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The Topp-Leone-Gompertz-G Power Series Class of Distributions with Applications

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Abstract

A new generalized class of distributions called the Topp-Leone-Gompertz-G Power Series (TL-Gom-GPS) distribution is presented. The TL-Gom-GPS distribution has the ability to capture diverse data patterns such as heavy-tailed, light-tailed and skewed distributions, as well as hazard rate functions that have monotonic and non-monotonic shapes. Some statistical properties of the new class of distributions are explored. For this new class of distributions, we derived the expansion of the density function, the hazard rate and quantile functions, moments and generating function, probability weighted moments, distribution of order statistics and Rényi entropy. Monte Carlo simulations are conducted to show the performance of the estimation methods. Finally, the usefulness and flexibility of the new class of distributions is examined by means of applications to real data sets.

Key Words: Topp-Leone Gompertz-G; Power Series Distribution; Maximum Likelihood Estimation; Statistical Properties; Monte Carlo Simulations.

Mathematical Subject Classification: 62E99, 60E05, 62E15.

1. Introduction

The power series class of distributions are of tremendous importance in the areas of finance, reliability and actuarial sciences. Some of the generalizations of the power series distributions in the literature include the Burr XII power series (BXIIPS) distribution by Silva and Cordeiro (2015), generalized exponential power series (GEPS) distribution (Mahmoudi and Jafari, 2012), a compound class of Weibull and power series distributions by Morais and Barreto-Souza (2011), Lindley-Burr XII (LBXII) power seiries distribution by Makubate et al. (2021), generalized linear failure rate power series distribution by Harandi and Alamatsaz (2016), odd Weibull-Topp-Leone-G power series distribution by Oluyede et al. (2021) and Gompertz-power series distribution by Jafari and Tahmasebi (2016).

The Topp-Leone generated family of distributions was introduced by Al-Shomrani et al. (2016) and Gompertz-G family of distributions was developed by Alizadeh et al. (2017). Later (Oluyede et al., 2022) developed Topp-Leone Gompertz-G (TL-Gom-G) distribution based on the works by Al-Shomrani et al. (2016) and Alizadeh et al. (2017). The Gompertz distribution is used in many areas including biology, gerontology, computer science, and marketing to analyze survival rates (Eliwa et al., 2020) and (Jafari and Tahmasebi, 2016). The TL-Gom-G family of distributions was found be useful for modeling heavy-tailed data, monotonic and non-monotonic hazard rate functions

(Oluyede et al., 2022). The power series class of distributions are extremely important in the areas of finance, reliability and actuarial sciences hence by compounding the TL-Gom-G distributions and the power series distributions, we obtain a more flexible and important family of distributions. With the components from the Gompertz, power series, and Topp-Leone distributions, this distribution is unique in its formulation. The TL-Gom-GPS class of distributions provides a flexible framework for modeling a wide range of phenomena, with applications in finance, reliability engineering, actuarial science, and beyond. The cumulative distribution function (cdf) and probability density function (pdf) of the TL-Gom-G family of distributions is given by

$$F_{TL-Gom-G}(x;b,\gamma,\xi) = \left[1-\left(\exp\left[\frac{1}{\gamma}(1-(1-G(x;\xi))^{-\gamma})\right]\right)^2\right]^b \tag{1}$$

and

$$\begin{aligned} f_{TL-Gom-G}(x;b,\gamma,\xi) &= 2bg(x;\xi)(1-G(x;\xi))^{-\gamma-1}\left(\exp\left[\frac{1}{\gamma}(1-(1-G(x;\xi))^{-\gamma})\right]\right)^2 \\ &\times \left[1-\left(\exp\left[\frac{1}{\gamma}(1-(1-G(x;\xi))^{-\gamma})\right]\right)^2\right]^{b-1}, \end{aligned} \tag{2}$$

respectively, for $x > 0$, $b, \gamma > 0$ and parameter vector ξ from baseline cdf G.
Suppose N is a discrete random variable following a power series distribution and assumed to be truncated at zero with probability mass function (pmf) given by

$$P(N=n) = \frac{a_n\theta^n}{C(\theta)}, \quad n = 1, 2, ..., \tag{3}$$

where $C(\theta) = \sum_{n=1}^{\infty} a_n\theta^n$ is finite, $\theta > 0$ and $\{a_n\}_{n\geq 1}$ a sequence of positive real numbers. The power series family of distributions includes Poisson, binomial, geometric and logarithmic distributions (Johnson et al., 1994).

The motivations for developing this new generalized class of distributions are as follows:

- To develop a distribution that can model data with different levels of skewness and kurtosis.
- To obtain flexible distribution for modelling various types of data in practice.
- To come up with a model with the hazard function that exhibit both monotonic and non-monotonic shapes.

The results of this paper are organised as follows: Section 2 contains the model, the Topp-Leone Gompertz-G Power Series (TL-Gom-GPS) class of distributions and its sub-classes. In Section 3, we present some statistical properties of the TL-Gom-GPS class of distributions including expansion of the probability density function, hazard rate and quantile functions, moments, generating functions, probability weighted moments, order statistics and Rényi entropy. Maximum likelihood estimation and other methods of estimation of the model parameters are given in Section 4. Examples of the special cases of the new class of distributions are given in Section 5. Monte Carlo simulation results for five different estimation methods are presented in Section 6. Section 7 contains applications of the new model to two real data sets. Finally, concluding remarks are given in Section 8.

2. The New Model

In this section, we present the proposed class and its sub-classes of distributions. Let X be a random variable such that X_i, for $i = 1, 2, ..., N$ denote the time to failure of a device due to the i^{th} defect with the assumption that the X_i's are independent and identically distributed (iid) TL-Gom-G random variables and $X_{(1)}$ =min$(X_1, X_2, ...X_N)$, with the distribution of N given by equation (3). The TL-Gom-GPS class of distributions is defined by the marginal distribution of $X_{(1)}$, say $F_\theta(x)$, and is given by

$$\begin{aligned} F_\theta(x) &= 1-\frac{C(\theta S(x;b,\gamma,\xi))}{C(\theta)} \\ &= 1-\frac{C\left(\theta\left(1-\left[1-\left(\exp\left[\frac{1}{\gamma}(1-(1-G(x;\xi))^{-\gamma})\right]\right)^2\right]^b\right)\right)}{C(\theta)}, \end{aligned} \tag{4}$$

where $S(x;b,\gamma,\xi)$ is the survival function of the TL-Gom-G distribution. The corresponding pdf is given by

$$\begin{aligned} f_\theta(x) &= 2b\theta g(x;\xi)(1-G(x;\xi))^{-\gamma-1}\left(\exp\left[\frac{1}{\gamma}(1-(1-G(x;\xi))^{-\gamma})\right]\right)^2 \\ &\times \left[1-\left(\exp\left[\frac{1}{\gamma}(1-(1-G(x;\xi))^{-\gamma})\right]\right)^2\right]^{b-1} \\ &\times \frac{C'\left(\theta\left(1-\left[1-\left(\exp\left[\frac{1}{\gamma}(1-(1-G(x;\xi))^{-\gamma})\right]\right)^2\right]^b\right)\right)}{C(\theta)} \end{aligned} \tag{5}$$

for $b,\gamma,\theta>0$ and parameter vector ξ.

Some sub-classes cases of the TL-Gom-GPS class of distributions are given in Table 1.

Table 1: Some sub-classes of the TL-Gom-GPS class of distributions

Distribution	a_n	$C(\theta)$	cdf
TL-Gom-G Poisson	$(n!)^{-1}$	$e^\theta-1$	$1-\frac{e^{\left(\theta\left(1-\left[1-\left(\exp[\frac{1}{\gamma}(1-(1-G(x;\xi))^{-\gamma})]\right)^2\right]^b\right)\right)}-1}{e^\theta-1}$
TL-Gom-G Geometric	1	$\theta(1-\theta)^{-1}$	$1-\frac{(1-\theta)\left(1-\left[1-\left(\exp[\frac{1}{\gamma}(1-(1-G(x;\xi))^{-\gamma})]\right)^2\right]^b\right)}{\left(1-\theta\left(1-\left[1-\left(\exp[\frac{1}{\gamma}(1-(1-G(x;\xi))^{-\gamma})]\right)^2\right]^b\right)\right)}$
TL-Gom-G Logarithmic	n^{-1}	$-\log(1-\theta)$	$1-\frac{\log\left(1-\theta\left(1-\left[1-\left(\exp[\frac{1}{\gamma}(1-(1-G(x;\xi))^{-\gamma})]\right)^2\right]^b\right)\right)}{\log(1-\theta)}$
TL-Gom-G Binomial	$\binom{m}{n}$	$(1+\theta)^m-1$	$1-\frac{\left(1+\theta\left(1-\left[1-\left(\exp[\frac{1}{\gamma}(1-(1-G(x;\xi))^{-\gamma})]\right)^2\right]^b\right)\right)^m-1}{(1+\theta)^m-1}$

3. Some Statistical Properties

In this section, we present some statistical properties including, linear representation of the density function, hazard and quantile functions, moments, generating function, probability weighted moments, distribution of order statistics and Rényi entropy of the TL-Gom-GPS class of distributions.

3.1. Expansion of the Density Function

In this subsection, we obtain the linear representations of the TL-Gom-GPS class of distributions. Using the following generalized binomial and Taylor series expansions:

$$(1-z)^s = \sum_{k=0}^{\infty}(-1)^k \binom{s}{k} z^k, \quad \text{for } |z|<1, \quad \text{and } e^z = \sum_{k=0}^{\infty}\frac{z^k}{k!},$$

we can write the pdf of the TL-Gom-GPS class of distributions as

$$f_\theta(x) = \sum_{t=0}^{\infty} W_{t+1} g^*_{t+1}(x;\xi), \tag{6}$$

where

$$\begin{aligned} W_{t+1} &= \sum_{n=1}^{\infty}\sum_{i,j,k,l=0}^{\infty}\frac{2b\theta n a_n \theta^n}{C(\theta)}\binom{n-1}{i}\binom{b(i+1)-1}{i}\frac{(2i+2)^k(-1)^{i+j+l+t}}{\gamma^k k!(t+1)} \\ &\times \binom{-\gamma(l+1)-1}{t}\binom{k}{l}, \end{aligned} \tag{7}$$

and $g^*_{t+1}(x;\xi) = (t+1)g(x;\xi)G^t(x;\xi)$ is the exponentiated-G (Exp-G) densities with power parameter (t+1).

Consequently, the pdf of the TL-Gom-GPS class of distributions can be expressed as an infinite linear combination of Exp-G densities and the statistical properties of the TL-Gom-GPS class of distributions can be obtained directly from those of the Exp-G family of distributions. See Appendix for more details of the derivation.

3.2. Hazard Rate and Quantile Functions

The hazard rate and quantile functions of the TL-Gom-GPS class of distributions are given in this subsection. The hazard rate function (hrf) is given by

$$\begin{aligned} h_\theta(x) = \frac{f_\theta(x)}{S_\theta(x)} &= 2b\theta g(x;\xi)(1-G(x;\xi))^{-\gamma-1}\left(\exp\left[\frac{1}{\gamma}(1-(1-G(x;\xi))^{-\gamma})\right]\right)^2 \\ &\times \left[1-\left(\exp[\frac{1}{\gamma}(1-(1-G(x;\xi))^{-\gamma})]\right)^2\right]^{b-1} \\ &\times \frac{C'\left(\theta\left(1-\left[1-\left(\exp\left[\frac{1}{\gamma}(1-(1-G(x;\xi))^{-\gamma})\right]\right)^2\right]^b\right)\right)}{C\left(\theta\left(1-\left[1-\left(\exp\left[\frac{1}{\gamma}(1-(1-G(x;\xi))^{-\gamma})\right]\right)^2\right]^b\right)\right)}. \end{aligned}$$

The quantile function is used to generate random numbers in simulations for a specified baseline distribution, and power series distribution. The quantile function of the TL-Gom-GPS class of distributions is obtained by inverting the non-linear equation $F_\theta(x) = u$, $0 \leqslant u \leqslant 1$. Note that

$$1-\frac{C\left(\theta\left(1-\left[1-\left(\exp\left[\frac{1}{\gamma}(1-(1-G(x;\xi))^{-\gamma})\right]\right)^2\right]^b\right)\right)}{C(\theta)} = u$$

and

$$C\left(\theta\left(1-\left[1-\left(\exp\left[\frac{1}{\gamma}(1-(1-G(x;\xi))^{-\gamma})\right]\right)^{2}\right]^{b}\right)\right)=C(\theta)(1-u).$$

This is equivalent to

$$1-\left[1-\left(\exp\left[\frac{1}{\gamma}(1-(1-G(x;\xi))^{-\gamma})\right]\right)^{2}\right]^{b}=\frac{C^{-1}(C(\theta)(1-u))}{\theta},$$

so that

$$1-(1-G(x;\xi))^{-\gamma}=\gamma\ln\left(\left[1-\left[1-\frac{C^{-1}(C(\theta)(1-u))}{\theta}\right]^{\frac{1}{b}}\right]^{\frac{1}{2}}\right).$$

Consequently, we obtain the quantile function of the TL-Gom-GPS class of distributions as

$$Q_{\theta}(u)=G^{-1}\left[1-\left(1-\gamma\ln\left(\left[1-\left[1-\frac{C^{-1}(C(\theta)(1-u))}{\theta}\right]^{\frac{1}{b}}\right]^{\frac{1}{2}}\right)\right)^{-\frac{1}{\gamma}}\right]. \tag{8}$$

By numerical methods with the aid of statistical software such as R, SAS and MATLAB, random numbers can be generated using equation (8) for specified baseline cdf G and the function C(θ).

