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The Gamma Odd Burr X-G Family of Distributions with Applications

Bakang Tlhaloganyang[1*], Whatmore Sengweni[1], Broderick Oluyede[1]

*Corresponding Author

1 Department of Mathematics & Statistical Sciences,
Botswana International University of Science and Technology, Botswana. tlhaloganyangs@gmail.com

Abstract

A new family of distributions called Gamma Odd Burr X-G (GOBX-G) distribution is introduced in this paper. Its structural properties such as the survival function, hazard function, density expansion, quantile function, moments and generating functions, incomplete moments, probability weighted moments, Rényi entropy and order statistics were derived. Maximum likelihood technique is used to estimate the parameters of this model. The flexibility and applicability of this model is demonstrated using real life datasets.

Key Words: Family of distributions; Gamma distribution; Odd Burr X-G distribution; Weibull distribution; Maximum likelihood estimation.

Mathematical Subject Classification: 62E99, 60E05.

1. Introduction

Lifetime models are very useful for describing and predicting real world phenomena found in medicine, sciences, economics and many other areas. The usefulness of these models is usually seen in products lifetime evaluations such as in reliability and survival analysis. Numerous studies have been conducted to find suitable lifetime models that best describes the real world phenomena. For the past decade, researches mainly focused in developing new family of distributions through addition of extra parameter/s in the traditional and common families of continuous distributions. These added parameters have shown to be producing much flexible models that fits real world datasets better than traditional models (Barreto-Souza et al., 2013).

Some of these families of distributions includes the Exponentiated Generalized Power Series (Oluyede et al., 2020), Exponentiated Half Logistic-Log-Logistic Weibull (Chamunorwa et al., 2021), Exponentiated Half-logistic Odd Lindley-G (Sengweni et al., 2021), Weibull odd Burr III-G (Peter et al., 2021b) and Gamma Kumaraswamy-G (Arshad et al., 2020). Peter et al. (2021a) studied Gamma Odd Burr III-G distribution, Al-Babtain et al. (2021a) studied transmuted Burr X-G (TBX-G), Arshad et al. (2020) studied Gamma Kumaraswamy-G and Lahcene (2021) also developed an extended-Gamma family of distributions. Additionally, Burr X distribution was also extended to Exponentiated Generalized Burr X distribution (Khaleel et al., 2018), Weibull Burr X distribution (Ibrahim et al., 2017), Exponentiated Burr X distribution (Ahmed et al., 2021), Beta Burr X distribution (Merovci et al., 2016), Power Burr Type X distribution (Usman and Ilyas, 2020) and Type I Half-Logistic Burr X distribution (Shrahili et al., 2019). Khaleel et al. (2016) also developed a Gamma Burr type X distribution via the Gamma generator and a two parameter Burr X distribution introduced by Surles and Padgett (2001). Other recently developed models includes the discrete Poisson-Lindley and discrete Lindley distributions by Al-Babtain et al. (2021b), the Extended Power-Lindley distribution by Al-Babtain et al. (2021c) and a two parameter Burr-Hatke distribution by Afify et al. (2021).

The motivation behind developing this family of distributions is to improve the general performance of classical distributions, specifically the Gamma and the Odd Burr X distributions. We aim to provide models which can handle skewed and heavy tailed data sets compared to other competitive models. Thus, the generalization of the Gamma distribution and Odd Burr X distribution provided a very flexible family of distributions which has a density function with different shapes such as the left-skewed, almost symmetric, reversed-J shape and right-skewed. The hazard rate function of this new family of distribution can capture increasing, decreasing, bathtub and upside-down shapes. For this reason, we developed a generalized lifetime family of distributions called the Gamma Odd Burr X-G (GOBX-G) distribution made up of the combination of the Gamma generator proposed by Ristić and Balakrishnan (2012) and the Odd Burr X-G family of distributions proposed by Yousof et al. (2017).

The rest of the paper is organized as follows; the new family of distributions is introduced in Section 2 together with its survival function, hazard function, cumulative hazard function, series expansion, and quantile function. Section 3 covers special cases for selected baseline distributions. Section 4 covers derivation of some mathematical properties such as moments, generating functions, incomplete moments, Rényi entropy, order statistics and probability weighted moments. Section 5 covers maximum likelihood estimation. In Section 6, simulation study is conducted to evaluate the performance of the proposed maximum likelihood estimators. Real life applications are considered in Section 7 to evaluate the flexibility of our model followed by some concluding remarks in Section 8.

2. The New Model

Yousof et al. (2017) developed a one parameter Odd Burr X-G family of distributions by using a one parameter Burr type X distribution introduced by Burr (1942). The cumulative distribution function (cdf) and probability density function (pdf) of Odd Burr X-G distribution is given by

$$\mathrm{F}(x;\theta,\xi) = 2\theta \int_0^{\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}} t\exp(-t^2)\left[1-\exp(-t^2)\right]^{\theta-1} dt = \left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]\right\}^{\theta}$$

and

$$\mathrm{f}(x;\theta,\xi) = \frac{2\theta \mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}^3(x;\xi)}\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]\right\}^{\theta-1},$$

respectively, where $\overline{\mathrm{G}}(x;\xi) = 1-\mathrm{G}(x;\xi)$; $\mathrm{G}(x;\xi)$ and $\mathrm{g}(x;\xi)$ are the cdf and pdf of any baseline distribution. Ristić and Balakrishnan (2012) developed the Gamma-G family of distributions with cdf and pdf given by

$$\mathrm{F}(x;\delta) = 1-\frac{1}{\Gamma(\delta)}\int_0^{-\log(\mathrm{G}(x))} t^{\delta-1}e^{-t}dt,\ \delta>0 \tag{1}$$

and

$$\mathrm{f}(x;\delta) = \frac{1}{\Gamma(\delta)}\left[-\log\left(\mathrm{G}(x)\right)\right]^{\delta-1} g(x)\ , x\in\mathbb{R}. \tag{2}$$

In equation (1) and (2), if we let $\mathrm{G}(x) = \mathrm{F}(x;\theta,\xi)$ and $\mathrm{g}(x) = \mathrm{f}(x;\theta,\xi)$, we get a new family of distributions called the Gamma Odd Burr X-G (GOBX-G) family of distributions with the cdf and pdf given as

$$\begin{aligned} \mathrm{F}(x;\delta,\theta,\xi) &= 1-\frac{1}{\Gamma(\delta)}\int_0^{-\log\left(\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]\right\}^{\theta}\right)} t^{\delta-1}e^{-t}dt \\ &= 1-\frac{\gamma\left(\delta,-\theta\log\left(1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]\right)\right)}{\Gamma(\delta)} \end{aligned} \tag{3}$$

and

$$\begin{aligned} \mathrm{f}(x;\delta,\theta,\xi) &= \frac{2\theta \mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{\Gamma(\delta)\,\overline{\mathrm{G}}^3(x;\xi)}\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]\right\}^{\theta-1} \\ &\times \left[-\theta\log\left(1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]\right)\right]^{\delta-1}, \end{aligned} \tag{4}$$

respectively, for $x,\delta,\theta>0$ and ξ being a vector of parameters from the baseline distribution, where $\gamma(\delta,z)=\int_0^z t^{\delta-1}e^{-t}dt$ is the lower incomplete gamma function. The survival function of the GOBX-G family of distributions is given as

$$\mathrm{S}(x;\delta,\theta,\xi)=\overline{\mathrm{F}}(x;\delta,\theta,\xi)=\frac{\gamma\left(\delta,-\theta\log\left(1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]\right)\right)}{\Gamma(\delta)}.$$

The hazard rate function (hrf) and the cumulative hazard function (chf) of GOBX-G family of distributions are respectively given by

$$\begin{aligned} \mathrm{h}(x;\delta,\theta,\xi) &= \frac{2\theta \mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}^3(x;\xi)}\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]\right\}^{\theta-1} \\ &\times \left[-\theta\log\left(1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]\right)\right]^{\delta-1} \\ &\times \left\{\gamma\left(\delta,-\theta\log\left(1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]\right)\right)\right\}^{-1} \end{aligned}$$

and

$$\mathrm{H}(x;\delta,\theta,\xi)=-\ln\left[\frac{\gamma\left(\delta,-\theta\log\left(1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]\right)\right)}{\Gamma(\delta)}\right].$$

2.1. Series Expansion of the GOBX-G Density Function

A series expansion of the GOBX-G density follows in this section. If we let $y=\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]$, then we can write equation (4) as

$$\mathrm{f}(x;\delta,\theta,\xi)=\frac{2\theta^{\delta}\mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{\Gamma(\delta)\,\overline{\mathrm{G}}^3(x;\xi)}\;y\,(1-y)^{\theta-1}\left[-\log(1-y)\right]^{\delta-1}.$$

Using series representation $-\log(1-y) = \sum_{i=0}^{\infty} \frac{y^{i+1}}{i+1}$, for $y \in (0,1)$, we have

$$[-\log(1-y)]^{\delta-1} = y^{\delta-1}\left[\sum_{m=0}^{\infty}\binom{\delta-1}{m}y^m\left(\sum_{s=0}^{\infty}\frac{y^s}{s+2}\right)^m\right].$$

Thus, we can write the GOBX-G density as

$$\mathrm{f}(x;\delta,\theta,\xi) = \frac{2\theta^\delta \mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{\Gamma(\delta)\,\overline{\mathrm{G}}^3(x;\xi)}\; y^\delta\,(1-y)^{\theta-1}\left[\sum_{m=0}^{\infty}\binom{\delta-1}{m}y^m\left(\sum_{s=0}^{\infty}\frac{y^s}{s+2}\right)^m\right].$$

Next, let $a_s = (s+2)^{-1}$ and considering the result of a power series raised to a positive integer we get

$$\left(\sum_{s=0}^{\infty}a_s y^s\right)^m = \sum_{s=0}^{\infty}b_{s,m}y^s\,, \tag{5}$$

where $b_{s,m} = (sa_0)^{-1}\sum_{l=1}^{\infty}[m(l+1)-s]a_l b_{s-l,m}$ and $b_{0,m} = a_0^m$ (see Grandshteyn and Ryzhik (1980)). The GOBX-G density function then simplifies to

$$\begin{aligned}\mathrm{f}(x;\delta,\theta,\xi) &= \frac{2\theta^\delta \mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{\Gamma(\delta)\,\overline{\mathrm{G}}^3(x;\xi)}\; y^\delta\,(1-y)^{\theta-1}\left[\sum_{m,s=0}^{\infty}\binom{\delta-1}{m}y^m b_{s,m}y^s\right]\\ &= \frac{2\theta^\delta \mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{\Gamma(\delta)\,\overline{\mathrm{G}}^3(x;\xi)}\;(1-y)^{\theta-1}\left[\sum_{m,s=0}^{\infty}\binom{\delta-1}{m}y^{m+s+\delta}b_{s,m}\right].\end{aligned}$$

Applying the generalized binomial series representation $(1-z)^n = \sum_{i=0}^{\infty}(-1)^i\binom{n}{i}z^i$ which is valid for $|z|<1$, we can express the GOBX-G density as

$$\begin{aligned}\mathrm{f}(x;\delta,\theta,\xi) &= \frac{2\theta^\delta \mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{\Gamma(\delta)\,\overline{\mathrm{G}}^3(x;\xi)}\left[\sum_{m,r,s=0}^{\infty}(-1)^r\binom{\theta-1}{r}\binom{\delta-1}{m}y^{m+s+\delta+r}b_{s,m}\right]\\ &= \frac{2\theta^\delta \mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{\Gamma(\delta)\,\overline{\mathrm{G}}^3(x;\xi)}\sum_{m,r,s=0}^{\infty}(-1)^r\,b_{s,m}\binom{\theta-1}{r}\binom{\delta-1}{m}\\ &\quad\times\exp\left[-(m+s+\delta+r)\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right].\end{aligned}$$

Applying the exponential series representation $\exp(-z) = \sum_{k=0}^{\infty}\frac{(-1)^k z^k}{k!}$, the density further expand to

$$\begin{aligned}\mathrm{f}(x;\delta,\theta,\xi) &= \frac{2\theta^\delta \mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{\Gamma(\delta)\,\overline{\mathrm{G}}^3(x;\xi)}\sum_{m,r,s=0}^{\infty}(-1)^r\,b_{s,m}\binom{\theta-1}{r}\binom{\delta-1}{m}\\ &\quad\times\sum_{k=0}^{\infty}\frac{(-1)^k(m+s+\delta+r)^k}{k!}\frac{\mathrm{G}^{2k}(x;\xi)}{\overline{\mathrm{G}}^{2k}(x;\xi)}\\ &= \frac{2\theta^\delta}{\Gamma(\delta)}\sum_{k,m,r,s=0}^{\infty}\frac{(-1)^{r+k}b_{s,m}(m+s+\delta+r)^k}{k!}\binom{\theta-1}{r}\binom{\delta-1}{m}\frac{\mathrm{g}(x;\xi)\mathrm{G}^{2k+1}(x;\xi)}{\overline{\mathrm{G}}^{2k+3}(x;\xi)}.\end{aligned}$$

