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A New Flexible Probability Model: Theory, Estimation and Modeling Bimodal Left Skewed Data

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Abstract
In this work, we introduced a new three-parameter Nadarajah-Haghighi model. We derived explicit expressions for some of its statistical properties. The Farlie Gumbel Morgenstern, modified Farlie Gumbel Morgenstern, Clayton, Renyi and Ali-Mikhail-Haq copulas are used for deriving some bivariate type extensions. We consider maximum likelihood, Cramér-vonMises, ordinary least squares, weighted least squares, Anderson Darling, right tail Anderson Darling and left tail Anderson Darling estimation procedures to estimate the unknown model parameters. Simulation study for comparing estimation methods is performed. An application for comparing methods as also presented. The maximum likelihood estimation method is the best method. However, the other methods performed well. Another application for comparing the competitive models is investigated.

Key Words: Nadarajah Haghighi model; Farlie Gumbel Morgenstern, Anderson Darling, Maximum Likelihood Estimation, Ordinary Least Squares, Generating Function, Moments.

Mathematical Subject Classification: 62N01; 62N02; 62E10.

1. Introduction

Among several parametric probability models, the exponential (Exp) distribution is the most widely applied statistical model. One of the reasons for its importance is that the Exp distribution has constant hazard rate function (HRF). A random variable (RV) T is said to have the Exp distribution if its cumulative distribution function (CDF) is given by

$$G_\lambda(t) = 1 - exp(-\lambda t) \ |_{(t>0)}. \tag{1}$$

The Exp distribution is the probability distribution of the time between events in a Poisson point process in which events occur continuously and independently at a constant average rate. It is a particular case of the well-known gamma (Gam) distribution. The r^{th} ordinary and incomplete moments, the moment generating function (MGF), Quantile function (QF) and several other properties of the Exp distribution can be expressed in terms of elementary functions; see, for example Balakrishnan and Basu (1995) and Marshall and Olkin (2007). Recently, a new useful generalization of the Exp distribution as an alternative to the Exp, Weibull (W) Gam, and exponentiated exponential (ExpExp) distributions was proposed by Nadarajah and Haghighi (2011). A RV W is said to have the Weibull Nadarajah Haghighi (WNH) distribution if its PDF is given as

$$h_a(w) = a(1 + w)^{a-1}\nabla_a(w) \ |_{(w\geq 0)}, \tag{2}$$

where $\nabla_a(w) = exp[1 - (1 + w)^a]$ refers to the reliability function of the NH distribution. Nadarajah and Haghighi (2011) proved that the PDF in (2) always has the zero mode. A RV W is said to have the GWNH distribution if its CDF is given as

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$$F_{\underline{\delta}}(w) = (1 - exp\{-[\nabla_a^{-1}(w) - 1]^b\})^\gamma, \tag{3}$$

where $\underline{\boldsymbol{\delta}} = (\gamma, b, a)$ is the vector of parameters. Therefore, the PDF of the GWNH model reduces to

$$f_{\underline{\delta}}(w) = b\gamma a \frac{\nabla_a^{-b}(w)[1 - \nabla_a(w)]^{b-1}}{(1 + w)^{1-a}\, exp[\nabla_a^{-1}(w) - 1]^b}(1 - exp\{-[\nabla_a^{-1}(w) - 1]^b\})^{\gamma-1}. \tag{4}$$

The additional shape parameters γ and b can allow us to study the tail behavior of the PDF (4) with greater flexibility (see Figure 1 and Figure 2). Further, the GWNH due to its wide flexibility in accommodating all shapes of the HRF (monotonically increasing, monotonically decreasing, **J**-shape, upside-down-bathtub, bathtub and constant) as given in Figure 2.

