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Approximate MLEs for the location and scale parameters of the Poisson-half-logistic distribution

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Abstract

The application of compound distributions has recently increased due to the flexibility in fitting actual data in various fields such as economics, insurance, etc. Poisson-half-logistic distribution is one of these distributions with an increasing-constant hazard rate that can be used in parallel systems and complementary risk models. Because of the complexity of the form of this distribution, it is not possible to obtain classical parameter estimates (such as MLE) by the analytical method for the location and scale parameters. We present a simple way of deriving explicit estimators by approximating the likelihood equations appropriately. This paper presents the AMLE (Approximate Maximum Likelihood Estimator) method to estimate the location and scale parameters. Using simulation, we show that this method is as efficient as the maximum likelihood estimators (MLEs). We obtain the variance of estimators from the inverse of the observed Fisher information matrix, and we see that when sample size increases, bias and variance of these estimators, and hence MSEs of parameters, decrease. Some pivotal quantities are proposed for finding confidence intervals for location and scale parameters based on asymptotic normality. From the coverage probability, the MLEs do not work well, especially for the small sample sizes; thus, simulated percentiles based on the Monte Carlo method are used to improve the coverage probability. Finally, we present a numerical example to illustrate the methods of inference developed here.

Key Words: Compounding distribution; Poisson-half-logistic distribution; Approximate Maximum Likelihood Estimator; Monte Carlo simulation.

Mathematical Subject Classification: 60E05, 62E15.

1. Introduction

The random variable X has a standard Poisson-half-logistic (PHL) distribution (Abdel-Hamid (2016)) if the cumulative distribution function (cdf) and probability density function (pdf) are,

$$F(z) = \frac{e^{\theta G(z)}-1}{e^{\theta}-1}, \tag{1}$$

$$\begin{aligned} f(z) &= \frac{\theta g(z) e^{\theta G(z)}}{e^{\theta}-1} \\ &= \frac{2\theta e^{-z+\theta G(z)}}{(e^{\theta}-1)\left[1+e^{-z}\right]^2}, \qquad z>0, \qquad (\theta>0), \end{aligned} \tag{2}$$

respectively, where $G(z) = \frac{1-e^{-z}}{1+e^{-z}}$ and $g(z) = \frac{2e^{-z}}{(1+e^{-z})^2}$ are the cdf and pdf of standard half logistic distribution respectively. We consider a version of (2) with location (μ) and scale (σ) parameters. If $X = \sigma Z + \mu$, then X has the Poisson-half-logistic distribution with the following pdf,

$$\begin{aligned} f(x;\mu,\sigma) &= \frac{\theta g(\frac{x-\mu}{\sigma}) e^{\theta G(\frac{x-\mu}{\sigma})}}{\sigma(e^{\theta}-1)} \\ &= \frac{2\theta e^{-\frac{x-\mu}{\sigma}+\theta G(\frac{x-\mu}{\sigma})}}{\sigma\left(e^{\theta}-1\right)\left[1+e^{-\frac{x-\mu}{\sigma}}\right]^2}, \quad x>\mu, \quad (\sigma,\theta>0). \end{aligned} \tag{3}$$

We denote this distribution with $PHL(\mu, \sigma, \theta)$. In this research, we consider estimating the location and scale parameters of the Poisson-half-logistic distribution. The contents of this paper are organized as follows. In Section (2), we discuss MLEs of the location and scale parameters of the Poisson-half-logistic distribution and provide explicit estimators by appropriately approximating the likelihood equations. Also, section (3), provides expressions for the observed Fisher information matrix. In section (4), some characteristics of the distribution are calculated. In Section (5), we provide the results of a simulation study to evaluate the performance of the approximate estimators and the MLEs determined by numerical methods. Finally, in Section (6), we present a numerical example to illustrate all the methods of inference discussed in the privies sections.

2. Estimation of the location and scale parameters

2.1. Maximum Likelihood Estimation

Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from the $PHL(\mu, \sigma, \theta)$, and let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be the corresponding order statistics. The likelihood function based on this ordered sample is then

$$L(\mu,\sigma) = n! \prod_{i=1}^{n} f(x_{(i)};\mu,\sigma). \tag{4}$$

Using the relation $f(x;\mu,\sigma) = \frac{1}{\sigma} f(\frac{x-\mu}{\sigma})$, the likelihood function may be rewritten as

$$L(\mu,\sigma) = n!\sigma^{-n} \prod_{i=1}^{n} f(z_{(i)}) = n!\sigma^{-n}\theta^{n} \prod_{i=1}^{n} \frac{g(z_{(i)}) e^{\theta G(z_{(i)})}}{e^{\theta}-1},$$

where $z_{(i)} = (x_{(i)} - \mu)/\sigma$.

