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The Multimodal Extension of the Balakrishnan Alpha Skew Normal Distribution: Properties and Applications

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Abstract

In this paper, we introduced a new family of distributions by generalizing existing multimodal skew-normal distribution. Statistical properties of the new family of distributions are studied in detail. In particular, explicit expressions of the density and distribution function, moments, skewness, kurtosis and the moments generating function are derived. Furthermore, estimation of the parameters using the maximum likelihood method of the new family of distributions is considered. Finally, the paper ends with an illustration of real-life data sets and then comparing the value of Akaike Information Criterion and Bayesian information criterion of the new distribution with some other known distributions. For the nested models, the Likelihood Ratio Test is carried out.

Key Words: AIC, BIC, Balakrishnan, Skew distribution, Simulation

Mathematical Subject Classification: 60E05, 62E10

1. Introduction

In addition to the skewness, there are many real-life situations where the observed data exhibits more than one mode, i.e. multimodal. For example, in measurement of speed distribution of road traffic vehicular data is multimodal (Torok and Zefreh, 2016), where the authors found that the emission level for applying combined multimodal speed distribution function was lower as compared to unimodal speed distribution. Haze particle size in a chemically oxidising atmosphere follows multimodal distribution (Fan *et al.*, 2021). Measurement of circular data such as wind and wave directional data are multimodal. In astronomy, the velocity of the density follows multimodal pattern if the required galaxies are clustered (Roeder, 1990). However, modeling such types of phenomena with skew normal distribution may not be appropriate. A mixture of distributions may be appropriate to analyze these types of dataset, but the increasing number of parameters in a finite mixture model has always been challenging in applications.

In 2002, Balakrishnan as a discussion of Arnold and Beaver (2002) proposed the generalized form of a skew-normal distribution with the probability density function (pdf) given by

$$f_Z(z;\lambda,n) = g(z)\left[G(\lambda z)\right]^n \Big/ C_n(\lambda), \tag{1}$$

where n represents the positive integer and $C_n(\lambda)=\left[G^n(\lambda U)\right]$, $U \sim N(0,1)$. This distribution has been studied and generalized by some researchers, including Sharafi and Behboodian (2008), Yadegari *et al*. (2008), Bahra *et al.* (2009), Hasanalipour and Sharafi (2012), Asgharzadeh *et al*. (2016).

Considering the idea of Hung and Chen (2007), Chakraborty *et al.* (2015) studied a multimodal extension of skew-normal distribution in the presence of both positive and negative skewness in the dataset. The pdf of skewed multimodal extension of the normal distribution is given by

$$f(z)=\left(\sigma\sqrt{2\pi}\right)^{-1}\exp\left(\frac{-z^2}{2\sigma^2}\right)\left\{\frac{1+\left\{\sin\left(\lambda z/\sigma\right)\right\}}{\alpha}\right\};\ (z,\lambda)\in R, \tag{2}$$

where asymmetric parameter $\alpha>0$. This distribution function is flexible enough to manage the data sets with multiple modes and manage to fit the different degrees of symmetry and asymmetry as well.

A new class of alpha-skew-Laplace distribution popularly known as Balakrishnan alpha skew Laplace distribution was introduced by Shah *et al.* (2019). Further, they studied some of the extensions and generalizations of the same distribution. The pdf of the Balakrishnan-alpha-skew-normal distribution of Hazarika *et al.* (2020) is given by

$$f_Z(z;\alpha)=\frac{[(1-\alpha z)^2+1]^2}{C_2(\alpha)}\varphi(z);\ z\in R, \tag{3}$$

Where $C_2(\alpha)=4+8\alpha^2+3\alpha^4$ and $\varphi(.)$ is the pdf of standard normal distribution.

This article mainly concentrates on introducing a new family of distribution that is flexible enough to multimodal behaviors of real-life phenomenon. Therefore, this distribution plays a unifying role to model real-life data where multiple modes may occur.

2. A Multimodal Balakrishnan Alpha Skew Normal Distribution

In this section we introduce a new form of multimodal skew normal distribution and investigate some of its basic properties.

Definition 1: If a random variable (r.v.) Z has a density function

$$f(z;\alpha,\lambda,\delta)=\frac{1}{C_2(\alpha,\lambda,\delta)}\left(1+\frac{Sin(\lambda z)}{\delta}\right)[(1-\alpha z)^2+1]^2\varphi(z),\ z\in R, \tag{4}$$

where $C_2(\alpha,\lambda,\delta)=C_2(\alpha)+\dfrac{4\alpha\lambda e^{-\frac{\lambda^2}{2}}(-2-3\alpha^2+\alpha^2\lambda^2)}{\delta}$ and $C_2(\alpha)$ is as defined before, then it is said to be multimodal Balakrishnan alpha skew normal distribution with parameters $\alpha\in R$, $\lambda\in R$ and $\delta\geq 1$. We denote it as $MMBASN_2(\alpha,\lambda,\delta)$. The normalizing constant $C_2(\alpha,\lambda,\delta)$ is calculated as follows:

$$C_2(\alpha,\lambda,\delta)=\int_{-\infty}^{\infty}\left(1+\frac{Sin(\lambda z)}{\delta}\right)[(1-\alpha z)^2+1]^2\varphi(z)\,dz$$

$$=\int_{-\infty}^{\infty}[(1-\alpha z)^2+1]^2\varphi(z)\,dz+\int_{-\infty}^{\infty}\left(\frac{Sin(\lambda z)}{\delta}\right)[(1-\alpha z)^2+1]^2\varphi(z)\,dz$$

$$=C_2(\alpha)+\int_{-\infty}^{\infty}\left(\frac{\exp(i\lambda z)-\exp(-i\lambda z)}{2i\delta}\right)[(1-\alpha z)^2+1]^2\varphi(z)\,dz$$

$$=C_2(\alpha)+\int_{-\infty}^{\infty}\left(\frac{\exp(i\lambda z)}{2i\delta}\right)[(1-\alpha z)^2+1]^2\varphi(z)\,dz-\int_{-\infty}^{\infty}\left(\frac{\exp(-i\lambda z)}{2i\delta}\right)[(1-\alpha z)^2+1]^2\varphi(z)\,dz$$

$$= C_2(\alpha) + \frac{1}{2i\delta}\left[C_2(\alpha)\{\phi_Y(\lambda) - \phi_Y(-\lambda)\}\right] \qquad (5)$$

where $\phi_Y(.)$ is the characteristic function of $Y \sim BASN_2(\alpha)$ distribution. We have,

$$\phi_Y(\lambda) = C_2(\alpha) + \alpha\lambda(2i + \alpha\lambda)(-4 + 2i\alpha\lambda - 6\alpha^2 + \alpha^2\lambda^2) \text{ and}$$

$$\phi_Y(-\lambda) = C_2(\alpha) + \alpha\lambda(-2i + \alpha\lambda)(-4 - 2i\alpha\lambda - 6\alpha^2 + \alpha^2\lambda^2)$$

Putting these in the Equation (5), we get

$$C_2(\alpha,\lambda,\delta) = C_2(\alpha) + \frac{4\alpha\lambda e^{-\frac{\lambda^2}{2}}(-2 - 3\alpha^2 + \alpha^2\lambda^2)}{\delta}.$$

