1}}{\sum_{x=1}^{k-1} P(x)p^x}\right)\right]$$
$$= F(k)\left\{\left(\frac{p}{1-p}\right)\left(\frac{1-p^k}{\sum_{x=1}^{k} P(x)p^x}\right) - \left(\frac{p}{1-p}\right)\left(\frac{1-p^{k-1}}{\sum_{x=1}^{k-1} P(x)p^x}\right)\right\}.$$

From the above equality, after some computation, we have

$$\frac{f(k)}{F(k)} = \frac{\left(\frac{p}{1-p}\right)\left(\frac{1-p^k}{\sum_{x=1}^{k} P(x)p^x}\right) - \left(\frac{p}{1-p}\right)\left(\frac{1-p^{k-1}}{\sum_{x=1}^{k-1} P(x)p^x}\right)}{[P(x)]^{-1} - \left(\frac{p}{1-p}\right)\left(\frac{1-p^{k-1}}{\sum_{x=1}^{k-1} P(x)p^x}\right)}$$
$$= \frac{P(k)p^k}{\sum_{x=1}^{k} P(x)p^x},$$

which is the reverse hazard function, (19) corresponding to the pmf (18), so X has pmf (18). □

(F) PXGD Distribution

Proposition 2.6. *Let* $X : \Omega \to \mathbb{N}^*$ *be a random variable. The pmf of* X *is* (22) *if and only if*

$$E\left\{[P(X)]^{-1} \mid X > k\right\} = \frac{\theta}{Q(k)}. \tag{72}$$

Proof. If X has pmf (22), then for $k \in \mathbb{N}^*$, the left-hand side of (72), using infinite geometric sum formula, will be

$$\frac{2(1+\theta)^{k+4}}{Q(k)} \sum_{x=k+1}^{\infty}\left(\frac{\theta^2}{2(1+\theta)^4}\right)(1+\theta)^{-x} = \frac{\theta^2(1+\theta)^k}{Q(k)} \sum_{x=k+1}^{\infty}(1+\theta)^{-x}$$
$$= \frac{\theta^2(1+\theta)^k}{Q(k)}\left(\frac{(1+\theta)^{-(k+1)}}{1-(1+\theta)^{-1}}\right) = \frac{\theta}{Q(k)}.$$

Conversely, if (72) holds, then

$$\sum_{x=k+1}^{\infty}\left\{[P(x)]^{-1}\ f(x)\right\} = (1-F(k))\left(\frac{\theta}{Q(k)}\right) = (1-F(k+1)+f(k+1))\left(\frac{\theta}{Q(k)}\right). \tag{73}$$

From (73), we also have

$$\sum_{x=k+2}^{\infty}\left\{[P(x)]^{-1}\ f(x)\right\} = (1-F(k+1))\left(\frac{\theta}{Q(k+1)}\right). \tag{74}$$

Now, subtracting (74) from (73), yields

$$[P(k+1)]^{-1} f(k+1) = (1-F(k+1)) \left\{ \left(\frac{\theta}{Q(k)} \right) - \left(\frac{\theta}{Q(k+1)} \right) \right\} + f(k+1) \left(\frac{\theta}{Q(k)} \right),$$

or

$$f(k+1) \left[[P(k+1)]^{-1} - \left(\frac{\theta}{Q(k)} \right) \right] = (1-F(k+1)) \left\{ \left(\frac{\theta}{Q(k)} \right) - \left(\frac{\theta}{Q(k+1)} \right) \right\}.$$

From the above equality, after some computation, we have

$$\frac{f(k+1)}{1-F(k+1)} = \frac{\left(\frac{\theta}{Q(k)} \right) - \left(\frac{\theta}{Q(k+1)} \right)}{[P(k+1)]^{-1} - \left(\frac{\theta}{Q(k)} \right)} = \left(\frac{\theta^2 P(k+1)}{Q(k+1)} \right),$$

which is the hazard function, (24) corresponding to the pmf (22), so X has pmf (22). □

(G) DGLi Distribution

Proposition 2.7. *Let* $X : \Omega \to \mathbb{N}^*$ *be a random variable. The pmf of* X *is* (26) *if and only if*

$$E\left\{ [P(X) - \eta P(X+1)]^{-1} \mid X > k \right\} = \frac{\alpha}{(1-\eta) P(k+1)}. \tag{75}$$

Proof. If X has pmf (26), then for $k \in \mathbb{N}^*$, the left-hand side of (75), using infinite geometric sum formula, will be

$$\frac{\alpha(1-\ln\eta)}{P(k+1)\eta^{k+1}} \sum_{x=k+1}^{\infty} \left(\frac{1}{1-\ln\eta} \right) \eta^x = \frac{\alpha(1-\ln\eta)}{P(k+1)\eta^{k+1}} \left(\frac{1}{1-\ln\eta} \right) \sum_{x=k+1}^{\infty} \eta^x$$
$$= \frac{\alpha}{P(k+1)\eta^{k+1}} \left(\frac{\eta^{k+1}}{1-\eta} \right) = \frac{\alpha}{(1-\eta) P(k+1)}.$$

Conversely, if (75) holds, then

$$\sum_{x=k+1}^{\infty} \left\{ [P(x) - \eta P(x+1)]^{-1} \; f(x) \right\} = (1-F(k)) \frac{\alpha}{(1-\eta) P(k+1)}$$
$$= (1-F(k+1) + f(k+1)) \left(\frac{\alpha}{(1-\eta) P(k+1)} \right). \tag{76}$$

From (76), we also have

$$\sum_{x=k+2}^{\infty} \left\{ [P(x) - \eta P(x+1)]^{-1} \; f(x) \right\} = (1-F(k+1)) \left(\frac{\alpha}{(1-\eta) P(k+2)} \right). \tag{77}$$

Now, subtracting (77) from (76), yields

$$[P(k+1) - \eta P(k+2)]^{-1} f(k+1)$$
$$= (1-F(k+1)) \left\{ \left(\frac{\alpha}{(1-\eta) P(k+1)} \right) - \left(\frac{\alpha}{(1-\eta) P(k+2)} \right) \right\}$$
$$+ f(k+1) \left(\frac{\alpha}{(1-\eta) P(k+1)} \right),$$

or

$$f\left(k+1\right)\left[\left[P\left(k+1\right)-\eta P\left(k+2\right)\right]^{-1}-\left(\frac{\alpha}{\left(1-\eta\right)P\left(k+1\right)}\right)\right]$$
$$=\left(1-F(k+1)\right)\left\{\left(\frac{\alpha}{\left(1-\eta\right)P\left(k+1\right)}\right)-\left(\frac{\alpha}{\left(1-\eta\right)P\left(k+2\right)}\right)\right\}.$$

From the above equality, after some computation, we have

$$\frac{f\left(k+1\right)}{1-F\left(k+1\right)}=\frac{\left(\frac{\alpha}{(1-\eta)P(k+1)}\right)-\left(\frac{\alpha}{(1-\eta)P(k+2)}\right)}{\left[P\left(k+1\right)-\eta P\left(k+2\right)\right]^{-1}-\left(\frac{\alpha}{(1-\eta)P(k+1)}\right)}$$
$$=\frac{\alpha\left[P\left(k+1\right)-\eta P\left(k+2\right)\right]}{P\left(k+2\right)\eta},$$

which is the hazard function, (28) corresponding to the pmf (26), so X has pmf (26). □