3.3. Moments and Generating Function

The moments and generating function for the TL-Gom-GPS class of distributions are presented in this subsection. The n^{th} moment for the TL-Gom-GPS class of distributions is given by

$$E(X^{n}) = \int_{-\infty}^{\infty}x^{n}f_{\theta}(x)dx=\sum_{t=0}^{\infty}W_{t+1}E(Y_{t+1}^{n}),$$

where $E(Y_{t+1}^{n})$ is the n^{th} moment of Y_{t+1} which follows an Exp-G distribution with power parameter $(t+1)$ and W_{t+1} is given in equation (7). The moment generating function for the TL-Gom-GPS class of distributions is given by

$$M_{X}(h)=E\left(e^{hX}\right)=\sum_{t=0}^{\infty}W_{t+1}E\left(e^{hY_{t+1}}\right),$$

where $E\left(e^{hY_{t+1}}\right)$ is the moment generating function of the Exp-G family of distributions with power parameter $(t+1)$ and W_{t+1} is given in equation (7).

3.4. Probability Weighted Moments

The probability weighted moments (PWMs) offer an alternate technique for estimating distribution parameters in situations where the inverse form of a distribution cannot be clearly stated. The PWMs for the TL-Gom-GPS class of distributions are presented in this subsection. The PWMs for a random variable X following the TL-Gom-GPS distributions is given by

$$\eta_{a,r} = E(X^a[F(X)]^r) = \int_{-\infty}^{\infty} x^a f_\theta(x)[F_\theta(x)]^r dx.$$

Note that

$$f_\theta(x)[F_\theta(x)]^r = \sum_{t=0}^{\infty} A_{t+1} g^*_{t+1}(x;\xi),$$

where

$$\begin{aligned} A_{t+1} &= \sum_{n,z=1}^{\infty} \sum_{q,i,j,k,l=0}^{\infty} \frac{2b\theta n a_n \theta^{n+z} d_{z,q}}{(C(\theta))^{q+1}} (-1)^{q+i+j+l+t} \binom{r}{q} \binom{b(i+1)-1}{j} \\ &\times \frac{(2i+2)^k}{\gamma^k k!(t+1)} \binom{k}{l} \binom{-\gamma(l+1)-1}{t}, \end{aligned}$$

and $g^*_{t+1}(x;\xi) = (t+1)g(x;\xi)G^t(x;\xi)$ is the Exp-G density with the power parameter $(t+1)$ and parameter vector ξ. Therefore, the PWMs of the TL-Gom-GPS class of distributions is given by

$$\eta_{a,r} = \sum_{t=0}^{\infty} A_{t+1} \int_{-\infty}^{\infty} x^a g^*_{t+1}(x;\xi) dx.$$

See the Appendix for details of the derivation.

3.5. Distribution of Order Statistics and Rényi Entropy

In this subsection, we present the pdf of the i^{th} order statistic and Rényi entropy for the TL-Gom-GPS class of distributions.

3.5.1. Distribution of Order Statistics

Order statistics are important in many areas of probability and statistics, particularly in reliability, survival analysis, and selection of experiments. Let $X_1, X_2, ..., X_n$ be independent and identically distributed random variables from the TL-Gom-GPS class of distributions. The pdf of the i^{th} order statistic is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-1)!} \sum_{r=0}^{n-i} \sum_{t=0}^{\infty} (-1)^r \binom{n-i}{r} D_{t+1} g^*_{t+1}(x;\xi), \tag{9}$$

where $g^*_{t+1}(x;\xi) = (t+1)g(x;\xi)G^t(x;\xi)$ is the Exp-G density with power parameter $(t+1)$ and

$$\begin{aligned} D_{t+1} &= \sum_{n,z=1}^{\infty} \sum_{q,i,j,k,l=0}^{\infty} \frac{2b\theta n a_n \theta^{n+z} d_{z,q}}{(C(\theta))^{q+1}} (-1)^{q+i+j+l+t} \binom{i+r-1}{q} \binom{b(i+1)-1}{j} \\ &\times \frac{(2i+2)^k}{\gamma^k k!(t+1)} \binom{k}{l} \binom{-\gamma(l+1)-1}{t}. \end{aligned}$$

Details of the derivations can be found in the Appendix section.

3.5.2. Rényi Entropy

Rényi entropy is a mathematical concept that measures uncertainty and disorder within a distribution. Rényi entropy (Rényi, 1961) of a random variable X following TL-Gom-GPS class of distributions is defined by

$$I_R(\omega) = (1-\omega)^{-1} \log \left[\int_0^\infty f_\theta^\omega(x)dx \right] \text{ for } \omega > 0 \ and \ \omega \neq 1.$$

Rényi entropy of the TL-Gom-GPS class of distributions can be expressed as

$$I_R(\omega) = (1-\omega)^{-1} \log \left[\sum_{t=0}^{\infty} \Psi_t^* \exp\left((1-\omega) I_{REG}\right) \right], \quad (10)$$

where $I_{REG} = \int_0^\infty \left(\left(\frac{t}{\omega} + 1 \right) g(x;\xi) G^{\frac{t}{\omega}}(x;\xi) \right)^\omega dx$ is Rényi entropy of Exp-G densities with power parameter $(\frac{t}{\omega} + 1)$ and

$$\begin{aligned} \Psi_t^* &= \sum_{n=1}^{\infty} \sum_{i,j,k,l=0}^{\infty} \frac{n d_{\omega,n} (2b)^\omega \theta^{\theta+n-1}}{(C(\theta))^\omega} \binom{\omega(n-1)}{i} \binom{b(i+\omega)-\omega}{i} \binom{k}{l} \\ &\times \frac{(2i+2\omega)^k (-1)^{i+j+l+t}}{\gamma^k k!} \binom{-\gamma(l+\omega)-\omega}{t} \frac{1}{\left(\frac{t}{\omega}+1\right)^\omega}. \end{aligned}$$

Details of the derivations are given in the Appendix section.

4. Methods of Estimation

This section presents different estimation methods, including maximum likelihood estimation (MLE), ordinary least squares (OLS), weighted least squares (WLS), Anderson-Darling (AD), and Cramér-von Mises (CVM) method. Let $X_1, X_2, ..., X_n$ be a random sample from TL-Gom-GPS class of distributions and suppose $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ denote the corresponding order statistics

4.1. Maximum Likelihood Estimation

Let $X \sim TL-Gom-GPS(b,\gamma,\theta,\xi)$ and $\Psi = (b,\gamma,\theta,\xi)^T$ be the parameter vector. The log-likelihood function, ℓ_n of a random sample of size n from the $TL-Gom-GPS(b,\gamma,\theta,\xi)$ class of distributions is given by

$$\begin{aligned} \ell_n(\Psi) &= n\log(2b\theta) + \sum_{i=1}^{n} \log[g(x_i;\xi)] - (\gamma+1) \sum_{i=1}^{n} \log[(1-G(x_i;\xi))] \\ &+ (b-1) \sum_{i=1}^{n} \log\left[1 - \left(\exp\left[\frac{1}{\gamma}(1-(1-G(x_i;\xi))^{-\gamma})\right]\right)^2\right] \\ &+ \frac{2}{\gamma} \sum_{i=1}^{n} \left[(1-(1-G(x_i;\xi))^{-\gamma})\right] - \sum_{i=1}^{n} \log[C(\theta)] \\ &+ \sum_{i=1}^{n} \log\left[C'\left(\theta\left(1-\left[1-\left(\exp\left[\frac{1}{\gamma}(1-(1-G(x_i;\xi))^{-\gamma})\right]\right)^2\right]^b\right)\right)\right]. \end{aligned}$$

Elements of the score vector $U(\mathbf{\Psi})=\left(\frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \gamma}, \frac{\partial \ell_n}{\partial \theta}, \frac{\partial \ell_n}{\partial \xi_k}\right)$ are presented in the appendix.

The Fisher Information Matrix (FIM) of the TL-Gom-GPS class of distributions is given by $\mathbf{I}(\Psi) = [\mathbf{I}_{\psi_i,\psi_j}]_{(3+q)\times(3+q)} = E\left(-\frac{\partial^2 \ell}{\partial\psi_i \partial\psi_j}\right), i,j = 1,2,...,(3+q)$, can be numerically obtained via the mle2 package in R software or MATHLAB. The expectations in the FIM can be numerically obtained and the total Fisher information matrix $n\mathbf{I}(\Delta)$ can be approximated by $\mathbf{J}_n(\hat{\Psi}) \approx \left[-\frac{\partial^2 \ell}{\partial\psi_i \partial\psi_j}\Big|_{\Psi=\hat{\Psi}}\right]_{(3+q)\times(3+q)}, i,j = 1,2,....,(3+q)$. The multivariate normal distribution

$N_{(3+q)}(\underline{0}, \boldsymbol{J}(\hat{\boldsymbol{\Psi}})^{-1})$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions. Note that, the approximate $100(1-\eta)\%$ two sided confidence intervals for b, γ, θ and ξ are given by:

$$\hat{b}\pm Z^{\frac{\eta}{2}}\sqrt{I_{bb}^{-1}(\hat{\Psi})}, \quad \hat{\gamma}\pm Z_{\frac{\eta}{2}}\sqrt{I_{\gamma\gamma}^{-1}(\hat{\Psi})}, \quad \hat{\theta}\pm Z_{\frac{\eta}{2}}\sqrt{I_{\theta\theta}^{-1}(\hat{\Psi})} \quad \hat{\xi}_k\pm Z_{\frac{\eta}{2}}\sqrt{I_{\xi_{kk}}^{-1}(\hat{\Psi})},$$

respectively, where $I_{bb}^{-1}(\hat{\Psi})$, $I_{\gamma\gamma}^{-1}(\hat{\Psi})$, $I_{\theta\theta}^{-1}(\hat{\Psi})$ and $I_{\xi_{kk}}^{-1}(\hat{\Psi})$ are the diagonal elements of $I_n^{-1}(\hat{\Psi}) = (nI(\hat{\Psi}))^{-1}$, and $Z_{\frac{\eta}{2}}$ is the upper $(\frac{\eta}{2})^{th}$ percentile of the standard normal distribution.