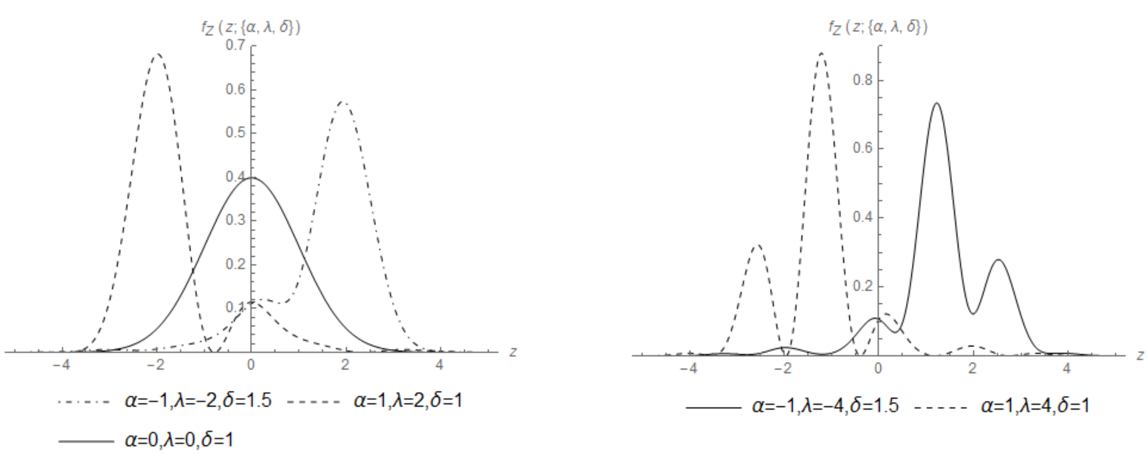
2.1. Special Cases of $MMBASN_2(\alpha,\lambda,\delta)$:

- If $\delta = 1$, then we get $f(z;\alpha,\lambda) = (1 + Sin(\lambda z))[(1-\alpha z)^2 + 1]^2 \varphi(z) / C_2(\alpha,\lambda)$ where $C_2(\alpha,\lambda) = C_2(\alpha) + 4\alpha\lambda e^{-\frac{\lambda^2}{2}}(-2 - 3\alpha^2 + \alpha^2\lambda^2)$.
- If $\alpha = 0$, then we get the standard $MMSN(\lambda,\delta)$ distribution of Chakraborty *et al.* (2015) given by $f(z;\lambda,\delta) = \left(1 + \frac{Sin(\lambda z)}{\delta}\right)\varphi(z)$.
- If $\lambda = 0$, then we get the $BASN(\alpha)$ distribution of Hazarika *et al.* (2019) given by $f(z;\alpha) = [(1-\alpha z)^2 + 1]^2 \varphi(z) / C_2(\alpha)$.
- If $\alpha = \lambda = 0$, then we get the standard normal distribution given by $f(z) = \varphi(z)$.
- If $\alpha \to \pm\infty$, then we get $f(z;\lambda,\delta) = \frac{z^4}{3}\left(1 + \frac{Sin(\lambda z)}{\delta}\right)\varphi(z)$.
- If $\delta \to \pm\infty$, then we again get the $BASN_2(\alpha)$ distribution of Hazarika *et al.* (2019) given by $f(z;\alpha) = [(1-\alpha z)^2 + 1]^2 \varphi(z) / C_2(\alpha)$.
- If $Z \sim MMBASN_2(\alpha,\lambda,\delta)$, then $-Z \sim -MMBASN_2(\alpha,\lambda,\delta)$.

2.2. Plots of the Density Function

The pdf of $MMBASN_2(\alpha,\lambda,\delta)$ distribution for different choices of the parameters α, λ and δ are plotted in Figure 1.

It is observed from Figure 1 of the plots of the pdf of $MMBASN_2(\alpha,\lambda,\delta)$ that as the value of λ increases the number of peaks also increases, as the value of δ increases the curve tends to normal curve and the skewness is positive (negative) according as $\alpha > (<) 0$. Also, the plot of pdf approaches to normal as λ and α tends to zero.



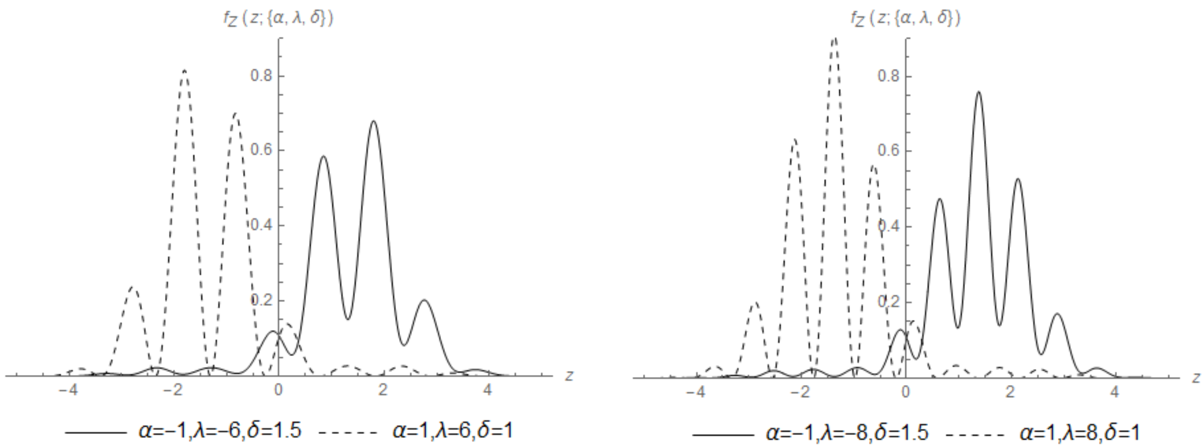


Figure 1: Plots of the pdf of $MMBASN_2(\alpha,\lambda,\delta)$

2.3. Cumulative Distribution Function

Theorem 1: The cumulative distribution function (cdf) of $MMBASN_2(\alpha,\lambda,\delta)$ distribution is given by

$$F_Z(z)=\frac{1}{C_2(\alpha,\lambda,\delta)}\left[C_2(\alpha)F_Y(z)+\frac{C_2(\alpha)C_2(\alpha,b)}{2i\delta}\{\phi_X(\lambda)-\phi_X(-\lambda)\}\right] \tag{6}$$

where $C_2(\alpha,b)=\Phi(b)+\dfrac{\alpha(8-8b\alpha+8\alpha^2+4b^2\alpha^2-3b\alpha^3-b^3\alpha^3)}{C_2(\alpha)}\varphi(b)$, $\varphi(.)$ and $\Phi(.)$ are the pdf and cdf of standard normal distribution, $F_Y(z)$ is the cdf of $Y\sim BASN_2(\alpha)$ distribution, $\phi_X(.)$ is the characteristics function of $X\sim Truncated\ BASN_2(\alpha)$ distribution (see *Remark 1 below*).

Proof: $F_Z(z)=\int_{-\infty}^{z}\frac{1}{C_2(\alpha,\lambda,\delta)}\left(1+\frac{Sin(\lambda z)}{\delta}\right)[(1-\alpha z)^2+1]^2\varphi(z)\,dz$

$$=\frac{1}{C_2(\alpha,\lambda,\delta)}\left[\int_{-\infty}^{z}[(1-\alpha z)^2+1]^2\varphi(z)\,dz+\int_{-\infty}^{z}\left(\frac{Sin(\lambda z)}{\delta}\right)[(1-\alpha z)^2+1]^2\varphi(z)\,dz\right]$$

$$=\frac{1}{C_2(\alpha,\lambda,\delta)}\left[C_2(\alpha)F_Y(z)+\int_{-\infty}^{z}\left(\frac{\exp(i\lambda z)-\exp(-i\lambda z)}{2i\delta}\right)[(1-\alpha z)^2+1]^2\varphi(z)\,dz\right]$$

$$=\frac{1}{C_2(\alpha,\lambda,\delta)}\left[C_2(\alpha)F_Y(z)+\frac{1}{2i\delta}\left[\int_{-\infty}^{z}\exp(i\lambda z)[(1-\alpha z)^2+1]^2\varphi(z)\,dz-\int_{-\infty}^{z}\exp(-i\lambda z)[(1-\alpha z)^2+1]^2\varphi(z)\,dz\right]\right]$$

$$=\frac{1}{C_2(\alpha,\lambda,\delta)}\left[C_2(\alpha)F_Y(z)+\frac{1}{2i\delta}\left[C_2(\alpha)C_2(\alpha,b)\{\phi_X(\lambda)-\phi_X(-\lambda)\}\right]\right]$$

The plot of cdf of $MMBASN_2(\alpha,\lambda,\delta)$ distribution is given in Figure 2 for different values of the parameters α, λ and δ.