(H) BNDL Distribution

Proposition 2.8. *Let $X:\Omega\to\mathbb{N}^*$ be a random variable. The pmf of X is* (30) *if and only if*

$$E\left\{\left[\frac{(2-p)^{X+2}}{1+X+2p-p^2}\right]\mid X>k\right\}=\frac{(2-p)^{k+2}}{p(3+k+p-p^2)}. \tag{78}$$

Proof. If X has pmf (30), then for $k\in\mathbb{N}^*$, the left-hand side of (78), using infinite geometric sum formula, will be

$$\left(\frac{\left(p+1\right)\left(2-p\right)^{k+2}}{\left(1-p\right)^{k+1}\left(3+k+p-p^2\right)}\right)\sum_{x=k+1}^{\infty}\left(\frac{1}{\left(p+1\right)}\right)\left(1-p\right)^x$$
$$=\left(\frac{\left(p+1\right)\left(2-p\right)^{k+2}}{\left(1-p\right)^{k+1}\left(3+k+p-p^2\right)}\right)\left(\frac{\left(1-p\right)^{k+1}}{p\left(p+1\right)}\right)$$
$$=\frac{\left(2-p\right)^{k+2}}{p(3+k+p-p^2)}.$$

Conversely, if (78) holds, then

$$\sum_{x=k+1}^{\infty}\left\{\left[\frac{\left(2-p\right)^{x+2}}{1+x+2p-p^2}\right]f\left(x\right)\right\}=\left(1-F(k)\right)\left(\frac{\left(2-p\right)^{k+2}}{p(3+k+p-p^2)}\right)$$
$$=\left(1-F\left(k+1\right)+f\left(k+1\right)\right)\left(\frac{\left(2-p\right)^{k+2}}{p(3+k+p-p^2)}\right). \tag{79}$$

From (79), we also have

$$\sum_{x=k+2}^{\infty}\left\{\left[\frac{\left(2-p\right)^{x+2}}{1+x+2p-p^2}\right]f\left(x\right)\right\}=\left(1-F\left(k+1\right)\right)\left(\frac{\left(2-p\right)^{k+3}}{p(3+\left(k+1\right)+p-p^2)}\right). \tag{80}$$

Now, subtracting (80) from (79), yields

$$\begin{aligned}
&\left[\frac{(2-p)^{k+3}}{1+(k+1)+2p-p^2}\right] f(k+1) \\
&= (1-F(k+1))\left\{\left(\frac{(2-p)^{k+2}}{p(3+k+p-p^2)}\right)-\left(\frac{(2-p)^{k+3}}{p(3+(k+1)+p-p^2)}\right)\right\} \\
&+ f(k+1)\left(\frac{(2-p)^{k+2}}{p(3+k+p-p^2)}\right),
\end{aligned}$$

or

$$\begin{aligned}
&f(k+1)\left[\left(\frac{(2-p)^{k+3}}{1+(k+1)+2p-p^2}\right)-\left(\frac{(2-p)^{k+2}}{p(3+k+p-p^2)}\right)\right] \\
&= (1-F(k+1))\left\{\left(\frac{(2-p)^{k+2}}{p(3+k+p-p^2)}\right)-\left(\frac{(2-p)^{k+3}}{p(3+(k+1)+p-p^2)}\right)\right\}.
\end{aligned}$$

From the above equality, after some computation, we have

$$\begin{aligned}
\frac{f(k+1)}{1-F(k+1)} &= \frac{\left(\frac{(2-p)^{k+2}}{p(3+k+p-p^2)}\right)-\left(\frac{(2-p)^{k+3}}{p(3+(k+1)+p-p^2)}\right)}{\left(\frac{(2-p)^{k+3}}{1+(k+1)+2p-p^2}\right)-\left(\frac{(2-p)^{k+2}}{p(3+k+p-p^2)}\right)} \\
&= \frac{1+(k+1)+2p-p^2}{(1-p)\left(3+(k+1)+p-p^2\right)},
\end{aligned}$$

which is the hazard function, (32) corresponding to the pmf (30), so X has pmf (30). □

(I) TRTG Distribution

Proposition 2.9. *Let* $X:\Omega\to\mathbb{N}^*$ *be a random variable. The pmf of* X *is* (34) *if and only if*

$$E\left\{[P(X)]^{-1} \mid X>k\right\} = \frac{1}{(1-p)\left[1+\theta\log\left(p^{x+1}\right)\right]}. \tag{81}$$

Proof. If X has pmf (34), then for $k\in\mathbb{N}^*$, the left-hand side of (81), using infinite geometric sum formula, will be

$$\begin{aligned}
\left(\frac{1}{p^{k+1}\left(1+\theta\log\left(p^{k+1}\right)\right)}\right)\sum_{x=k+1}^{\infty} p^x &= \left(\frac{1}{p^{k+1}\left(1+\theta\log\left(p^{k+1}\right)\right)}\right)\left(\frac{p^{k+1}}{1-p}\right) \\
&= \frac{1}{(1-p)\left[1+\theta\log\left(p^{x+1}\right)\right]}.
\end{aligned}$$

Conversely, if (81) holds, then

$$\begin{aligned}
\sum_{x=k+1}^{\infty}\left\{[P(x)]^{-1} f(x)\right\} &= (1-F(k))\left(\frac{1}{(1-p)\left[1+\theta\log\left(p^{k+1}\right)\right]}\right) \\
&= (1-F(k+1)+f(k+1))\frac{1}{(1-p)\left[1+\theta\log\left(p^{k+1}\right)\right]}. \qquad (82)
\end{aligned}$$

From (82), we also have

$$\sum_{x=k+2}^{\infty}\left\{[P(x)]^{-1}\ f(x)\right\}=(1-F(k+1))\left(\frac{1}{(1-p)\left[1+\theta\log\left(p^{k+2}\right)\right]}\right). \tag{83}$$

Now, subtracting (83) from (82), yields

$$\begin{aligned}&\left[[P(k+1)]^{-1}\right]f(k+1)\\&=(1-F(k+1))\left\{\left(\frac{1}{(1-p)\left[1+\theta\log\left(p^{k+1}\right)\right]}\right)-\left(\frac{1}{(1-p)\left[1+\theta\log\left(p^{k+2}\right)\right]}\right)\right\}\\&+f(k+1)\left(\frac{1}{(1-p)\left[1+\theta\log\left(p^{k+1}\right)\right]}\right),\end{aligned}$$

or

$$\begin{aligned}&f(k+1)\left[[P(k+1)]^{-1}-\left(\frac{1}{(1-p)\left[1+\theta\log\left(p^{k+1}\right)\right]}\right)\right]\\&=(1-F(k+1))\left\{\left(\frac{1}{(1-p)\left[1+\theta\log\left(p^{k+1}\right)\right]}\right)-\left(\frac{1}{(1-p)\left[1+\theta\log\left(p^{k+2}\right)\right]}\right)\right\}.\end{aligned}$$