For simplicity, from now on, we will use z_i and x_i instead of $z_{(i)}$ and $x_{(i)}$. The log-likelihood function may then be written as

$$\ell = \ln(L(\mu,\sigma)) \propto n\ln(\theta) - n\ln(\sigma) + \sum_{i=1}^{n} \ln(g(z_i)) + \theta \sum_{i=1}^{n} G(z_i) - n\ln(e^{\theta}-1). \tag{5}$$

Taking derivatives from (5) with respect to μ and σ and simplified equations with replacing $\frac{g'(.)}{g(.)} = -G(.)$, $\frac{\partial z_i}{\partial \mu} = -\frac{1}{\sigma}$ and $\frac{\partial z_i}{\partial \sigma} = -\frac{1}{\sigma} z_i$ we obtain the likelihood equations as

$$\frac{\partial \ell}{\partial \mu} = \frac{1}{\sigma} \sum_{i=1}^{n} G(z_i) - \frac{\theta}{\sigma} \sum_{i=1}^{n} g(z_i) = 0, \tag{6}$$

$$\frac{\partial \ell}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma} \sum_{i=1}^{n} z_i G(z_i) - \frac{\theta}{\sigma} \sum_{i=1}^{n} z_i g(z_i) = 0. \tag{7}$$

Equations (6) and (7) do not yield explicit solutions and hence must be solved numerically to obtain the MLEs, say $\hat{\mu}$ and $\hat{\sigma}$. Such methods require a starting value near the global maximum.

2.2. Approximate Maximum Likelihood Estimation

Equations (6) and (7) do not admit the explicit solutions for the μ and σ so we will expand the function $G(z_i)$ and $g(z_i)$ in the Taylor series around the point $E(Z_i) = \nu_i$. See David and Nagaraja (2004) and Arnold and Balakrishnan (2012) for reasoning. Various authors have previously applied a similar method for different distributions, see, for example, Balakrishnan and Asgharzadeh(2005), Asgharzadeh(2006), Balakrishnan and Hossain(2007), Asgharzadeh et al.(2013), Gui and Guo(2018) and Rasekhi et al.(2022).
If $F(z_i)$ is the cdf in(4), then it is known that

$$F(Z_i) = U_i,$$

where U_i is the ith order statistic from a sample of size n from the uniform $U(0,1)$ distribution. We then have

$$Z_i = F^{-1}(U_i),$$

and hence

$$\nu_i = E(Z_i) \approx F^{-1}(\alpha_i),$$

where $\alpha_i = E(U_i) = \frac{i}{n+1}$.
By expanding the function $G(z_i)$ and $g(z_i)$ around the point ν_i and keeping only the first two terms, we may then approximate this functions by

$$\begin{aligned} G(z_i) &\approx G(\nu_i) + g(\nu_i)(z_i - \nu_i) \\ &= \alpha_i - \beta_i z_i, \qquad (8) \\ g(z_i) &\approx g(\nu_i) + g'(\nu_i)(z_i - \nu_i) \\ &= \gamma_i + \delta_i z_i, \qquad (9) \end{aligned}$$

where $\alpha_i = G(\nu_i) - \nu_i g(\nu_i)$, $\beta_i = -g(\nu_i) < 0$, $\gamma_i = g(\nu_i) - g'(\nu_i)\nu_i = g(\nu_i)(1 + \nu_i G(\nu_i)) > 0$ and $\delta_i = g'(\nu_i) = -G(\nu_i)g(\nu_i) < 0$.

Putting the equations (8) and (9) into the equations (6) and (7)and simplify, we obtain the following approximate likelihood equations

$$\frac{\partial \ln L}{\partial \mu} \approx \frac{1}{\sigma}\sum_{i=1}^{n}(\alpha_i - \beta_i z_i) - \frac{\theta}{\sigma}\sum_{i=1}^{n}(\gamma_i + \delta_i z_i) = 0, \qquad (10)$$

$$\frac{\partial \ln L}{\partial \sigma} \approx -\frac{n}{\sigma} + \frac{1}{\sigma}\sum_{i=1}^{n} z_i(\alpha_i - \beta_i z_i) - \frac{\theta}{\sigma}\sum_{i=1}^{n} z_i(\gamma_i + \delta_i zi) = 0. \qquad (11)$$

Upon solving equations (10) and (11) for μ and σ with replacing $z_i = \frac{x_i - \mu}{\sigma}$, we drive the AMLE of μ and σ as follows;

$$\tilde{\mu} = K + L\tilde{\sigma}, \qquad (12)$$

where

$$\begin{aligned} K &= \frac{\sum_{i=1}^{n}(\beta_i + \theta\delta_i)x_i}{\sum_{i=1}^{n}(\beta_i + \theta\delta_i)}, \\ L &= -\frac{\sum_{i=1}^{n}(\alpha_i - \theta\gamma_i)}{\sum_{i=1}^{n}(\beta_i + \theta\delta_i)}. \end{aligned}$$

If in equation (11) we replace $\tilde{\mu} = K + L\tilde{\sigma}$ and simplified we obtain the quadratic equation

$$n\sigma^2 - A\sigma + B = 0, \qquad (13)$$

where

$$A=\sum_{i=1}^{n}\left(\alpha_i-\theta\gamma_i\right)\left(x_i-K\right),\quad B=\sum_{i=1}^{n}\left(\beta_i+\theta\delta_i\right)\left(x_i-K\right)^2.$$

Equation (13) is a quadratic equation in σ, with the two roots given by

$$\tilde{\sigma}=\frac{A\pm\sqrt{A^2-4nB}}{2n},$$

Since $B<0$, one of them drops out. Hence the ALME of σ is

$$\tilde{\sigma}=\frac{A+\sqrt{A^2-4nB}}{2n}. \tag{14}$$

The AMLE method has an advantage over the MLE method because of an explicit solution to estimate the parameters. The AMLEs can be used as starting values for the iterative solution of the likelihood equations (6) and (7) to obtain the MLEs.