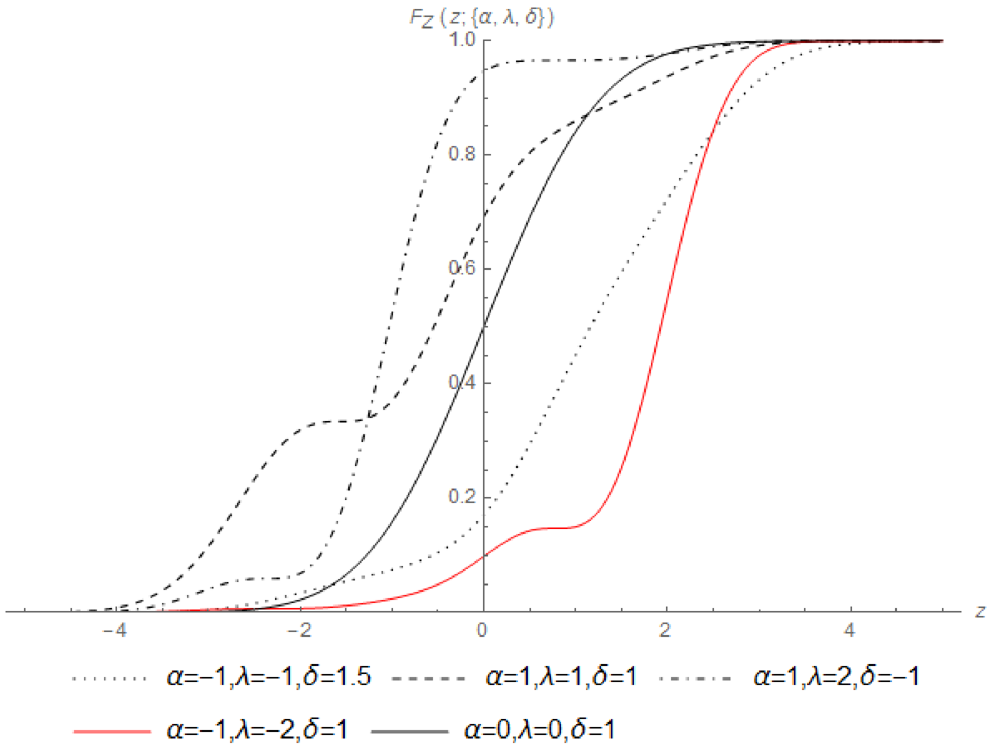


Figure 2: Plots of the cdf of $MMBASN_2(\alpha,\lambda,\delta)$

Remark 1: The pdf of $X \sim Truncated\ BASN_2(\alpha)$ distribution in the range $(-\infty,b)$ is given by

$$f(x;\alpha)=\frac{[(1-\alpha x)^2+1]^2\varphi(x)}{C_2(\alpha)C_2(\alpha,b)};\ -\infty<x<b \tag{7}$$

where, $C_2(\alpha,b)=\Phi(b)+\dfrac{\alpha(8-8b\alpha+8\alpha^2+4b^2\alpha^2-3b\alpha^3-b^3\alpha^3)}{C_2(\alpha)}\varphi(b)$, $\Phi(.)$ and $\varphi(.)$ are defined above.

The characteristics function of Equation (7) is

$$\phi_X(t)=\frac{e^{ibt}\left[\begin{array}{l}\sqrt{2\pi}\,e^{\frac{(b-it)^2}{2}}(4-8it\alpha-8\alpha^2(t^2-1)+4it\alpha^3(t^2-3)+\alpha^4(3-6t^2+t^4))\Phi(b-it)-\\ \alpha(-8+8\alpha(b+it)-4\alpha^2(2+b^2+ibt-t^2)+\alpha^3(b(b^2+3)+it(b^2+5)-bt^2-it^3))\end{array}\right]\varphi(b)}{C_2(\alpha)C_2(\alpha,b)}$$

Proof: see Appendix A.

2.4 Moment Generating Function (mgf) and Moment

Theorem 2: The moment generating function (mgf) of $MMBASN_2(\alpha,\lambda,\delta)$ distribution is given by

$$M_Z(t)=\frac{1}{C_2(\alpha,\lambda,\delta)}\left[C_2(\alpha)M_Y(t)+\frac{C_2(\alpha)}{2i\delta}\{M_Y(t+i\lambda)-M_Y(t-i\lambda)\}\right] \tag{8}$$

where $M_Y(.)$ is the mgf of $Y \sim BASN_2(\alpha)$ distribution.

Proof: $M_Z(t)=\displaystyle\int_{-\infty}^{\infty} e^{tz}\frac{1}{C_2(\alpha,\lambda,\delta)}\left(1+\frac{Sin(\lambda z)}{\delta}\right)[(1-\alpha z)^2+1]^2\varphi(z)\,dz$

$$=\frac{1}{C_2(\alpha,\lambda,\delta)}\left[\int_{-\infty}^{\infty} e^{tz}[(1-\alpha z)^2+1]^2\varphi(z)\,dz+\int_{-\infty}^{\infty} e^{tz}\left(\frac{Sin(\lambda z)}{\delta}\right)[(1-\alpha z)^2+1]^2\varphi(z)\,dz\right]$$

$$=\frac{1}{C_2(\alpha,\lambda,\delta)}\left[C_2(\alpha)M_Y(t)+\int_{-\infty}^{\infty} e^{tz}\left(\frac{\exp(i\lambda z)-\exp(-i\lambda z)}{2i\delta}\right)[(1-\alpha z)^2+1]^2\varphi(z)\,dz\right]$$

$$=\frac{1}{C_2(\alpha,\lambda,\delta)}\left[C_2(\alpha)M_Y(t)+\frac{1}{2i\delta}\left[\int_{-\infty}^{\infty}\exp[(t+i\lambda)z][(1-\alpha z)^2+1]^2\varphi(z)\,dz -\int_{-\infty}^{\infty}\exp[(t-i\lambda)z][(1-\alpha z)^2+1]^2\varphi(z)\,dz\right]\right]$$

$$=\frac{1}{C_2(\alpha,\lambda,\delta)}\left[C_2(\alpha)M_Y(t)+\frac{1}{2i\delta}[C_2(\alpha)\{M_Y(t+i\lambda)-M_Y(t-i\lambda)\}]\right]$$

Remark 2: The n^{th} moment of $MMBASN_2(\alpha,\lambda,\delta)$ distribution is

$$E(Z^n)=\frac{1}{C(\alpha,\lambda,\delta)}[4E(X^n)-8\alpha E(X^{n+1})+8\alpha^2E(X^{n+2})-4\alpha^3E(X^{n+3})+\alpha^4E(X^{n+4})]$$

where, $E(X^n)$, $E(X^{n+1})$, $E(X^{n+2})$, $E(X^{n+3})$ and $E(X^{n+4})$ are the n^{th}, $(n+1)^{th}$, $(n+2)^{th}$, $(n+3)^{th}$ and $(n+4)^{th}$ moments of standard $MMSN(\lambda,\delta)$ distribution (Chakraborty *et al.*, 2015). We have,

$$E(X^n)==\begin{cases}[(n-1)(n-3)...3.1]; & \text{if } n \text{ is even}\\ -\delta^{-1}\exp\left(\frac{-\lambda^2}{2}\right)\sum_{k=0}^{(n/2)}\frac{n!}{(n-2k)!k!}\left\{\frac{1}{2}\right\}^k(\lambda)^{n-2k}\sin\left\{\frac{\pi(n-2k)}{2}\right\}; & \text{if } n \text{ is odd}\end{cases} \quad (9)$$

Similarly, replacing n by $n+1$ in Equation (9) we get $E(X^{n+1})$, n by $n+2$ in Equation (9) we get $E(X^{n+2})$, n by $n+3$ in Equation (9) we get $E(X^{n+3})$, and n by $n+4$, in Equation (9) we get $E(X^{n+4})$.