Applying the generalized binomial series representation in $\mathrm{G}^{2k+1}(x;\xi)=\left[1-\overline{\mathrm{G}}(x;\xi)\right]^{2k+1}$, we get

$$\begin{aligned}\mathrm{f}(x;\delta,\theta,\xi)&=\frac{2\theta^{\delta}}{\Gamma(\delta)}\sum_{j,k,m,r,s=0}^{\infty}\frac{(-1)^{r+j+k}b_{s,m}(m+s+\delta+r)^{k}}{k!}\\&\quad\times\binom{\theta-1}{r}\binom{\delta-1}{m}\binom{2k+1}{j}\mathrm{g}(x;\xi)\overline{\mathrm{G}}^{j-(2k+3)}(x;\xi)\\&=\frac{2\theta^{\delta}}{\Gamma(\delta)}\sum_{j,k,m,p,r,s=0}^{\infty}\frac{(-1)^{r+j+k+p}b_{s,m}(m+s+\delta+r)^{k}}{k!}\binom{\theta-1}{r}\\&\quad\times\binom{\delta-1}{m}\binom{2k+1}{j}\binom{j-(2k+3)}{p}\mathrm{g}(x;\xi)\mathrm{G}^{p}(x;\xi)\ .\end{aligned}$$

The GOBX-G density function can finally be expressed as an infinite linear combination of Exponentiated G (Exp-G) density functions as

$$\mathrm{f}(x;\delta,\theta,\xi)=\sum_{p=0}^{\infty}\mathrm{w}_{p+1}\mathrm{h}_{p+1}(x;\xi)\ , \tag{6}$$

where

$$\mathrm{w}_{p+1}=\frac{2\theta^{\delta}}{\Gamma(\delta)}\sum_{j,k,m,r,s=0}^{\infty}\frac{(-1)^{r+j+k+p}b_{s,m}(m+s+\delta+r)^{k}}{(p+1)k!} \tag{7}$$

$$\times\binom{\theta-1}{r}\binom{\delta-1}{m}\binom{2k+1}{j}\binom{j-(2k+3)}{p} \tag{8}$$

and $\mathrm{h}_{p+1}(x;\xi)=(p+1)\mathrm{g}(x;\xi)\mathrm{G}^{p+1-1}(x;\xi)$ is the Exponentiated-G pdf with power parameter $p+1$. Thus, mathematical properties of GOBX-G family of distributions are evident from those of the Exp- G distributions.

2.2. Quantile Function

This section present the quantile function for the GOBX-G distribution. The quantile function can be derived from the cdf of GOBX-G distribution in equation (3) by solving the non-linear equation

$$u=1-\frac{\gamma\left(\delta,-\theta\log\left(1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^{2}\right]\right)\right)}{\Gamma(\delta)}$$

for $0<u<1$. Thus,

$$\gamma\left(\delta,-\theta\log\left(1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^{2}\right]\right)\right)=(1-u)\Gamma(\delta)$$

which follows that

$$-\theta\log\left(1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^{2}\right]\right)=\gamma^{-1}\left(\delta,(1-u)\Gamma(\delta)\right)\ .$$

This simplifies to

$$\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^{2}=-\ln\left[1-\exp\left(\frac{\gamma^{-1}\left(\delta,(1-u)\Gamma(\delta)\right)}{-\theta}\right)\right]\ .$$

Replacing $\overline{\mathrm{G}}(x;\xi)$ with $1-\mathrm{G}(x;\xi)$, we get

$$\frac{\mathrm{G}(x;\xi)}{1-\mathrm{G}(x;\xi)}=\left[-\ln\left[1-\exp\left(\frac{\gamma^{-1}\left(\delta,(1-u)\Gamma(\delta)\right)}{-\theta}\right)\right]\right]^{1/2}$$

which finally gives the quantile function of the GOBX-G family of distributions as

$$\mathrm{Q}(u)=\mathrm{G}^{-1}\left\{\frac{\left[-\ln\left[1-\exp\left(\frac{\gamma^{-1}(\delta,(1-u)\Gamma(\delta))}{-\theta}\right)\right]\right]^{1/2}}{1+\left[-\ln\left[1-\exp\left(\frac{\gamma^{-1}(\delta,(1-u)\Gamma(\delta))}{-\theta}\right)\right]\right]^{1/2}}\right\}. \tag{9}$$

3. Special Cases

In this section, GOBX-G family of distributions takes on some selected baseline distributions even though any baseline distribution can be considered. For our study, we consider Weibull distribution, ℓog-ℓogistic distribution and Uniform distribution.

3.1. Gamma Odd Burr X-Weibull (GOBX-W) Distribution

Considering the baseline distribution of the GOBX-G family as Weibull distribution with pdf $\mathrm{g}(x;\alpha)=\alpha x^{\alpha-1}e^{-x^{\alpha}}$ and cdf $\mathrm{G}(x;\alpha)=1-e^{-x^{\alpha}}$, we respectively obtain the pdf, cdf and hrf of the GOBX-W distribution as

$$\begin{aligned}\mathrm{f}(x;\delta,\theta,\alpha)=&\frac{2\theta\alpha x^{\alpha-1}e^{2x^{\alpha}}(1-e^{-x^{\alpha}})}{\Gamma(\delta)}\exp\left[-\left(e^{x^{\alpha}}-1\right)^{2}\right]\left\{1-\exp\left[-\left(e^{x^{\alpha}}-1\right)^{2}\right]\right\}^{\theta-1}\\&\times\left[-\theta\log\left(1-\exp\left[-\left(e^{x^{\alpha}}-1\right)^{2}\right]\right)\right]^{\delta-1},\end{aligned} \tag{10}$$

$$\mathrm{F}(x;\delta,\theta,\alpha)=1-\frac{\gamma\left(\delta,-\theta\log\left(1-\exp\left[-\left(e^{x^{\alpha}}-1\right)^{2}\right]\right)\right)}{\Gamma(\delta)},$$

and

$$\begin{aligned}\mathrm{h}(x;\delta,\theta,\alpha)=&\frac{2\theta\alpha x^{\alpha-1}e^{2x^{\alpha}}(1-e^{-x^{\alpha}})}{\Gamma(\delta)}\exp\left[-\left(e^{x^{\alpha}}-1\right)^{2}\right]\left\{1-\exp\left[-\left(e^{x^{\alpha}}-1\right)^{2}\right]\right\}^{\theta-1}\\&\times\left[-\theta\log\left(1-\exp\left[-\left(e^{x^{\alpha}}-1\right)^{2}\right]\right)\right]^{\delta-1}\\&\times\left[1-\frac{\gamma\left(\delta,-\theta\log\left(1-\exp\left[-\left(e^{x^{\alpha}}-1\right)^{2}\right]\right)\right)}{\Gamma(\delta)}\right]^{-1}\end{aligned}$$

for $\delta,\theta,\alpha>0$. Plots of the pdf and hrf of GOBX-W distribution are given in Figure 1.

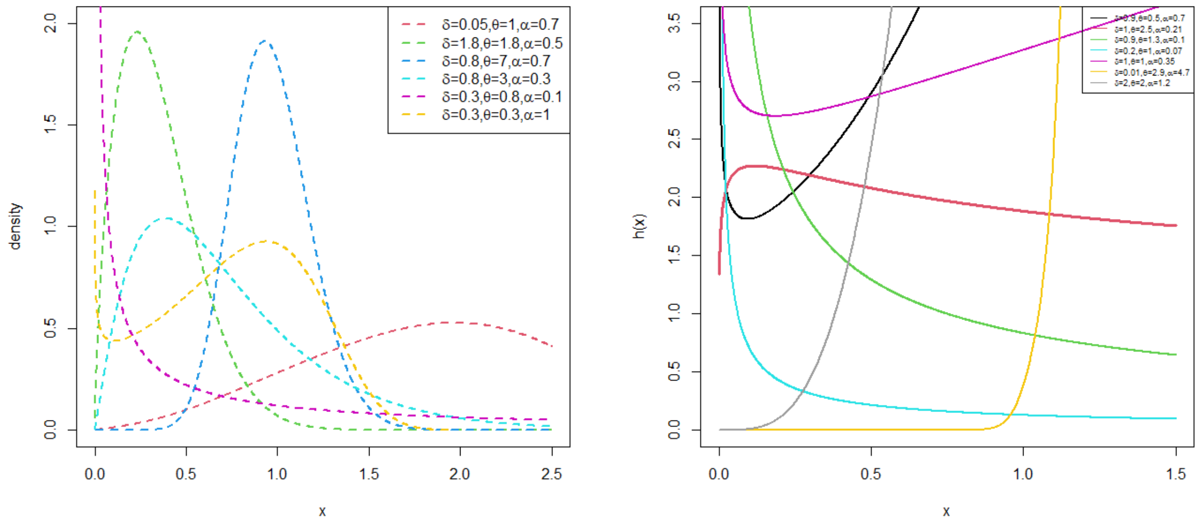

Figure 1: Plots of the pdf and hrf for the GOBX-W distribution

Figure 1 demonstrate different shapes of GOBX-W distribution for the pdf and hrf at different parameter values. The pdf takes various shapes which includes the reverse-J, left-skew, right-skew and unimodal shapes. The hrf also takes different shapes such as the increasing, decreasing, bathtub and upside down bathtub shapes.

3.2. Gamma Odd Burr X-Log-Logistic (GOBX-L) Distribution

Next, considering baseline distribution of the GOBX-G family as ℓog-ℓogistic distribution with pdf $\mathrm{g}(x;\beta) = \beta x^{\beta-1}(1+x^{\beta})^{-2}$ and cdf $\mathrm{G}(x;\beta) = 1-(1+x^{\beta})^{-1}$, we respectively obtain the pdf, cdf and hrf of the GOBX-L distribution as

$$\mathrm{f}(x;\delta,\theta,\beta) = \frac{2\theta\beta x^{\beta-1}(1+x^{\beta})\left[1-(1+x^{\beta})^{-1}\right]}{\Gamma(\delta)} e^{-x^{2\beta}}\left[1-e^{-x^{2\beta}}\right]^{\theta-1}\left[-\theta\log\left(1-e^{-x^{2\beta}}\right)\right]^{\delta-1}, \tag{11}$$

$$\mathrm{F}(x;\delta,\theta,\beta) = 1-\frac{\gamma\left(\delta,-\theta\log\left(1-e^{-x^{2\beta}}\right)\right)}{\Gamma(\delta)},$$

and

$$\mathrm{h}(x;\delta,\theta,\beta) = \frac{2\theta\beta x^{\beta-1}(1+x^{\beta})\left[1-(1+x^{\beta})^{-1}\right]}{\Gamma(\delta)} e^{-x^{2\beta}}\left[1-e^{-x^{2\beta}}\right]^{\theta-1} \times \left[-\theta\log\left(1-e^{-x^{2\beta}}\right)\right]^{\delta-1}\left[1-\frac{\gamma\left(\delta,-\theta\log\left(1-e^{-x^{2\beta}}\right)\right)}{\Gamma(\delta)}\right]^{-1}$$

for $\delta,\theta,\beta>0$. Plots of the pdf and hrf of GOBX-L distribution are given in Figure 2.

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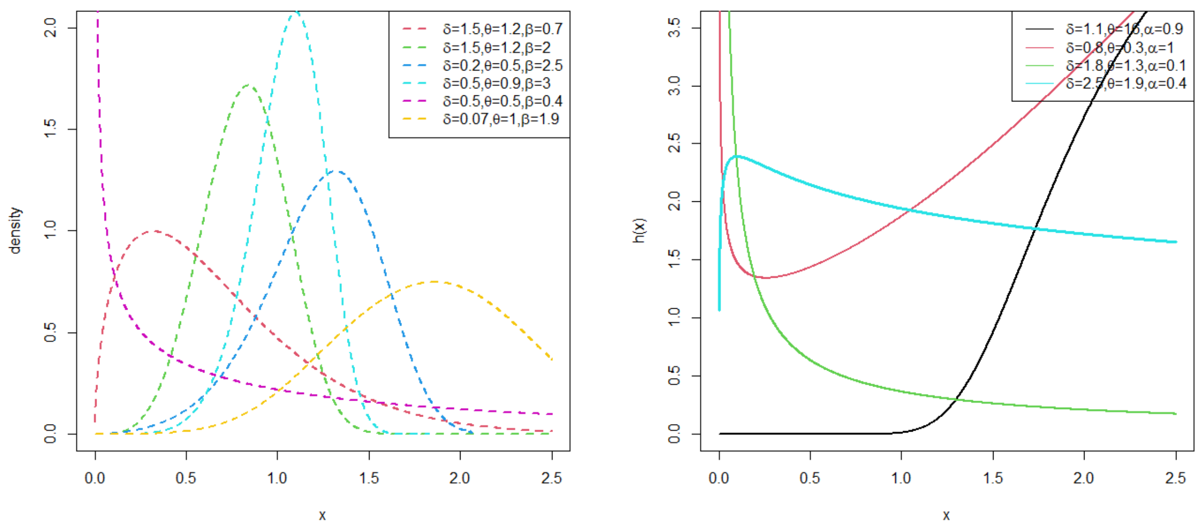


Figure 2: Plots of the pdf and hrf for the GOBX-L distribution

Figure 2 also shows flexible nature of the GOBX-L pdf and hrf for different underlying parameter values. The pdfs of GOBX-L distribution follow different shapes including unimodal, almost symmetric, reverse-J, left-skewed and right skewed. The hrf also exhibit decreasing, increasing, bathtub and upside down bathtub.