From the above equality, after some computation, we have

$$\begin{aligned}\frac{f(k+1)}{1-F(k+1)}&=\frac{\left(\frac{1}{(1-p)\left[1+\theta\log\left(p^{k+1}\right)\right]}\right)-\left(\frac{1}{(1-p)\left[1+\theta\log\left(p^{k+2}\right)\right]}\right)}{[P(x)]^{-1}-\left(\frac{1}{(1-p)\left[1+\theta\log\left(p^{k+1}\right)\right]}\right)}\\&=1-\frac{2p+\theta\log\left(p^{k+1}\right)-1}{p\left[1+\theta\log\left(p^{k+2}\right)\right]},\end{aligned}$$

which is the hazard function, (36) corresponding to the pmf (34), so X has pmf (34). □

(J) QB Distribution

Proposition 2.10. *Let* $X:\Omega\rightarrow I$ *be a random variable. The pmf of* X *is* (38) *if and only if*

$$E\left\{[P(X)]^{-1}\ |\ X\leq k\right\}=\frac{1-p^{k+1}}{(1-p)\sum_{x=0}^{k}p^{x}P(x)}. \tag{84}$$

Proof. If X has pmf (38), then for $k\in I$, the left-hand side of (84), using infinite geometric sum formula, will be

$$\begin{aligned}\left(\frac{1}{\sum_{x=0}^{k}p^{x}P(x)}\right)\sum_{x=0}^{k}p^{x}&=\left(\frac{1}{\sum_{x=0}^{k}p^{x}P(x)}\right)\left(\frac{1-p^{k+1}}{1-p}\right)\\&=\frac{1-p^{k+1}}{(1-p)\sum_{x=0}^{k}p^{x}P(x)}.\end{aligned}$$

Conversely, if (84) holds, then

$$\sum_{x=0}^{k}\left\{[P(x)]^{-1}\ f(x)\right\}=F(k)\left(\frac{1-p^{k+1}}{(1-p)\sum_{x=0}^{k}p^{x}P(x)}\right). \tag{85}$$

From (85), we also have

$$\sum_{x=0}^{k-1}\left\{[P(x)]^{-1}\ f(x)\right\} = F(k-1))\left(\frac{1-p^k}{(1-p)\sum_{x=0}^{k-1}p^xP(x)}\right)$$
$$= (F(k)-f(k))\left(\frac{1-p^k}{(1-p)\sum_{x=0}^{k-1}p^xP(x)}\right). \tag{86}$$

Now, subtracting (86) from (85), yields

$$\left[[P(k)]^{-1}\right] f(k) = F(k)\left\{\left(\frac{1-p^{k+1}}{(1-p)\sum_{x=0}^{k}p^xP(x)}\right)-\left(\frac{1-p^k}{(1-p)\sum_{x=0}^{k-1}p^xP(x)}\right)\right\}$$
$$+ f(k)\left(\frac{1-p^k}{(1-p)\sum_{x=0}^{k-1}p^xP(x)}\right),$$

or

$$f(k)\left[[P(k)]^{-1}-\left(\frac{1-p^k}{(1-p)\sum_{x=0}^{k-1}p^xP(x)}\right)\right]$$
$$= F(k)\left\{\left(\frac{1-p^{k+1}}{(1-p)\sum_{x=0}^{k}p^xP(x)}\right)-\left(\frac{1-p^k}{(1-p)\sum_{x=0}^{k-1}p^xP(x)}\right)\right\}.$$

From the above equality, after some computation, we have

$$\frac{f(k)}{F(k)} = \frac{\left(\frac{1-p^{k+1}}{(1-p)\sum_{x=0}^{k}p^xP(x)}\right)-\left(\frac{1-p^k}{(1-p)\sum_{x=0}^{k-1}p^xP(x)}\right)}{[P(k)]^{-1}-\left(\frac{1-p^k}{(1-p)\sum_{x=0}^{k-1}p^xP(x)}\right)}$$
$$= \frac{p^kP(k)}{\sum_{u=0}^{k}p^uP(u)},$$

which is the reverse hazard function, (39) corresponding to the pmf (38), so X has pmf (38). □

(K) Bin-PLD Distribution

Proposition 2.11. *Let* $X:\Omega\to\mathbb{N}^*$ *be a random variable. The pmf of* X *is* (42) *if and only if*

$$E\left\{[P(X)]^{-1}\mid X>k\right\} = \frac{\theta^2}{(1-p)\,Q(k)}. \tag{87}$$

Proof. If X has pmf (42), then for $k\in\mathbb{N}^*$, the left-hand side of (87), using infinite geometric sum formula, will be

$$\left(\frac{(1+\theta)}{Q(k)\,p^{k+1}}\right)\left(\frac{\theta^2}{(1+\theta)}\right)\sum_{x=k+1}^{\infty}p^x = \left(\frac{\theta^2}{Q(k)\,p^{k+1}}\right)\left(\frac{p^{k+1}}{1-p}\right) = \frac{\theta^2}{(1-p)\,Q(k)}.$$

Conversely, if (87) holds, then

$$\sum_{x=k+1}^{\infty}\left\{[P(x)]^{-1}\ f(x)\right\}=(1-F(k))\left(\frac{\theta^2}{(1-p)\,Q(k)}\right)$$
$$=(1-F(k+1)+f(k+1))\left(\frac{\theta^2}{(1-p)\,Q(k)}\right). \tag{88}$$

From (88), we also have

$$\sum_{x=k+2}^{\infty}\left\{[P(x)]^{-1}\ f(x)\right\}=(1-F(k+1))\left(\frac{\theta^2}{(1-p)\,Q(k+1)}\right). \tag{89}$$

Now, subtracting (89) from (88), yields

$$\left[[P(k+1)]^{-1}\right]f(k+1)=(1-F(k+1))\left\{\left(\frac{\theta^2}{(1-p)\,Q(k)}\right)-\left(\frac{\theta^2}{(1-p)\,Q(k+1)}\right)\right\}$$
$$+f(k+1)\left(\frac{\theta^2}{(1-p)\,Q(k)}\right),$$

or

$$f(k+1)\left[[P(k+1)]^{-1}-\left(\frac{\theta^2}{(1-p)\,Q(k)}\right)\right]$$
$$=(1-F(k+1))\left\{\left(\frac{\theta^2}{(1-p)\,Q(k)}\right)-\left(\frac{\theta^2}{(1-p)\,Q(k+1)}\right)\right\}.$$

From the above equality, after some computation, we have

$$\frac{f(k+1)}{1-F(k+1)}=\frac{\left(\frac{\theta^2}{(1-p)Q(k)}\right)-\left(\frac{\theta^2}{(1-p)Q(k+1)}\right)}{[P(k+1)]^{-1}-\left(\frac{\theta^2}{(1-p)Q(k)}\right)}=\frac{\theta^2P(k+1)}{Q(k+1)\,p},$$

which is the hazard function, (44) corresponding to the pmf (42), so X has pmf (42). □

(L) DOW-G Distribution

Proposition 2.12. *Let* $X:\Omega\rightarrow\mathbb{N}^*$ *be a random variable. The pmf of* X *is* (46) *if and only if*

$$E\left\{\left[p^{\left(\frac{G(X;\mathbf{\Phi})}{1-G(X;\mathbf{\Phi})}\right)^{\beta}}+p^{\left(\frac{G(X+1;\mathbf{\Phi})}{1-G(X+1;\mathbf{\Phi})}\right)^{\beta}}\right]\mid X>k\right\}=p^{\left(\frac{G(k+1;\mathbf{\Phi})}{1-G(k+1;\mathbf{\Phi})}\right)^{\beta}}. \tag{90}$$