Thus,

$$E[Z]=\frac{-4\alpha\delta(2+3\alpha^2)+\lambda e^{-\frac{\lambda^2}{2}}[4-8\alpha^2(\lambda^2-3)+\alpha^4(15-10\lambda^2+\lambda^4)]}{\delta C(\alpha,\lambda,\delta)}$$

$$E[Z^2]=\frac{\delta(4+24\alpha^2+15\alpha^4)-4\alpha\lambda e^{-\frac{\lambda^2}{2}}[6-2\lambda^2+\alpha^2(15-10\lambda^2+\lambda^4)]}{\delta C(\alpha,\lambda,\delta)}$$

$$E[Z^3]=\frac{e^{-\frac{\lambda^2}{2}}\left(-12\alpha\delta e^{\frac{\lambda^2}{2}}(2+5\alpha^2)-\lambda\{4(\lambda^2-3)-8\alpha^2(15-10\lambda^2+\lambda^4)+\alpha^4(-105+105\lambda^2-21\lambda^4+\lambda^6)\}\right)}{\delta C(\alpha,\lambda,\delta)}$$

$$E[Z^4]=\frac{3\delta(4+40\alpha^2+35\alpha^4)+4\alpha\lambda e^{-\frac{\lambda^2}{2}}[-30+20\lambda^2-2\lambda^4+\alpha^2(-105+105\lambda^2-21\lambda^4+\lambda^6)]}{\delta C(\alpha,\lambda,\delta)}$$

and the variance is given by

$$Var(Z)=\frac{\left(\begin{array}{l}\delta C(\alpha,\lambda,\delta)(\delta(4+24\alpha^2+15\alpha^4)-4\alpha\lambda e^{-\frac{\lambda^2}{2}}(6-2\lambda^2+\alpha^2(15-10\lambda^2+\lambda^4)))-\\(-4\alpha\delta(2+3\alpha^2)+\lambda e^{-\frac{\lambda^2}{2}}(4-8\alpha^2(\lambda^2-3)+\alpha^4(15-10\lambda^2+\lambda^4)))^2\end{array}\right)}{\delta^2 C(\alpha,\lambda,\delta)^2}$$

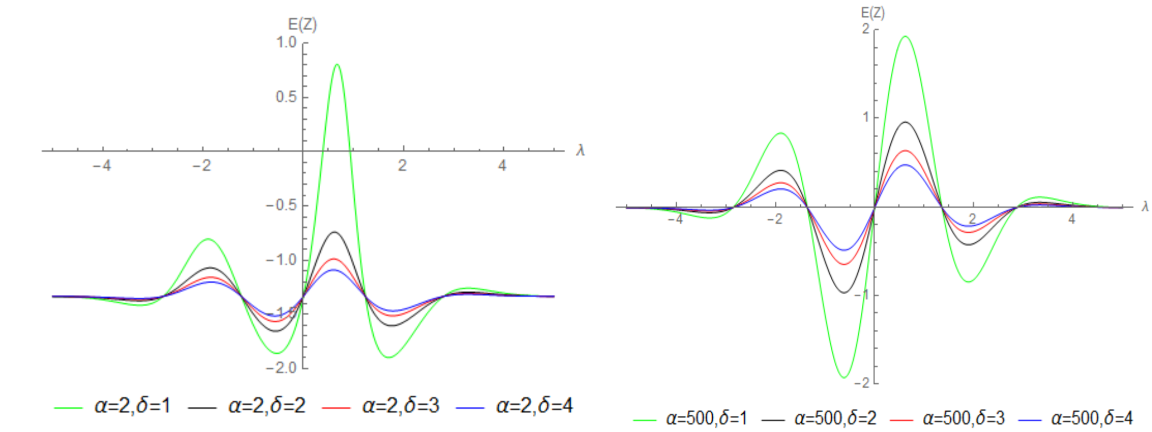


Figure 3: Plots of the mean of $MMBASN_2(\alpha,\lambda,\delta)$

It is observed from Figure 3 that for fixed values of α when $|\lambda|$ and δ increases, the mean of $MMBASN_2(\alpha,\lambda,\delta)$ tends to -1.3 and for large values of α when $|\lambda|$ and δ increases, the mean of $MMBASN_2(\alpha,\lambda,\delta)$ tends to zero which is the mean of the standard normal distribution.

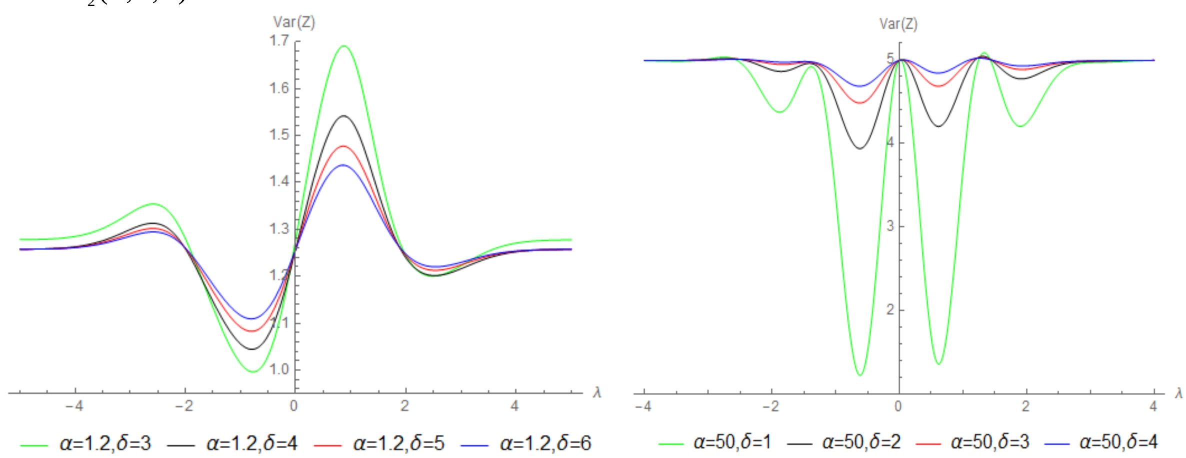


Figure 4: Plots of the variance of $MMBASN_2(\alpha,\lambda,\delta)$

It is observed from Figure 4 that for fixed values of α when $|\lambda|$ and δ increases, the variance of $MMBASN_2(\alpha,\lambda,\delta)$ tends to 1.28 and for large values of α when $|\lambda|$ and δ increases, the variance of $MMBASN_2(\alpha,\lambda,\delta)$ tends to 5.