4.2. Ordinary Least Squares

In this subsection, the ordinary least squares (OLS) estimates of the parameters of the TL-Gom-GPS class of distributions are presented. The OLS estimates of the parameters of the TL-Gom-GPS class of distributions are derived by minimizing the function

$$\kappa(b,\gamma,\theta,\xi) = \sum_{i=1}^{n}\left[F(x_i,b,\gamma,\theta,\xi) - \frac{i}{n+1}\right]^2$$

with respect to the parameters b, γ, θ and parameter vector ξ. The OLS estimates of the parameters can be obtained by solving the non-linear equations

$$\sum_{i=1}^{n}\left[F(x_i,b,\gamma,\theta,\xi) - \frac{i}{n+1}\right]\Delta_r(x_i,b,\gamma,\theta,\xi) = 0, \quad r = 1,2,3,4,$$

where

$$\Delta_1(x_i,b,\gamma,\theta,\xi) = \frac{\partial}{\partial b}F(x_i,b,\gamma,\theta,\xi), \quad \Delta_2(x_i,b,\gamma,\theta,\xi) = \frac{\partial}{\partial \gamma}F(x_i,b,\gamma,\theta,\xi),$$
$$\Delta_3(x_i,b,\gamma,\theta,\xi) = \frac{\partial}{\partial \theta}F(x_i,b,\gamma,\theta,\xi), \quad \text{and} \quad \Delta_4(x_i,b,\gamma,\theta,\xi) = \frac{\partial}{\partial \xi_s}F(x_i,b,\gamma,\theta,\xi). \tag{11}$$

4.3. Weighted Least Squares

The weighted least squares (WLS) estimates of the parameters of the TL-Gom-GPS class of distributions are obtained by minimizing the function

$$W(b,\gamma,\theta,\xi) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_i,\gamma,\alpha,\theta,\zeta) - \frac{i}{n+1}\right]^2 \tag{12}$$

with respect to the parameters b, γ, θ and parameter vector ξ. The WLS estimates can be obtained by solving the non-linear equations:

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_i,b,\gamma,\theta,\xi) - \frac{i}{n+1}\right]\Delta_r(x_i,b,\gamma,\theta,\xi) = 0, \quad r = 1,2,3,4, \tag{13}$$

where $\Delta_1(x_i,b,\gamma,\theta,\xi)$, $\Delta_2(x_i,b,\gamma,\theta,\xi)$, $\Delta_3(x_i,\gamma,\alpha,\theta,\zeta)$, and $\Delta_4(x_i,b,\gamma,\theta,\xi)$ are given in equation (11).

4.4. Anderson-Darling

The Anderson-Darling (AD) estimates of the parameters of the TL-Gom-GPS class of distributions are obtained by minimizing the function

$$AD(b,\gamma,\theta,\xi) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\big(\log\big(F(x_{(i:n)};b,\gamma,\theta,\xi)\big) + \log\big(1 - F(x_{(n+1-i:n)};b,\gamma,\theta,\xi)\big)\big) \tag{14}$$

with respect to the parameters γ, α, θ, and parameter vector ζ. The Anderson-Darling estimates are obtained by solving the non-linear equations

$$\left(\frac{\partial AD(b,\gamma,\theta,\xi)}{\partial b}, \frac{\partial AD(b,\gamma,\theta,\xi)}{\partial \gamma}, \frac{\partial AD(b,\gamma,\theta,\xi)}{\partial \theta}, \frac{\partial AD(b,\gamma,\theta,\xi)}{\partial \xi_k}\right)^T = \mathbf{0}, \tag{15}$$

using non-linear optimization techniques.

4.5. Cramér-von Mises

In this subsection, we present the Cramér-von Mises (CVM) method of estimation. The Cramér-von Mises estimates of the parameters of the TL-Gom-GPS class of distributions are derived by minimizing the function

$$CVM(b,\gamma,\theta,\xi) = \frac{1}{12n}\sum_{i=1}^{n}\left(F(x_{(i:n)};b,\gamma,\theta,\xi) - \frac{2i-1}{2n}\right)^2 \tag{16}$$

with respect to the parameters b, γ, θ and parameter vector ξ.

5. Some Special Cases

Some of the special models of the TL-Gom-GPS class of distributions are presented in this section. We consider the cases when the baseline cdf $G(x;\xi)$ are Weibull, log-logistic and exponential distributions.

5.1. Topp-Leone-Gompertz-Weibull Power Series (TL-Gom-WPS) Distribution

If we take the baseline distribution to be Weibull distribution with the cdf and pdf given by $G(x;\lambda) = 1 - e^{-x^\lambda}$ and $g(x;\lambda) = \lambda x^{\lambda-1}e^{-x^\lambda}$, for $\lambda, x > 0$, we obtain the cdf and pdf of the of the TL-Gom-WPS distribution as

$$F_{TL-Gom-WPS}(x;b,\gamma,\lambda,\theta) = 1 - \frac{C\left(\theta\left(1-\left[1-\left(\exp\left[\frac{1}{\gamma}(1-(e^{-x^\lambda})^{-\gamma})\right]\right)^2\right]^b\right)\right)}{C(\theta)},$$

and

$$\begin{aligned} f_{TL-Gom-WPS}(x;b,\gamma,\lambda,\theta) &= 2b\theta\lambda x^{\lambda-1}(e^{-x^\lambda})^{-\gamma-1}\left(\exp\left[\frac{1}{\gamma}(1-(e^{-x^\lambda})^{-\gamma})\right]\right)^2 \\ &\times \left[1-\left(\exp\left[\frac{1}{\gamma}(1-(e^{-x^\lambda})^{-\gamma})\right]\right)^2\right]^{b-1} \\ &\times \frac{C'\left(\theta\left(1-\left[1-\left(\exp\left[\frac{1}{\gamma}(1-(e^{-x^\lambda})^{-\gamma})\right]\right)^2\right]^b\right)\right)}{C(\theta)}, \end{aligned}$$

respectively, for $b, \gamma, \lambda, \theta > 0$.

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5.1.1. Topp-Leone-Gompertz-Weibull Poisson (TL-Gom-WP) Distribution

We consider when the power series distribution is a Poisson distribution. The cdf and pdf of the Topp-Leone-Gompertz-Weibull Poisson (TL-Gom-WP) distribution are given by

$$F_{TL-Gom-WP}(x;b,\gamma,\lambda,\theta)=1-\frac{\exp\left(\theta\left(1-\left[1-\left(\exp\left[\frac{1}{\gamma}(1-(e^{-x^{\lambda}})^{-\gamma})\right]\right)^{2}\right]^{b}\right)\right)-1}{e^{\theta}-1}$$

and

$$\begin{aligned} f_{TL-Gom-WP}(x;b,\gamma,\lambda,\theta) &= 2b\theta\lambda x^{\lambda-1}(e^{-x^{\lambda}})^{-\gamma}\left(\exp\left[\frac{1}{\gamma}(1-(e^{-x^{\lambda}})^{-\gamma})\right]\right)^{2} \\ &\times \left[1-\left(\exp\left[\frac{1}{\gamma}(1-(e^{-x^{\lambda}})^{-\gamma})\right]\right)^{2}\right]^{b-1} \\ &\times \frac{\exp\left(\theta\left(1-\left[1-\left(\exp\left[\frac{1}{\gamma}(1-(e^{-x^{\lambda}})^{-\gamma})\right]\right)^{2}\right]^{b}\right)\right)}{e^{\theta}-1}, \end{aligned}$$

respectively, for $b,\gamma,\lambda,\theta>0$.

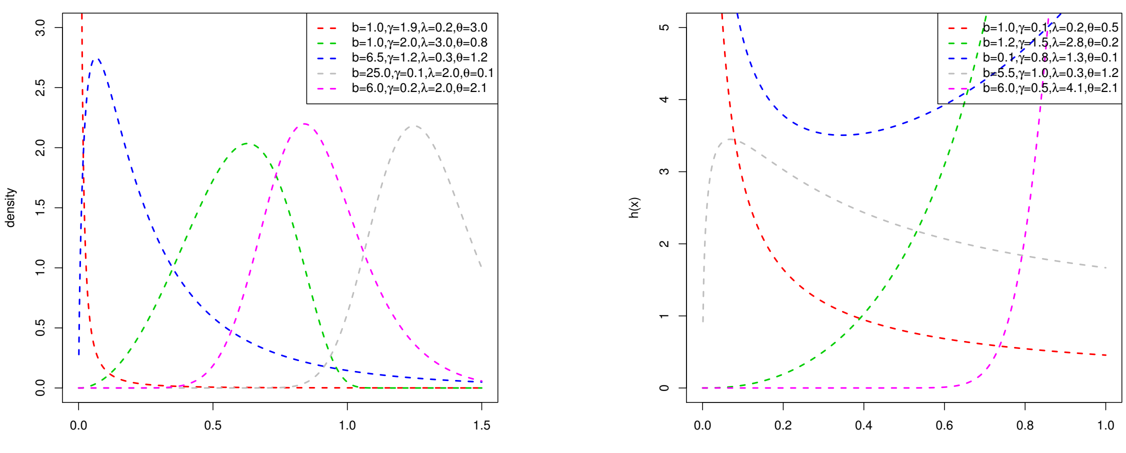

Figure 1: Pdf and hrf plots for the TL-Gom-WP distribution

Figure 1 shows the pdf and hrf of the TL-Gom-WP distribution. The pdf can take several shapes including right-skewed, left-skewed, almost symmetric and reverse-J shapes, whereas the hrf displays increasing, decreasing, upside-down bathtub and bathtub shapes. Table 2 gives some quantile values of the TL-Gom-WP distribution for different parameter values. The 3D plots for TL-Gom-WP distribution are given in Figures 2 and 3. The plots indicate that TL-Gom-WP distribution can model data with different levels of kurtosis and skewness.

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Table 2: Quantiles for TL-Gom-WP Distribution

	$(b,\gamma,\lambda,\theta)$				
u	$(1,2,1.3,0.2)$	$(0.7,1,3,1.5)$	$(2.4,1,1,2)$	$(2.1,1,1.9,1)$	$(1.5,1,1.2,3)$
0.1	0.0933	0.1978	0.1492	0.3655	0.0867
0.2	0.1607	0.2816	0.2110	0.4485	0.1335
0.3	0.2226	0.3504	0.2642	0.5117	0.1757
0.4	0.2825	0.4137	0.3158	0.5674	0.2177
0.5	0.3429	0.4756	0.3696	0.6208	0.2622
0.6	0.4058	0.5393	0.4291	0.6752	0.3122
0.7	0.4742	0.6083	0.4999	0.7344	0.3721
0.8	0.5539	0.6884	0.5928	0.8043	0.4518
0.9	0.6607	0.7945	0.7405	0.9006	0.5815

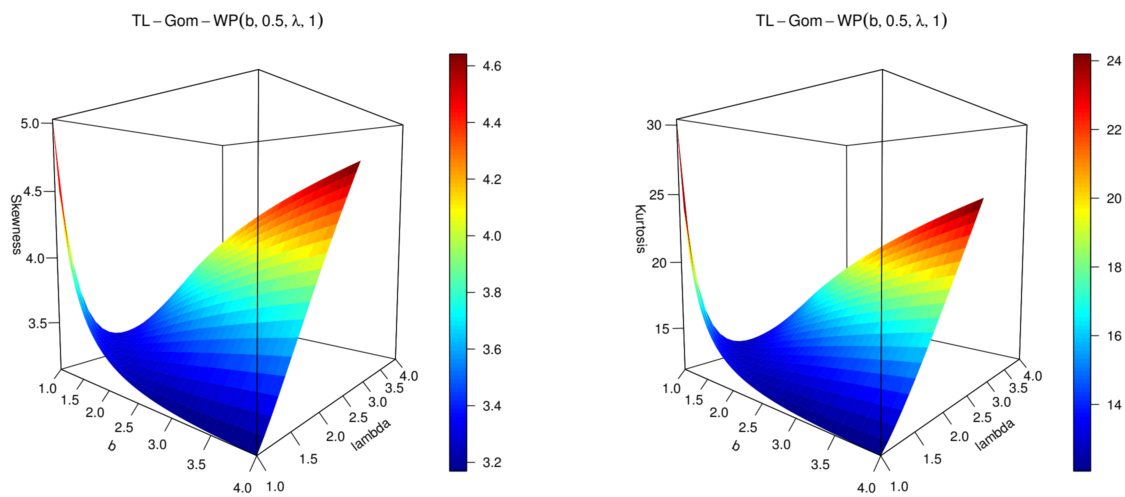

Figure 2: Plots of skewness and kurtosis for TL-Gom-WP distribution

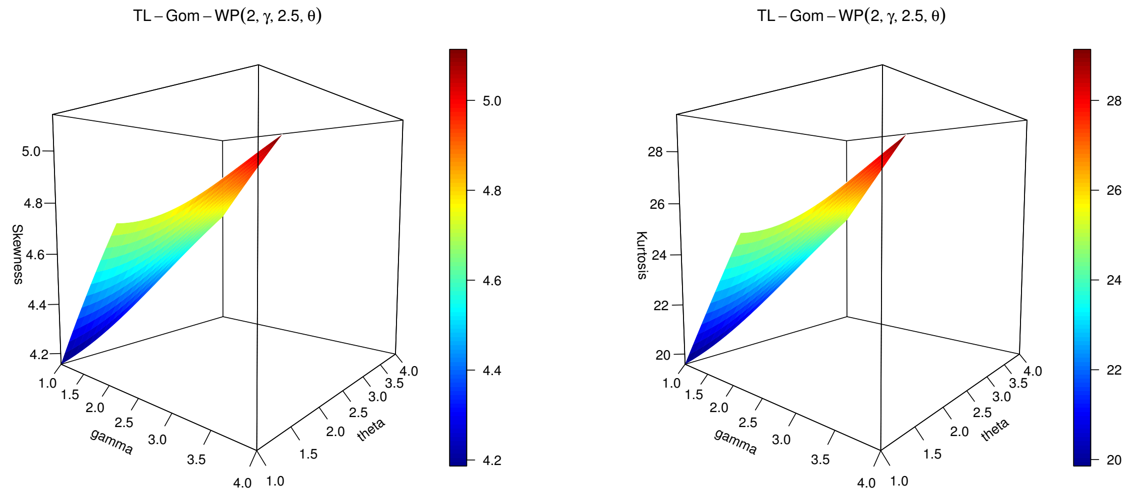

Figure 3: Plots of skewness and kurtosis for TL-Gom-WP distribution

5.2. Topp-Leone-Gompertz-Log-logistic Power Series (TL-Gom-LLoPS) Distribution

Suppose we take the baseline distribution to be log-logistic distribution with the cdf and pdf given by $G(x;c) = 1-(1+x^c)^{-1}$ and $g(x;c) = cx^{c-1}(1+x^c)^{-2}$, for $c, x > 0$, we obtain the cdf and pdf of the of the TL-Gom-LLoPS distribution as