It should be mentioned that the approximate MLEs in equations (12) and (14) depend on the parameter θ. When the parameter θ is unknown; we may use the profile likelihood function to obtain its estimate. For fixed θ, the approximate MLEs $\tilde{\mu}=\tilde{\mu}(\theta)$ and $\tilde{\sigma}=\tilde{\sigma}(\theta)$ are given by equations (12) and (14), respectively. Thus, the profile log-likelihood function is

$$\begin{aligned}\ell_p(\theta) &= \ln L_p(\theta)=\ln L(\tilde{\mu}(\theta),\tilde{\sigma}(\theta),\theta)\\ &\propto n\ln(\theta)-n\ln(\sigma(\theta))+\sum_{i=1}^{n}\ln(g(z_i))+\theta\sum_{i=1}^{n}G(z_i)-n\ln(e^{\theta}-1),\end{aligned}$$

where $z_i=\frac{x_i-\tilde{\mu}(\theta)}{\tilde{\sigma}(\theta)}$, $\tilde{\mu}(\theta)=K+L\tilde{\sigma}(\theta)$ and $\tilde{\sigma}(\theta)$ is given in equation (14).

Now the maximum profile likelihood estimate $\tilde{\theta}_p$ may be obtained by maximizing the log-likelihood $\ell(\tilde{\mu}(\theta),\tilde{\sigma}(\theta),\theta)$ with respect to θ. Iterative numerical methods are required to obtain $\tilde{\theta}_p$.

3. Observed Fisher information

In this section, the observed Fisher information matrix is computed based on the likelihood and the approximate likelihood equations. then, some pivotal quantities based on the limiting normal distribution are presented and the behavior of these quantities is examined based on a Monte Carlo simulation study.

To evaluate the accuracy of estimators, it is necessary to obtain the asymptotic variance-covariance matrix of the estimators. This matrix is obtained from the inverse of the observed Fisher information matrix. We define the elements of the observed Fisher information matrix for maximum likelihood estimators (MLE) as follows

$$J=\left[\begin{pmatrix}-\frac{\partial^2\ln L}{\partial^2\mu} & -\frac{\partial^2\ln L}{\partial\mu\partial\sigma}\\ -\frac{\partial^2\ln L}{\partial\mu\partial\sigma} & -\frac{\partial^2\ln L}{\partial^2\sigma}\end{pmatrix}\right]^{-1}=\left[\frac{1}{\sigma^2}\begin{pmatrix}V_1 & V_2\\ V_2 & V_3\end{pmatrix}\right]^{-1}=\sigma^2\begin{pmatrix}V^{11} & V^{12}\\ V^{12} & V^{22}\end{pmatrix},$$

where

$$\begin{aligned}V^{11} &= \frac{V_3}{V_1V_3-V_2^2},\\ V^{22} &= \frac{V_1}{V_1V_3-V_2^2},\\ V^{12} &= -\frac{V_2}{V_1V_3-V_2^2}.\end{aligned}$$

and similarly, V_*^{11}, V_*^{12}, V_*^{22} can be obtained from the observed Fisher information for the approximate likelihood equations (AMLE).

Using equations (6) and (7), we obtain components of observed Fisher information matrix for the MLE estimators as bellow

$$\begin{aligned}
\frac{\partial^2 \ln L}{\partial^2 \mu} &= -\frac{1}{\sigma^2}\sum_{i=1}^{n} g(z_i)\left[1+\theta G(z_i)\right],\\
\frac{\partial^2 \ln L}{\partial \mu \partial \sigma} &= -\frac{1}{\sigma^2}\sum_{i=1}^{n}\left[G(z_i)(1+\theta z_i g(z_i)) + g(z_i)(z_i-\theta)\right],\\
\frac{\partial^2 \ln L}{\partial^2 \sigma} &= -\frac{1}{\sigma^2}\left[-n + 2\sum_{i=1}^{n}\left(z_i(G(z_i)-\theta g(z_i)) + z_i^2 g(z_i)(1+\theta G(z_i))\right)\right].
\end{aligned}$$

Similarly, from the equations(10) and (11) for the AMLE estimators we obtain

$$\begin{aligned}
\frac{\partial^2 \ln L}{\partial^2 \mu} &\approx \frac{1}{\sigma^2}\sum_{i=1}^{n}(\beta_i+\theta\delta_i),\\
\frac{\partial^2 \ln L}{\partial \mu \partial \sigma} &\approx \frac{1}{\sigma^2}\sum_{i=1}^{n}\left[2z_i(\beta_i+\theta\delta_i) - (\alpha_i-\theta\gamma_i)\right],\\
\frac{\partial^2 \ln L}{\partial^2 \sigma} &\approx \frac{1}{\sigma^2}\left[n - 2\sum_{i=1}^{n}\left((\alpha_i-\theta\gamma_i)z_i + 3(\beta_i+\theta\delta_i)z_i^2\right)\right].
\end{aligned}$$

The approximate asymptotic variance-covariance matrices are valid only if asymptotic normality holds. Moreover, certain regularity conditions must be satisfied, see, for example, Ferguson(1996).
These conditions are:

- $(\mu,\sigma) \neq (\mu_0,\sigma_0)$ if and only if $L(\mu,\sigma) \neq L(\mu_0,\sigma_0)$;
- $L(\mu,\sigma)$ is continuous in (μ,σ) for almost all $\mathbf{y} = (Y_1, Y_2, \cdots, Y_n)$;
- there exists and integrable function $D(\mathbf{y})$ such that $|\ln L| < D(\mathbf{y})$ for all (μ,σ);
- the MLEs of (μ,σ) must be in the interior of $(-\infty,\infty)\times(0,\infty)$;
- $L(\mu,\sigma) > 0$ and is twice continuously differentiable in (μ,σ) in some neighborhood N of $(-\infty,\infty)\times(0,\infty)$;
- $\int \sup_{(\mu,\sigma)\in N} \left\| \begin{pmatrix} \partial L/\partial\mu \\ \partial L/\partial\sigma \end{pmatrix} \right\| d\mathbf{y} < \infty$;
- $\int \sup_{(\mu,\sigma)\in N} \left\| \begin{pmatrix} \frac{\partial^2 L}{\partial^2\mu} & \frac{\partial^2 L}{\partial\mu\partial\sigma} \\ \frac{\partial^2 L}{\partial\mu\partial\sigma} & \frac{\partial^2 L}{\partial^2\sigma} \end{pmatrix} \right\| d\mathbf{y} < \infty$;
- $E\left[\begin{pmatrix} \partial L/\partial\mu \\ \partial L/\partial\sigma \end{pmatrix}\begin{pmatrix} \partial L/\partial\mu \\ \partial L/\partial\sigma \end{pmatrix}'\right]$ at $(\mu,\sigma) = (\hat{\mu},\hat{\sigma})$exists and is non-singular;
- $E\left[\sup_{(\mu,\sigma)\in N} \left\| \begin{pmatrix} \frac{\partial^2 L}{\partial^2\mu} & \frac{\partial^2 L}{\partial\mu\partial\sigma} \\ \frac{\partial^2 L}{\partial\mu\partial\sigma} & \frac{\partial^2 L}{\partial^2\sigma} \end{pmatrix} \right\|\right] < \infty$;

4. Coverage probability and percentage point

To compute confidence intervals (CIs) for the location and scale parameters, one has to obtain pivotal quantities. since $\begin{pmatrix} \hat{\mu} \\ \hat{\sigma} \end{pmatrix}$, are asymptotic normally distributed, we have

$$P_1 = \frac{\hat{\mu}-\mu}{\hat{\sigma}\sqrt{V^{11}}} \sim N(0,1), \tag{15}$$

$$P_2 = \frac{\hat{\mu}-\mu}{\sigma\sqrt{V^{11}}} \sim N(0,1), \tag{16}$$

$$P_3 = \frac{\hat{\sigma}-\sigma}{\hat{\sigma}\sqrt{V^{22}}} \sim N(0,1). \tag{17}$$

as $n \to \infty$. Equations (15) and (17) holds because $\hat{\mu} \to N(\mu, \sigma^2 V^{11})$ and $\hat{\sigma} \to N(\sigma, \sigma^2 V^{22})$ in distribution and $\hat{\sigma} \to \sigma$ in probability respectively, and also equation(16) establishes because $\hat{\mu} \to N(\mu, \sigma^2 V^{11})$ as $n \to \infty$. Therefore $P_i(i = 1, 2, 3)$ can be taken as pivotal quantities because their distributions do not depend on the unknown location and scale parameters. Through Monte Carlo simulations, we computed the coverage probabilities 95% as bellow

$$P(-1.96 < P_i < 1.96) =, \quad i = 1, 2, 3 \tag{18}$$

via a simulation study. Similary, for the pivotal quantities $Q_i, i = 1, 2, 3$ based on the AMLES,

$$Q_1 = \frac{\tilde{\mu}-\mu}{\tilde{\sigma}\sqrt{V_*^{11}}}, \quad Q_2 = \frac{\tilde{\mu}-\mu}{\sigma\sqrt{V_*^{11}}}, \quad Q_3 = \frac{\tilde{\sigma}-\sigma}{\tilde{\sigma}\sqrt{V_*^{22}}} \tag{19}$$

We can also easily determine the asymptotic γ percentage points of the distributions of P_i and Q_i. These percentage points cannot be determined explicitly note that, for finite sample sizes. Hence, we used Monte Carlo simulations in order to determine the γ percentage point, m_γ , where, for example, for the distribution of P_1, we have

$$P\left[\frac{\hat{\mu}-\mu}{\hat{\sigma}\sqrt{V^{11}}} \leq m_\gamma\right] = \gamma.$$

Using values of m_γ, we obtain CIs for the parameters σ. For example, using the values of $m_{0.025}$ and $m_{0.975}$, we have

$$P\left\{m_{0.025} \leq \frac{\hat{\sigma}-\sigma}{\hat{\sigma}\sqrt{V^{22}}} \leq m_{0.975}\right\} = 0.95, \tag{20}$$

A 95% confidence interval for a σ is

$$\left(\hat{\sigma} - m_{0.975}\hat{\sigma}\sqrt{V^{22}}, \hat{\sigma} + m_{0.025}\hat{\sigma}\sqrt{V^{22}}\right). \tag{21}$$

The above confidence intervals obtains for simulated data in section (5). Note, this confidence intervals depends on the value of θ, we apply profile likelihood function and methods based on moments to estimate θ.