3.3. Gamma Odd Burr X-Uniform (GOBX-U) Distribution

For a uniform baseline distribution with pdf $\mathrm{g}(x;\mu) = 1/\mu$ and cdf $\mathrm{G}(x;\mu) = x/\mu$, $0 < x < \mu$, we respectively obtain the pdf, cdf and hrf of the GOBX-U distribution as

$$\mathrm{f}(x;\delta,\theta,\mu) = \frac{2\theta\mu x}{\Gamma(\delta)\ (\mu-x)^3}\exp\left[-\left(\frac{x}{\mu-x}\right)^2\right]\left\{1-\exp\left[-\left(\frac{x}{\mu-x}\right)^2\right]\right\}^{\theta-1} \times \left[-\theta\log\left(1-\exp\left[-\left(\frac{x}{\mu-x}\right)^2\right]\right)\right]^{\delta-1}, \tag{12}$$

$$\mathrm{F}(x;\delta,\theta,\mu) = 1 - \frac{\gamma\left(\delta, -\theta\log\left(1-\exp\left[-\left(\frac{x}{\mu-x}\right)^2\right]\right)\right)}{\Gamma(\delta)}$$

and

$$\mathrm{h}(x;\delta,\theta,\mu) = \frac{2\theta\mu x}{\Gamma(\delta)\ (\mu-x)^3}\exp\left[-\left(\frac{x}{\mu-x}\right)^2\right]\left\{1-\exp\left[-\left(\frac{x}{\mu-x}\right)^2\right]\right\}^{\theta-1} \times \left[-\theta\log\left(1-\exp\left[-\left(\frac{x}{\mu-x}\right)^2\right]\right)\right]^{\delta-1} \times \left[1-\frac{\gamma\left(\delta, -\theta\log\left(1-\exp\left[-\left(\frac{x}{\mu-x}\right)^2\right]\right)\right)}{\Gamma(\delta)}\right]^{-1}$$

for $\delta, \theta, \mu > 0$. Plots of the pdf and hrf of GOBX-U distribution are given in Figure 3.

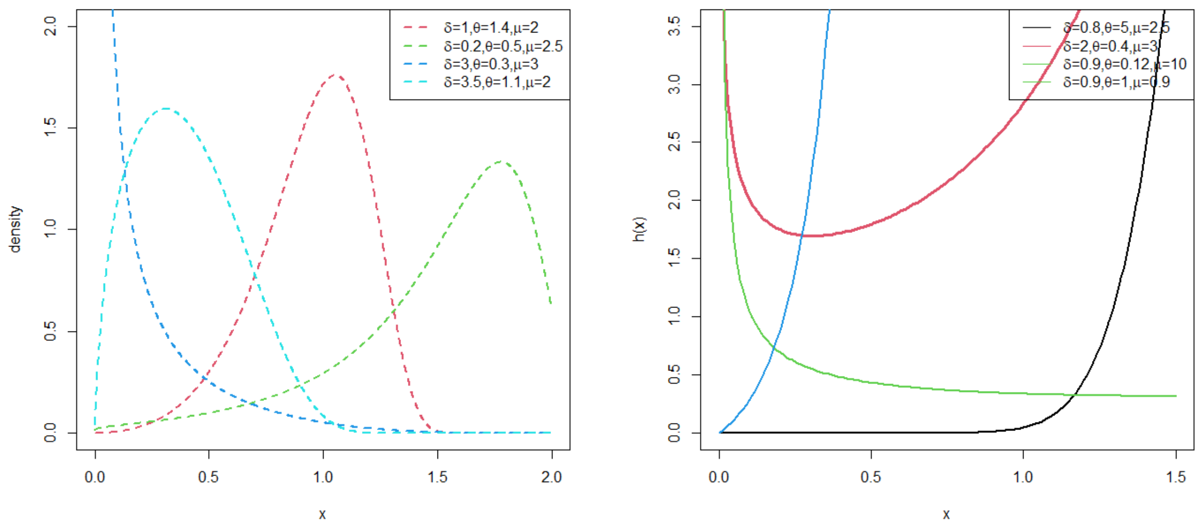


Figure 3: Plots of the pdf and hrf for the GOBX-U distribution

Figure 3 shows the GOBX-U distribution pdf and hrf. The pdf takes on different shapes which includes left-skew, right-skew, almost symmetric and reverse-J. Additionally, the hrf also show increasing, decreasing and bathtub shapes. Table 1 presents some values generated from the GOBX-W distribution for different values of the parameters δ, θ and α.

Table 1: Table of quantiles for selected parameter values of the GOBX-W distribution

	(δ, θ, α)				
u	(1.5,0.9,1.01)	(2.0,2.5,0.9)	(1.5,.5,1.5)	(0.9,5.0,1.8)	(1.2,4.5,1.2)
0.1	0.1664	0.3579	0.1227	0.8253	0.6916
0.2	0.251	0.4318	0.2066	0.8683	0.751
0.3	0.3214	0.4871	0.2815	0.8986	0.7935
0.4	0.3857	0.5352	0.3521	0.9241	0.8295
0.5	0.4478	0.5805	0.4209	0.9475	0.863
0.6	0.5109	0.626	0.4901	0.9706	0.8961
0.7	0.5782	0.6747	0.5626	0.995	0.9314
0.8	0.6558	0.7314	0.6432	1.0232	0.9722
0.9	0.7594	0.8092	0.7453	1.0614	1.0282

4. Other Statistical and Mathematical Properties

In this section, some structural properties of GOBX-G family of distributions are presented. These includes moments and generating functions, conditional moments, entropy, distribution of order statistics and probability weighted moments.

4.1. Moments and Generating functions

The r^{th} ordinary moment of GOBX-G family of distributions can be derived from equation (6) as

$$\mu_r' = \mathrm{E}(X^r) = \sum_{p=0}^{\infty} \mathrm{w}_{p+1} \mathrm{E}\left(\mathrm{Y}_{p+1}\right) = \sum_{p=0}^{\infty} \mathrm{w}_{p+1}(p+1) \int_0^1 u^{p+1-1} \mathrm{Q}_G^r(u;\xi) du \;, \tag{13}$$

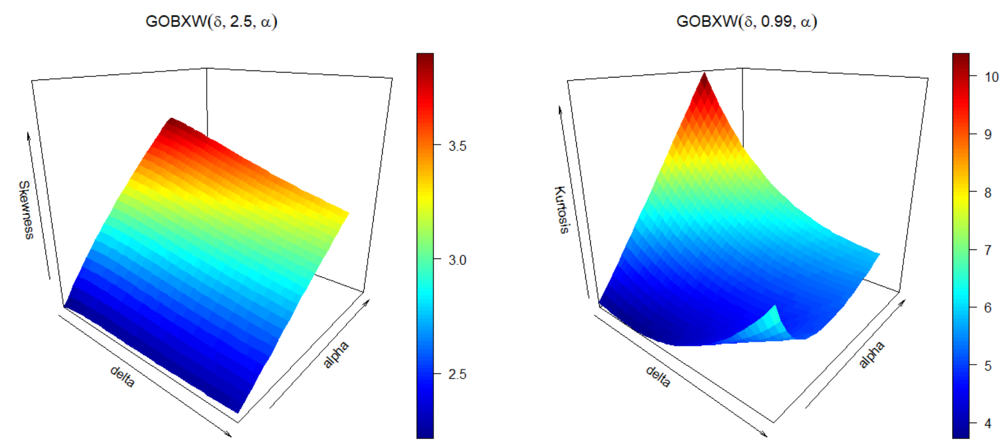
where Y_{p+1} is the Exp-G random variable with power parameter $p+1$ and $\mathrm{Q}_G(u;\xi)$ is the quantile function of the baseline distribution with the cdf $\mathrm{G}(x;\xi)$. To obtain the skewness and the kurtosis of GOBX-G family of distributions, we use the n^{th} central moment, say M_n, given as

$$M_n = \mathrm{E}(X-\mu_1')^n = \sum_{r=0}^{\infty}\binom{n}{r}(-\mu_1')^{n-r}\mathrm{E}(X^r)$$
$$= \sum_{p=0}^{\infty}\sum_{r=0}^{\infty}\binom{n}{r}(-\mu_1')^{n-r}\mathrm{w}_{p+1}\mathrm{E}\left(\mathrm{Y}_{p+1}\right) \ .$$

The moment generating function of the GOBX-G family of distributions can be obtained as

$$\mathrm{M}_X(t) = \mathrm{E}(e^{tX}) = \sum_{p=0}^{\infty}\mathrm{w}_{p+1}\mathrm{M}_{p+1}(t) \ ,$$

where $\mathrm{M}_{p+1}(t)$ is the moment generating function of the Exp-G random variable Y_{p+1}. Figure 4 and 5 shows 3D plots of skewness and kurtosis for the GOBX-W distribution for some selected parameter values. Figure 4 shows that, when θ is fixed, skewness increases as α increases whereas kurtosis decrease with a decrease in δ. An increase in θ leads to increase in skewness and kurtosis when α is held constant as shown in figure 5. This shows that all the three parameters play a role in the variation of skewness an kurtosis.



(a) (b)

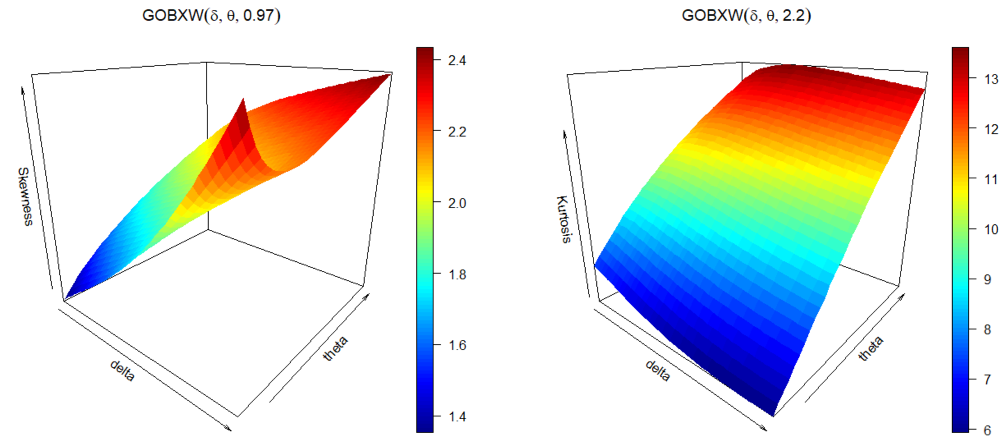
Figure 4: 3D plots of skewness and kurtosis for the GOBX-W distribution for some selected parameter values

4.2. Incomplete Moments

The s^{th} $(s>0)$ incomplete moments for the GOBX-G family of distributions follow from equation (6) as

$$\varphi_s(t) = \int_{-\infty}^{t} x^s f(x)dx = \sum_{p=0}^{\infty}\mathrm{w}_{p+1}\int_{-\infty}^{t} x^s\mathrm{h}_{p+1}(x;\xi)dx \ ,$$

where $\int_{-\infty}^{t} x^s\mathrm{h}_{p+1}(x;\xi)dx$ is the s^{th} incomplete moment of the Exp-G random variable Y_{p+1}. The incomplete moments are critical as they can be used to derive the mean deviations, Bonferroni, Lorenz and Zenga curves which are usually used in many fields such as demography, engineering and medicine. The mean deviation about the mean $\mu = \mathrm{E}(X)$ and median $M = \mathrm{Median}(X)$ can respectively be derived as $\mathrm{E}|X-\mu_1'| = 2\mu_1'\mathrm{F}(\mu_1') - 2\varphi_1(\mu_1')$ and $\mathrm{E}|X-M| = \mu_1' - 2\varphi_1(M)$, where μ_1' can be evaluated from equation (13), M= $Q(0.5)$ from equation (9) and $\mathrm{F}(\mu_1')$ from equation (3).