Proof. If X has pmf (46), then for $k\in\mathbb{N}^*$, the left-hand side of (90), using telescoping sum formula, will be

$$\left(p^{-\left(\frac{G(k+1;\mathbf{\Phi})}{1-G(k+1;\mathbf{\Phi})}\right)^{\beta}}\right)\sum_{x=k+1}^{\infty}\left[p^{2\left(\frac{G(x;\mathbf{\Phi})}{1-G(x;\mathbf{\Phi})}\right)^{\beta}}-p^{2\left(\frac{G(x+1;\mathbf{\Phi})}{1-G(x+1;\mathbf{\Phi})}\right)^{\beta}}\right]$$
$$=\left(p^{-\left(\frac{G(k+1;\mathbf{\Phi})}{1-G(k+1;\mathbf{\Phi})}\right)^{\beta}}\right)\left[p^{2\left(\frac{G(k+1;\mathbf{\Phi})}{1-G(k+1;\mathbf{\Phi})}\right)^{\beta}}\right]=p^{\left(\frac{G(k+1;\mathbf{\Phi})}{1-G(k+1;\mathbf{\Phi})}\right)^{\beta}}.$$

Conversely, if (90) holds, then

$$\begin{aligned}&\sum_{x=k+1}^{\infty}\left\{\left[p^{\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^{\beta}}+p^{\left(\frac{G(x+1;\Phi)}{1-G(x+1;\Phi)}\right)^{\beta}}\right]f(x)\right\}\\&=(1-F(k))\left(p^{\left(\frac{G(k+1;\Phi)}{1-G(k+1;\Phi)}\right)^{\beta}}\right)\\&=(1-F(k+1)+f(k+1))\left(p^{\left(\frac{G(k+1;\Phi)}{1-G(k+1;\Phi)}\right)^{\beta}}\right).\end{aligned}\tag{91}$$

From (91), we also have

$$\sum_{x=k+2}^{\infty}\left\{\left[p^{\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^{\beta}}+p^{\left(\frac{G(x+1;\Phi)}{1-G(x+1;\Phi)}\right)^{\beta}}\right]f(x)\right\}=(1-F(x+1))\left(p^{\left(\frac{G(k+2;\Phi)}{1-G(k+2;\Phi)}\right)^{\beta}}\right).\tag{92}$$

Now, subtracting (92) from (91), yields

$$\begin{aligned}&\left[p^{\left(\frac{G(k+1;\Phi)}{1-G(k+1;\Phi)}\right)^{\beta}}+p^{\left(\frac{G(k+2;\Phi)}{1-G(k+2;\Phi)}\right)^{\beta}}\right]f(k+1)\\&=(1-F(k+1))\left\{p^{\left(\frac{G(k+1;\Phi)}{1-G(k+1;\Phi)}\right)^{\beta}}-p^{\left(\frac{G(k+2;\Phi)}{1-G(k+2;\Phi)}\right)^{\beta}}\right\}\\&+f(k+1)\left(p^{\left(\frac{G(k+1;\Phi)}{1-G(k+1;\Phi)}\right)^{\beta}}\right),\end{aligned}$$

or

$$f(k+1)\left[p^{\left(\frac{G(k+2;\Phi)}{1-G(k+2;\Phi)}\right)^{\beta}}\right]=(1-F(k+1))\left\{p^{\left(\frac{G(k+1;\Phi)}{1-G(k+1;\Phi)}\right)^{\beta}}-p^{\left(\frac{G(k+2;\Phi)}{1-G(k+2;\Phi)}\right)^{\beta}}\right\}.$$

From the above equality, after some computation, we have

$$\frac{f(k+1)}{1-F(k+1)}=\frac{p^{\left(\frac{G(k+1;\Phi)}{1-G(k+1;\Phi)}\right)^{\beta}}-p^{\left(\frac{G(k+2;\Phi)}{1-G(k+2;\Phi)}\right)^{\beta}}}{p^{\left(\frac{G(k+2;\Phi)}{1-G(k+2;\Phi)}\right)^{\beta}}}=\frac{p^{\frac{G(k+1;\Phi)}{1-G(k+1;\Phi)}}}{p^{\frac{G(k+2;\Phi)}{1-G(k+2;\Phi)}}}-1,$$

which is the hazard function, (48) corresponding to the pmf (46), so X has pmf (46). □

(M) DEBuH Distribution

Proposition 2.13. *Let* $X:\Omega\to\mathbb{N}^*$ *be a random variable. The pmf of* X *is* (50) *if and only if*

$$E\left\{\left[\left(1-\frac{p^{X+1}}{X+2}\right)^{\beta}+\left(1-\frac{p^{X}}{X+1}\right)^{\beta}\right]\mid X\le k\right\}=\left(1-\frac{p^{k+1}}{k+2}\right)^{\beta}.\tag{93}$$

Proof. If X has pmf (50), then for $k\in\mathbb{N}^*$, the left-hand side of (93), using telescoping sum formula, will be

$$\begin{aligned}&\sum_{x=0}^{k}\left[\left(1-\frac{p^{x+1}}{x+2}\right)^{2\beta}-\left(1-\frac{p^{x}}{x+1}\right)^{2\beta}\right]\\&=\left(1-\frac{p^{k+1}}{k+2}\right)^{-\beta}\left(1-\frac{p^{k+1}}{k+2}\right)^{2\beta}=\left(1-\frac{p^{k+1}}{k+2}\right)^{\beta}.\end{aligned}$$

Conversely, if (93) holds, then

$$\sum_{x=0}^{k}\left\{\left[\left(1-\frac{p^{x+1}}{x+2}\right)^{\beta}+\left(1-\frac{p^{x}}{x+1}\right)^{\beta}\right]\,f\left(x\right)\right\}=F\left(k\right)\left(1-\frac{p^{k+1}}{k+2}\right)^{\beta}. \tag{94}$$

From (94), we also have

$$\begin{aligned}\sum_{x=0}^{k-1}\left\{\left[\left(1-\frac{p^{x+1}}{x+2}\right)^{\beta}+\left(1-\frac{p^{x}}{x+1}\right)^{\beta}\right]\,f\left(x\right)\right\}&=F\left(k-1\right)\left(1-\frac{p^{k}}{k+1}\right)^{\beta}\\&=\left(F\left(k\right)-f\left(k\right)\right)\left(1-\frac{p^{k}}{k+1}\right)^{\beta}.\end{aligned} \tag{95}$$

Now, subtracting (95) from (94), yields

$$\begin{aligned}&\left[\left(1-\frac{p^{k+1}}{k+2}\right)^{\beta}+\left(1-\frac{p^{k}}{k+1}\right)^{\beta}\right]f\left(k\right)\\&=F\left(k\right)\left\{\left(1-\frac{p^{k+1}}{k+2}\right)^{\beta}-\left(1-\frac{p^{k}}{k+1}\right)^{\beta}\right\}+f\left(k\right)\left(1-\frac{p^{k}}{k+1}\right)^{\beta},\end{aligned}$$

or

$$f\left(k\right)\left[\left(1-\frac{p^{k+1}}{k+2}\right)^{\beta}\right]=F\left(k\right)\left\{\left(1-\frac{p^{k+1}}{k+2}\right)^{\beta}-\left(1-\frac{p^{k}}{k+1}\right)^{\beta}\right\}.$$