5. Simulation results

In this section, we discuss the results of a simulation study comparing the performance of the AMLEs with the corresponding MLEs. We consider the simulation of the values of a random variable X with pdf (3). One way to simulate values of X is to use the following representations due to the inverse transformation method:

1. Generate $U_i \sim Uniform(0, 1), \quad i = 1, \cdots, n$

2. set

$$X_i = \sigma \ln\left(\frac{1 + \frac{1}{\theta}\ln(U_i e^\theta + 1 - U_i)}{1 - \frac{1}{\theta}\ln(U_i e^\theta + 1 - U_i)}\right) + \mu.$$

We generated N=1000 sample form standard Poisson-half-logistic distribution of sample sizes n=(30, 40, 50, 100, 200) for $\theta = 3, 5, 10$. We computed the AMLEs from (12) and (14). The MLEs of the parameters were then obtained by solving the nonlinear Equations (6) and (7) using the Maple 14 package. Table (1) provides the average values of the estimates, their variances, and their MSEs. The values of the variances and covariances were determined by inverting the observed Fisher information matrix. Table(1) also provides the average values of the maximum profile likelihood estimates, $\widetilde{\theta}_p$ and $\widehat{\theta}_p$. From Table(1), we observe that the AMLEs and the MLEs are almost identical in terms of both bias and variance. The AMLEs are almost as efficient as the MLEs for all sample sizes. As the sample size n increases, the bias and variance of the estimators reduce appreciably. This is expected because of asymptotic normality. For more clarify, the MSEs of the MLEs and AMLEs computed over one thousand replications are plotted in Figure (1). From Figure (1) we observe that the MSEs for the AMLEs are only slightly less than those for MLEs. As expected, the MSEs for both estimators decrease with respect to n. Confidence intervals based on P_i's and Q_i' (i=1,2,3) for location and scale parameters are listed in Table (2). We have also obtained coverage probability of 95% confidence intervals for pivotal quantities based on the MLE and AMLE of location and scale parameters for some representative values of θ in Table(3).

6. Illustrative example

This section presents a numerical example to show the inference methods discussed in previous sections. As Lawless(2011 , p. 98) indicated, the following data arise in tests on the endurance of deep groove ball bearings. The observations are the number of million revolutions before failure for each of 23 ball bearings. The 23 failure times are `17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.40, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.`

To obtain an initial guess value of θ, since the skewness is independent of the location and scale parameters, we obtain the MME of θ by equating the sample skewness with the population skewness. Sample skewness is 0.92057 and $\gamma_1 = E\left[(\frac{X-\mu_X}{\sigma_X})^3\right]$ is population skewness. From a numerical solution of the following equation

$$0.92057 = E[(\frac{X-\mu_X}{\sigma_X})^3] = \frac{E(X^3) - 3E(X^2)E(X) + 2E^3(X)}{[E(X^2) - E^2(X)]^{\frac{3}{2}}},$$

$\theta = 4.9783$ is obtained, where $\mu_X = E(X)$, $\sigma_X = \sqrt{Var(X)}$ and $E(X^r)$, for $r = 1, 2, 3$, are the rth moment of X, Let $\theta = 4.9783$ be the initial guess of θ, the MLE and AMLE of μ and σ can be calculated. for $n = 23$ and $\theta = 4.9783$, we have

$$K = 52.31358 \qquad , L = -2.02135 \qquad , A = 242.66860, \qquad B = -8879.58774.$$

Then the equations (12) and (14),

$$\widetilde{\sigma} = \frac{A + \sqrt{A^2 - 4nB}}{2n} = 25.61991, \qquad and \qquad \widetilde{\mu} = K + L\widetilde{\sigma} = 0.52668.$$

The MLEs of μ and σ are then computed by solving the nonlinear equations (6) and (7). Using the Newton–Raphson iterative procedure: $\widehat{\mu} = 0.713527$ and $\widehat{\sigma} = 25.3670$. Here, the AMLEs were used as the starting values. We observe that the AMLEs are very close to the MLEs.

When the parameter θ is unknown, the maximum profile likelihood estimate $\widetilde{\theta}_p$ obtains by maximizing the loglikelihood $\ell(\widetilde{\mu}(\theta), \widetilde{\sigma}(\theta), \theta)$ with respect to θ. Figure (2) provides the plot of the profile likelihood function with respect to θ for the above data. From Figure (2), we observe that the maximum profile likelihood estimate of θ should be $\tilde{\theta}_p = 4.92139$.

7. Conclusion

In this paper, the Poisson-half-logistic distribution is considered as a combination of distributions. Characteristics of this distribution have been concerned and analyzed by several researchers. Estimates of the maximum likelihood