$$F_\theta(x) = 1 - \frac{C\left(\theta\left(1-\left[1-\left(\exp\left[\frac{1}{\gamma}(1-(1+x^c)^\gamma)\right]\right)^2\right]^b\right)\right)}{C(\theta)},$$

and

$$\begin{aligned} f_\theta(x) &= 2b\theta c x^{c-1}(1+x^c)^{\gamma-1}\left(\exp\left[\frac{1}{\gamma}(1-(1+x^c)^\gamma)\right]\right)^2 \\ &\times \left[1-\left(\exp[\frac{1}{\gamma}(1-(1+x^c)^\gamma)]\right)^2\right]^{b-1} \\ &\times \frac{C'\left(\theta\left(1-\left[1-\left(\exp\left[\frac{1}{\gamma}(1-(1+x^c)^\gamma)\right]\right)^2\right]^b\right)\right)}{C(\theta)}, \end{aligned}$$

respectively, for $b, \gamma, c, \theta > 0$.

5.2.1. Topp-Leone-Gompertz-Log-logistic Poisson (TL-Gom-LLoP) Distribution

The cdf and pdf of the Topp-Leone-Gompertz-log-logistic Poisson (TL-Gom-LLoP) distribution are given by

$$F_{TL-Gom-LLoP}(x;b,\gamma,c,\theta) = 1 - \frac{\exp\left(\theta\left(1-\left[1-\left(\exp\left[\frac{1}{\gamma}(1-(1+x^c)^\gamma)\right]\right)^2\right]^b\right)\right)-1}{e^\theta - 1}$$

and

$$
\begin{aligned}
f_{TL-Gom-LLoP}(x;b,\gamma,c,\theta) &= 2b\theta c x^{c-1}(1+x^c)^{\gamma-1}\left(\exp\left[\frac{1}{\gamma}(1-(1+x^c)^{\gamma})\right]\right)^2 \\
&\times \left[1-\left(\exp[\frac{1}{\gamma}(1-(1+x^c)^{\gamma})]\right)^2\right]^{b-1} \\
&\times \frac{\exp\left(\theta\left(1-\left[1-\left(\exp\left[\frac{1}{\gamma}(1-(1+x^c)^{\gamma})\right]\right)^2\right]^b\right)\right)}{e^{\theta}-1},
\end{aligned}
$$

respectively, for $b,\gamma,c,\theta>0$.

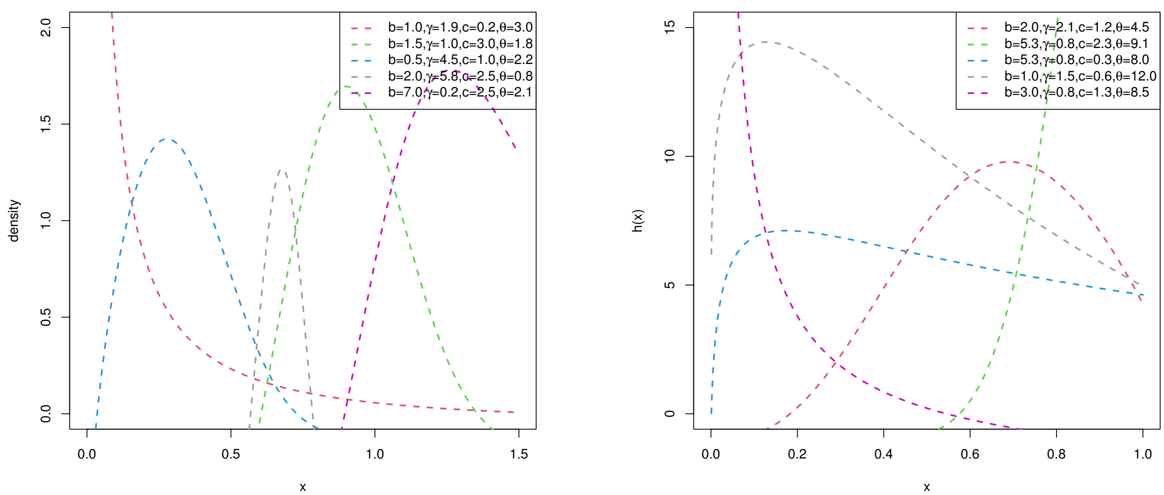

Figure 4: Plots of pdf and hrf for the TL-Gom-LLoP distribution

The TL-Gom-LLoP distribution has the pdf that displays right-skewed, left-skewed, almost symmetric and reverse-J shapes, whereas the hrf exhibit upside-down bathtub, decreasing and increasing shapes.

Table 3: Some Quantiles for TL-Gom-LLoP Distribution

		(b,γ,c,θ)			
u	$(1,2,1.3,0.2)$	$(0.7,1,3,1.5)$	$(2.4,1,1,2)$	$(2.1,1,1.9,1)$	$(1.5,1,1.2,3)$
0.1	0.0949	0.1981	0.1609	0.3802	0.0887
0.2	0.1666	0.2826	0.2349	0.4755	0.1386
0.3	0.2352	0.3529	0.3024	0.5517	0.1851
0.4	0.3047	0.4186	0.3714	0.6222	0.2329
0.5	0.3780	0.4843	0.4471	0.6929	0.2855
0.6	0.4585	0.5538	0.5359	0.7688	0.3468
0.7	0.5512	0.6319	0.6486	0.8559	0.4239
0.8	0.6668	0.7279	0.8091	0.9662	0.5333
0.9	0.8359	0.8668	1.0969	1.1339	0.7295

Quantiles for selected parameters values of the TL-Gom-LLoP distribution are given in Table 3. Figures 5 and 6 shows that TL-Gom-LLoP distribution can model data sets with different levels of skewness and kurtosis.

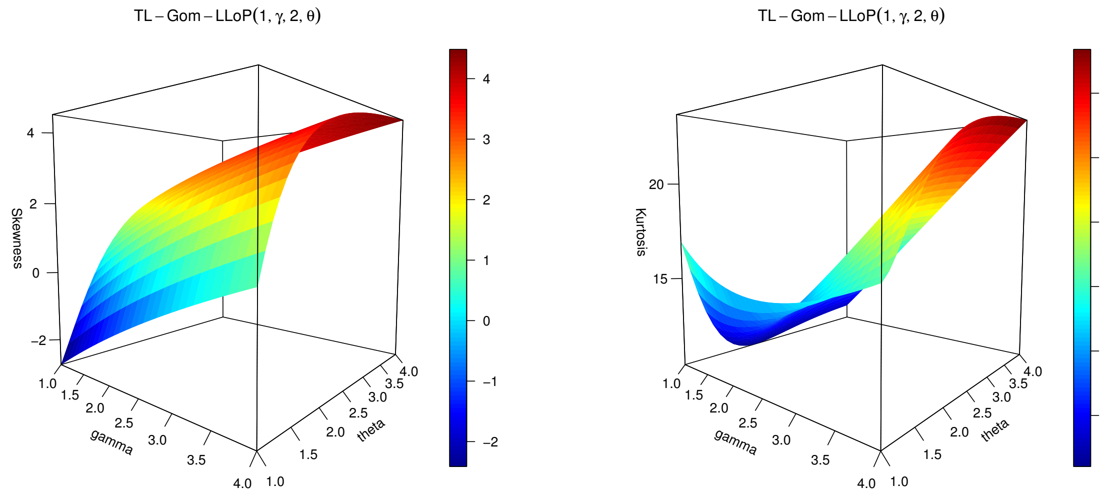

Figure 5: Plots of skewness and kurtosis for TL-Gom-LLoP distribution

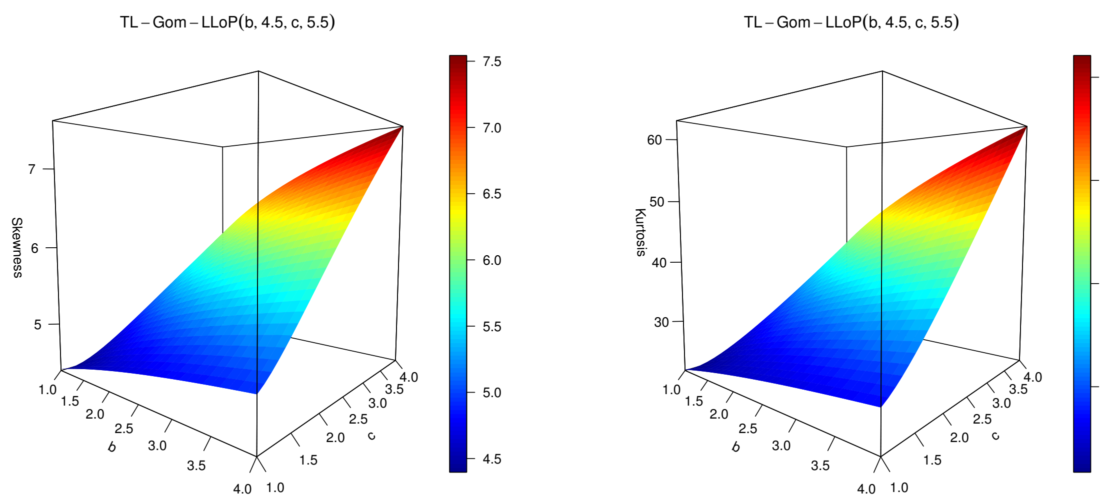

Figure 6: Plots of skewness and kurtosis for TL-Gom-LLoP distribution

5.3. Topp-Leone-Gompertz-Exponential Power Series (TL-Gom-EPS) Distribution

If we consider the baseline distribution to be the exponential distribution with the cdf and pdf given by $G(x;\lambda) = 1 - e^{-\lambda x}$ and $g(x;\lambda) = \lambda e^{-\lambda x}$, for $\lambda, x > 0$, we obtain the cdf and pdf of the of the TL-Gom-EPS distribution as

$$F_{TL-Gom-EPS}(x;b,\gamma,\lambda,\theta) \quad = \quad 1 - \frac{C\left(\theta\left(1-\left[1-\left(\exp\left[\frac{1}{\gamma}(1-e^{-\lambda x})^{-\gamma}\right)\right]\right)^{2}\right]^{b}\right)\right)}{C(\theta)},$$

and

$$\begin{aligned} f_{TL-Gom-EPS}(x;b,\gamma,\lambda,\theta) &= 2b\theta\lambda(e^{-\lambda x})^{-\gamma-1}\left(\exp\left[\frac{1}{\gamma}(1-e^{-\lambda x})^{-\gamma})\right]\right)^2 \\ &\times \left[1-\left(\exp\left[\frac{1}{\gamma}(1-e^{-\lambda x})^{-\gamma})\right]\right)^2\right]^{b-1} \\ &\times \frac{C'\left(\theta\left(1-\left[1-\left(\exp\left[\frac{1}{-\gamma}(1-e^{-\lambda x})^{-\gamma})\right]\right)^2\right]^b\right)\right)}{C(\theta)}, \end{aligned}$$

respectively, for $b,\gamma,\lambda,\theta > 0$.