From the above equality, after some computation, we have

$$\frac{f\left(k\right)}{F\left(k\right)}=\frac{\left(1-\frac{p^{k+1}}{k+2}\right)^{\beta}-\left(1-\frac{p^{k}}{k+1}\right)^{\beta}}{\left(1-\frac{p^{k+1}}{k+2}\right)^{\beta}}=1-\frac{\left(1-\frac{p^{k}}{k+1}\right)^{\beta}}{\left(1-\frac{p^{k+1}}{k+2}\right)^{\beta}},$$

which is the hazard function, (52) corresponding to the pmf (50), so X has pmf (50). □

(N) BNPWE Distribution

Proposition 2.14. *Let* $X:\Omega\rightarrow\mathbb{N}^{*}$ *be a random variable. The pmf of* X *is* (54) *if and only if*

$$E\left\{\left[\left(\alpha+\beta+\alpha\theta\right)^{X+1}\right]\mid X>k\right\}=\frac{\alpha\left(1+\theta\right)\left(\alpha+\beta+\alpha\theta\right)^{k+1}}{1-\beta}. \tag{96}$$

Proof. If X has pmf (54), then for $k\in\mathbb{N}^{*}$, the left-hand side of (96), using infinite geometric sum formula, will be

$$\begin{aligned}\sum_{x=k+1}^{\infty}\left[\alpha\left(1+\theta\right)\beta^{x}\right]&=\left(\frac{\alpha+\beta+\alpha\theta}{\beta}\right)^{k+1}\alpha\left(1+\theta\right)\sum_{x=k+1}^{\infty}\beta^{x}\\&=\left(\frac{\alpha+\beta+\alpha\theta}{\beta}\right)^{k+1}\alpha\left(1+\theta\right)\left(\frac{\beta^{k+1}}{1-\beta}\right)\\&=\frac{\alpha\left(1+\theta\right)\left(\alpha+\beta+\alpha\theta\right)^{k+1}}{1-\beta}.\end{aligned}$$

Conversely, if (96) holds, then

$$\begin{aligned}
&\sum_{x=k+1}^{\infty}\left\{\left[(\alpha+\beta+\alpha\theta)^{x+1}\right]\ f(x)\right\} \\
&=(1-F(k))\left(\frac{\alpha(1+\theta)(\alpha+\beta+\alpha\theta)^{k+1}}{1-\beta}\right) \\
&=(1-F(k+1)+f(k+1))\left(\frac{\alpha(1+\theta)(\alpha+\beta+\alpha\theta)^{k+1}}{1-\beta}\right). \qquad (97)
\end{aligned}$$

From (97), we also have

$$\sum_{x=k+2}^{\infty}\left\{\left[(\alpha+\beta+\alpha\theta)^{x+1}\right]\ f(x)\right\}=(1-F(k+1))\left(\frac{\alpha(1+\theta)(\alpha+\beta+\alpha\theta)^{k+2}}{1-\beta}\right). \qquad (98)$$

Now, subtracting (98) from (97), yields

$$\begin{aligned}
&\left[(\alpha+\beta+\alpha\theta)^{k+2}\right]f(k+1) \\
&=(1-F(k+1))\left(\frac{\alpha(1+\theta)}{1-\beta}\right)\left\{(\alpha+\beta+\alpha\theta)^{k+1}-(\alpha+\beta+\alpha\theta)^{k+2}\right\} \\
&+f(k+1)\left(\frac{\alpha(1+\theta)}{1-\beta}\right)(\alpha+\beta+\alpha\theta)^{k+1},
\end{aligned}$$

or

$$\begin{aligned}
&f(k+1)\left[(\alpha+\beta+\alpha\theta)^{k+2}-\left(\frac{\alpha(1+\theta)}{1-\beta}\right)(\alpha+\beta+\alpha\theta)^{k+1}\right] \\
&=(1-F(k+1))\left(\frac{\alpha(1+\theta)}{1-\beta}\right)\left\{(\alpha+\beta+\alpha\theta)^{k+1}-(\alpha+\beta+\alpha\theta)^{k+2}\right\}.
\end{aligned}$$

From the above equality, after some computation, we have

$$\frac{f(k+1)}{1-F(k+1)}=\alpha(1+\theta)\beta^{-1},$$

which is the hazard function, (56) corresponding to the pmf (54), so X has pmf (54). □

2.2. Characterizations of distributions based on reverse hazard

This subsection deals with three distributions listed in the Introduction. The characterization presented here is in terms of the reverse hazard function.

Proposition 2.15. *Let* $X:\Omega\rightarrow\mathbb{N}$ *be a random variable. The pmf of* X *is* (18) *if and only if its reverse hazard function,* r_F*, satisfies the difference equation*

$$r_F(k+1)-r_F(k)=\frac{P(k+1)\,p^{k+1}}{\sum_{x=1}^{k+1}P(x)\,p^{x}}-\frac{P(k)\,p^{k}}{\sum_{x=1}^{k}P(x)\,p^{x}},\quad k\in\mathbb{N}, \qquad (99)$$

with the initial condition $r_F(1)=1$.

Proof. If X has pmf (18), then clearly (99) holds. Now, if (99) holds, then for every $x\in\mathbb{N}$, we have

$$\sum_{k=1}^{x-1}\left\{r_F(k+1)-r_F(k)\right\}=\sum_{k=1}^{x-1}\left\{\frac{P(k+1)\,p^{k+1}}{\sum_{x=1}^{k+1}P(x)\,p^{x}}-\frac{P(k)\,p^{k}}{\sum_{x=1}^{k}P(x)\,p^{x}}\right\},$$

or, using telescoping sum

$$r_F(x) - r_F(1) = \frac{P(x)\,p^x}{\sum_{u=1}^{x} P(u)\,p^u} - 1,$$

or in view of the initial condition

$$r_F(x) = \frac{P(x)\,p^x}{\sum_{u=1}^{x} P(u)\,p^u}, \quad x \in \mathbb{N},$$

which is the reverse hazard function corresponding to the pmf (18). □

Proposition 2.16. *Let* $X : \Omega \to I$ *be a random variable. The pmf of* X *is* (38) *if and only if its reverse hazard function,* r_F*, satisfies the difference equation*

$$r_F(k+1) - r_F(k) = \frac{P(k+1)\,p^{k+1}}{\sum_{x=1}^{k+1} P(x)\,p^x} - \frac{P(k)\,p^k}{\sum_{x=1}^{k} P(x)\,p^x}, \quad k \in I, \tag{100}$$

with the initial condition $r_F(0) = (1-p)^m$.

Proof. If X has pmf (38), then clearly (100) holds. Now, if (100) holds, then for every $x \in I \setminus \{0\}$, we have

$$\sum_{k=0}^{x-1} \{r_F(k+1) - r_F(k)\} = \sum_{k=0}^{x-1} \left\{ \frac{P(k+1)\,p^{k+1}}{\sum_{x=1}^{k+1} P(x)\,p^x} - \frac{P(k)\,p^k}{\sum_{x=1}^{k} P(x)\,p^x} \right\},$$

or, using telescoping sum

$$r_F(x) - r_F(0) = \frac{P(x)\,p^x}{\sum_{u=1}^{x} P(u)\,p^u} - (1-p)^m,$$

or in view of the initial condition

$$r_F(x) = \frac{P(x)\,p^x}{\sum_{u=1}^{x} P(u)\,p^u}, \quad x \in I,$$

which is the reverse hazard function corresponding to the pmf (38). □

Proposition 2.17. *Let* $X : \Omega \to \mathbb{N}^*$ *be a random variable. The pmf of* X *is* (50) *if and only if its reverse hazard function,* r_F*, satisfies the difference equation*

$$r_F(k+1) - r_F(k) = \frac{\left(1 - \frac{p^k}{k+1}\right)^\beta}{\left(1 - \frac{p^{k+1}}{k+2}\right)^\beta} - \frac{\left(1 - \frac{p^{k+1}}{k+2}\right)^\beta}{\left(1 - \frac{p^{k+2}}{k+3}\right)^\beta}, \quad k \in \mathbb{N}^*, \tag{101}$$

with the initial condition $r_F(0) = 1$.