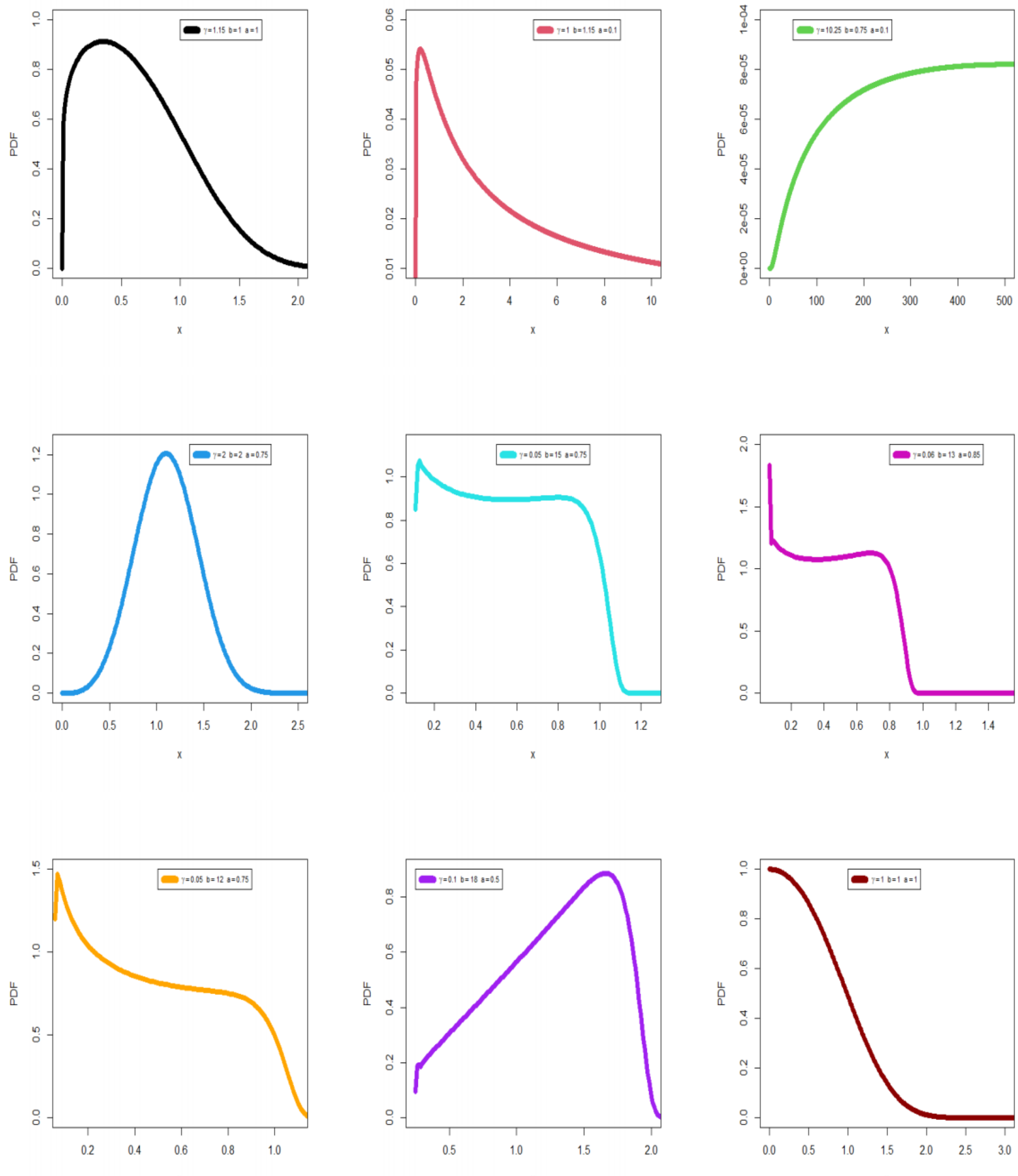

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Figure 1: Plots of the GWNH PDF for selected parameter values.