(a) (b)

Figure 5: 3D plots of skewness and kurtosis for the GOBX-W distribution for some selected parameter values

4.3. Rényi Entropy

This subsection presents the Rényi entropy as a measures of the variation of uncertainty. Rényi entropy is an extension of Shannon entropy and it is defined as

$$\mathrm{I}_R(v) = \frac{1}{1-v}\log\left[\int_0^\infty f^v(x;\delta,\theta,\xi)dx\right] \ ,$$

for $v > 0$, $v \neq 1$. If we let $y = \exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]$ in the GOBX-G density given in equation (4), then we have

$$\mathrm{f}^v(x;\delta,\theta,\xi) = \frac{(2\theta^\delta)^v \mathrm{g}^v(x;\xi)\mathrm{G}^v(x;\xi)}{[\Gamma(\delta)]^v\ \overline{\mathrm{G}}^{3v}(x;\xi)}\ y^v\ (1-y)^{v(\theta-1)}\ [-\log\,(1-y)]^{v(\delta-1)}\ .$$

Using series representation $-\log(1-y) = \sum\limits_{i=0}^{\infty}\frac{y^{i+1}}{i+1}$, for $y \in (0,1)$, we have

$$[-\log(1-y)]^{v(\delta-1)} = y^{v(\delta-1)}\left[\sum_{m=0}^{\infty}\binom{v(\delta-1)}{m}y^m\left(\sum_{s=0}^{\infty}\frac{y^s}{s+2}\right)^m\right] \ ,$$

so that $\mathrm{f}^v(x;\delta,\theta,\xi)$ can be written as

$$\mathrm{f}^v(x;\delta,\theta,\xi) = \frac{(2\theta^\delta)^v \mathrm{g}^v(x;\xi)\mathrm{G}^v(x;\xi)}{[\Gamma(\delta)]^v\ \overline{\mathrm{G}}^{3v}(x;\xi)}\ y^{v(\delta-1)+1}\ (1-y)^{v(\theta-1)}\left[\sum_{m=0}^{\infty}\binom{v(\delta-1)}{m}y^m\left(\sum_{s=0}^{\infty}\frac{y^s}{s+2}\right)^m\right] \ .$$

If we let $c_s = (s+2)^{-1}$ and considering the result of a power series raised to a positive integer (Grandshteyn and Ryzhik, 1980), we get

$$\left(\sum_{s=0}^{\infty}c_s y^s\right)^m = \sum_{s=0}^{\infty}d_{s,m}y^s \ ,$$

where $d_{s,m} = (sc_0)^{-1}\sum_{l=1}^{\infty}[m(l+1)-s]c_l d_{s-l,m}$ and $d_{0,m} = c_0^m$, so that

$$\mathrm{f}^v(x;\delta,\theta,\xi) = \frac{(2\theta^\delta)^v \mathrm{g}^v(x;\xi)\mathrm{G}^v(x;\xi)}{[\Gamma(\delta)]^v\ \overline{\mathrm{G}}^{3v}(x;\xi)}\ (1-y)^{v(\theta-1)}\left[\sum_{m,s=0}^{\infty}\binom{v(\delta-1)}{m}y^{w_*}d_{s,m}\right],$$

where $w_* = m+s+v(\delta-1)+1$. Applying the generalized binomial series representation, we get

$$\mathrm{f}^v(x;\delta,\theta,\xi) = \frac{(2\theta^\delta)^v \mathrm{g}^v(x;\xi)\mathrm{G}^v(x;\xi)}{[\Gamma(\delta)]^v\ \overline{\mathrm{G}}^{3v}(x;\xi)}\sum_{m,r,s=0}^{\infty}(-1)^r\ d_{s,m}\binom{v(\theta-1)}{r}\binom{v(\delta-1)}{m} \times \exp\left[-(w_*+r)\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right].$$

Applying the exponential series representation, we have

$$\mathrm{f}^v(x;\delta,\theta,\xi) = \frac{(2\theta^\delta)^v}{(\Gamma(\delta))^v}\sum_{k,m,r,s=0}^{\infty}\frac{(-1)^{r+k}d_{s,m}(w_*+r)^k}{k!}\binom{v(\theta-1)}{r}\binom{v(\delta-1)}{m} \times \frac{\mathrm{g}^v(x;\xi)\mathrm{G}^{2k+v}(x;\xi)}{\overline{\mathrm{G}}^{2k+3v}(x;\xi)}.$$

Applying the generalized binomial series representation in $\mathrm{G}^{2k+v}(x;\xi) = \left[1-\overline{\mathrm{G}}(x;\xi)\right]^{2k+v}$, we obtain

$$\begin{aligned}\mathrm{f}^v(x;\delta,\theta,\xi) &= \frac{(2\theta^\delta)^v}{(\Gamma(\delta))^v}\sum_{j,k,m,r,s=0}^{\infty}\frac{(-1)^{r+j+k}d_{s,m}(w_*+r)^k}{k!}\\ &\quad\times\binom{v(\theta-1)}{r}\binom{v(\delta-1)}{m}\binom{2k+v}{j}\mathrm{g}^v(x;\xi)\overline{\mathrm{G}}^{j-(2k+3v)}(x;\xi)\\ &= \frac{(2\theta^\delta)^v}{(\Gamma(\delta))^v}\sum_{j,k,m,p,r,s=0}^{\infty}\frac{(-1)^{r+j+k+p}d_{s,m}(w_*+r)^k}{k!}\binom{v(\theta-1)}{r}\\ &\quad\times\binom{v(\delta-1)}{m}\binom{2k+v}{j}\binom{j-(2k+3v)}{p}\mathrm{g}^v(x;\xi)\mathrm{G}^p(x;\xi).\end{aligned}$$

Finally, the Rényi entropy for the GOBX-G family of distributions is given as

$$\begin{aligned}\mathrm{I}_R(v) &= \frac{1}{1-v}\log\left[\frac{(2\theta^\delta)^v}{(\Gamma(\delta))^v}\sum_{j,k,m,p,r,s=0}^{\infty}\frac{(-1)^{r+j+k+p}d_{s,m}(w_*+r)^k}{\left[\frac{2k+j}{v}+2\right]^v k!}\binom{v(\theta-1)}{r}\binom{v(\delta-1)}{m}\right.\\ &\quad\left.\times\binom{2k+v}{j}\binom{j-(2k+3v)}{p}\int_0^\infty\left(\left[\frac{p}{v}+1\right]\mathrm{g}(x;\xi)\left[\mathrm{G}(x;\xi)\right]^{\frac{p}{v}}\right)^v dx\right]\\ &= \frac{1}{1-v}\log\left[\sum_{p=0}^{\infty}W_{\frac{p}{v}+1}\exp\left[(1-v)I_{REG}\right]\right],\end{aligned}$$

where $I_{REG} = (1-v)^{-1}\log\left[\int\limits_0^\infty \left(\left[\frac{p}{v}+1\right] \mathrm{g}(x;\xi)\left[\mathrm{G}(x;\xi)\right]^{\frac{p}{v}}\right)^v dx\right]$ is the Rényi entropy for the Exp-G family with power parameter $\frac{p}{v}+1$ and

$$W_{\frac{p}{v}+1} = \frac{(2\theta^\delta)^v}{(\Gamma(\delta))^v} \sum_{j,k,m,r,s=0}^{\infty} \frac{(-1)^{r+j+k+p} d_{s,m}(w_* + r)^k}{\left[\frac{p}{v}+1\right]^v k!} \times \binom{v(\theta-1)}{r}\binom{v(\delta-1)}{m}\binom{2k+v}{j}\binom{j-(2k+3v)}{p}.$$

4.4. Order Statistics

In this subsection, the distribution of the i^{th} order statistics for the GOBX-G family of distributions is provided. Let $X_1, X_2, X_3, ..., X_n$ be independent and identically distributed GOBX-G random variables. The pdf of the i^{th} order statistic, say $X_{i:n}$ is given as

$$\begin{aligned}\mathrm{f}_{i:n}(x;\delta,\theta,\xi) &= \frac{n!\mathrm{f}(x;\delta,\theta,\xi)}{(i-1)!(n-i)!}\sum_{t=0}^{n-i}(-1)^t\binom{n-i}{t}\mathrm{F}^{t+i-1}(x;\delta,\theta,\xi)\\ &= \frac{n!\mathrm{f}(x;\delta,\theta,\xi)}{(i-1)!(n-i)!}\sum_{t=0}^{n-i}(-1)^t\binom{n-i}{t}\left[1-\frac{\gamma\left(\delta,-\theta\log\left(1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]\right)\right)}{\Gamma(\delta)}\right]^{t+i-1}.\end{aligned}$$

Using the binomial theorem, we obtain

$$\mathrm{f}_{i:n}(x;\delta,\theta,\xi) = \frac{n!\mathrm{f}(x;\delta,\theta,\xi)}{(i-1)!(n-i)!}\sum_{j=0}^{\infty}\sum_{t=0}^{n-i}\frac{(-1)^t}{[\Gamma(\delta)]^j}\binom{n-i}{t}\binom{t+i-1}{j} \times\left[\gamma\left(\delta,-\theta\log\left(1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]\right)\right)\right]^j. \quad (14)$$

Applying the power series $\gamma(\delta,x)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{k+\delta}}{(k+\delta)k!}$ (see Abramowitz and Stegun (1972)), we obtain

$$\mathrm{f}_{i:n}(x;\delta,\theta,\xi) = \frac{n!\mathrm{f}(x;\delta,\theta,\xi)}{(i-1)!(n-i)!}\sum_{j=0}^{\infty}\sum_{t=0}^{n-i}\frac{(-1)^t}{[\Gamma(\delta)]^j}\binom{n-i}{t}\binom{t+i-1}{j} \times\left[\sum_{k=0}^{\infty}\frac{(-1)^k\left(-\theta\log\left(1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]\right)\right)^{k+\delta}}{(k+\delta)k!}\right]^j.$$

Next, let $c_k = \frac{(-1)^k}{(k+\delta)k!}$ and use the result on a power series raised to a positive integer (Grandshteyn and Ryzhik, 1980), then we have

$$\left[\sum_{k=0}^{\infty}c_k\left(-\theta\log\left(1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]\right)\right)^{k+\delta}\right]^j = \sum_{k=0}^{\infty}d_{k,j}\left(-\theta\log\left(1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]\right)\right)^{k+\delta}, \quad (15)$$

where $d_{0,j} = c_0^j$ and $d_{k,j} = (kc_0)^{-1} \sum_{l=1}^{\infty} [j(l+1) - k] c_l d_{k-l,j}$. Replacing $\mathrm{f}(x;\delta,\theta,\xi)$ in the above expression for $\mathrm{f}_{i:n}(x;\delta,\theta,\xi)$, we get

$$\begin{aligned}\mathrm{f}_{i:n}(x;\delta,\theta,\xi) &= \frac{2n!\mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{(i-1)!(n-i)!\overline{\mathrm{G}}^3(x;\xi)} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{t=0}^{n-i} \frac{\theta^{k+2\delta}(-1)^t}{[\Gamma(\delta)]^{j+1}} d_{k,j} \binom{n-i}{t}\binom{t+i-1}{j} \\ &\times \exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]\left\{1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]\right\}^{\theta-1} \\ &\times \left(-\log\left(1-\exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]\right)\right)^{k+2\delta-1}.\end{aligned}$$

Using the previously stated series expansion of $-\log(1-y) = \sum_{i=0}^{\infty} \frac{y^{i+1}}{i+1}$ and letting $y = \exp\left[-\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]$, we obtain

$$[-\log(1-y)]^{k+2\delta-1} = y^{k+2\delta-1}\left[\sum_{m=0}^{\infty}\binom{k+2\delta-1}{m} y^m \left(\sum_{s=0}^{\infty}\frac{y^s}{s+2}\right)^m\right] \tag{16}$$

which when we consider the previous definition of $b_{s,m}$ in equation (5), we get

$$\begin{aligned}\mathrm{f}_{i:n}(x;\delta,\theta,\xi) &= \frac{2n!\mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{(i-1)!(n-i)!\overline{\mathrm{G}}^3(x;\xi)} \sum_{j,k=0}^{\infty}\sum_{t=0}^{n-i} \frac{\theta^{k+2\delta}(-1)^t}{[\Gamma(\delta)]^{j+1}} d_{k,j} \binom{n-i}{t}\binom{t+i-1}{j} \\ &\times y^{k+2\delta}(1-y)^{\theta-1}\sum_{m,s=0}^{\infty}\binom{k+2\delta-1}{m} y^m b_{s,m} y^s \\ &= \frac{2n!\mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{(i-1)!(n-i)!\overline{\mathrm{G}}^3(x;\xi)} \sum_{j,k,m,s=0}^{\infty}\sum_{t=0}^{n-i} \frac{\theta^{k+2\delta}(-1)^t}{[\Gamma(\delta)]^{j+1}} d_{k,j} b_{s,m} \\ &\times \binom{n-i}{t}\binom{t+i-1}{j}\binom{k+2\delta-1}{m} y^{k+2\delta+m+s}(1-y)^{\theta-1}.\end{aligned}$$