5.3.1. Topp-Leone-Gompertz-Exponential Poisson (TL-Gom-EP) Distribution

The cdf and pdf of the Topp-Leone Gompertz-exponential Poisson (TL-Gom-EP) distribution are given by

$$F_{TL-Gom-EP}(x;b,\gamma,\lambda,\theta) = 1-\frac{\exp\left(\theta\left(1-\left[1-\left(\exp\left[\frac{1}{\gamma}(1-e^{-\lambda x})^{-\gamma})\right]\right)^2\right]^b\right)\right)-1}{e^\theta-1},$$

and

$$\begin{aligned} f_{TL-Gom-EP}(x;b,\gamma,\lambda,\theta) &= 2b\theta\lambda(e^{-\lambda x})^{-\gamma-1}\left(\exp\left[\frac{1}{\gamma}(1-e^{-\lambda x})^{-\gamma})\right]\right)^2 \\ &\times \left[1-\left(\exp\left[\frac{1}{\gamma}(1-e^{-\lambda x})^{-\gamma})\right]\right)^2\right]^{b-1} \\ &\times \frac{\exp\left(\theta\left(1-\left[1-\left(\exp\left[\frac{1}{\gamma}(1-e^{-\lambda x})^{-\gamma})\right]\right)^2\right]^b\right)\right)}{e^\theta-1}, \end{aligned}$$

respectively, for $b,\gamma,\lambda,\theta > 0$.

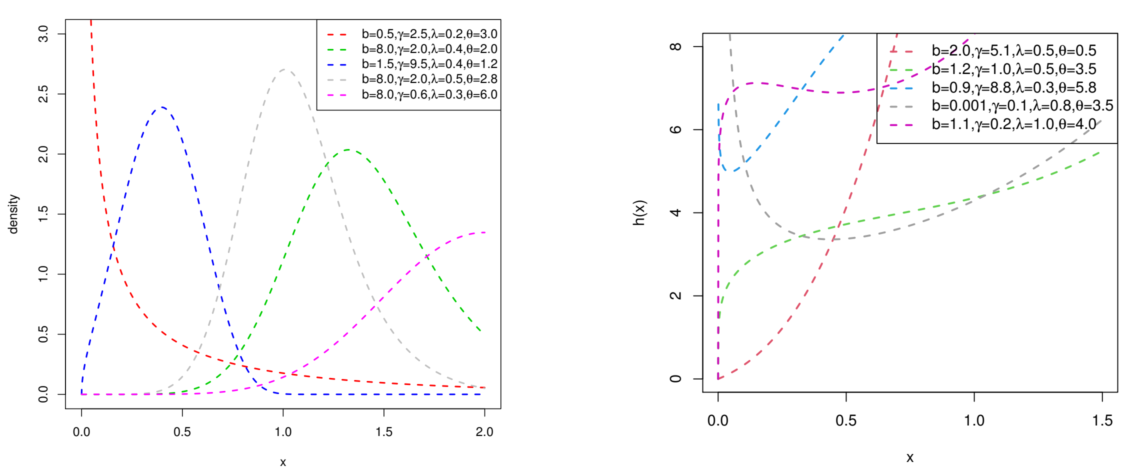

Figure 7: Plots of pdf and hrf for the TL-Gom-EP distribution

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Table 4: Some Quantiles for TL-Gom-EP Distribution

	(b, γ, c, θ)				
u	$(1, 2, 1.3, 0.2)$	$(0.7, 1, 3, 1.5)$	$(2.4, 1, 1, 2)$	$(2.1, 1, 1.9, 1)$	$(1.5, 1, 1.2, 3)$
0.1	0.0352	0.0026	0.1492	0.0778	0.0443
0.2	0.0714	0.0075	0.2110	0.1147	0.0743
0.3	0.1091	0.0143	0.2642	0.1473	0.1034
0.4	0.1488	0.0236	0.3158	0.1793	0.1337
0.5	0.1913	0.0359	0.3696	0.2127	0.1672
0.6	0.2381	0.0523	0.4291	0.2496	0.2061
0.7	0.2916	0.0750	0.4999	0.2928	0.2545
0.8	0.3569	0.1087	0.5928	0.3479	0.3212
0.9	0.4488	0.1672	0.7405	0.4314	0.4348

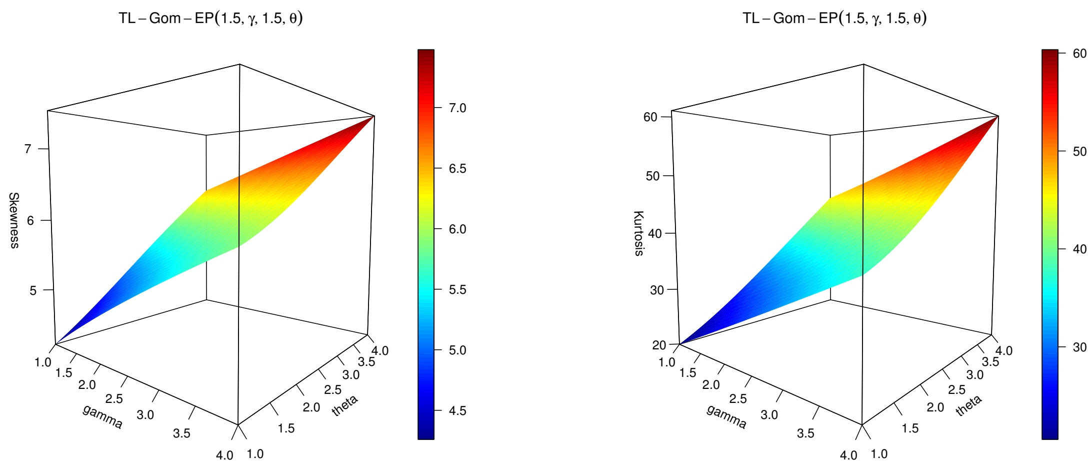

Figure 8: Plots of skewness and kurtosis for TL-Gom-EP distribution

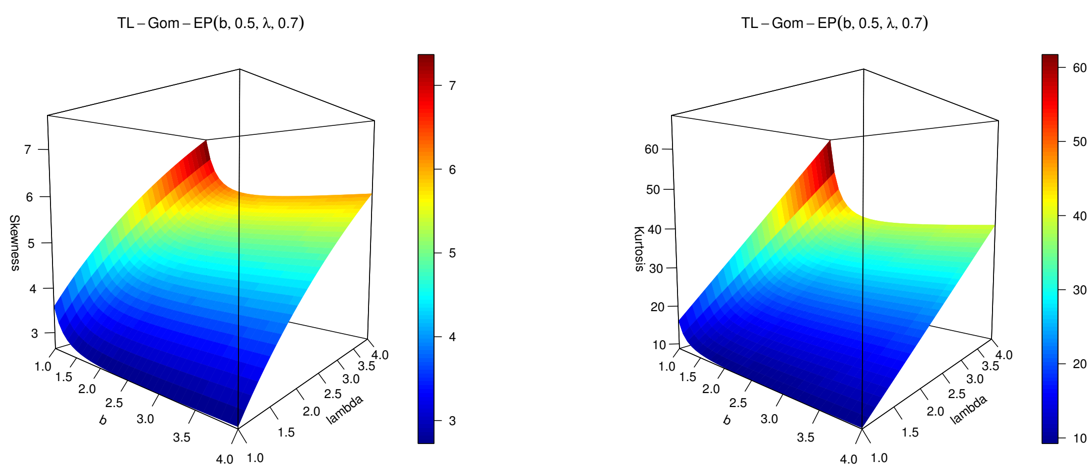

Figure 9: Plots of skewness and kurtosis for TL-Gom-EP distribution

The TL-Gom-EP distribution has the pdf that displays right-skewed, left-skewed, almost symmetric and reverse-J shapes, whereas the hrf can be upside-down bathtub followed by bathtub, bathtub and increasing shapes. Quantiles for selected parameters values of the TL-Gom-EP distribution are given in Table 4. Figure 8 shows that TL-Gom-EP distribution can model data sets with different levels of skewness and kurtosis.

6. Simulation Results

In this section, we present some simulation results for the case of the TL-Gom-GPS class of distributions. The simulation study was performed in order to examine the performance of the root mean square errors (RMSEs) and average bias (ABIAS) of estimators of the parameters of the TL-Gom-GPS class of distributions. We performed various simulations for different sample sizes and different parameter values. Equation (8) was used to generate random samples from the special cases of the TL-Gom-GPS class of distributions via the R package. The simulations were repeated for $N = 3000$ times each with sample sizes $n = 25, 50, 100, 200, 400$. Specifically, the average bias and root mean square errors of the parameter say, $\hat{\xi}$, are computed as
RMSE $(\hat{\xi}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\xi}_i - \xi)^2}{N}}$ and ABIAS$(\hat{\xi}) = \frac{\sum_{i=1}^{N}(\hat{\xi}_i - \xi)}{N}$, respectively.

Table 5: Simulation Results for (1.5, 0.1, 0.002, 0.1)