2.5. Skewness and Kurtosis

Remark 3: The skewness or Pearson's β_1 coefficient of $MMBASN_2(\alpha,\lambda,\delta)$ distribution is

$$\beta_1 = \frac{\left[\begin{array}{l} 3\delta C_2(\alpha,\lambda,\delta)\{d_2\delta - 4\alpha\lambda d_3 e^{-\frac{\lambda^2}{2}}\}\{-4\alpha\delta d_1 + \lambda d_4 e^{-\frac{\lambda^2}{2}}\} - 2\{-4\alpha\delta d_1 + \lambda d_4 e^{-\frac{\lambda^2}{2}}\}^3 \\ +\delta^2 C_2(\alpha,\lambda,\delta)^2\{12\alpha(2+5\alpha^2)\delta + \lambda d_5 e^{-\frac{\lambda^2}{2}}\} \end{array}\right]^2}{\left[\delta C_2(\alpha,\lambda,\delta)\{d_2\delta - 4\alpha\lambda d_3 e^{-\frac{\lambda^2}{2}}\} - \{-4\alpha\delta d_1 + \lambda d_4 e^{-\frac{\lambda^2}{2}}\}^2\right]^3}$$

where $d_1 = 2+3\alpha^2$, $d_2 = 4+24\alpha^2+15\alpha^4$, $d_3 = 6-2\lambda^2+\alpha^2(15-10\lambda^2+\lambda^4)$,
$d_4 = 4-8\alpha^2(-3+\lambda^2)+\alpha^4(15-10\lambda^2+\lambda^4)$ and
$d_5 = 4(-3+\lambda^2)-8\alpha^2(15-10\lambda^2+\lambda^4)+\alpha^4(-105+105\lambda^2-21\lambda^4+\lambda^6)$.

When $\alpha=\lambda=\delta=1$, the value of $\beta_1 = 0.00376325$, when $\alpha=\lambda=0$, $\beta_1=0$ and the distribution is symmetric normal, and when $\lambda<(>)0$, then the distribution is positively (negatively) skewed. The values of β_1 are plotted in Figure 5 against λ $(-5\le\lambda\le5)$ for different values of α and δ.

The kurtosis or Pearson's β_2 coefficient of $MMBASN_2(\alpha,\lambda,\delta)$ distribution is

$$\beta_2 = \frac{\left[\begin{array}{l} 6\delta C_2(\alpha,\lambda,\delta)\{d_2\delta-4\alpha\lambda d_3 e^{-\frac{\lambda^2}{2}}\}\{-4\alpha\delta d_1+\lambda d_4 e^{-\frac{\lambda^2}{2}}\}^2-3\{-4\alpha\delta d_1+\lambda d_4 e^{-\frac{\lambda^2}{2}}\}^4 \\ +\delta^3 C_2(\alpha,\lambda,\delta)^3\{3d_6\delta+4\alpha\lambda d_7 e^{-\frac{\lambda^2}{2}}\}-4\delta^2 C_2(\alpha,\lambda,\delta)^2 e^{-\lambda^2}\{4\alpha\delta d_1 e^{\frac{\lambda^2}{2}}+\lambda d_4\} \\ \{12\alpha(2+5\alpha^2)\delta e^{\frac{\lambda^2}{2}}+\lambda d_5\} \end{array}\right]}{\left[-\delta C_2(\alpha,\lambda,\delta)\{d_2\delta-4\alpha\lambda d_3 e^{-\frac{\lambda^2}{2}}\}+\{-4\alpha\delta d_1+\lambda d_4 e^{-\frac{\lambda^2}{2}}\}^2\right]^2}$$

Where, $d_6 = 4+40\alpha^2+35\alpha^4$ and $d_7 = -30+20\lambda^2-2\lambda^4+\alpha^2(-105+105\lambda^2-21\lambda^4+\lambda^6)$.
when $\alpha=\lambda=\delta=1$, the value of $\beta_2 = 2.32383$, and when $\alpha=\lambda=0$, $\beta_2=3$ and the distribution is symmetric normal curve (mesokurtic). The values of β_2 are plotted in Figure 6 against λ $(-5\le\lambda\le5)$ for different values of α and δ.

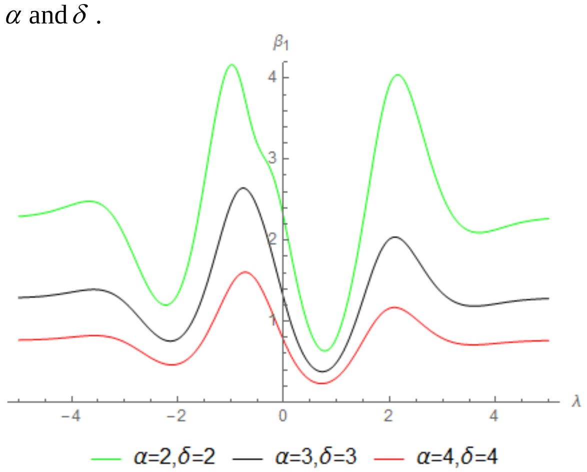


Figure 5: Plots of skewness

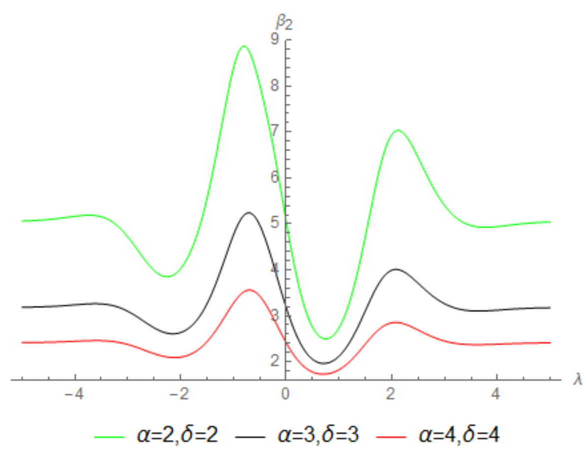


Figure 6: Plots of kurtosis

From Figure 5 and 6 it is observed that β_1 and β_2 increases when α and δ decreases and remains constant for $|\lambda|>5$.

3. Parameter Estimation of $MMBASN_2(\alpha,\lambda,\delta)$ Distribution

3.1 Location Scale Extension

The location and scale extension of $MMBASN_2(\alpha,\lambda,\delta)$ distribution is as follows. If $Z \sim MMBASN_2(\alpha,\lambda,\delta)$ then $Y=\mu+\sigma Z$ is said to be the location (μ) and scale (σ) extension of Z and has the density function given by

$$f(y;\mu,\sigma,\alpha,\lambda,\delta)=\left(\frac{1}{\delta C_2(\alpha,\lambda,\delta)}\right)\left[\delta+Sin\left\{\lambda\left(\frac{y-\mu}{\sigma}\right)\right\}\right]\left[\left\{1-\alpha\left(\frac{y-\mu}{\sigma}\right)\right\}^2+1\right]^2\varphi\left(\frac{y-\mu}{\sigma}\right) \quad (10)$$

We denote it by $Y\sim MMBASN_2(\mu,\sigma,\alpha,\lambda,\delta)$. where $y\in R$, $\mu\in R$, $\alpha\in R$, $\lambda\in R$, $\sigma>0$ and $\delta>0$.