Further expansion of the last part conducted using binomial theorem gives

$$\begin{aligned}\mathrm{f}_{i:n}(x;\delta,\theta,\xi) &= \frac{2n!\mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{(i-1)!(n-i)!\overline{\mathrm{G}}^3(x;\xi)} \sum_{j,k,m,r,s=0}^{\infty}\sum_{t=0}^{n-i} \frac{\theta^{k+2\delta}(-1)^{t+r}}{[\Gamma(\delta)]^{j+1}} d_{k,j} b_{s,m} \\ &\times \binom{n-i}{t}\binom{t+i-1}{j}\binom{k+2\delta-1}{m}\binom{\theta-1}{r} y^{v_*},\end{aligned}$$

where $v_* = k+2\delta+m+s+r$. Expanding $y^{v_*} = \exp\left[-v_*\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]$ based on exponential series representation gives

$$\begin{aligned}\mathrm{f}_{i:n}(x;\delta,\theta,\xi) &= \frac{2n!}{(i-1)!(n-i)!} \sum_{j,k,m,r,s,u=0}^{\infty}\sum_{t=0}^{n-i} \frac{\theta^{k+2\delta}(-1)^{t+r+u} v_*^u d_{k,j} b_{s,m}}{[\Gamma(\delta)]^{j+1} u!} \\ &\times \binom{n-i}{t}\binom{t+i-1}{j}\binom{k+2\delta-1}{m}\binom{\theta-1}{r} \frac{\mathrm{g}(x;\xi)\mathrm{G}^{2u+1}(x;\xi)}{\overline{\mathrm{G}}^{2u+3}(x;\xi)}.\end{aligned}$$

Expansion of the series $G^{2u+1}(x;\xi) = \left[1 - \overline{G}(x;\xi)\right]^{2u+1}$ gives

$$\begin{aligned} f_{i:n}(x;\delta,\theta,\xi) &= \frac{2n!}{(i-1)!(n-i)!} \sum_{j,k,m,p,r,s,u=0}^{\infty} \sum_{t=0}^{n-i} \frac{\theta^{k+2\delta}(-1)^{t+r+u+p} v_*^u d_{k,j} b_{s,m}}{[\Gamma(\delta)]^{j+1} u!} \binom{n-i}{t} \\ &\times \binom{t+i-1}{j} \binom{k+2\delta-1}{m} \binom{\theta-1}{r} \binom{2u+1}{p} g(x;\xi) \overline{G}^{p-(2u+3)}(x;\xi) \\ &= \frac{2n!}{(i-1)!(n-i)!} \sum_{j,k,m,p,q,r,s,u=0}^{\infty} \sum_{t=0}^{n-i} \frac{\theta^{k+2\delta}(-1)^{t+r+u+p+q} v_*^u d_{k,j} b_{s,m}}{[\Gamma(\delta)]^{j+1} u!} \binom{n-i}{t} \\ &\times \binom{t+i-1}{j} \binom{k+2\delta-1}{m} \binom{\theta-1}{r} \binom{2u+1}{p} \binom{p-(2u+3)}{q} g(x;\xi) G^q(x;\xi) \ . \end{aligned}$$

Alternatively, the GOBX-G order statistic can be expressed as

$$f_{i:n}(x;\delta,\theta,\xi) = \sum_{q=0}^{\infty} m_{q+1} h_{q+1}^*(x;\xi) \ , \tag{17}$$

where $h_{q+1}^*(x;\xi) = (q+1) g(x;\xi) G^{q+1-1}(x;\xi)$ is the Exp-G distribution of power parameter $q+1$ and

$$\begin{aligned} m_{q+1} &= \frac{2n!}{(i-1)!(n-i)!} \sum_{j,k,m,p,r,s,u=0}^{\infty} \sum_{t=0}^{n-i} \frac{\theta^{k+2\delta}(-1)^{t+r+u+p+q} v_*^u d_{k,j} b_{s,m}}{(q+1)[\Gamma(\delta)]^{j+1} u!} \binom{n-i}{t} \\ &\times \binom{t+i-1}{j} \binom{k+2\delta-1}{m} \binom{\theta-1}{r} \binom{2u+1}{p} \binom{p-(2u+3)}{q} \ . \end{aligned}$$

4.5. Probability Weighted Moments

This section present the probability weighted moments with most of the derivations given under the order statistics section. The $(r,s)^{th}$ probability weighted moments for the GOBX-G family of distributions is given as

$$\omega_{(r,s)} = E[X^r F^s(X)] = \int_{-\infty}^{\infty} x^r f(x) F^s(x) dx \ .$$

With the help of derivations from equation (14) under order statistics section, we get

$$\begin{aligned} f(x)F^s(x) &= f(x) \sum_{j=0}^{\infty} [\Gamma(\delta)]^{-j} \binom{s}{j} \left[\gamma\left(\delta, -\theta \log\left(1-y\right)\right)\right]^{-j} \\ &= f(x) \sum_{j,k=0}^{\infty} [\Gamma(\delta)]^{j} \binom{s}{j} d_{j,k} [-\theta \log\left(1-y\right)]^{k+\delta} \ , \end{aligned}$$

where $y = \exp\left[-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^2\right]$ and $d_{j,k}$ is defined in equation (15). Plugging $f(x)$, we get

$$f(x)F^s(x) = \frac{2\theta g(x;\xi) G(x;\xi)}{\overline{G}^3(x;\xi)} y(1-y)^{\theta-1} \sum_{j,k=0}^{\infty} [\Gamma(\delta)]^{-(j+1)} \binom{s}{j} d_{j,k} [-\theta \log\left(1-y\right)]^{k+2\delta-1} \ .$$

Further expansion based on equation (16) yields

$$\begin{aligned}\mathrm{f}(x)\mathrm{F}^s(x) &= \frac{2\mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}^3(x;\xi)} \sum_{j,k,m,s=0}^{\infty} [\Gamma(\delta)]^{-(j+1)}\theta^{k+2\delta} d_{j,k} b_{s,m} \\ &\times \binom{s}{j}\binom{k+2\delta-1}{m} y^{k+2\delta+m+s}(1-y)^{\theta-1} \\ &= \frac{2\mathrm{g}(x;\xi)\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}^3(x;\xi)} \sum_{j,k,m,r,s=0}^{\infty} [\Gamma(\delta)]^{-(j+1)}(-1)^r\theta^{k+2\delta} d_{j,k} b_{s,m} \\ &\times \binom{s}{j}\binom{k+2\delta-1}{m}\binom{\theta-1}{r} y^{v_*} ,\end{aligned}$$

where $b_{s,m}$ is defined in equation (5) and $v_* = k+2\delta+m+s+r$. Series expansion of $y^{v_*} = \exp\left[-v_*\left(\frac{\mathrm{G}(x;\xi)}{\overline{\mathrm{G}}(x;\xi)}\right)^2\right]$ gives

$$\begin{aligned}\mathrm{f}(x)\mathrm{F}^s(x) &= 2\sum_{j,k,m,r,s,u=0}^{\infty} \frac{\theta^{k+2\delta}}{[\Gamma(\delta)]^{(j+1)}} \frac{(-1)^{r+u} v_*^u d_{j,k} b_{s,m}}{u!} \binom{s}{j}\binom{k+2\delta-1}{m} \\ &\times \binom{\theta-1}{r} \frac{\mathrm{g}(x;\xi)\mathrm{G}^{2u+1}(x;\xi)}{\overline{\mathrm{G}}^{2u+3}(x;\xi)} \\ &= 2\sum_{j,k,m,p,r,s,u=0}^{\infty} \frac{\theta^{k+2\delta}}{[\Gamma(\delta)]^{(j+1)}} \frac{(-1)^{r+u+p} v_*^u d_{j,k} b_{s,m}}{u!} \binom{s}{j}\binom{k+2\delta-1}{m} \\ &\times \binom{\theta-1}{r}\binom{2u+1}{p} \mathrm{g}(x;\xi)\overline{\mathrm{G}}^{p-(2u+3)}(x;\xi) \\ &= 2\sum_{j,k,m,p,q,r,s,u=0}^{\infty} \frac{\theta^{k+2\delta}}{[\Gamma(\delta)]^{(j+1)}} \frac{(-1)^{r+u+p+q} v_*^u d_{j,k} b_{s,m}}{u!} \binom{s}{j}\binom{k+2\delta-1}{m} \\ &\times \binom{\theta-1}{r}\binom{2u+1}{p}\binom{p-(2u+3)}{q} \mathrm{g}(x;\xi)\mathrm{G}^q(x;\xi) \\ &= \sum_{q=0}^{\infty} \eta_{q+1}\mathrm{h}^*_{q+1}(x;\xi) ,\end{aligned}$$

where $\mathrm{h}^*_{q+1}(x;\xi)$ is defined under equation (17) and

$$\begin{aligned}\eta_{q+1} &= 2\sum_{j,k,m,p,r,s,u=0}^{\infty} \frac{\theta^{k+2\delta}}{[\Gamma(\delta)]^{(j+1)}} \frac{(-1)^{r+u+p+q} v_*^u d_{j,k} b_{s,m}}{u!} \\ &\times \binom{s}{j}\binom{k+2\delta-1}{m}\binom{\theta-1}{r}\binom{2u+1}{p}\binom{p-(2u+3)}{q} .\end{aligned}$$

Consequently, we obtain the probability weighted moments for the GOBX-G family of distributions reduces to

$$\omega_{(r,s)} = \sum_{q=0}^{\infty} \eta_{q+1} \int_{-\infty}^{\infty} x^r \mathrm{h}^*_{q+1}(x;\xi)dx .$$

5. Maximum Likelihood Estimation

This section present the maximum likelihood technique for estimating the parameters of the GOBX-G family of distributions. Maximum likelihood is one of the most reliable estimation technique as it mostly produce accurate and consistent estimators (Zanakis and Kyparisis, 1986). Let X_1, X_2, X_3,...,X_n be a random sample of size n from the GOBX-G family of distributions and $\Theta = (\delta, \theta, \xi)^T$ be a vector of model parameters, then the log-likelihood function can be expressed as

$$\ell_n(\Theta) = n\ln(2) + n\ln(\theta) - n\ln(\Gamma(\delta)) + \sum_{i=1}^{n}\ln[\mathrm{g}(x_i;\xi)] + \sum_{i=1}^{n}\ln[\mathrm{G}(x_i;\xi)] - 3\sum_{i=1}^{n}\ln[\overline{\mathrm{G}}(x_i;\xi)] - \sum_{i=1}^{n} y_i^2$$
$$+ (\theta-1)\sum_{i=1}^{n}\ln\left(1-e^{-y_i^2}\right) + (\delta-1)\sum_{i=1}^{n}\ln\left[-\log\left(1-e^{-y_i^2}\right)\right],$$

where $y_i = \mathrm{G}(x_i;\xi)/\overline{\mathrm{G}}(x_i;\xi)$. The components of the score function $U_n(\Theta) = \left(\frac{\partial\ell_n(\Theta)}{\partial\delta}, \frac{\partial\ell_n(\Theta)}{\partial\theta}, \frac{\partial\ell_n(\Theta)}{\partial\xi_k}\right)^T$ are given in the appendix and R software is used to derive the solution to $U_n(\Theta) = 0$ under a specified baseline distribution. The multivariate normal distribution $N_q(\underline{\mathbf{0}}, J(\hat{\Theta})^{-1})$, where $\underline{\mathbf{0}} = (\underline{0}, \underline{0}, \underline{0})^T$ and $J(\hat{\Theta})^{-1}$ are the mean vector and the observed information matrix evaluated at $\hat{\Theta}$, can be used to construct the confidence intervals and regions for model parameters. The $100(1-\varphi)\%$ two sided confidence intervals for δ, θ and ε_k are $\hat{\delta} \pm \mathrm{Z}_{\frac{\varphi}{2}}\sqrt{I^{-1}_{\delta\delta}(\hat{\Theta})}$, $\hat{\theta} \pm \mathrm{Z}_{\frac{\varphi}{2}}\sqrt{I^{-1}_{\theta\theta}(\hat{\Theta})}$ and $\hat{\varepsilon}_k \pm \mathrm{Z}_{\frac{\varphi}{2}}\sqrt{I^{-1}_{\varepsilon_k\varepsilon_k}(\hat{\Theta})}$, respectively, where $I^{-1}_{\delta\delta}(\hat{\Theta})$, $I^{-1}_{\theta\theta}(\hat{\Theta})$ and $I^{-1}_{\varepsilon_k\varepsilon_k}(\hat{\Theta})$ are the diagonal elements of $I_n^{-1}(\hat{\Theta}) = (nI(\hat{\Theta}))^{-1}$ and $\mathrm{Z}_{\frac{\varphi}{2}}$ is the standard normal $\left(\frac{\varphi}{2}\right)^{th}$ percentile.