Table 1: Simulation results for the parameters estimation

θ	Method	n	μ	σ	θ_p	$var(\mu)$	$var(\sigma)$	$MSE(\mu)$	$MSE(\sigma)$
$\theta = 3$	MLE	30	0.33731	0.89011	2.46025	0.11211	0.01894	0.22589	0.03101
	AMLE	30	0.33730	0.89010	2.46024	0.11210	0.01894	0.22587	0.03102
	MLE	40	0.31399	0.89347	2.39368	0.09277	0.01547	0.19135	0.02682
	AMLE	40	0.31105	0.89784	2.45274	0.08571	0.01450	0.18246	0.02493
	MLE	50	0.31188	0.89181	2.38981	0.07392	0.01228	0.17118	0.02395
	AMLE	50	0.29280	0.90390	2.45520	0.06969	0.01181	0.15542	0.02104
	MLE	100	0.29258	0.90328	2.40005	0.03745	0.00626	0.12305	0.01561
	AMLE	100	0.28332	0.90866	2.44255	0.03534	0.00599	0.11561	0.01433
	MLE	200	0.28765	0.90565	2.39999	0.01878	0.00314	0.10152	0.01204
	AMLE	200	0.28011	0.90950	2.43664	0.01777	0.00301	0.09623	0.01120
$\theta = 5$	MLE	30	0.15399	0.94779	4.85989	0.13062	0.02017	0.15758	0.02289
	AMLE	30	0.14242	0.95597	4.90413	0.12675	0.01945	0.15012	0.02139
	MLE	40	0.11497	0.96047	4.86373	0.09972	0.01545	0.11294	0.01701
	AMLE	40	0.10576	0.96715	4.89801	0.09748	0.01500	0.10867	0.01608
	MLE	50	0.10199	0.96429	4.86372	0.08027	0.01243	0.09067	0.01370
	AMLE	50	0.09420	0.96978	4.89214	0.07859	0.01211	0.08746	0.01302
	MLE	100	0.08657	0.97144	4.86349	0.04055	0.00627	0.04804	0.00708
	AMLE	100	0.08225	0.97434	4.87896	0.03994	0.00616	0.04671	0.00682
	MLE	200	0.07867	0.97456	4.86115	0.02038	0.00315	0.02657	0.00379
	AMLE	200	0.07629	0.97608	4.86998	0.02015	0.00310	0.02597	0.00367
$\theta = 10$	MLE	30	0.12445	0.96532	10.00923	0.17657	0.01986	0.19206	0.02106
	AMLE	30	0.13459	0.96537	10.06402	0.17675	0.01949	0.19486	0.02069
	MLE	40	0.06408	0.98069	10.01104	0.13597	0.01532	0.14008	0.01569
	AMLE	40	0.06194	0.98455	10.04823	0.13616	0.01509	0.13999	0.01533
	MLE	50	0.04689	0.98506	10.00871	0.10951	0.01232	0.11171	0.01254
	AMLE	50	0.04472	0.98826	10.04146	0.10945	0.01216	0.11145	0.01229
	MLE	100	0.02442	0.99281	10.00474	0.05536	0.00622	0.05596	0.00627
	AMLE	100	0.02257	0.99463	10.02489	0.05518	0.00617	0.05569	0.00619
	MLE	200	0.01184	0.99667	9.99996	0.02782	0.00312	0.02796	0.00313
	AMLE	200	0.01051	0.99769	10.01237	0.02772	0.00310	0.02783	0.00310

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Table 2: (2.5,97.5) percentage points of the pivotal quantities based on the MLEs and AMLEs

θ	n	P_1 / Q_1	P_2 / Q_2	P_3 / Q_3
	30	(-0.1816, 3.3348)	(-0.1999, 2.2677)	(-3.8679, 0.7152)
		(-0.2184, 2.9705)	(-0.2770, 2.1912)	(-3.2801, 0.8532)
	40	(-0.3041, 2.8761)	(-0.3022, 2.2952)	(-3.2885, 0.9046)
		(-0.3107, 2.9106)	(-0.3147, 2.3867)	(-3.3107, 0.9938)
$\theta = 3$	50	(-0.1326, 2.7729)	(-0.1431, 2.2713)	(-3.1388, 0.6077)
		(-0.1376, 2.8396)	(-0.2023, 2.3413)	(-3.1165, 0.6654)
	100	(0.0771, 3.0656)	(0.0778, 2.4848)	(-3.1345, 0.5682)
		(0.0715, 3.0913)	(0.0637, 2.5484)	(-3.0984, 0.6346)
	200	(0.4798, 3.8803)	(0.4491, 3.1976)	(-4.0742, 0.0701)
		(0.4867, 3.9426)	(0.4552, 3.2664)	(-4.0813, 0.1640)
	30	(-1.0361, 2.7501)	(-1.1910, 1.9765)	(-2.6462, 1.1687)
		(-1.0647, 2.7953)	(-1.2018, 2.0435)	(-2.6432, 1.2289)
	40	(-1.1985, 2.7306)	(-1.3766, 2.09146)	(-2.6562, 1.2734)
		(-1.1925, 2.7439)	(-1.3808, 2.0989)	(-2.5757, 1.3644)
$\theta = 5$	50	(-1.0533, 2.4424)	(-1.1297, 1.9828)	(-2.6737, 1.1311)
		(-1.0473, 2.4267)	(-1.1713, 1.9830)	(-2.6886, 1.1621)
	100	(-1.2655, 2.2032)	(-1.4267, 1.9011)	(-2.2756, 1.4007)
		(-1.2457, 2.2005)	(-1.4451, 1.8921)	(-2.2589, 1.4441)
	200	(-1.2616, 2.6434)	(-1.3622, 2.3623)	(-2.8169, 1.3187)
		(-1.2731, 2.6354)	(-1.3519, 2.3801)	(-2.8132, 1.3454)
	30	(-1.3329, 2.7519)	(-1.5948, 2.0755)	(-2.50230, 1.4466)
		(-1.3245, 2.7563)	(-1.5885, 2.05997)	(-2.4942, 1.4847)
	40	(-1.3854, 2.7667)	(-1.7322, 2.1365)	(-2.5664, 1.5163)
		(-1.3976, 2.7724)	(-1.7496, 2.1578)	(-2.5589, 1.5252)
$\theta = 10$	50	(-1.3892, 2.3558)	(-1.5843, 1.9419)	(-2.4604, 1.3578)
		(-1.3962, 2.3492)	(-1.5981, 1.9395)	(-2.4578, 1.4335)
	100	(-1.7443, 2.0619)	(-2.0354, 1.7978)	(-2.1167, 1.6824)
		(-1.7659, 2.0860)	(-2.0623, 1.8136)	(-2.0738, 1.7495)
	200	(-1.9235, 2.4619)	(-2.1496, 2.1106)	(-2.4104, 1.7494)
		(-1.9257, 2.4702)	(-2.1344, 2.1141)	(-2.4056, 1.7495)