		MLE		**OLS**		**WLS**		**AD**		**CVM**	
Parameter	n	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS
b	25	$0.0272^{(2)}$	$6.2512^{(3)}$	$0.0095^{(1)}$	$8.3042^{(4)}$	$0.4437^{(5)}$	$0.1493^{(1)}$	$0.0385^{(3)}$	$8.4089^{(5)}$	$0.4097^{(4)}$	$0.2023^{(2)}$
	50	$0.0052^{(3}$	$1.5965^{(3)}$	$0.0051^{(2)}$	$3.8633^{(4)}$	$0.2814^{(4)}$	$0.0190^{(1)}$	$0.0044^{(1)}$	$4.9220^{(5)}$	$0.3438^{(5)}$	$0.1300^{(3)}$
	100	$0.0039^{(2)}$	$2.2226^{(3)}$	$0.0058^{(3)}$	$5.6029^{(5)}$	$0.1479^{(4)}$	$0.0651^{(1)}$	$0.0032^{(1)}$	$5.3569^{(4)}$	$0.2768^{(5)}$	$0.1398^{(2)}$
	200	$0.0002^{(1)}$	$1.1913^{(5)}$	$0.0016^{(2)}$	$1.0532^{(4)}$	$0.0646^{(4)}$	$0.0049^{(2)}$	$0.0021^{(3)}$	$0.0012^{(1)}$	$0.1960^{(5)}$	$0.1051^{(3)}$
	400	$0.0001^{(1)}$	$0.0005^{(2)}$	$0.0005^{(2)}$	$0.0002^{(1.5)}$	$0.0394^{(4)}$	$0.0015^{(3)}$	$0.0007^{(3)}$	$0.0002^{(1.5)}$	$0.1541^{(5)}$	$0.0762^{(5)}$
	Sum of ranks		39		28.5		29		27.5		25
γ	25	$0.0514^{(1)}$	$0.0074^{(1)}$	$0.1084^{(3)}$	$0.0154^{(3)}$	$0.1679^{(4)}$	$0.0416^{(4)}$	$0.0694^{(2)}$	$0.0089^{(2)}$	$2.5099^{(5)}$	$1.6217^{(5)}$
	50	$0.0549^{(1)}$	$0.0075^{(1)}$	$0.0737^{(3)}$	$0.0125^{(3)}$	$0.1172^{(4)}$	$0.0411^{(4)}$	$0.0611^{(2)}$	$0.0086^{(2)}$	$1.6862^{(5)}$	$1.0657^{(5)}$
	100	$0.0396^{(1)}$	$0.0068^{(1)}$	$0.0705^{(3)}$	$0.0136^{(3)}$	$0.1063^{(4)}$	$0.0361^{(4)}$	$0.0529^{(2)}$	$0.0069^{(2)}$	$1.0954^{(5)}$	$0.7470^{(5)}$
	200	$0.0376^{(2)}$	$0.0067^{(2)}$	$0.0606^{(3)}$	$0.0073^{(3)}$	$0.0997^{(4)}$	$0.0264^{(4)}$	$0.0370^{(1)}$	$0.0064^{(1)}$	$0.8859^{(5)}$	$0.5495^{(5)}$
	400	$0.0394^{(2)}$	$0.0027^{(3)}$	$0.0459^{(3)}$	$0.0004^{(1)}$	$0.0789^{(4)}$	$0.0146^{(4)}$	$0.0370^{(1)}$	$0.0006^{(2)}$	$0.5067^{(5)}$	$0.3280^{(5)}$
	Sum of ranks		50		28		40		17		15
λ	25	$0.0001^{(1)}$	$0.00007^{(1)}$	$0.0107^{(4)}$	$0.0084^{(3)}$	$0.0095^{(3)}$	$0.0087^{(5)}$	$0.0109^{(5)}$	$0.0085^{(4)}$	$0.0082^{(2)}$	$0.0081^{(2)}$
	50	$0.0001^{(1)}$	$0.00003^{(1)}$	$0.0083^{(2.5)}$	$0.0082^{(3)}$	$0.0090^{(4)}$	$0.0085^{(4.5)}$	$0.0097^{(5)}$	$0.0085^{(4.5)}$	$0.0083^{(2.5)}$	$0.0081^{(2)}$
	100	$0.0001^{(1)}$	$0.00005^{(1)}$	$0.0083^{(3)}$	$0.0082^{(3)}$	$0.0089^{(4)}$	$0.0085^{(5)}$	$0.0095^{(5)}$	$0.0083^{(4)}$	$0.0081^{(2)}$	$0.0081^{(2)}$
	200	$0.0001^{(1)}$	$0.00004^{(1)}$	$0.0082^{(3)}$	$0.0082^{(3.5)}$	$0.0088^{(4)}$	$0.0085^{(5)}$	$0.0091^{(5)}$	$0.0082^{(3.5)}$	$0.0081^{(2)}$	$0.0081^{(2)}$
	400	$0.0000^{(1)}$	$0.00002^{(1)}$	$0.0081^{(3)}$	$0.0081^{(3)}$	$0.0088^{(5)}$	$0.0083^{(5)}$	$0.0085^{(4)}$	$0.0082^{(4)}$	$0.0080^{(2)}$	$0.0080^{(2)}$
	Sum of ranks		20.5		31		44.5		44		10
θ	25	$0.0549^{(2)}$	$0.0075^{(1)}$	$0.0814^{(3)}$	$0.0132^{(2)}$	$0.1169^{(4)}$	$0.0421^{(3)}$	$0.0379^{(1)}$	$1.8737^{(5)}$	$0.6920^{(5)}$	$0.6504^{(4)}$
	50	$0.0515^{(2)}$	$0.0075^{(2)}$	$0.0738^{(3)}$	$0.0136^{(3)}$	$0.1060^{(4)}$	$0.0413^{(4)}$	$0.0407^{(1)}$	$0.0062^{(1)}$	$0.6211^{(5)}$	$0.5614^{(5)}$
	100	$0.0394^{(2)}$	$0.0068^{(2)}$	$0.0705^{(3)}$	$0.0126^{(3)}$	$0.1045^{(4)}$	$0.0329^{(4)}$	$0.0371^{(1)}$	$0.0054^{(1)}$	$0.6150^{(5)}$	$0.5457^{(5)}$
	200	$0.0397^{(2)}$	$0.0067^{(1)}$	$0.0607^{(3)}$	$0.0074^{(2)}$	$0.0971^{(4)}$	$0.0260^{(4)}$	$0.0348^{(1)}$	$0.0094^{(3)}$	$0.5793^{(5)}$	$0.5175^{(5)}$
	400	$0.0377^{(2)}$	$0.0028^{(2)}$	$0.0459^{(3)}$	$0.0004^{(1)}$	$0.0789^{(4)}$	$0.0146^{(4)}$	$0.0328^{(1)}$	$0.0052^{(3)}$	$0.5068^{(5)}$	$0.4379^{(5)}$
	Sum of ranks		49		26		39		18		18

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Table 6: Rankings of Estimation Methods for TL-Gom-EP Distribution

Parameters	MLE	OLS	WLS	AD	CVM
$b = 1.5$	1	2	5	2	4
$\gamma = 0.1$	1	3	4	2	5
$\lambda = 0.002$	1	3	5	4	2
$\theta = 0.1$	2	3	4	1	5
Sum of ranks	5	11	18	9	16
Total rank	1	3	5	2	4

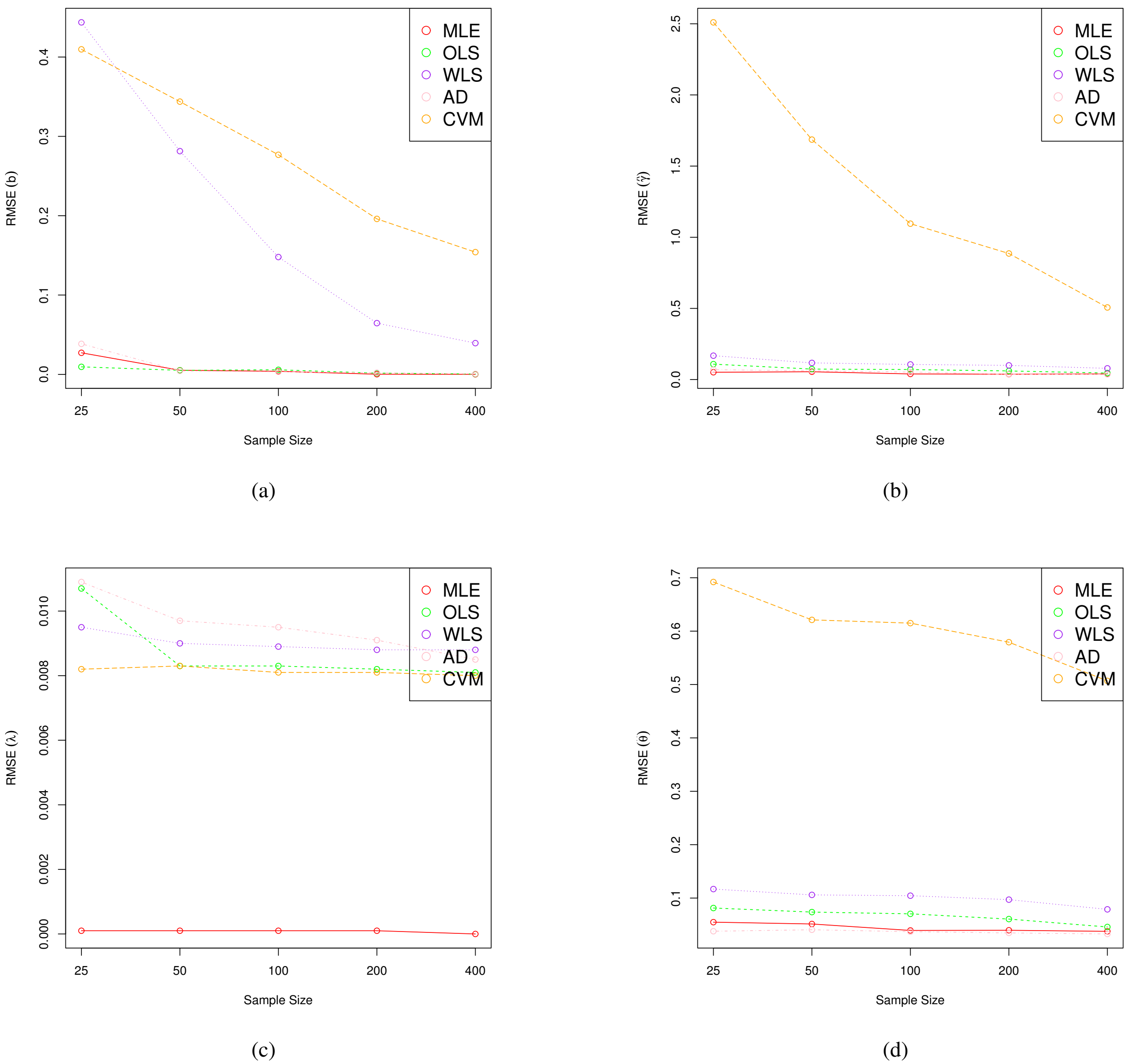


Figure 10: Plots of RMSE's for the TL-Gom-EP distribution

 DOI: http://dx.doi.org/10.18187/pjsor.v20i2.4032

Table 7: Simulation Results for (3.0, 0.8, 0.01, 0.3)