6. Simulation Study

In this section, simulation results are presented for different sample sizes of $n = 100, 200, 400, 600$ and 800 to examine the accuracy and consistency of the maximum likelihood estimators (MLEs) for each of the parameter of the GOBX-W distribution. The simulation was repeated $N = 1000$ times and the mean estimates, average bias (Abias) and the root mean square errors (RMSEs) were evaluated. For consistent MLEs, it is expected that as the sample size n increases, the mean estimates gets closer to the true parameters and, the RSMEs and Abias also decay to zero. Table 2 and 3 show the mean estimates together with their respective RSMEs and Abias. The Abias and RMSEs for the estimated parameter, say, $\hat{\theta}$, are respectively given as:

$$\mathrm{Abias}(\hat{\theta}) = \frac{\sum_{i=1}^{N}\hat{\theta}_i}{N} - \theta, \quad \text{and} \quad \mathrm{RMSE}(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2}{N}}.$$

Table 2: Monte Carlo simulation results for GOBX-W distribution

		$(2.1, 1.8, 0.5)$			$(1.2, 1.2, 0.5)$		
Parameter	Sample Size	Mean	RMSE	Abias	Mean	RMSE	Abias
δ	100	2.3447	0.5550	0.2447	1.2704	0.2154	0.0704
	200	2.2161	0.3079	0.1161	1.2372	0.1361	0.0372
	400	2.1435	0.1569	0.0435	1.2143	0.0862	0.0143
	600	2.1268	0.1225	0.0268	1.2112	0.0700	0.0112
	800	2.1151	0.1039	0.0151	1.2050	0.0599	0.0050
θ	100	2.2127	2.7293	0.4127	1.3616	0.8506	0.1616
	200	1.9436	0.9222	0.1436	1.2683	0.4627	0.0683
	400	1.8576	0.5194	0.0576	1.2263	0.2836	0.0263
	600	1.8244	0.3971	0.0244	1.2076	0.2159	0.0076
	800	1.8146	0.3264	0.0146	1.2007	0.1812	0.0007
α	100	0.6907	0.6368	0.1907	0.5531	0.2351	0.0531
	200	0.5917	0.4098	0.0917	0.5268	0.1507	0.0268
	400	0.5261	0.1627	0.0261	0.5111	0.0761	0.0111
	600	0.5151	0.0885	0.0151	0.5082	0.0594	0.0082
	800	0.5109	0.0732	0.0109	0.5062	0.0499	0.0062

Table 3: Monte Carlo simulation results for GOBX-W distribution

		$(1.8, 1.4, 1.1)$			$(1.8, 1.2, 1.1)$		
Parameter	Sample Size	Mean	RMSE	Abias	Mean	RMSE	Abias
δ	100	1.9797	0.4030	0.1797	1.9782	0.3947	0.1782
	200	1.8869	0.2474	0.0869	1.9002	0.2764	0.1002
	400	1.8329	0.1282	0.0329	1.8356	0.1331	0.0356
	600	1.8206	0.1029	0.0206	1.8232	0.1073	0.0232
	800	1.8110	0.0883	0.0110	1.8130	0.0913	0.0130
θ	100	1.6186	1.2441	0.2186	1.3523	0.9380	0.1523
	200	1.4809	0.6234	0.0809	1.2483	0.5204	0.0483
	400	1.4305	0.3696	0.0305	1.2216	0.3128	0.0216
	600	1.4088	0.2824	0.0088	1.2024	0.2394	0.0024
	800	1.4051	0.2369	0.0051	1.1997	0.2013	-0.0003
α	100	1.4724	1.2224	0.3724	1.4926	1.1828	0.3926
	200	1.2634	0.6989	0.1634	1.3357	0.8756	0.2357
	400	1.1553	0.2988	0.0553	1.1660	0.3271	0.0660
	600	1.1348	0.1869	0.0348	1.1435	0.2090	0.0435
	800	1.1247	0.1521	0.0247	1.1315	0.1686	0.0315

From the results in Table 2 and 3, it is clear that as the sample size increases, the mean estimates gets closer to true parameters whereas the respective RSMEs and Abias decay to zero indicating consistent MLEs.

7. Applications

This section presents two real data applications to show the flexibility of GOBX-W distribution compared with other existing distributions. The first data consists of $n = 59$ monthly the actual taxes revenue (in 1000 million Egyptian pounds) in Egypt between January 2006 and November 2010 extracted from Nassar and Nada (2011) and given as: 5.9, 20.4, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1,6.7, 17.0, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10.0, 4.1, 36.0, 8.5, 8.0, 9.2, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7.0, 8.6, 12.5, 10.3, 11.2, 6.1, 8.4, 11.0, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, 10.8. This data have mean, median, minimum and maximum values of 13.49, 10.60, 4.10 and 39.20, respectively. The second data consists of $n = 46$ active repair times (hours) for an airborne communication transceiver referenced by Sultan and Al-Moisheer (2015) and are given as: 0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5. This data have mean, median, minimum and maximum values of 3.61, 1.75, 0.2 and 24.5, respectively. The GOBX-W distribution was compared with other existing models with the use of R-software to run model parameter estimates and goodness-of-fit measures. The goodness-of-fit measures used for model performance comparison are; -2 log-likelihood ($-2\ln(L)$), Akaike Information Criterion ($AIC = 2p - 2\ln(L)$), Bayesian Information Criterion ($BIC = p\ln(n) - 2\ln(L)$), Consistent Akaike Information Criterion $\left(AICC = AIC + 2\frac{p(p+1)}{n-p-1}\right)$, where $L = L(\hat{\Theta})$ is the value of the likelihood function at the parameter estimates, n is the number of observations and p is the number of estimated parameters for a given function. Model fit was also investigated using graphical plots such as the fitted densities, empirical cdfs, plots of hrfs, Kaplan-Meier, TTT plots and probability plots with $F(x_{(j)}; \hat{\delta}, \hat{\theta}, \hat{\xi})$ plotted against $\dfrac{j - 0.375}{n + 0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measure of closeness termed as Sum of Squares (SS), (Chambers et al., 1983), was computed as

$$SS = \sum_{j=1}^{n} \left[F(x_{(j)}; \hat{\delta}, \hat{\theta}, \hat{\xi}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2 .$$

In addition, the Cramér-von Mises (W^*) and Andersen-Darling (A^*) as described by Chen and Balakrishnan (1995) were obtained together with the Kolmogorov-Smirnov (K-S) statistics and their p-values. For all these goodness-of-fit statistics, smaller values suggests a very good fit of the model to the data. Additionally, large p-values also represent a good fit for the model. The new GOBX-W distribution was compared to Weibull (W) distribution, Weibull Lomax (WL) distribution (Jamal et al., 2019), Gamma Weibull (GW) distribution (Pogány and Saxena, 2010), Gamma Burr X (GBX) distribution (Khaleel et al., 2016), Exponentiated Half Logistic (EHL) distribution (Cordeiro et al., 2017) and Burr X (BX) distribution (Khaleel et al., 2016) with pdfs

$$f_{WL}(x) = \alpha ab(1+bx)^{a\alpha-1}(1-(1+bx)^{-a})^{\alpha-1}\exp\left(-\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right)\right),$$

for $a, b, \alpha > 0$,

$$f_{GW}(x) = \frac{k\lambda^{-k-\beta}x^{\beta+k-1}e^{-\lambda^{-k}x^k}}{\Gamma(1+\frac{\beta}{k})},$$

for $k, \beta, \lambda > 0$,

$$f_{GBX}(x) = \frac{2\theta\lambda^2}{\Gamma(\delta)}xe^{-(\lambda x)^2}\left(1-e^{-(\lambda x)^2}\right)^{\theta-1}\left[-\theta\log\left(1-e^{-(\lambda x)^2}\right)\right]^{\delta-1},$$

for $\theta, \lambda, \delta > 0$,

$$f_{EHL}(x) = \frac{2a\lambda e^{-\lambda x}}{1-e^{-2\lambda x}}\left[\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right]^a,$$

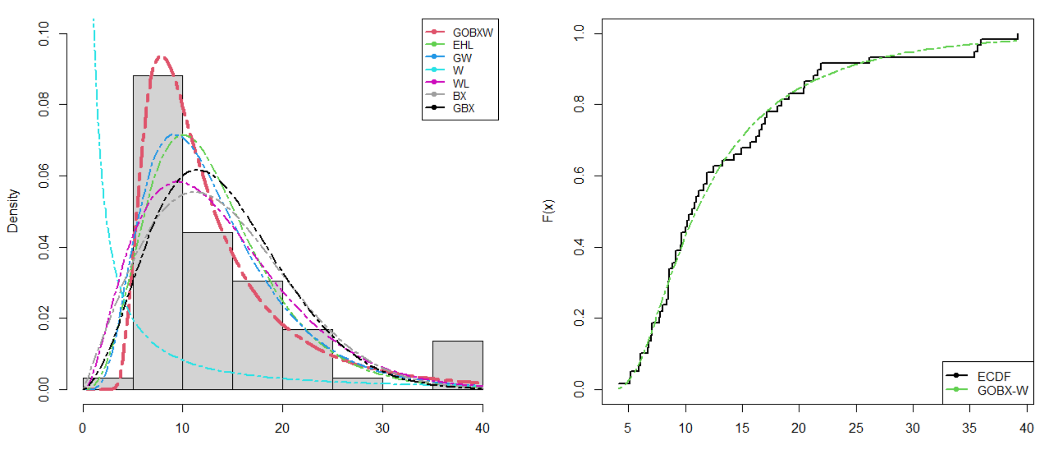
for $a, \lambda > 0$, and

$$f_{BX}(x) = 2\theta\alpha^2 xe^{-(\alpha x)^2}\left(1-e^{-(\alpha x)^2}\right)^{\theta-1},$$

for $\theta, \alpha > 0$ and $x > 0$, respectively.

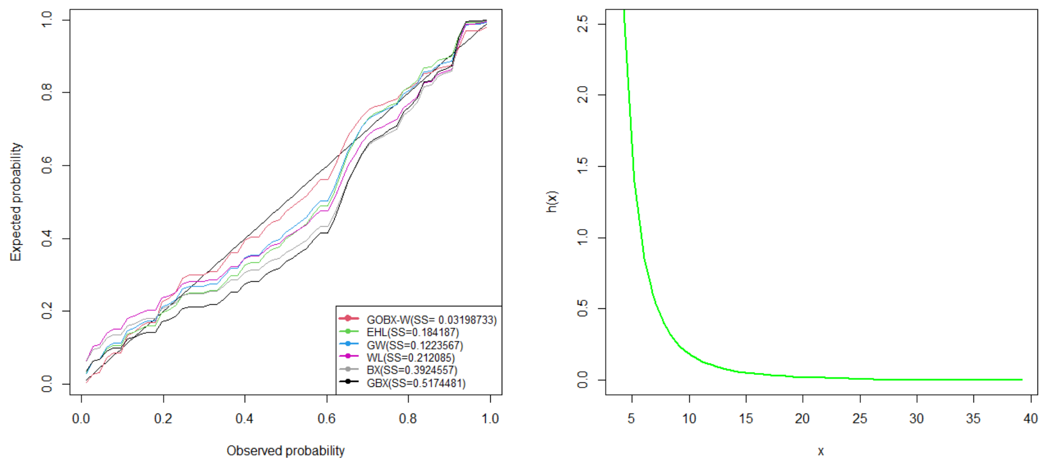
7.1. Taxes Revenue Data

This subsection contain parameter estimates (standard error in parenthesis), goodness-of-fit statistics, plots of the fitted densities, empirical cdf, Kaplan-Meier and TTT, hrf plots and probability plots for the taxes revenue data.