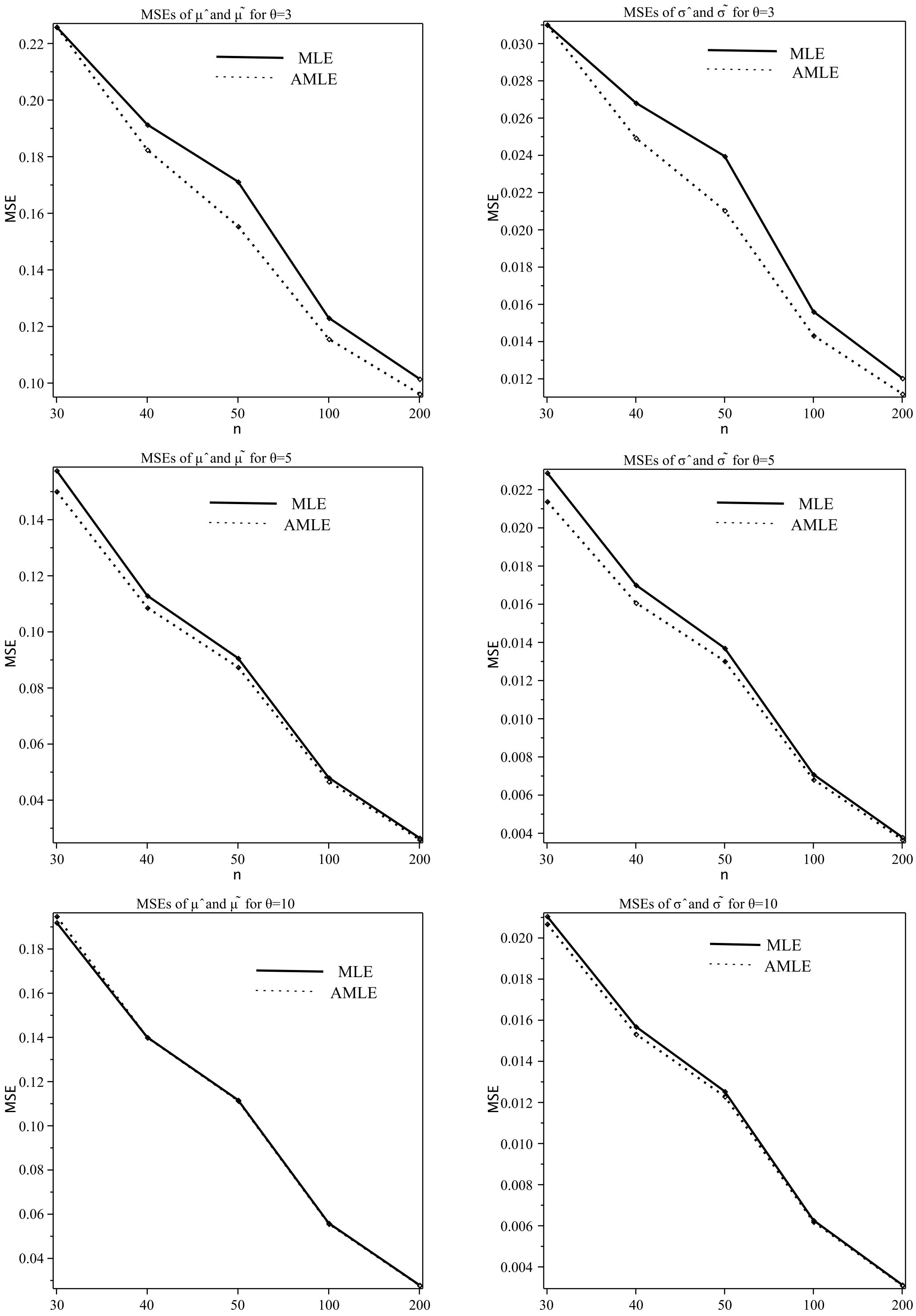

Figure 1: MSEs of μ and σ for different value of θ.

of this distribution, whether the shape parameter is known or unknown, do not have an explicit form and need to use numerical methods to estimate the parameters. These numerical methods are sensitive to the initial value of the parameters, and the results might be changed with a slight change in initial values.

In this study, assuming the parameter θ is known, the approximate maximum likelihood method for estimating the parameters is proposed. This method's obtained AMLE of location and scale parameters have an explicit form, and their values can be obtained quickly.

Table 3: Coverage probabilities of 95% confidence intervals for the pivotal quantities based on the MLEs and AMLEs of parameters.