3.2 Maximum Likelihood Estimation

Let $y_1,y_2,...,y_n$ be a random sample from the distribution of the random variable $Y\sim MMBASN_2(\mu,\sigma,\alpha,\lambda,\delta)$ so that the log-likelihood function for $\theta=(\mu,\sigma,\alpha,\lambda,\delta)$ is given by

$$l(\theta)=2\sum_{i=1}^{n}\log\left[\left\{1-\alpha\left(\frac{y_i-\mu}{\sigma}\right)\right\}^2+1\right]-n\log C_2(\alpha,\lambda,\delta)-n\log(\sigma)-\frac{n}{2}\log(2\pi)-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{y_i-\mu}{\sigma}\right)^2$$
$$-n\log(\delta)+\sum_{i=1}^{n}\log\left[\delta+Sin\left\{\lambda\left(\frac{y_i-\mu}{\sigma}\right)\right\}\right] \quad (11)$$

By taking the partial derivatives of the Equation (11) with respect to the parameters $\theta=(\mu,\sigma,\alpha,\lambda,\delta)$, the following normal equations are obtained:

$$\frac{\partial l(\theta)}{\partial\mu}=\sum_{i=1}^{n}-\frac{\lambda\cos\left(\frac{\lambda(y_i-\mu)}{\sigma}\right)}{D_2\,\sigma}-\sum_{i=1}^{n}-\frac{(y_i-\mu)}{\sigma^2}+2\sum_{i=1}^{n}\frac{2D_1\alpha}{\sigma(1+D_1^2)}$$

$$\frac{\partial l(\theta)}{\partial\sigma}=-\frac{n}{\sigma}+\sum_{i=1}^{n}-\frac{\lambda\cos\left(\frac{\lambda(y_i-\mu)}{\sigma}\right)(y_i-\mu)}{D_2\,\sigma^2}-\sum_{i=1}^{n}-\frac{(y_i-\mu)^2}{\sigma^3}+2\sum_{i=1}^{n}\frac{2D_1\alpha(y_i-\mu)}{\sigma^2(1+D_1^2)}$$

$$\frac{\partial l(\theta)}{\partial\alpha}=-\frac{n[4\alpha(4+3\alpha^2)+\frac{4\lambda(-2-9\alpha^2+3\alpha^2\lambda^2)e^{-\frac{\lambda^2}{2}}}{\delta}]}{C_2(\alpha,\lambda,\delta)}-2\sum_{i=1}^{n}\frac{2D_1(y_i-\mu)}{\sigma(1+D_1^2)}$$

$$\frac{\partial l(\theta)}{\partial\lambda}=-\frac{n\left(-\frac{4\alpha[2-2\lambda^2+\alpha^2(3-6\lambda^2+\lambda^4)]e^{-\frac{\lambda^2}{2}}}{\delta}\right)}{C_2(\alpha,\lambda,\delta)}+\sum_{i=1}^{n}\frac{\cos\left(\frac{\lambda(y_i-\mu)}{\sigma}\right)(y_i-\mu)}{D_2\,\sigma}$$

$$\frac{\partial l(\theta)}{\partial\delta}=-\frac{n}{\delta}+\frac{4n\alpha\lambda(-2-3\alpha^2+\alpha^2\lambda^2)e^{-\frac{\lambda^2}{2}}}{\delta^2C_2(\alpha,\lambda,\delta)}+\sum_{i=1}^{n}\frac{1}{D_2}$$

where $D_1=\left(1-\frac{\alpha(y_i-\mu)}{\sigma}\right)$ and $D_2=\delta+\sin\left(\frac{\lambda(y_i-\mu)}{\sigma}\right)$.

Simultaneous solutions of the above equations give the desired estimate of the parameters, but solving the above normal equations is not mathematically tractable. Hence, one should apply some numerical optimization routine to get the solutions.

			$\mu=0\quad\sigma=1\quad\delta=3$									
			$\hat{\mu}$		$\hat{\sigma}$		$\hat{\delta}$		$\hat{\alpha}$		$\hat{\lambda}$	
λ	α	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE

-2	-2	100	-0.460	0.564	0.573	0.554	-0502	0.474	-0.541	0.435	-0.546	0.487
		300	-0.376	0.470	-0503	0.511	0.477	0.490	0.439	0.399	0.313	0.398
		500	0.309	0.400	0.432	0.419	-0.434	0.300	0.201	0.124	-0.210	0.200
	-1	100	-0.363	0.333	0.400	0.372	-0.473	0.333	0.457	0.403	0.470	0.399
		300	-0.312	0.341	0.371	0.309	-0.408	0.279	0.411	0.389	-0.412	0.402
		500	0.207	0.198	-0.234	0.201	0.322	0.300	-0.335	0.311	-0.306	0.179
	1	100	0.410	0.376	0.431	0.334	0.431	0.333	-0.434	0.396	-0.397	0.312
		300	-0.326	0.345	-0.380	0.401	0.397	0.345	-0.400	0.309	-0.301	0.222
		500	0.223	0.213	0.190	0.118	-0.211	0.113	0.221	0.183	0.201	0.143
	2	100	-0.358	0.289	0.371	0.259	0.388	0.321	-0.344	0.402	-0.437	0.354
		300	-0.398	0.301	-0.300	0.307	-0.316	0.298	0.367	0.265	-0.402	0.333
		500	0.187	0.109	0.151	0.180	-0.300	0.203	-0.127	0.210	0.130	0.191
1	-2	100	0.500	0.332	0.433	0.442	-0.464	0.412	-0.408	0.387	-0.445	0.355
		300	-0.327	0.301	-0.413	0.367	0.389	0.322	-0.333	0.341	-0.365	0.300
		500	-0.216	0.219	-0.181	0.291	-0.260	0.120	-0.241	0.200	-0.148	0.187
	-1	100	-0.434	0.447	0.455	0.377	0.487	0.332	-0.447	0.421	-0.498	0.333
		300	-0.401	0.376	0.411	0.381	-0.509	0.343	-0.309	0.422	0.305	0.302
		500	0.230	0.282	-0.201	0.109	0.232	0.217	-0.180	0.098	-0.201	0.170
	1	100	-0.398	0.453	0.444	0.432	-0.556	0.437	0.477	0.334	0.488	0.478
		300	-0.400	0.322	-0.403	0.334	-0.410	0.387	-0.395	0.287	0.404	0.332
		500	0.155	0.099	-0.210	0.211	0.204	0.298	-0.151	0.180	-0.432	0.187
	2	100	-0.388	0.430	-0.322	0.410	0.484	0.337	0.440	0.379	-0.399	0.330
		300	-0.309	0.297	-0.301	0.222	-0.398	0.341	-0.385	0.286	-0.308	0.287
		500	0.178	0.101	-0.123	0.080	0.154	0.099	0.217	0.172	0.170	0.111
2	-2	100	-0.550	0.556	0.498	0.339	-0.511	0.444	0.461	0.398	0.400	0.549
		300	-0.321	0.368	-0.379	0.321	0.340	0.343	0.388	0.351	-0.330	0.419
		500	0.236	0.190	0.301	0.198	0.191	0.160	-0.272	0.180	-0.186	0.223
	-1	100	0.475	0.444	0.414	0.388	0.452	0.444	0.477	0.334	-0.511	0.443
		300	-0.400	0.309	-0.325	0.332	-0.432	0.401	-0.404	0.390	-0.423	0.398
		500	-0.140	0.170	-0.109	0.194	0.289	0.196	-0.173	0.190	0.234	0.214
	1	100	0.399	0.445	0.602	0.368	-0.465	0.339	0.502	0.356	0.449	0.393
		300	-0.409	0.365	-0.321	0.343	-0.359	0.298	0.421	0.330	-0.390	0.222
		500	-0.197	0.190	0.200	0.135	-0.291	0.193	-0.181	0.214	0.100	0.090
	2	100	0.500	0.486	0.456	0.334	0.390	0.340	-0.465	0.546	0.487	0.435
		300	-0.339	0.380	-0.379	0.291	-0.300	0.298	-0.432	0.421	-0.332	0.386
		500	-0.305	0.199	-0.211	0.184	0.111	0.185	-0.265	0.357	-0.210	0.288

Table 1: Results of Simulation

Table 2: Results of Simulation

$\mu = 1 \quad \sigma = 2 \quad \delta = 2$				
$\hat{\mu}$	$\hat{\sigma}$	$\hat{\delta}$	$\hat{\alpha}$	$\hat{\lambda}$