Proof. If X has pmf (50), then clearly (101)holds. Now, if (101) holds, then for every $x \in \mathbb{N}$, we have

$$\sum_{k=0}^{x-1} \{r_F(k+1) - r_F(k)\} = \sum_{k=0}^{x-1} \left\{ \frac{\left(1 - \frac{p^k}{k+1}\right)^\beta}{\left(1 - \frac{p^{k+1}}{k+2}\right)^\beta} - \frac{\left(1 - \frac{p^{k+1}}{k+2}\right)^\beta}{\left(1 - \frac{p^{k+2}}{k+3}\right)^\beta} \right\},$$

or, using telescoping sum

$$r_F(x) - r_F(0) = -\frac{\left(1 - \frac{p^x}{x+1}\right)^\beta}{\left(1 - \frac{p^{x+1}}{x+2}\right)^\beta},$$

or in view of the initial condition

$$r_F(x) = 1 - \frac{\left(1 - \frac{p^x}{x+1}\right)^\beta}{\left(1 - \frac{p^{x+1}}{x+2}\right)^\beta}, \quad x \in \mathbb{N}^*,$$

which is the reverse hazard function corresponding to the pmf (50). □

2.3. Characterizations of distributions based on hazard function

This subsection is devoted to 10 distribution listed in the Introduction. The characterizations presented here are in terms of the hazard function.

Proposition 2.18. *Let* $X : \Omega \to \mathbb{N}^*$ *be a random variable. The pmf of* X *is* (2) *if and only if its hazard function satisfies the difference equation*

$$h_F(k+1) - h_F(k) = \frac{(1+2(k+1))^\delta (2+(k+1))^{2\delta}}{(1+(k+1))^{2\delta}(3+2(k+1))^\delta} - \frac{(1+2k)^\delta (2+k)^{2\delta}}{(1+k)^{2\delta}(3+2k)^\delta}, \tag{102}$$

$x \in \mathbb{N}$, *with the initial condition* $h_F(1) = \left(\frac{27}{20}\right)^\delta - 1$.

Proof. If X has pmf (2), then clearly (102) holds. Now, if (102) holds, then for every $x \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{k=1}^{x-1}\{h_F(k+1) - h_F(k)\} &= \sum_{k=1}^{x-1}\left\{\frac{(1+2(k+1))^\delta (2+(k+1))^{2\delta}}{(1+(k+1))^{2\delta}(3+2(k+1))^\delta} - \frac{(1+2k)^\delta (2+k)^{2\delta}}{(1+k)^{2\delta}(3+2k)^\delta}\right\} \\ &= \frac{(1+2k)^\delta (2+k)^{2\delta}}{(1+k)^{2\delta}(3+2k)^\delta} - \left(\frac{27}{20}\right)^\delta,\end{aligned}$$

or

$$h_F(x) - h_F(1) = \frac{(1+2x)^\delta (2+x)^{2\delta}}{(1+x)^{2\delta}(3+2x)^\delta} - \left(\frac{27}{20}\right)^\delta,$$

or, in view of the initial condition

$$h_F(x) = \frac{(1+2x)^\delta (2+x)^{2\delta}}{(1+x)^{2\delta}(3+2x)^\delta} - 1, \quad x \in \mathbb{N},$$

which is the hazard function, (4), corresponding to the pmf (2). □

Proposition 2.19. *Let* $X : \Omega \to \mathbb{N}^*$ *be a random variable. The pmf of* X *is* (10) *if and only if its hazard function satisfies the difference equation*

$$h_F(k+1) - h_F(k) = \frac{\left\{1 - \left[\frac{1-p^{k+1}}{1-\overline{\alpha}p^{k+1}}\right]\right\}^b}{\left\{1 - \left[\frac{1-p^{k+2}}{1-\overline{\alpha}p^{k+2}}\right]\right\}^b} - \frac{\left\{1 - \left[\frac{1-p^{k}}{1-\overline{\alpha}p^{k}}\right]\right\}^b}{\left\{1 - \left[\frac{1-p^{k+1}}{1-\overline{\alpha}p^{k+1}}\right]\right\}^b}, \quad x \in \mathbb{N}^* \tag{103}$$

with the initial condition $h_F(0) = \left\{1 - \left[\frac{1-p}{1-\overline{\alpha}p}\right]\right\}^{-b} - 1$.

Proof. If X has pmf (10), then clearly (103) holds. Now, if (103) holds, then for every $x \in \mathbb{N}^*$, we have

$$\sum_{k=0}^{x-1}\left\{h_F\left(k+1\right)-h_F\left(k\right)\right\}=\sum_{k=0}^{x-1}\left\{\frac{\left\{1-\left[\frac{1-p^{k+1}}{1-\overline{\alpha}p^{k+1}}\right]\right\}^b}{\left\{1-\left[\frac{1-p^{k+2}}{1-\overline{\alpha}p^{k+2}}\right]\right\}^b}-\frac{\left\{1-\left[\frac{1-p^{k}}{1-\overline{\alpha}p^{k}}\right]\right\}^b}{\left\{1-\left[\frac{1-p^{k+1}}{1-\overline{\alpha}p^{k+1}}\right]\right\}^b}\right\}$$
$$=\frac{\left\{1-\left[\frac{1-p^{x}}{1-\overline{\alpha}p^{x}}\right]\right\}^b}{\left\{1-\left[\frac{1-p^{x+1}}{1-\overline{\alpha}p^{x+1}}\right]\right\}^b}-\frac{1}{\left\{1-\left[\frac{1-p}{1-\overline{\alpha}p}\right]\right\}^b},$$

or

$$h_F\left(x\right)-h_F\left(0\right)=\frac{\left\{1-\left[\frac{1-p^{x}}{1-\overline{\alpha}p^{x}}\right]\right\}^b}{\left\{1-\left[\frac{1-p^{x+1}}{1-\overline{\alpha}p^{x+1}}\right]\right\}^b}-\frac{1}{\left\{1-\left[\frac{1-p}{1-\overline{\alpha}p}\right]\right\}^b},$$

or, in view of the initial condition

$$h_F\left(x\right)=\frac{\left\{1-\left[\frac{1-p^{x}}{1-\overline{\alpha}p^{x}}\right]\right\}^b}{\left\{1-\left[\frac{1-p^{x+1}}{1-\overline{\alpha}p^{x+1}}\right]\right\}^b}-1,\quad x\in\mathbb{N}^*,$$

which is the hazard function, (12), corresponding to the pmf (10). □

Proposition 2.20. *Let* $X:\Omega\to\mathbb{N}$ *be a random variable. The pmf of* X *is* (14) *if and only if its hazard function satisfies the difference equation*

$$h_F\left(k+1\right)-h_F\left(k\right)=\frac{\theta\left(\theta\beta+\beta-1+\theta(k+1)\right)}{\left(\theta\beta+\theta+\beta+\theta(k+1)\right)}-\frac{\theta\left(\theta\beta+\beta-1+\theta k\right)}{\left(\theta\beta+\theta+\beta+\theta k\right)},k\in\mathbb{N},\tag{104}$$

with the initial condition $h_F\left(1\right)=\frac{\theta(\theta\beta+\beta-1+\theta)}{(\theta\beta+\theta+\beta+\theta)}$.