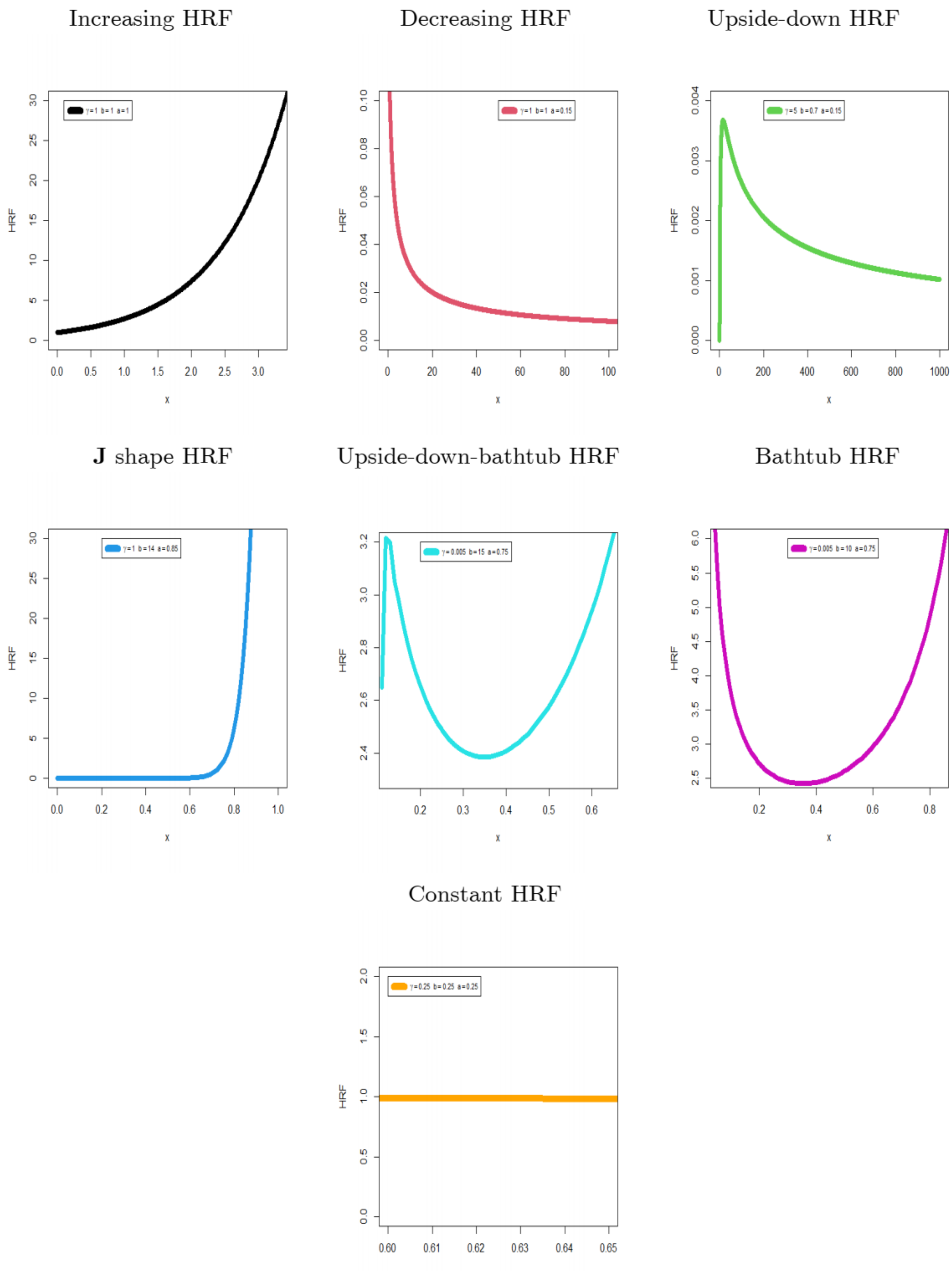

Figure 2: Plots of the GWNH HRF for selected parameter values.

Several extensions of the NH model can be found in the statistical literature such as the exponentiated NH model (ENH) model (Lemonte (2013)), gamma NH (GNH) (Ortega et al. (2015)), Poisson gamma NH (PGNH) (Ortega et al. (2015)), transmuted NH (TNH) (Ahmed et al. (2015)), Kumaraswamy NH (KNH) (Lima (2015)), The modified NH (MNH) (El Damcese and Ramadan (2015)), Marshall-Olkin NH (MONH) (Lemonte et al. (2016)), The beta NH distribution (Cicero et al. (2016)), Topp-Leone NH (TLNH) (Yousof and Korkmaz (2017)), Lindley NH (LNH) (Yousof el al. (2017)), extended exponentiated NH (ExENH) (Alizadeh et al. (2018)), beta NH (BNH) (Dias (2016)), inverted NH (I-NH) (Tahir et al. (2018)), the Burr X NH (BXNH) (Elsayed and Yousof (2019)), the odd NH-G (NH-G) family (Nascimento et al. (2019)), the Topp-Leone exponentiated NH model for modeling extreme values (Almazah et al. (2021)) and the Lindely exponentiated NH (Shehata and Yousof (2021)). In this paper, we present a new three parameter NH version called the generalized Weibull Nadarajah Haghighi (GWNH) distribution.

Table 1 gives some special cases of the GWNH model. The HRF of the GWNH model can be derived from $h_{\underline{\boldsymbol{\delta}}}(w)=f_{\underline{\boldsymbol{\delta}}}(w)/\left[1-F_{\underline{\boldsymbol{\delta}}}(w)\right]$ and henceforth the RV W having CDF as in (3) is denoted by $W\sim$ GWNH ($\underline{\boldsymbol{\delta}}$).

Table 1: Some special cases of the GWNH model.

γ	b	a	Reduced Model
1			WNH
1		1	WExp
	2		BXNH
1	2		Reighley-NH
	2	1	Burr X-Exp
1	1		quasi W-NH
	1		quasi GWNH
	1	1	quasi GWExp
γ	b	a	reduced model