θ	n	P_1	Q_1	P_2	Q_2	P_3	Q_3
	30	0.8510	0.8460	0.9450	0.9260	0.8400	0.8300
	40	0.8520	0.8440	0.9410	0.9299	0.8220	0.8310
$\theta = 3$	50	0.8500	0.8310	0.9260	0.9220	0.8360	0.8330
	100	0.7010	0.6980	0.8230	0.8100	0.7500	0.7580
	200	0.4490	0.4480	0.5430	0.5370	0.5870	0.6040
	30	0.8950	0.8950	0.9600	0.9550	0.9050	0.9050
	40	0.9200	0.9200	0.9560	0.9560	0.9160	0.9220
$\theta = 5$	50	0.9260	0.9280	0.9540	0.9510	0.9210	0.9250
	100	0.9250	0.9250	0.9510	0.9480	0.9300	0.9330
	200	0.9060	0.9070	0.9330	0.9330	0.9190	0.9220
	30	0.8880	0.8880	0.9400	0.9400	0.9140	0.9140
	40	0.9300	0.9310	0.9430	0.9430	0.9310	0.9320
$\theta = 10$	50	0.9300	0.9290	0.9540	0.9540	0.9360	0.9370
	100	0.9360	0.9360	0.9420	0.9440	0.9450	0.9460
	200	0.9340	0.9330	0.9470	0.9460	0.9450	0.9460

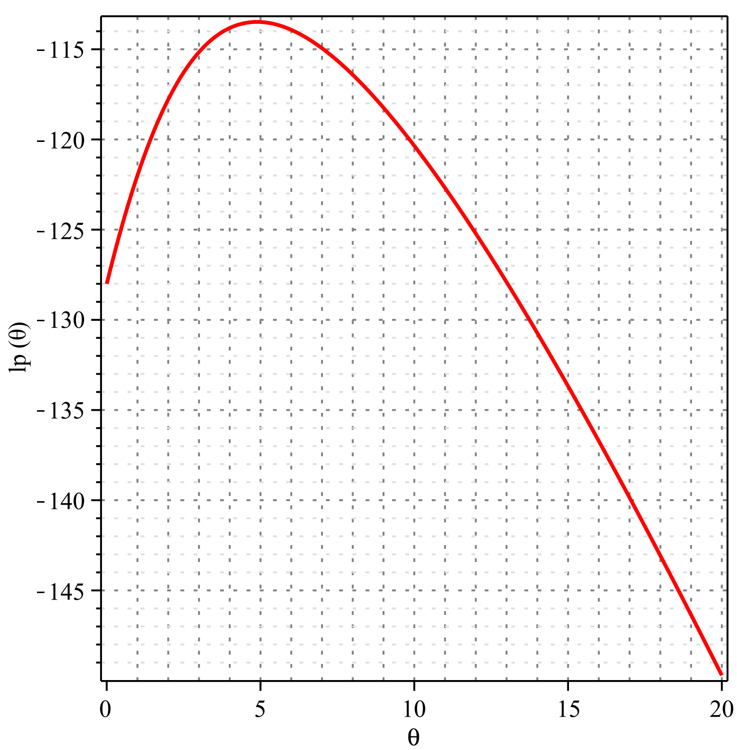

Figure 2: Plot of profile likelihood function, $\ell_p(\theta)$

Data from the Poisson-half-logistic distribution were simulated to compare the AMLE and the MLE methods to estimate the parameters. The results presented in Table (1) showed that the estimates by the MLE and the AMLE methods are very close, and the values of variance and MSE parameters decrease with increasing sample size.
A practical example was also provided to illustrate the efficiency of the proposed method. In this example, to obtain the parameter θ, we used the equalizing of data skewness and distribution skewness. By placing it as the known value of θ, we estimate the location and scale parameters. It is observed that the results are very close to each other, and distribution also fits convenient.
The other methods of estimations, including percentile, least squares and weighted least squares can be suggested. The idea of this study can also be extended to many distributions.

References

1. Abdel-Hamid, A. H. (2016). Properties, estimations and predictions for a poisson-half-logistic distribution based on progressively type-ii censored samples. *Applied Mathematical Modelling*, 40(15-16):7164–7181.
2. Arnold, B. C. and Balakrishnan, N. (2012). *Relations, bounds and approximations for order statistics*, volume 53. Springer Science & Business Media.
3. Asgharzadeh, A. (2006). Point and interval estimation for a generalized logistic distribution under progressive type ii censoring. *Communications in Statistics-Theory and Methods*, 35(9):1685–1702.
4. Asgharzadeh, A., Esmaily, L., and Nadarajah, S. (2013). Approximate mles for the location and scale parameters of the skew logistic distribution. *Statistical Papers*, 54(2):391–411.
5. Balakrishnan, N. and Asgharzadeh, A. (2005). Inference for the scaled half-logistic distribution based on progressively type-ii censored samples. *Communications in Statistics-Theory and Methods*, 34(1):73–87.
6. Balakrishnan, N. and Hossain, A. (2007). Inference for the type ii generalized logistic distribution under progressive type ii censoring. *Journal of Statistical Computation and Simulation*, 77(12):1013–1031.
7. David, H. A. and Nagaraja, H. N. (2004). *Order statistics*. John Wiley & Sons.
8. Ferguson, T. (1996). A course in large sample theory chapman and hall. *New York*.
9. Gui, W. and Guo, L. (2018). Statistical inference for the location and scale parameters of the skew normal distribution. *Indian Journal of Pure and Applied Mathematics*, 49(4):633–650.
10. Lawless, J. F. (2011). *Statistical models and methods for lifetime data*. John Wiley & Sons.
11. Rasekhi, M., Saber, M. M., Hamedani, G., Abd El-Raouf, M., Aldallal, R., and Gemeay, A. M. (2022). Approximate maximum likelihood estimations for the parameters of the generalized gudermannian distribution and its characterizations. *Journal of Mathematics*, 2022.