λ	α	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
-2	-2	100	0.431	0.320	0.541	0.487	-0.431	0.390	0.430	0.454	0512	0.487
		300	-0.311	0.226	-0.390	0.402	-0.404	0.287	-0.476	0.368	-0.470	0.409
		500	0.122	0.160	-0.220	0.098	0.227	0.210	0.190	0.281	-0.404	0.309
	-1	100	-0.400	0.425	0.431	0.399	-0.445	0.434	-0.459	0.370	-0.478	0.433
		300	0.327	0.300	0.331	0.249	-0.350	0.354	-0.410	0.351	-0.418	0.280
		500	0.145	0.187	-0.211	0.181	-0.293	0.189	-0.208	0.103	0.333	0.309
	1	100	-0.552	0.387	-0.468	0.399	-0.394	0.333	0.453	0.452	-0.439	0.356
		300	0.421	0.376	0.350	0.299	-0.307	0.246	-0.413	0.338	0.387	0.309
		500	0.207	0.139	-0.297	0.197	0.112	0.087	0.218	0.219	0.210	0.110
	2	100	-0.476	0.330	0.398	0.344	0.500	0.443	-0.362	0.476	-0.365	0.421
		300	-0.370	0.297	-0.306	0.290	-0.410	0.376	0.343	0.234	-0.319	0.209
		500	-0.218	0.189	-0.114	0.189	0.311	0.211	0.120	0.132	0.298	0.103
1	-2	100	0512	0.475	0.544	0.511	0.488	0.430	-0.548	0.434	-0.498	0.343
		300	-0.467	0.496	-0513	0.550	-0.370	0.320	0.447	0.396	-0.333	0.311
		500	-0.380	0.307	0.451	0.410	0.308	0.290	-0.223	0.129	0.217	0.210
	-1	100	-0.570	0.356	0.404	0.371	-0.415	0.400	0.400	0.413	-0.454	0.440
		300	0.418	0.270	0.376	0.313	-0.320	0.301	0.413	0.380	0.411	0.370
		500	-0.324	0.308	-0.235	0.216	0.101	0.098	-0.337	0.318	-0.239	0.272
	1	100	0.439	0.364	0.435	0.333	-0.606	0.530	-0.439	0.390	-0.390	0.450
		300	-0.390	0.305	-0.388	0.413	-0.329	0.327	0.409	0.319	-0.434	0.328
		500	0.271	0.115	0.167	0.110	0.203	0.198	0.227	0.183	-0.159	0.143
	2	100	-0.489	0.329	0.366	0.250	-0.421	0.411	0.345	0.402	0.367	0.439
		300	-0.326	0.290	-0.310	0.301	-0.380	0.398	-0.366	0.265	-0.319	0.299
		500	0.222	0.156	0.159	0.189	0.171	0.234	-0.120	0.210	-0.179	0.154
2	-2	100	0.446	0.488	-0.543	0.510	0.476	0.416	-0.568	0.330	-0.550	0.556
		300	0.413	0.308	-0513	0.461	-0.388	0.324	-0.370	0.327	-0.343	0.376
		500	-0.217	0.206	-0.435	0.420	-0.264	0.128	-0.310	0.178	-0.110	0.267
	-1	100	0.408	0.389	0.407	0.372	0.477	0.336	-0.476	0.378	-0.440	0.309
		300	-0.417	0.413	0.376	0.309	-0.543	0.349	-0.345	0.338	-0.419	0.412
		500	-0.314	0.189	-0.238	0.201	-0.233	0.234	-0.112	0.190	0.316	0.189
	1	100	0.399	0.323	0.434	0.368	0.586	0.443	0.502	0.398	-0.390	0.318
		300	-0.343	0.232	-0.381	0.465	-0.440	0.300	-0.398	0.349	0.301	0.226
		500	-0.223	0.147	0.194	0.119	0.214	0.278	-0.208	0.230	-0.231	0.153
	2	100	-0.497	0.394	0.376	0.269	-0.477	0.334	0.486	0.387	-0.437	0.359
		300	-0.412	0.331	-0.366	0.389	-0.399	0.348	0.399	0.276	-0.405	0.373
		500	-0.138	0.190	-0,138	0.351	0.159	0.123	-0.210	0.156	-0.134	0.190

3.3 Simulation Study of the $MMBASN_2(\alpha,\lambda,\delta)$

To assess the performance of the estimators, we have conducted a simulation study using the well-known simulation methods Metropolis Hasting. The calculation was relayed by assuming the three different numbers of sample sizes $n = 100, 300$ and 500 with different combinations of the parameters. The results are based on the 1000 replicated samples generated from the $MMBASN_2(\alpha,\lambda,\delta)$. For each generated sample, the maximum likelihood estimates are assessed using the GenSA package in R software. After taking the average of the estimated parameters, we measured the bias and MSE of the respective parameters. The formula for Bias and MSE is given as follows:

$$Bias(\hat{\theta}) = E(\hat{\theta}) - \theta \text{ and } MSE(\hat{\theta}) = E[(\hat{\theta} - \theta)^2]$$

where θ is the true value of the parameter.

The simulation results in Table 1 and Table 2 show that the bias and MSE decrease as the sample size increases, as expected. Therefore, it follows the fact that the parameters of the studied distribution are asymptotically consistent for the large and the moderate numbers of sample sizes.

4. Real Life Applications

In the following section, we provide two applications of the new proposed distribution using real data for illustrative purposes. These applications show the flexibility and usefulness of the new proposed distribution. For these data sets, we compare the fits of the proposed distribution, that is, the multimodal Balakrishnan alpha skew normal distribution $MMBASN_2(\mu,\sigma,\alpha,\lambda,\delta)$ to other well-known distributions such as the normal distribution, the alpha-skew-normal distribution $ASN(\mu,\sigma,\alpha)$ of Elal-Olivero (2010), the alpha-skew-logistic distribution $ASLG(\mu,\alpha,\beta)$ of Hazarika and Chakraborty (2014), the alpha-skew-laplace distribution $ASLa(\mu,\alpha,\beta)$ of Harandi and Alamatsaz (2013), the alpha-beta-skew-normal distribution $ABSN(\alpha,\beta,\mu,\sigma)$ of Sharafi *et al.* (2017), beta-skew-normal distribution $BSN(\mu,\sigma,\beta)$ of Mameli and Musio (2013), the generalized alpha skew normal distribution $GASN(\mu,\sigma,\alpha,\lambda)$ of Sharafi *et al.* (2017), Balakrishnan alpha skew normal distribution $BASN(\mu,\sigma,\alpha)$ of Hazarika *et al.* (2020) and the multimodal skew normal distribution $MMSN(\mu,\sigma,\lambda,\delta)$ of Chakraborty *et al.* (2015). Using GenSA package in R, the MLE of the parameters are obtained by using numerical optimization routine. In order to compare the models, we consider the analytical measure, that is, Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC).

Illustration I:

The dataset gives the body mass index (BMI) of 202 Australian athletes and can be obtained from Cook and Weisberg (1994). Table 3 shows the MLE's, log-likelihood and AIC of the above-mentioned distributions.

It is found from Table 3 that the proposed $MMBASN_2(\mu,\sigma,\alpha,\lambda,\delta)$ distribution provides best fit to the data set in terms of AIC and BIC values.

Illustration II:

The data set on Environmental Performance Index (EPI) has been considered for fitting of the proposed distribution. On the basis of 25 performance indicators tracked across ten policy categories covering both Environmental public health and ecosystem vitality, the 2010 EPI ranked 163 countries (see the website *http://epi.yale.edu/* for the data set). Table 4 shows the MLE's, log-likelihood and AIC and BIC of the above-mentioned distributions.

Table 3: MLE's, log-likelihood, AIC and BIC for body mass index (BMI) of 202 Australian athletes.