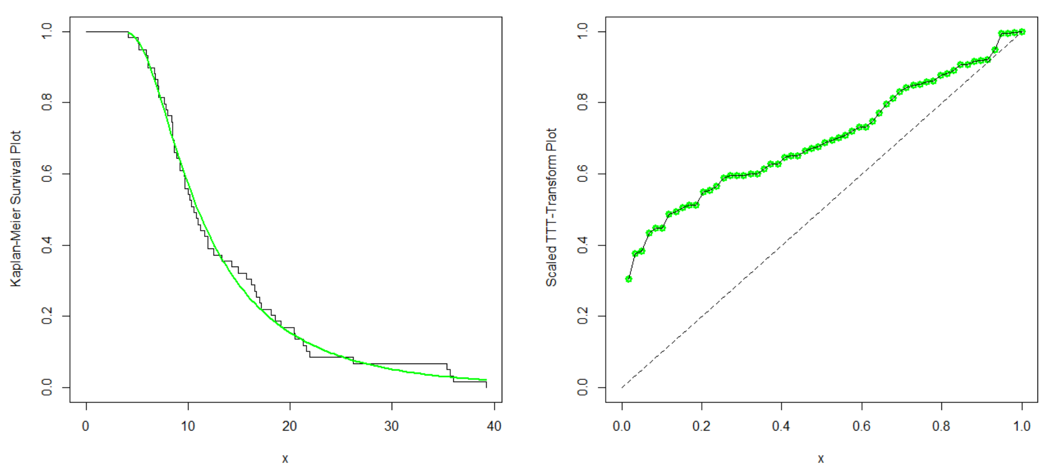
(a) (b)

Figure 6: Fitted densities and empirical cdf plots for monthly actual taxes revenue data



(a) (b)

Figure 7: Probability plot and hrf plot for monthly actual taxes revenue data



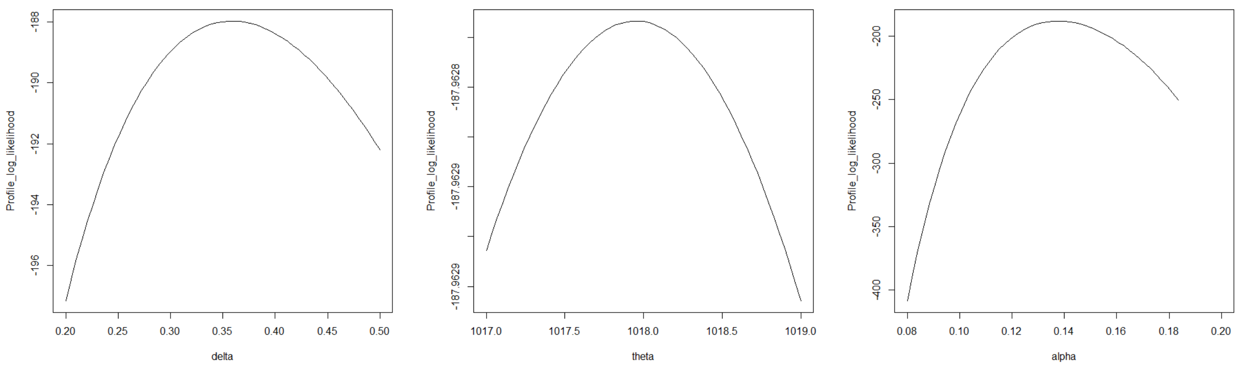
(a) (b)

Figure 8: Kaplan-Meier and TTT plots for monthly actual taxes revenue data

Figures 6(a), 6(b), 7(a), 7(b), 8(a) and 8(b) illustrates how best the GOBX-W distribution fits the monthly actual taxes revenue data. The fitted density shows that GOBX-W distribution can accommodate skewed data. The fitted hrf exhibits decreasing shape. The estimated variance-covariance matrix for GOBX-W model in revenue data is given by

$$\begin{bmatrix} 1.49 \times 10^{-2} & 4.52 \times 10^{-6} & -1.18 \times 10^{-3} \\ 4.52 \times 10^{-6} & 1.38 \times 10^{-9} & -3.58 \times 10^{-7} \\ -1.18 \times 10^{-3} & -3.58 \times 10^{-7} & 1.07 \times 10^{-4} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\delta \in [3.59 \times 10^{-1} \pm 2.39 \times 10^{-1}], \theta \in [1.02 \times 10^{3} \pm 7.27 \times 10^{-5}]$ and $\alpha \in [1.38 \times 10^{-1} \pm 2.03 \times 10^{-2}]$.



(a) (b) (c)

Figure 9: Profile plots of δ, θ and α for monthly actual taxes revenue data

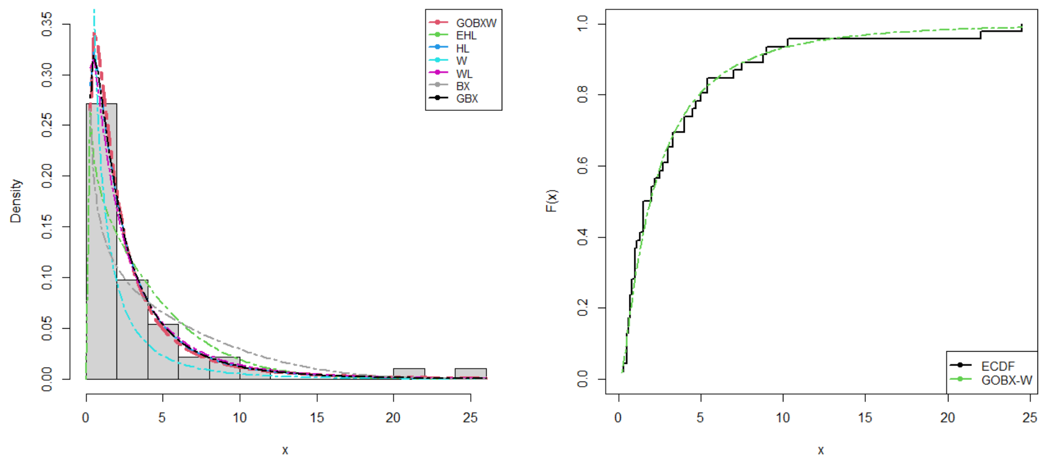
Figure 9(a), 9(b) and 9(c) shows profile plots of the MLEs of δ, θ and α. It can be seen that the parameters attained the absolute maximum for monthly actual taxes revenue data.

Table 4: Parameter estimation and goodness-of-fit statistics of the GOBX-W model and various models for the revenue data set

Distribution	Estimates			-2LogL	AIC	AICC	BIC	W^*	A^*	K-S	P-value	SS
	δ	θ	α									
GOBX-W	3.588×10^{-1}	1.018×10^{3}	1.377×10^{-1}	375.9	381.9	382.4	388.2	0.03356	0.2240	0.0587	0.9871	0.0320
	(1.220×10^{-1})	(3.710×10^{-5})	(1.034×10^{-2})									
	a	b	α									
WL	0.2558	0.9385	3.9034	389.1	395.1	395.5	401.3	0.2259	1.4233	0.1332	0.2463	0.2121
	(0.1315)	(1.3999)	(1.6510)									
	k	β	λ									
GW	3.548×10^{-1}	9.9433	9.201×10^{-4}	381.5	387.5	387.9	393.7	0.1326	0.7959	0.1060	0.5211	0.1224
	(2.874×10^{-2})	(2.871×10^{-6})	(9.126×10^{-4})									
	θ	λ	δ									
GBX	1.0741	0.1412	0.2269	394.2	400.2	400.6	406.4	0.2737	1.7546	0.1943	0.0232	0.5174
	(0.4418)	(0.0562)	(0.1836)									
	a	λ										
EHL	3.4904	0.1917	-	384.8	388.8	389.0	392.9	0.1720	1.0405	0.1215	0.3486	0.18418
	(0.8301)	(0.0227)										
	θ	α										
BX	1.0309	0.0644	-	395.3	399.3	399.6	403.5	0.3112	1.9904	0.1763	0.0510	0.3925
	(0.1844)	(0.0056)										
	λ											
W	0.3163	-	-	594.7	596.7	596.8	598.8	0.1329	0.7973	0.7956	< 0.0001	12.3219
	(0.0261)											

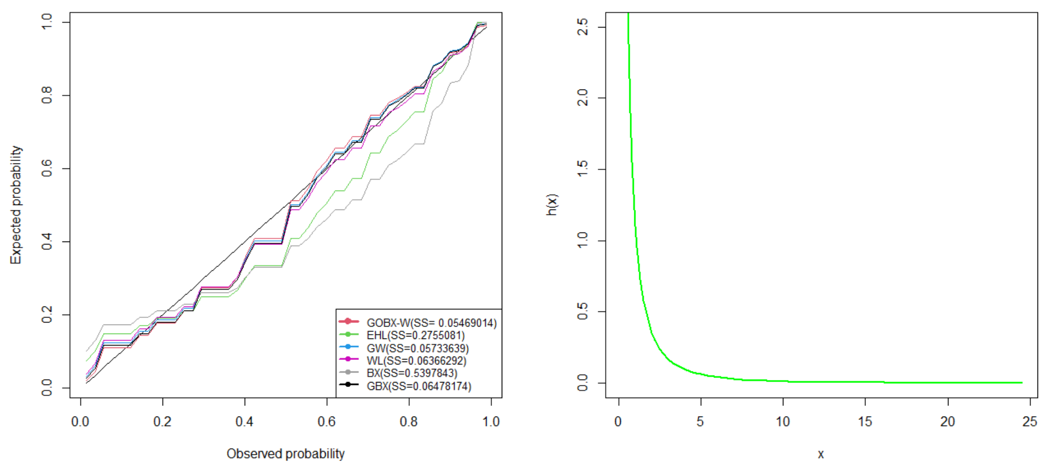
7.2. Repair Time Data

This subsection contain parameter estimates (standard error in parenthesis), goodness-of-fit statistics, plots of the fitted densities, empirical cdf, Kaplan-Meier and TTT, hrf plots and probability plots for the repair time data.



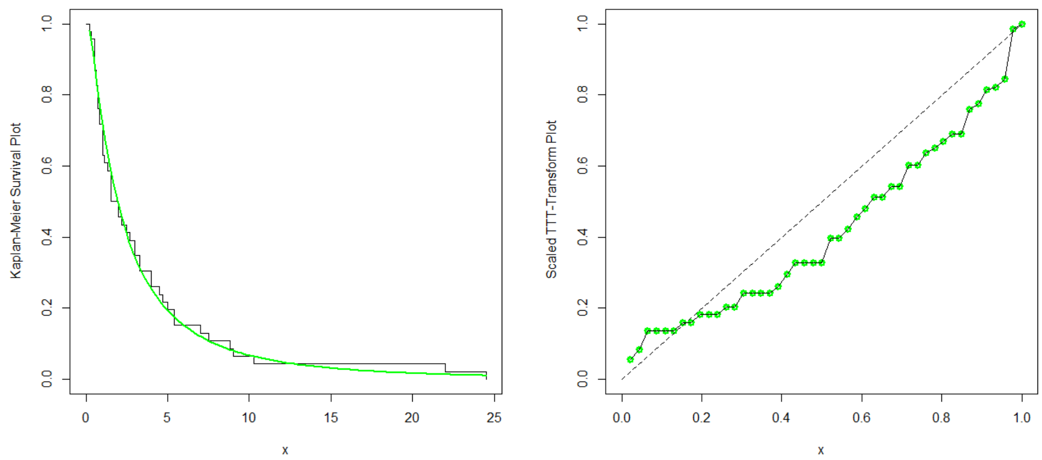
(a) (b)

Figure 10: Fitted densities and empirical cdf plots for repair time data



(a) (b)

Figure 11: Probability plot and hrf plot for repair time data



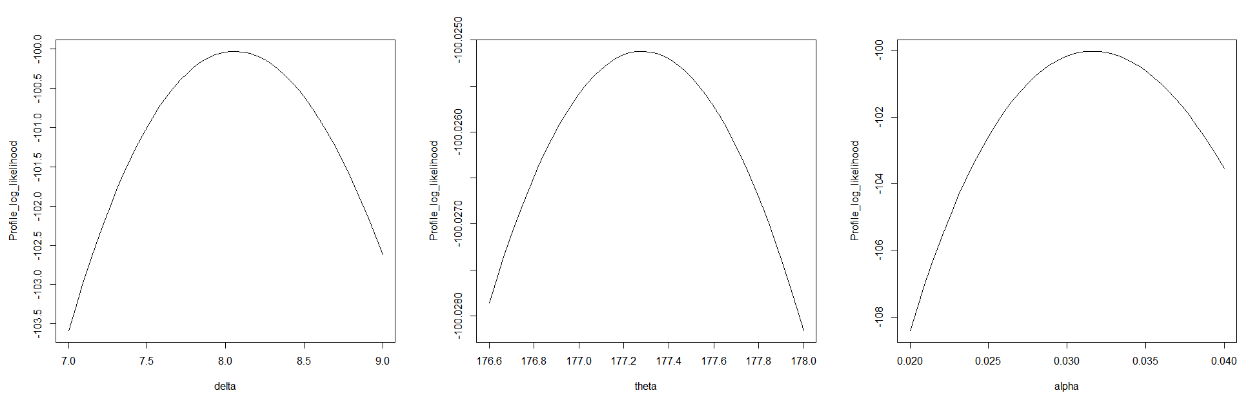
(a) (b)

Figure 12: Kaplan-Meier and TTT plots for repair time data

Figures 10(a), 10(b), 11(a), 11(b), 12(a) and 12(b) also shows GOBX-W distribution providing a better fit to the repair time data. The fitted density further illustrates the flexibility of GOBX-W distribution with fitting skewed data. The estimated variance-covariance matrix for GOBX-W model in repair time data is given by

$$\begin{bmatrix} 2.09\times10^{-1} & -1.01\times10^{-2} & -7.36\times10^{-4} \\ -1.01\times10^{-2} & 4.91\times10^{-4} & 3.56\times10^{-5} \\ -7.36\times10^{-4} & 3.56\times10^{-5} & 1.18\times10^{-5} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\delta \in [8.06 \pm 0.89]$, $\theta \in [1.77\times10^{2} \pm 4.34\times10^{-2}]$ and $\alpha \in [3.16\times10^{-2} \pm 6.75\times10^{-3}]$.