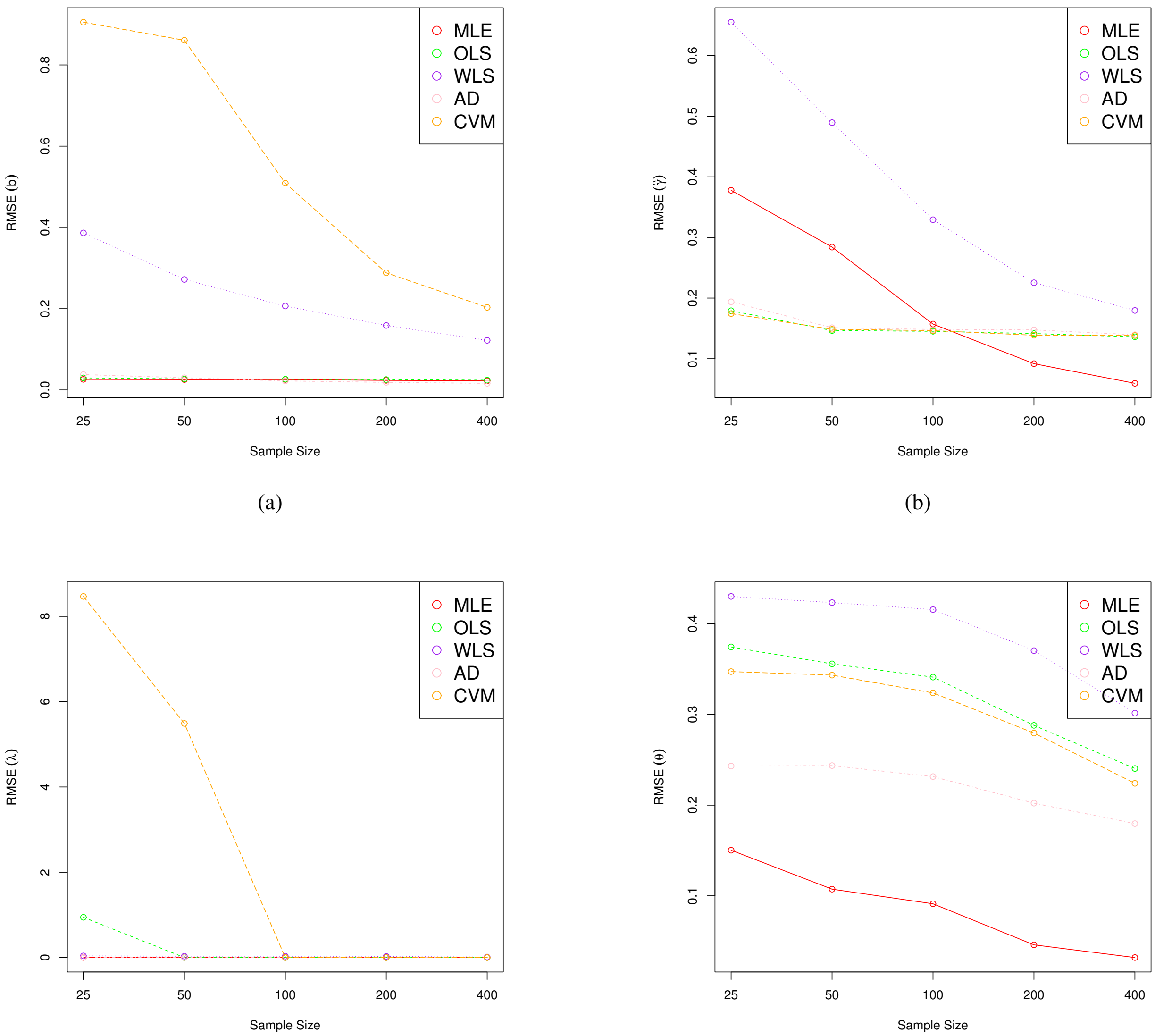
		MLE		OLS		WLS		AD		CVM	
Parameter	n	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS
b	25	$0.0261^{(1)}$	$0.4396^{(5)}$	$0.0297^{(2)}$	$0.0080^{(2)}$	$0.3865^{(4)}$	$0.1929^{(4)}$	$0.0379^{(3)}$	$0.0085^{(3)}$	$0.9051^{(5)}$	$0.0071^{(1)}$
	50	$0.0256^{(1)}$	$0.3341^{(5)}$	$0.0272^{(2)}$	$0.0053^{(2)}$	$0.2719^{(4)}$	$0.0952^{(4)}$	$0.0301^{(3)}$	$0.0082^{(3)}$	$0.8607^{(5)}$	$0.0052^{(1)}$
	100	$0.0258^{(2)}$	$0.2386^{(5)}$	$0.0264^{(3)}$	$0.0048^{(2)}$	$0.2066^{(4)}$	$0.0428^{(4)}$	$0.0222^{(1)}$	$0.0069^{(3)}$	$0.5089^{(5)}$	$0.0043^{(1)}$
	200	$0.0236^{(2)}$	$0.1313^{(5)}$	$0.0256^{(3)}$	$0.0041^{(1)}$	$0.1587^{(4)}$	$0.0215^{(4)}$	$0.0187^{(1)}$	$0.0054^{(3)}$	$0.2883^{(5)}$	$0.0042^{(2)}$
	400	$0.0225^{(2)}$	$0.0905^{(5)}$	$0.0241^{(3)}$	$0.0033^{(2)}$	$0.1220^{(4)}$	$0.0186^{(4)}$	$0.0163^{(1)}$	$0.0006^{(1)}$	$0.2032^{(5)}$	$0.0040^{(3)}$
	Sum of ranks		16		22		40		22		50
γ	25	$0.3778^{(4)}$	$0.0718^{(1)}$	$0.1791^{(2)}$	$0.1013^{(4)}$	$0.6550^{(5)}$	$0.5228^{(5)}$	$0.1938^{(3)}$	$0.0863^{(2)}$	$0.1744^{(1)}$	$0.0871^{(3)}$
	50	$0.2841^{(4)}$	$0.0178^{(1)}$	$0.1464^{(1)}$	$0.0954^{(4)}$	$0.4894^{(5)}$	$0.2021^{(5)}$	$0.1511^{(3)}$	$0.0801^{(2)}$	$0.1485^{(2)}$	$0.0840^{(3)}$
	100	$0.1571^{(4)}$	$0.0240^{(1)}$	$0.1452^{(1)}$	$0.0740^{(4)}$	$0.3292^{(5)}$	$0.1274^{(5)}$	$0.1482^{(3)}$	$0.0603^{(2)}$	$0.1463^{(2)}$	$0.0611^{(3)}$
	200	$0.0918^{(1)}$	$0.0015^{(1)}$	$0.1414^{(2)}$	$0.0277^{(2)}$	$0.2253^{(5)}$	$0.1136^{(5)}$	$0.1475^{(4)}$	$0.0541^{(4)}$	$0.1387^{(3)}$	$0.0471^{(3)}$
	400	$0.0594^{(1)}$	$0.0035^{(1)}$	$0.1361^{(2)}$	$0.0378^{(5)}$	$0.1796^{(5)}$	$0.0223^{(4)}$	$0.1395^{(4)}$	$0.0098^{(2)}$	$0.1382^{(3)}$	$0.0165^{(3)}$
	Sum of ranks		19		27		49		29		26
λ	25	$0.0016^{(2)}$	$0.0002^{(1)}$	$0.9434^{(4)}$	$5.4953^{(3)}$	$0.0389^{(3)}$	$0.0032^{(2)}$	$0.0013^{(1)}$	$7.9389^{(5)}$	$8.4672^{(4)}$	$8.6631^{(5)}$
	50	$0.0017^{(3)}$	$0.0000^{(1)}$	$0.0007^{(2)}$	$5.3064^{(4)}$	$0.0319^{(4)}$	$0.0016^{(2)}$	$0.0004^{(1)}$	$2.5446^{(3)}$	$5.4888^{(5)}$	$8.1454^{(5)}$
	100	$0.0011^{(4)}$	$0.0000^{(1)}$	$0.0001^{(1)}$	$4.1993^{(4)}$	$0.0318^{(5)}$	$0.0015^{(2)}$	$0.0004^{(2.5)}$	$2.4152^{(3)}$	$0.0004^{(2.5)}$	$5.0095^{(5)}$
	200	$0.0005^{(4)}$	$0.0000^{(1)}$	$0.0003^{(2.5)}$	$1.6430^{(4)}$	$0.0271^{(5)}$	$0.0013^{(3)}$	$0.0002^{(1)}$	$0.0001^{(2)}$	$0.0003^{(2.5)}$	$5.7539^{(5)}$
	400	$0.0003^{(3.5)}$	$0.0000^{(1)}$	$0.0003^{(3.5)}$	$0.0001^{(2.5)}$	$0.0121^{(5)}$	$0.0003^{(4)}$	$0.0001^{(1.5)}$	$0.0001^{(2.5)}$	$0.0001^{(1.5)}$	$2.0374^{(5)}$
	Sum of ranks		21.5		30.5		35		22.5		40.5
θ	25	$0.1503^{(1)}$	$0.0616^{(1)}$	$0.3745^{(4)}$	$0.2894^{(4)}$	$0.4302^{(5)}$	$0.3872^{(5)}$	$0.2431^{(2)}$	$0.1781^{(2)}$	$0.3473^{(3)}$	$0.2695^{(3)}$
	50	$0.1073^{(1)}$	$0.0261^{(1)}$	$0.3558^{(4)}$	$0.3136^{(4)}$	$0.4234^{(5)}$	$0.3854^{(5)}$	$0.2436^{(2)}$	$0.1728^{(2)}$	$0.3434^{(3)}$	$0.2635^{(3)}$
	100	$0.0912^{(1)}$	$0.0022^{(1)}$	$0.3412^{(4)}$	$0.2559^{(4)}$	$0.4157^{(5)}$	$0.3066^{(5)}$	$0.2315^{(2)}$	$0.1336^{(2)}$	$0.3238^{(3)}$	$0.2195^{(3)}$
	200	$0.0459^{(1)}$	$0.0054^{(1)}$	$0.2880^{(4)}$	$0.1474^{(5)}$	$0.3704^{(5)}$	$0.1328^{(4)}$	$0.2022^{(2)}$	$0.0552^{(2)}$	$0.2795^{(3)}$	$0.1194^{(3)}$
	400	$0.0319^{(1)}$	$0.0028^{(1)}$	$0.2404^{(4)}$	$0.0246^{(3)}$	$0.3015^{(5)}$	$0.1198^{(5)}$	$0.1796^{(2)}$	$0.0431^{(4)}$	$0.2241^{(3)}$	$0.0128^{(2)}$
	Sum of ranks		10		40		49		22		29

Tables 5 and 7 list the RMSEs and ABIAS of the TL-Gom-EP distribution. From the results in tables, the RMSEs and ABIAS decrease with increasing sample size for different estimation methods listed in the table. The tables includes a row labelled sum of ranks which represents partial sum of ranks. The superscript (in parentheses) next to each estimator indicates its rank for a specific measure.

Table 8: Rankings of Estimation Methods for TL-Gom-EP Distribution

Parameters	MLE	OLS	WLS	AD	CVM
$b = 3.0$	1	2.5	4	2.5	5
$\gamma = 0.8$	1	3	5	4	2
$\lambda = 0.01$	1	3	4	2	5
$\theta = 0.3$	1	4	5	2	3
Sum of ranks	5	12.5	18	10.5	15
Total rank	1	3	5	2	4

Tables 6 and 8 present partial and total ranks for each estimation techniques employed in estimating the TL-Gom-EP distribution across the model parameter values. From the tables we see that the MLE is best performing method followed by the AD method for the selected parameter values of the TL-Gom-EP distribution. Furthermore, the line plots in Figures 10 and 11 show significant reduction of RMSEs as the sample size increase for selected estimation methods.



(a) (b)

(c) (d)

Figure 11: Plots of RMSE's for the TL-Gom-EP distribution

7. Applications

In this section, applications of the TL-Gom-EP and TL-Gom-EL distributions to actual data sets are presented to demonstrate the models usefulness and versatility. We compared the new TL-Gom-EPS distributions with the non-nested models including the Topp-Leone-Weibull Poisson (TL-WP) distribution by Chipepa et al. (2021), Log-logistic-Weibull Poisson (LLoGWP) distribution by Oluyede et al. (2016), the Marshall-Olkin-Gompertz-Weibull (MO-Gom-W) distribution by Chipepa and Oluyede (2021) and the odd log-logistic exponentiated Weibull (OLLEW) distribution by Afify et al. (2018). The pdf's of the above distributions are listed in the Appendix.

7.1. Breaking Stress of Carbon Fibres Data

The first dataset correspond to breaking stress of carbon fibres data cited by El-Bassiouny et al. (2015), and given as; 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97 ,1.36, 0.98, 2.76, 4.91 ,3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.

The estimated variance-covariance matrix of the TL-Gom-EP model for carbon fibres data is given by

$$\begin{bmatrix} 1.2966\times10^{-02} & -5.1857\times10^{-04} & -9.7543\times10^{-06} & 7.0402\times10^{-02} \\ -5.1857\times10^{-04} & 9.6163\times10^{-05} & 6.9438\times10^{-06} & -1.2166\times10^{-02} \\ -9.7543\times10^{-06} & 6.9438\times10^{-06} & 7.2676\times10^{-07} & -8.6541\times10^{-04} \\ 7.0402\times10^{-02} & -1.2166\times10^{-02} & -8.6541\times10^{-07} & 1.5414\times10^{+00} \end{bmatrix}$$

and the approximate 95% confidence interval for the parameters are given by $b \in (1.0116, 1.4579)$, $\gamma \in (93.1229, 93.16122)$, $\lambda \in (0.0083, 0.0117)$ and $\theta \in (2.3361, 7.2029)$.

The maximum likelihood estimates (MLE's) of the TL-Gom-EPS distributions with (standard errors in parentheses) are presented in Tables 9 and 10. The goodness-of-fit statistics given in the tables including: -2log-likelihood(-2logL), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramér-Von Mises (W^*), Anderson-Darling (A^*), Kolmogorov-Smirnov (K-S) statistic and its P-Value.

Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al.,1983) are given in Figures 12 and 14. For the probability plots, we plotted $F(x_{(j)}; \hat{b}, \hat{\gamma}, \hat{\lambda}, \hat{\theta})$ against $\dfrac{j-0.375}{n+0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measure of closeness to the diagonal line is given by the sum of squares

$$SS = \sum_{j=1}^{n} \left[F(x_{(j)}; \hat{b}, \hat{\gamma}, \hat{\lambda}, \hat{\theta}) - \left(\frac{j-0.375}{n+0.25} \right) \right]^2.$$

Table 9: Estimates of Models for Carbon Fibres Data

	Estimates				Statistics							
Model	b	γ	λ	θ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	P-value
TL-Gom-EP	1.2348	93.1420	0.0099	4.7695	281.63	289.63	290.05	300.05	0.0703	0.6141	0.0558	0.9149
	(0.1047)	(0.0025)	(0.0009)	(1.5585)								
TL-Gom-EL	3.3573	1.6773	0.2047	0.2882	282.78	290.78	291.20	301.20	0.0721	0.4266	0.0636	0.8122
	(1.0705)	(0.9570)	(0.0763)	(5.7677×10^{-06})								
	α	β	b	θ								
TL-WP	1.8497×10^{-01}	$1.4227\times10^{+00}$	$2.1985\times10^{+00}$	1.3279×10^{-08}	294.47	302.47	302.89	312.89	0.1309	0.6679	0.1057	0.2135
	(4.2738×10^{-02})	(1.4246×10^{-01})	(4.0254×10^{-01})	(9.8272×10^{-03})								
	c	α	β	θ								
LLOGWP	2.6951	0.0001	1.0621×10^{-05}	7.6903	308.79	316.79	317.21	327.21	0.4637	2.5406	0.1319	0.0613
	(0.1679)	(7.4506×10^{-09})	(6.8425)	(0.9466)								
	δ	θ	λ	γ								
MO-Gom-W	461.0900	0.0230	0.0952	6.8646	284.29	292.29	292.72	302.72	0.0741	0.6366	0.0637	0.8120
	(0.0003)	(0.0183)	(0.0143)	(0.9236)								
	α	β	γ	θ								
OLLEW	18.4404	0.5347	0.5614	8.2657	289.19	297.19	297.62	307.62	0.1936	0.9894	0.0899	0.3930
	(9.9159)	(0.2005)	(0.1709)	(2.1152)								