Proof. If X has pmf (14), then clearly (104) holds. Now, if (104) holds, then for every $x\in\mathbb{N}$, we have

$$\sum_{k=1}^{x-1}\left\{h_F\left(k+1\right)-h_F\left(k\right)\right\}=\sum_{k=1}^{x-1}\left\{\frac{\theta\left(\theta\beta+\beta-1+\theta(k+1)\right)}{\left(\theta\beta+\theta+\beta+\theta(k+1)\right)}-\frac{\theta\left(\theta\beta+\beta-1+\theta k\right)}{\left(\theta\beta+\theta+\beta+\theta k\right)}\right\}$$
$$=\frac{\theta\left(\theta\beta+\beta-1+\theta x\right)}{\left(\theta\beta+\theta+\beta+\theta x\right)}-\frac{\theta\left(\theta\beta+\beta-1+\theta\right)}{\left(\theta\beta+\theta+\beta+\theta\right)},$$

or

$$h_F\left(x\right)-h_F\left(0\right)=\frac{\theta\left(\theta\beta+\beta-1+\theta x\right)}{\left(\theta\beta+\theta+\beta+\theta x\right)}-\frac{\theta\left(\theta\beta+\beta-1+\theta\right)}{\left(\theta\beta+\theta+\beta+\theta\right)},$$

or, in view of the initial condition

$$h_F\left(x\right)=\frac{\theta\left(\theta\beta+\beta-1+\theta x\right)}{\left(\theta\beta+\theta+\beta+\theta x\right)},\quad x\in\mathbb{N},$$

which is the hazard function, (16), corresponding to the pmf (14). □

Proposition 2.21. *Let* $X:\Omega\to\mathbb{N}^*$ *be a random variable. The pmf of* X *is* (22) *if and only if its hazard function satisfies the difference equation*

$$h_F\left(k+1\right)-h_F\left(k\right)=\theta^2\left\{\frac{P\left(k+1\right)}{Q\left(k+1\right)}-\frac{P\left(k+1\right)}{Q\left(k+1\right)}\right\},\quad k\in\mathbb{N}^*,\tag{105}$$

with the initial condition $h_F(0) = \frac{\theta^2 P(0)}{Q(0)}$.

Proof. If X has pmf (22), then clearly (105) holds. Now, if (105) holds, then for every $x \in \mathbb{N}$, we have

$$\sum_{k=0}^{x-1} \{h_F(k+1) - h_F(k)\} = \theta^2 \sum_{k=0}^{x-1} \left\{ \frac{P(k+1)}{Q(k+1)} - \frac{P(k)}{Q(k)} \right\} = \theta^2 \left\{ \frac{P(x)}{Q(x)} - \frac{P(0)}{Q(0)} \right\},$$

or

$$h_F(x) - h_F(0) = \theta^2 \left\{ \frac{P(x)}{Q(x)} - \frac{P(0)}{Q(0)} \right\},$$

or, in view of the initial condition

$$h_F(x) = \theta^2 \left\{ \frac{P(x)}{Q(x)} \right\}, \quad x \in \mathbb{N}^*,$$

which is the hazard function, (24), corresponding to the pmf (22). □

Proposition 2.22. *Let* $X : \Omega \to \mathbb{N}^*$ *be a random variable. The pmf of* X *is* (26) *if and only if its hazard function satisfies the difference equation*

$$h_F(k+1) - h_F(k) = \frac{\alpha[P(k+1) - \eta P(k+2)]}{P(k+2)\eta} - \frac{\alpha[P(k) - \eta P(k+1)]}{P(k+1)\eta}, \quad k \in \mathbb{N}^*, \tag{106}$$

with the initial condition $h_F(0) = \frac{\alpha[P(0)-\eta P(1)]}{P(1)\eta}$.

Proof. If X has pmf (26), then clearly (106) holds. Now, if (106) holds, then for every $x \in \mathbb{N}$, we have

$$\sum_{k=0}^{x-1} \{h_F(k+1) - h_F(k)\} = \sum_{k=0}^{x-1} \left\{ \frac{\alpha[P(k+1) - \eta P(k+2)]}{P(k+2)\eta} - \frac{\alpha[P(k) - \eta P(k+1)]}{P(k+1)\eta} \right\},$$

or

$$h_F(x) - h_F(0) = \frac{\alpha[P(x) - \eta P(x+1)]}{P(x+1)\eta} - \frac{\alpha[P(0) - \eta P(1)]}{P(1)\eta},$$

or, in view of the initial condition

$$h_F(x) = \frac{\alpha[P(x) - \eta P(x+1)]}{P(x+1)\eta}, \quad x \in \mathbb{N}^*,$$

which is the hazard function, (28), corresponding to the pmf (26). □

Proposition 2.23. *Let* $X : \Omega \to \mathbb{N}^*$ *be a random variable. The pmf of* X *is* (30) *if and only if its hazard function satisfies the difference equation*

$$h_F(k+1) - h_F(k) = \left(\frac{\left(1 + (k+1) + 2p - p^2\right)}{(1-p)\left(3 + (k+1) + p - p^2\right)} \right) - \left(\frac{\left(1 + k + 2p - p^2\right)}{(1-p)\left(3 + k + p - p^2\right)} \right), \tag{107}$$

with the initial condition $h_F(0) = \frac{1+2p-p^2}{(1-p)(3+p-p^2)}$.

Proof. If X has pmf (30), then clearly (107) holds. Now, if (107) holds, then for every $x \in \mathbb{N}$, we have

$$\sum_{k=0}^{x-1} \{h_F(k+1) - h_F(k)\}$$
$$= \left(\frac{1}{1-p}\right) \sum_{k=0}^{x-1} \left\{ \left(\frac{(1 + (k+1) + 2p - p^2)}{(3 + (k+1) + p - p^2)} \right) - \left(\frac{(1 + k + 2p - p^2)}{(3 + k + p - p^2)} \right) \right\},$$

or

$$h_F(x) - h_F(0) = \left(\frac{1}{1-p}\right) \left\{ \left(\frac{(1 + x + 2p - p^2)}{(3 + x + p - p^2)} \right) - \left(\frac{(1 + 2p - p^2)}{(3 + p - p^2)} \right) \right\},$$

or, in view of the initial condition

$$h_F(x) = \left(\frac{(1 + x + 2p - p^2)}{(1-p)(3 + x + p - p^2)} \right), \quad x \in \mathbb{N}^*,$$

which is the hazard function, (32), corresponding to the pmf (30). □

Proposition 2.24. *Let $X : \Omega \to \mathbb{N}^*$ be a random variable. The pmf of X is* (34) *if and only if its hazard function satisfies the difference equation*

$$h_F(k+1) - h_F(k) = \left(\frac{2p + \theta \log(p^k) - 1}{p[1 + \theta \log(p^{k+1})]} \right) - \left(\frac{2p + \theta \log(p^{k+1}) - 1}{p[1 + \theta \log(p^{k+2})]} \right), \tag{108}$$

with the initial condition $h_F(0) = 1 - \frac{2p-1}{p(1+\theta \log p)}$.