(a) (b) (c)

Figure 13: Profile plots of δ, θ and α for the repair time data

Figure 13(a), 13(b) and 13(c) shows profile plots of the MLEs of δ, θ and α. It can be seen that the parameters attained the absolute maximum for the repair time data.

Table 5: Parameter estimation and goodness-of-fit statistics of the GOBX-W model and various models for the repair time data set

Distribution	Estimates			-2LogL	AIC	AICC	BIC	W^*	A^*	K-S	P-value	SS
	δ	θ	α									
GOBX-W	8.057	1.773×10^{2}	3.165×10^{-2}	200.1	206.1	206.6	211.5	0.0523	0.3315	0.0949	0.8017	0.0547
	(4.577×10^{-1})	(2.216×10^{-2})	(3.444×10^{-3})									
	a	b	α									
WL	0.2222	7.0310	2.12279	202.3	208.3	208.8	213.8	0.0697	0.4745	0.1059	0.6802	0.0637
	(0.1094)	(11.2110)	(0.91009)									
	k	β	λ									
GW	0.2648	2.7865	0.0002	201.7	207.7	208.2	213.1	0.0661	0.4379	0.0987	0.7619	0.0573
	(0.0226)	0.0001)	(0.0002)									
	θ	λ	δ									
GBX	3.5237	7.250×10^{-5}	62.530	201.3	207.3	207.9	212.8	0.0631	0.4181	0.1048	0.6923	0.0647
	(2.086×10^{-9})	(1.180×10^{-5})	(1.195×10^{-10})									
	a	λ										
EHL	0.744	0.311	-	215.3	219.3	219.6	223.0	0.2072	1.3926	0.1660	0.1584	0.2755
	(0.1378)	(0.0572)										
	θ	α										
BX	0.2906	0.0991	-	223.4	227.4	227.7	231.1	0.2782	1.9018	0.1813	0.0973	0.5397
	(0.0482)	(0.0145)										
	λ											
W	0.565	-	-	241.9	243.9	243.9	245.7	0.0977	0.6718	0.4478	< 0.0001	3.1535
	(0.0500)											

From the results in Table 4 and 5, GOBX-W distribution outperformed all the competing non-nested models. It had the lowest values of -2LogL, AIC, AICC, BIC, W^*, A^*, K-S and SS together with largest p-value when compared to competing models in all data sets.

8. Concluding Remarks

We developed a new family of distributions termed as the Gamma Odd Burr X-G (GOBX-G) distribution. Mathematical and statistical properties of this new family were studied. In addition, different baseline distributions were considered to show the flexibility of this family of distributions on real life applications. Simulation study was also carried out using Weibull distribution as the baseline and it showed that its maximum likelihood estimators are accurate and consistent. Based on two real data sets, Gamma Odd Burr X-Weibull (GOBX-W) distribution outperformed models it was competing with based on -2LogL, AIC, AICC, BIC, W^*, A^*, K-S and SS statistics which were very low compared to those of the non-nested models. Graphical plots were also added to show how best GOBX-W distribution fits real life datasets. Moreover, the hrf can be decreasing, increasing, upside down bathtub and bathtub shaped depending on the selected parameter values. Future work may focus on the application of the developed models in regression analysis, use of Bayesian inference, as well as estimation of the model parameters under various censoring schemes. One may also consider different parameter estimation techniques for the proposed family of distributions.

References

1. Abramowitz, M. and Stegun, I. (1972). Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Table.
2. Afify, A. Z., Aljohani, H. M., Alghamdi, A. S., Gemeay, A. M., and Sarg, A. M. (2021). A New Two-Parameter Burr-Hatke Distribution: Properties and Bayesian and Non-Bayesian Inference with Applications. *Journal of Mathematics*.
3. Ahmed, M. T., Khaleel, M. A., Oguntunde, P. E., and Abdal-Hammed, M. K. (2021). A New Version of the Exponentiated Burr X Distribution. In *Journal of Physics: Conference Series*, volume 1818. IOP Publishing.
4. Al-Babtain, A. A., Elbatal, I., Al-Mofleh, H., Gemeay, A. M., Afify, A. Z., and Sarg, A. M. (2021a). The Flexible Burr X-G Family: Properties, Inference, and Applications in Engineering Science. *Symmetry*, 13(3):474.
5. Al-Babtain, A. A., Gemeay, A. M., and Afify, A. Z. (2021b). Estimation Methods for the Discrete Poisson-Lindley and Discrete Lindley Distributions with Actuarial Measures and Applications in Medicine. *Journal of King Saud University-Science*, 33(2):101224.
6. Al-Babtain, A. A., Kumar, D., Gemeay, A. M., Dey, S., and Afify, A. Z. (2021c). Modeling Engineering Data Using Extended Power-Lindley Distribution: Properties and Estimation Methods. *Journal of King Saud University-Science*, 33(8):101582.

7. Arshad, R. M. I., Tahir, M. H., Chesneau, C., Jamal, F., Mansoor, M., Zubair, M., and Nasir, M. (2020). The Gamma Kumaraswamy-G Family of Distributions: Theory, Inference and Applications. *Statistics in Transition New Series*, 21(5):17–40.
8. Barreto-Souza, W., Lemonte, A. J., and Cordeiro, G. M. (2013). General Results for the Marshall and Olkin's Family of Distributions. *Anais da Academia Brasileira de Ciências*, 85:3–21.
9. Burr, I. W. (1942). Cumulative Frequency Functions. *The Annals of Mathematical Statistics*, 13(2):215–232.
10. Chambers, J., Cleveland, W., Kleiner, B., , and Tukey, J. (1983). Graphical Methods for Data Analysis. *Chapman and Hall*.
11. Chamunorwa, S., Makubate, B., Oluyede, B., and Chipepa, F. (2021). The Exponentiated Half Logistic-Log-Logistic Weibull distribution: Model, Properties and Applications. *Journal of Statistical Modelling: Theory and Applications*, pages 1–20.
12. Chen, G. and Balakrishnan, N. (1995). A General Purpose Approximate Goodness-of-Fit Test. *Journal of Quality Technology*, 27(2):154–161.
13. Cordeiro, G. M., Afify, A. Z., Yousof, H. M., Pescim, R. R., and Aryal, G. R. (2017). The Exponentiated Weibull-H Family of Distributions: Theory and Applications. *Mediterranean Journal of Mathematics*, 14(4):1–22.
14. Grandshteyn, I. and Ryzhik, I. (1980). Table of Integrals, Series, and Products. *New York: Academic Press*, 8:362.
15. Ibrahim, N. A., Khaleel, M. A., Merovci, F., Kilicman, A., and Shitan, M. (2017). Weibull Burr X Distribution Properties and Application. *Pakistan Journal of Statistics*, 33(5).
16. Jamal, F., Reyad, H., Nasir, M., Chesneau, C., Shah, M., and Ahmed, S. (2019). Topp-Leone Weibull-Lomax Distribution: Properties, Regression model and Applications. *Hal Open Science*, hal-02270561.
17. Khaleel, M. A., Ibrahim, N. A., Shitan, M., and Merovci, F. (2016). Some Properties of Gamma Burr Type X Distribution with Application. In *AIP Conference proceedings*, volume 1739, pages 1–8. AIP Publishing LLC.
18. Khaleel, M. A., Ibrahim, N. A., Shitan, M., and Merovci, F. (2018). New Extension of Burr Type X Distribution Properties with Application. *Journal of King Saud University-Science*, 30(4):450–457.
19. Lahcene, B. (2021). A New Extended-Gamma Family of Distributions: Properties and Applications. *Journal of Applied Mathematics and Computation*, 5(1):9–17.
20. Merovci, F., Khaleel, M. A., Ibrahim, N. A., and Shitan, M. (2016). The Beta Burr Type X Distribution Properties with Application. *SpringerPlus*, 5(1):1–18.
21. Nassar, M. and Nada, N. (2011). The Beta Generalized Pareto Distribution. *Journal of Statistics: Advances in Theory and Applications*, 6(1/2):1–17.
22. Oluyede, B. O., Mashabe, B., Fagbamigbe, A., Makubate, B., and Wanduku, D. (2020). The Exponentiated Generalized Power Series Family of Distributions: Theory, Properties and Applications. *Heliyon*, 6(8):e04653.
23. Peter, P., Oluyede, B., Bindele, H., Ndwapi, N., and Mabikwa, O. (2021a). The Gamma Odd Burr III-G Family of Distributions: Model, Properties and Applications. *Revista Colombiana de Estadística*, 44(2):331–368.
24. Peter, P. O., Oluyede, B., Ndwapi, N., and Bindele, H. (2021b). The Weibull Odd Burr III-G Family of Distributions with Properties and Applications. *Journal of Statistical Modelling: Theory and Applications*, 2(1):151–180.
25. Pogány, T. k. and Saxena, R. k. (2010). The Gamma-Weibull Distribution Revisited. *Anais da Academia Brasileira de Ciências*, 82(2):513–520.
26. Ristić, M. M. and Balakrishnan, N. (2012). The Gamma Exponentiated Exponential Distribution. *Journal of Statistical Computation and Simulation*, 82(8):1191–1206.
27. Sengweni, W., Oluyede, B., and Makubate, B. (2021). The Exponentiated Half-logistic Odd Lindley-G Family of Distributions with Applications. *Journal of Nonlinear Sciences & Applications (JNSA)*, 14(5):287–309.
28. Shrahili, M., Elbatal, M., and Muhammad, M. (2019). The Type I Half-Logistic Burr X Distribution: Theory and Practice. *J. Nonlinear Sci. Appl*, 12:262–277.
29. Sultan, K. and Al-Moisheer, A. (2015). Mixture of Inverse Weibull and Lognormal Distributions: Properties, Estimation, and Illustration. *Mathematical Problems in Engineering*.
30. Surles, J. and Padgett, W. (2001). Inference for Reliability and Stress-strength for a Scaled Burr Type X

Distribution. *Lifetime Data Analysis*, 7(2):187–200.
31. Usman, R. M. and Ilyas, M. (2020). The Power Burr Type X Distribution: Properties, Regression Modeling and Applications. *Punjab University Journal of Mathematics*, 52(8):27–44.
32. Yousof, H. M., Afify, A. Z., Hamedani, G., and Aryal, G. R. (2017). The Burr X Generator of Distributions for Lifetime Data. *Journal of Statistical Theory and Applications*, 16(3):288–305.
33. Zanakis, S. H. and Kyparisis, J. (1986). A Review of Maximum Likelihood Estimation Methods for the Three-Parameter Weibull Distribution. *Journal of Statistical Computation and Simulation*, 25(1-2):53–73.

A. Components of the Score Vector

The components of the score vector for the GOBX-G family of distributions are:

$$\frac{\partial \ell_n(\Theta)}{\partial \delta} = n\ln(\theta) - n\psi(\delta) + \sum_{i=1}^{n} \ln\left[-\log\left(1-e^{-y_i^2}\right)\right] \ ,$$

$$\frac{\partial \ell_n(\Theta)}{\partial \theta} = \frac{n\delta}{\theta} + \sum_{i=1}^{n} \ln\left(1-e^{-y_i^2}\right)$$

and

$$\begin{aligned}\frac{\partial \ell_n(\Theta)}{\partial \xi_k} &= \sum_{i=1}^{n} \frac{\mathrm{g}'(x;\xi)}{\mathrm{g}(x;\xi)} + \sum_{i=1}^{n} \frac{\mathrm{G}'(x;\xi)}{\mathrm{G}(x;\xi)} - 3\sum_{i=1}^{n} \frac{\overline{\mathrm{G}}\,'(x;\xi)}{\overline{\mathrm{G}}(x;\xi)} - 2\sum_{i=1}^{n} y_i \frac{\partial y_i}{\partial \xi_k} \\ &+ (\theta-1)\sum_{i=1}^{n} \frac{2y_i e^{-y_i^2}}{1-e^{-y_i^2}}\ \frac{\partial y_i}{\partial \xi_k} + (\delta-1)\sum_{i=1}^{n} \frac{2y_i e^{-y_i^2}}{\left(1-e^{-y_i^2}\right)\log\left(1-e^{-y_i^2}\right)}\ \frac{\partial y_i}{\partial \xi_k}\ ,\end{aligned}$$

where $\psi(\delta) = \partial\Gamma(\delta)/\partial\delta$, $\mathrm{g}'(x;\xi) = \partial \mathrm{g}(x;\xi)/\partial\xi_k$, $\mathrm{G}'(x;\xi) = \partial \mathrm{G}(x;\xi)/\partial\xi_k$, $\overline{\mathrm{G}}\,'(x;\xi) = \partial\overline{\mathrm{G}}(x;\xi)/\partial\xi_k$, ξ_k is the k^{th} element of the vector of parameters ξ and $y_i = \mathrm{G}(x_i;\xi)/\overline{\mathrm{G}}(x_i;\xi)$.

B. R code for Applications

All codes for applications, simulations and plots can be found at `https://drive.google.com/drive/folders/17XqzV07QZUDC0LYjysYT9qu1QWpaqYMK?usp=sharing`.