In Table 9, we see that the values of the goodness-of-fit statistics: AIC, AICC and BIC are the smallest for the TL-Gom-EP and TL-Gom-EP distributions. Also, the values of the goodness-of-fit statistics W^*, A^* and K-S indicate that these distributions fits the carbon fibres data better than the selected non-nested models with the same number of

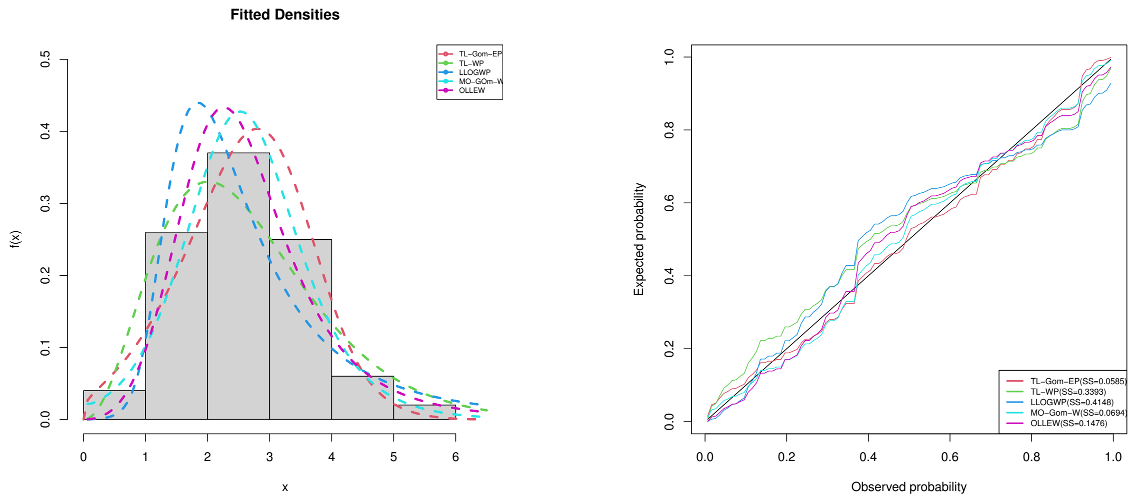


Figure 12: Fitted pdf and probability plots for carbon fibres data

parameters. The p-value of the K-S statistic is closer to 1 and the SS value from the probability plots in Figure 12 is also the smallest for the TL-Gom-EP distribution.

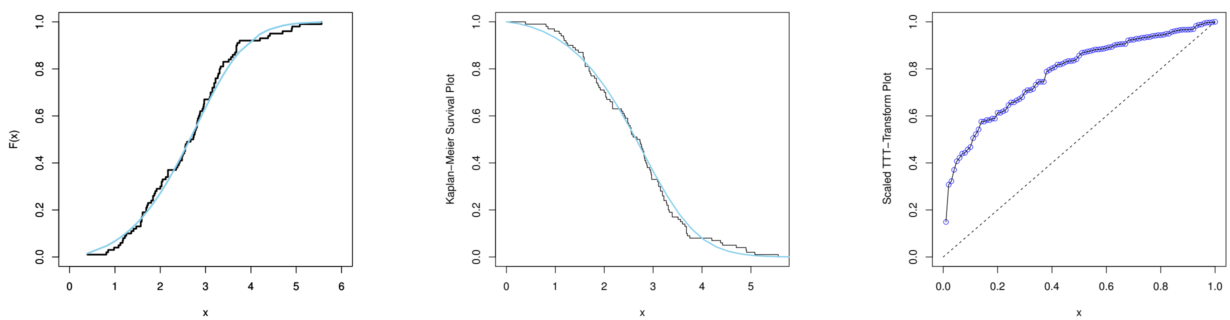


Figure 13: Estimated cdf, K-M survival and TTT-Transform plots for the TL-Gom-EP distribution for fibres data

Figure 13 gives the fitted cdf, survival function and TTT-Transform plots for the carbon fibres data. The fitted cdf in blue is close to the empirical cdf while the survival function in blue is closer to the Kaplan-Meier (K-M) survival curve which shows that indeed our model is better in explaining the carbon fibres data set.

7.2. Time-to-Failure (103 h) of Turbocharger Data

The second data is the time-to-failure of turbocharger of one type of engine by Xu et al. (2003). The data points are given by: 1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0.

The estimated variance-covariance matrix of the TL-Gom-EP model for turbocharger data is given by

$$\begin{bmatrix} 1.8257\times10^{-02} & 1.2041\times10^{-05} & 8.6715\times10^{-06} & -3.9368\times10^{-05} \\ 1.2041\times10^{-05} & 7.9419\times10^{-09} & 5.7193\times10^{-09} & -2.5965\times10^{-08} \\ 8.6715\times10^{-06} & 5.7193\times10^{-09} & 5.2274\times10^{-09} & -1.8699\times10^{-08} \\ -3.9368\times10^{-05} & -2.5965\times10^{-09} & -1.8699\times10^{-08} & 8.4891\times10^{-08} \end{bmatrix}$$

and the approximate 95% confidence interval for the parameters are given by
$b \in (0.2102, 1.8513)$, $\gamma \in (379.0598, 379.0602)$, $\lambda \in (0.0009, 0.0011)$ and $\theta \in (68.8144, 68.8156)$.

Table 10: Estimates of Models for Turbocharger Data

Model	Estimates				Statistics							
	b	γ	λ	θ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	P-value
TL-Gom-EP	1.5865	379.0700	0.0010	6.8815	159.96	167.96	169.10	174.71	0.0349	0.2669	0.0942	0.8700
	(0.1351)	(0.0009)	(0.0007)	(0.0003)								
TL-Gom-EL	0.7692	541.1200	0.0013	0.1019	159.50	167.50	168.70	174.30	0.0316	0.2185	0.0834	0.9436
	(0.1349)	(0.0003)	(0.0004)	(0.0001)								
	α	β	b	θ								
TL-WP	2.0318×10^{-02}	$1.9850\times10^{+00}$	$2.6192\times10^{+00}$	8.5500×10^{-09}	171.80	179.80	180.95	186.56	0.1633	1.1144	0.1143	0.6730
	(5.1960×10^{-02})	$(1.0634\times10^{+00})$	$(2.7043\times10^{+00})$	(1.0776×10^{-02})								
	c	α	β	θ								
LLOGWP	0.6565	0.0071	2.7666	9.2801	164.88	172.88	174.03	179.64	0.0653	0.4809	0.1043	0.7767
	(0.3631)	(0.0125)	(0.7791)	(2.5046)								
	δ	θ	λ	γ								
MO-Gom-W	1.3442	0.0071	0.8206	1.0282	160.18	168.18	169.33	174.94	0.0373	0.2853	0.1347	0.4622
	(1.3919)	(0.0089)	(0.2906)	(0.8137)								
	α	β	γ	θ								
OLLEW	12.5715	0.3613	1.1234	9.0836	175.78	183.78	2 184.92	190.5305	0.1947	1.2940	0.1369	0.4410
	(11.2282)	(0.3469)	(0.5705)	(6.7288)								

Table 10 gives the MLE's and goodness-of-fit statistics of the TL-Gom-EPS distributions for turbocharger data compared with other non-nested models. The values of the goodness-of-fit statistics: AIC, AICC and BIC are the smallest for the TL-Gom-EL distribution followed by the TL-Gom-EP distribution. The goodness-of-fit statistics W^*, A^* and K-S values are also small for the TL-Gom-EL and TL-Gom-EP distributions as compared to the non-nested models listed in the table. This shows that indeed the TL-Gom-EP and TL-Gom-EL distributions fits the turbocharger data better. The p-value of the K-S statistic is the largest for TL-Gom-EL distribution and the sum of squares value from the probability plots in Figure 14 is the smallest for the TL-Gom-EP distribution.

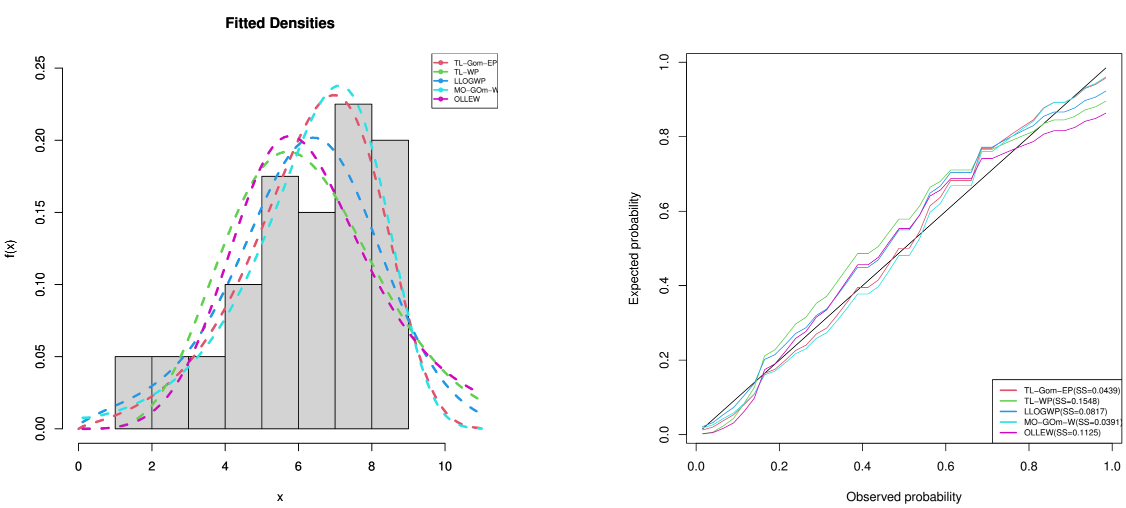


Figure 14: Fitted pdf and probability plots for turbocharger data

Figure 15 gives the estimated cdf, K-M survival and TTT-Transform plots for the turbocharger data. The plot of the estimated cdf for the TL-Gom-EP distribution is closer to the empirical cdf while the survival function is also close to the K-M curve, hence our model is better in explaining the turbocharger data. The TTT-transform plot shows that our model can fit data with increasing hrf.

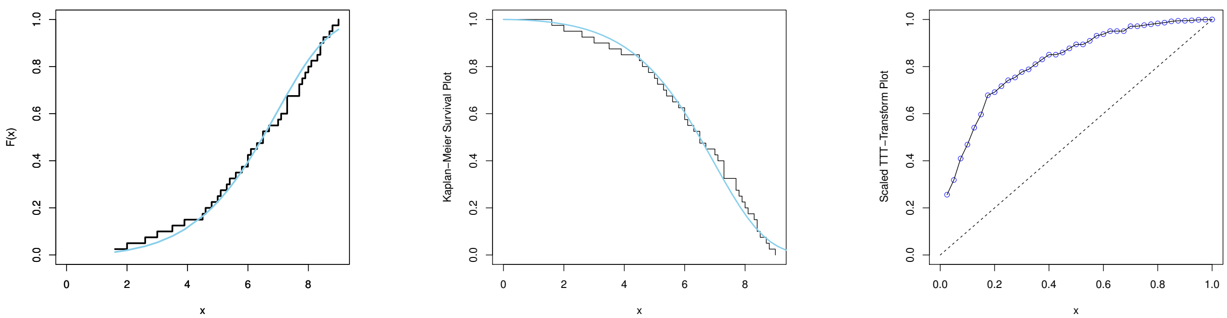


Figure 15: Estimated cdf, K-M survival and TTT-Transform plots for the TL-Gom-EP distribution for turbocharger data

8. Concluding Remarks

This paper introduces the Topp-Leone-Gompertz-G Power Series distribution, a new class of generalized distributions, and explores its statistical properties. Parameter estimation was performed using five different estimation methods and the MLE emerged the best estimation technique for the TL-Gom-GPS class of distributions. Various special cases of the distribution were investigated. Real-world applications demonstrated the effectiveness and versatility of the TL-Gom-GPS class of distributions, for the special cases of the Topp-Leone Gompertz-Exponential Poisson and Topp-Leone Gompertz-Exponential Logarithmic distributions. In the future research, we will explore the application of the developed models to censored data. In addition, we plan to explore Bayesian estimation technique and provide comparisons with other estimation methods via extensive simulations.

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