Proof. If X has pmf (34), then clearly (108) holds. Now, if (108) holds, then for every $x \in \mathbb{N}$, we have

$$\sum_{k=0}^{x-1} \{h_F(k+1) - h_F(k)\}$$
$$= \left(\frac{1}{p}\right) \sum_{k=0}^{x-1} \left\{ \left(\frac{2p + \theta \log(p^k) - 1}{[1 + \theta \log(p^{k+1})]} \right) - \left(\frac{2p + \theta \log(p^{k+1}) - 1}{[1 + \theta \log(p^{k+2})]} \right) \right\},$$

or

$$h_F(x) - h_F(0) = \left(\frac{1}{p}\right) \left\{ \left(\frac{2p - 1}{[1 + \theta \log(p)]} \right) - \left(\frac{2p + \theta \log(p^x) - 1}{[1 + \theta \log(p^{x+1})]} \right) \right\},$$

or, in view of the initial condition

$$h_F(x) = 1 - \left(\frac{2p + \theta \log(p^x) - 1}{p[1 + \theta \log(p^{x+1})]} \right), \quad x \in \mathbb{N}^*,$$

which is the hazard function, (36), corresponding to the pmf (34). □

Proposition 2.25. *Let $X : \Omega \to \mathbb{N}^*$ be a random variable. The pmf of X is* (42) *if and only if its hazard function satisfies the difference equation*

$$h_F(k+1) - h_F(k) = \frac{\theta^2 P(k+1)}{Q(k+1)p} - \frac{\theta^2 P(k)}{Q(k)p}, \tag{109}$$

with the initial condition $h_F(0) = \frac{\theta^2 P(0)}{Q(0)p}$.

Proof. If X has pmf (42), then clearly (109) holds. Now, if (109) holds, then for every $x \in \mathbb{N}$, we have

$$\sum_{k=0}^{x-1}\left\{h_F\left(k+1\right)-h_F\left(k\right)\right\}=\left(\frac{\theta^2}{p}\right)\sum_{k=0}^{x-1}\left\{\frac{P\left(k+1\right)}{Q\left(k+1\right)}-\frac{P\left(k\right)}{Q\left(k\right)}\right\},$$

or

$$h_F\left(x\right)-h_F\left(0\right)=\left(\frac{\theta^2}{p}\right)\left\{\frac{P\left(x\right)}{Q\left(x\right)}-\frac{P\left(0\right)}{Q\left(0\right)}\right\},$$

or, in view of the initial condition

$$h_F\left(x\right)=\frac{\theta^2 P\left(x\right)}{Q\left(x\right)p},\quad x\in\mathbb{N}^*,$$

which is the hazard function, (44), corresponding to the pmf (42). ☐

Proposition 2.26. *Let* $X:\Omega\rightarrow\mathbb{N}^*$ *be a random variable. The pmf of* X *is* (46) *if and only if its hazard function satisfies the difference equation*

$$h_F\left(k+1\right)-h_F\left(k\right)=\frac{p^{\left(\frac{G(k+1;\Phi)}{1-G(k+1;\Phi)}\right)^\beta}}{p^{\left(\frac{G(k+2;\Phi)}{1-G(k+2;\Phi)}\right)^\beta}}-\frac{p^{\left(\frac{G(k;\Phi)}{1-G(k;\Phi)}\right)^\beta}}{p^{\left(\frac{G(k+1;\Phi)}{1-G(k+1;\Phi)}\right)^\beta}},\tag{110}$$

with the initial condition $h_F\left(0\right)=p^{-\left(\frac{G(1;\Phi)}{1-G(1;\Phi)}\right)^\beta}-1.$

Proof. If X has pmf (46), then clearly (110) holds. Now, if (110) holds, then for every $x\in\mathbb{N}$, we have

$$\sum_{k=0}^{x-1}\left\{h_F\left(k+1\right)-h_F\left(k\right)\right\}=\sum_{k=0}^{x-1}\left\{\frac{p^{\left(\frac{G(k+1;\Phi)}{1-G(k+1;\Phi)}\right)^\beta}}{p^{\left(\frac{G(k+2;\Phi)}{1-G(k+2;\Phi)}\right)^\beta}}-\frac{p^{\left(\frac{G(k;\Phi)}{1-G(k;\Phi)}\right)^\beta}}{p^{\left(\frac{G(k+1;\Phi)}{1-G(k+1;\Phi)}\right)^\beta}}\right\},$$

or

$$h_F\left(x\right)-h_F\left(0\right)=\frac{p^{\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^\beta}}{p^{\left(\frac{G(x+1;\Phi)}{1-G(x+1;\Phi)}\right)^\beta}}-p^{-\left(\frac{G(1;\Phi)}{1-G(1;\Phi)}\right)^\beta},$$

or, in view of the initial condition

$$h_F\left(x\right)=\frac{p^{\left(\frac{G(x;\Phi)}{1-G(x;\Phi)}\right)^\beta}}{p^{\left(\frac{G(x+1;\Phi)}{1-G(x+1;\Phi)}\right)^\beta}}-1,\quad x\in\mathbb{N}^*,$$

which is the hazard function, (48), corresponding to the pmf (46). ☐

Proposition 2.27. *Let* $X:\Omega\rightarrow\mathbb{N}^*$ *be a random variable. The pmf of* X *is* (54) *if and only if its hazard function satisfies the difference equation*

$$h_F\left(k+1\right)=h_F\left(k\right)=\alpha\left(1+\theta\right)\beta^{-1},\tag{111}$$

with the initial condition $h_F\left(0\right)=\alpha\left(1+\theta\right)\beta^{-1}.$

Proof. If X has pmf (54), then clearly (111) holds. Now, if (111) holds, then for every $x\in\mathbb{N}^*$, then for $x=0$, we have $h_F\left(0\right)=\alpha\left(1+\theta\right)\beta^{-1}$ and $f\left(0\right)=F\left(0\right)=\alpha\left(1+\theta\right)\beta^{-1}$. Now, following the proof of the Proposition 3.3.1 on page 43 of (13), we arrive at pmf (54). ☐

Final Remark. After the completion of the present work, we came across the "Comment" by (17) regarding the reparametrization of the "Uniform Poisson-Ailamujia Discrete" (UPAD) distribution of (2), who proposed their distribution as

$$F\left(x;\alpha\right)=1-\left(1+2\alpha\right)^{-(x+1)},\quad x\in\mathbb{N}^*,$$

with the corresponding pmf

$$f(x;\alpha) = 2\alpha (1 + 2\alpha)^{-(x+1)}, \quad x \in \mathbb{N}^*,$$

where $\alpha > 0$ is a parameter.
Clearly, we can state the characterizations of UPAD similar to the ones in Subsections 2.1 and 2.3. (17) showed that taking $\theta(\alpha) = \frac{2\alpha}{1+2\alpha}$,the cdf $F(x;\alpha)$ can be expressed as a geometric distribution with parameter $\theta \in (0,1)$ which has been characterized in our previous work.

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