Distributions	μ	σ	λ	α	β	δ	$\log L$	AIC	BIC

$N(\mu,\sigma^2)$	22.956	2.857	--	--	--		-498.668	1001.336	1007.953
$BSN(\mu,\sigma,\beta)$	22.528	2.694	--	--	-0.058		-492.880	991.76	1001.685
$ASLa(\mu,\alpha,\beta)$	22.350	--	--	-0.14	2.07		-492.601	991.202	1001.127
$SN(\mu,\sigma,\lambda)$	19.969	4.133	2.313	--	--		-490.099	986.198	996.123
$ASLG(\mu,\alpha,\beta)$	21.933	--	--	-0.201	1.475		-489.094	984.188	994.113
$ASN(\alpha,\mu,\sigma)$	24.834	2.653	--	0.994	--		-488.690	983.38	993.305
$ABSN(\mu,\sigma,\alpha,\beta)$	23.998	2.853	--	0.817	-0.131		-486.743	981.486	994.719
$GASN(\mu,\sigma,\alpha,\lambda)$	26.879	3.109	-0.668	2.066	--	--	-485.788	979.576	992.809
$MMSN(\mu,\sigma,\lambda,\delta)$	24.604	3.298	-1.263	--	--	1.147	-484.697	977.394	990.627
$BASN(\mu,\sigma,\alpha)$	26.482	2.706	--	0.971	--		-484.773	975.546	991.217
$MMBASN_2(\mu,\sigma,\alpha,\lambda,\delta)$	26.200	2.644	-23.765	0.985	--	3.582	**-480.992**	**971.984**	**988.525**

Table 4: MLE's, log-likelihood, AIC and BIC for the EPI data.

Distributions	μ	σ	δ	λ	α	β	$\log L$	AIC	BIC
$ASLG(\mu,\beta,\alpha)$	59.185	--	--	--	0.034	7.242	-645.149	1296.299	1305.58
$ASLa(\mu,\beta,\alpha)$	56.3	--	--	--	-2.724	3.789	-644.169	1294.34	1303.621
$ASN(\mu,\sigma,\alpha)$	61.292	12.360	--	--	0.243	--	-641.275	1288.551	1297.832
$BASN(\mu,\sigma,\alpha)$	63.426	12.381	--	--	0.207	--	-641.273	1288.55	1297.83
$SN(\mu,\sigma,\lambda)$	51.633	14.086	--	0.749	--	--	-641.217	1288.435	1297.716
$ABSN\ (\mu,\sigma,\alpha,\beta)$	63.643	8.628	--	--	-0.056	0.210	-639.352	1286.704	1299.079
$N(\mu,\sigma^2)$	58.371	12.371	--	--	--	--	-641.284	1286.569	1292.756
$BSN(\mu,\sigma,\beta)$	63.739	8.761	--	--	--	0.199	-639.388	1284.776	1294.057
$GASN(\mu,\sigma,\alpha,\lambda)$	44.053	12.436	--	2.337	1.796	--	-636.238	1280.476	1292.851
$MMSN\,(\mu,\sigma,\lambda,\delta)$	59.194	12.398	3.246	3.036	--	--	-637.800	1283.600	1295.975
$MMBASN_2(\mu,\sigma,\alpha,\lambda,\delta)$	63.735	12.383	2.280	-25.885	0.222	--	**-634.214**	**1278.428**	**1293.897**

It is found from Table 4 that the proposed $MMBASN_2(\mu,\sigma,\alpha,\lambda,\delta)$ distribution provides best fit to the data set in terms of AIC and BIC values.

Table 5: The values of LR test statistic for different hypothesis.

Hypothesis	LR values		Degrees of Freedom	Critical values
	Dataset I	Dataset II		
$H_0 : \alpha = 0$ vs $H_1 : \alpha \neq 0$	7.41	7.172	1	6.635
$H_0 : \lambda = 0$ vs $H_1 : \lambda \neq 0$	7.562	14.117	2	9.210
$H_0 : \alpha = \lambda = 0$ vs $H_1 : \alpha = \lambda \neq 0$	35.352	14.14	3	11.345

4.1 Likelihood Ratio Test

Since $MMSN(\mu,\sigma,\lambda,\delta)$, $BASN(\alpha,\mu,\sigma)$ and $MMBASN_2(\mu,\sigma,\alpha,\lambda,\delta)$ distributions are nested models, the likelihood ratio (LR) test is used to discriminate between them. The LR test is carried out to test the following hypothesis as shown in Table 5.

The values of LR test statistic for different hypothesis exceed the critical value at 1% level of significance except the second one. In case of the first dataset, the value of the LR test statistic of the second hypothesis is not significant at 1% level of significance. Therefore, we conclude that $MMBASN_2(\mu,\sigma,\alpha,\lambda,\delta)$ distribution is most appropriate for the dataset II analyzed here.

5. Conclusion

In this study, a new family of skew distributions is introduced and is called the multimodal Balakrishnan alpha skew normal distribution which includes unimodal, bimodal as well as multimodal behaviors and has many modes. Some of its distributional properties are investigated. The numerical results from Table 3 and Table 4 above have shown that the $MMBASN_2(\mu,\sigma,\alpha,\lambda,\delta)$ distribution provides better fit compared to the other known distributions applied here. The likelihood ratio test confirms that the sample data comes from our proposed distribution.

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Appendix

A: Proof of Remark 1

The Characteristic function of *Truncated* $BASN_2(\alpha)$ distribution in the range $(-\infty, b)$ is given by

$$\phi_X(t) = \frac{e^{ibt}\begin{bmatrix}\sqrt{2\pi}\, e^{\frac{(b-it)^2}{2}}(4-8it\alpha-8\alpha^2(t^2-1)+4it\alpha^3(t^2-3)+\alpha^4(3-6t^2+t^4))\,\Phi(b-it)- \\ \alpha(-8+8\alpha(b+it)-4\alpha^2(2+b^2+ibt-t^2)+\alpha^3(b(b^2+3)+it(b^2+5)-bt^2-it^3))\end{bmatrix}\varphi(b)}{C_2(\alpha)\,C_2(\alpha,b)}$$

Proof:

$$\phi_X(t) = \int_{-\infty}^{b} e^{itx}\frac{[(1-\alpha x)^2+1]^2\varphi(x)}{C_2(\alpha)\,C_2(\alpha,b)}dx$$

$$= \frac{1}{C_2(\alpha)\,C_2(\alpha,b)}\int_{-\infty}^{b} e^{itx}[4-8ax+8a^2x^2-4a^3x^3+a^4x^4]\varphi(x)dx$$

$$= \frac{1}{C_2(\alpha)\,C_2(\alpha,b)}\begin{bmatrix}4\int_{-\infty}^{b} e^{itx}\varphi(x)dx-8a\int_{-\infty}^{b} x\,e^{itx}\varphi(x)dx+8a^2\int_{-\infty}^{b} x^2e^{itx}\varphi(x)dx \\ -4a^3\int_{-\infty}^{b} x^3e^{itx}\varphi(x)dx+a^4\int_{-\infty}^{b} x^4e^{itx}\varphi(x)dx\end{bmatrix}$$

$$= \frac{1}{C_2(\alpha)\,C_2(\alpha,b)}e^{ibt}\begin{bmatrix}\sqrt{2\pi}\, e^{\frac{(b-it)^2}{2}}(4-8it\alpha-8\alpha^2(t^2-1)+4it\alpha^3(t^2-3)+\alpha^4(3-6t^2+t^4))\,\Phi(b-it)- \\ \alpha(-8+8\alpha(b+it)-4\alpha^2(2+b^2+ibt-t^2)+\alpha^3(b(b^2+3)+it(b^2+5)-bt^2-it^3))\end{bmatrix}\varphi(b)$$