2. Copula and related results

We derive some new bivariate GWNH (BGWNH) type distributions using Farlie Gumbel Morgenstern (FGM) Copula (see Gumbel (1958), Gumbel (1960 and 1961), Johnson and Kotz (1975) and Johnson and Kotz (1977)), modified FGM Copula (see Rodriguez-Lallena and Ubeda-Flores (2004)), Clayton Copula and Renyi's entropy (Pougaza and Djafari (2011)) and Ali-Mikhail-Haq copula (see Ali et al(1978)). For other copulas and related derivations see Ali et al. (2020a,b), Elgohari et al. (2021), Elgohari and Yousof (2020a,b and 2021), Shehata and Yousof (2021a,b), Shehata et al. (2021) and Aboraya et al. (2022). The Multivariate GWNH (MGWNH) type is also presented. However, future works may be allocated to the study of these new models. First, we consider the joint CDF (JCDF) of the FGM family where

$$C_{\Upsilon}(m,v)=mv(1+\Upsilon m^{\cdot}v^{\cdot})|_{m^{\cdot}=1-m,v^{\cdot}=1-v},$$

and the marginal function $m=F_1$, $v=F_2$, $\Upsilon\in(-1,1)$ is a dependence parameter and for every $m,v\in(0,1)$, $C(m,0)=C(0,v)=0$ which is "grounded minimum" and $C(m,1)=m$ and $C(1,v)=v$ which is "grounded maximum", $C(m_1,v_1)+C(m_2,v_2)-C(m_1,v_2)-C(m_2,v_1)\geq 0$.

2.1 Via FGM family

A Copula is continuous in m and v ; actually, it satisfies the stronger Lipschitz condition, where

$$|C(m_2,v_2)-C(m_1,v_1)|\leq|m_2-m_1|+|v_2-v_1|.$$

For $0\leq m_1\leq m_2\leq 1$ and $0\leq v_1\leq v_2\leq 1$, we have

$$Pr(m_1\leq m\leq m_2,v_1\leq v\leq v_2)=C(m_1,v_1)+C(m_2,v_2)-C(m_1,v_2)-C(m_2,v_1)\geq 0.$$

Then, setting $m^{\cdot}=1-F_{\underline{\boldsymbol{\delta}}_1}(m_1)|_{[m^{\cdot}=(1-m)\in(0,1)]}$ and $v^{\cdot}=1-F_{\underline{\Phi}_2}(m_2)|_{[v^{\cdot}=(1-v)\in(0,1)]}$, we can esaily get the JCDF of the GWNH using the FGM family as

$$C_{\Upsilon}(m,v)=\left(1-exp\left\{-\left[\nabla_{a_1}^{-1}(m)-1\right]^{b_1}\right\}\right)^{\gamma_1}\left(1-exp\left\{-\left[\nabla_{a_2}^{-1}(v)-1\right]^{b_2}\right\}\right)^{\gamma_2}$$
$$\times\left(1+\Upsilon\left\{\begin{array}{l}\left[1-\left(1-exp\left\{-\left[\nabla_{a_1}^{-1}(m)-1\right]^{b_1}\right\}\right)^{\gamma_1}\right]\\ \times\left[1-\left(1-exp\left\{-\left[\nabla_{a_2}^{-1}(v)-1\right]^{b_2}\right\}\right)^{\gamma_2}\right]\end{array}\right\}\right).$$

2.2 Via modified (MFGM) family

The MFGM copula is defined as $C_\Upsilon(m,v) = mv[1+\Upsilon\omega(m)C(v)]|_{\Upsilon\in(-1,1)}$ or $C_\Upsilon(m,v) = mv + \Upsilon W_m K_v|_{\Upsilon\in(-1,1)}$, where $W_m = m\omega(m)$ and $K_v = vC(v)$ and $\omega(m)$ and $C(v)$ are two continuous functions on $(0,1)$ with $\omega(0)=\omega(1)=C(0)=C(1)=0$. Let

$$a_1(W_m) = inf\left\{W_m{:}\frac{\partial}{\partial m}W_m\right\}|_{\kappa_{1,m}} < 0, a_2(W_m) = sup\left\{W_m{:}\frac{\partial}{\partial m}W_m\right\}|_{\kappa_{1,m}} < 0,$$

$$b_1(K_v) = inf\left\{K_v{:}\frac{\partial}{\partial v}K_v\right\}|_{\kappa_{2,v}} > 0, b_2(K_v) = sup\left\{K_v{:}\frac{\partial}{\partial v}K_v\right\}|_{\kappa_{2,v}} > 0.$$

Then,

$$1 \le min\{a_1(W_m)a_2(W_m), b_1(K_v)b_2(K_v)\} < \infty,$$

where

$$m\frac{\partial}{\partial m}\omega(m) = \frac{\partial}{\partial m}W_m - \omega(m),$$

$$\kappa_{1,m} = \left\{m{:}\, m \in (0,1)|_{\frac{\partial}{\partial m}W_m \ \text{exists}}\right\}$$

and

$$\kappa_{2,v} = \left\{v{:}\, v \in (0,1)|_{\frac{\partial}{\partial v}K_v \ \text{exists}}\right\}.$$

The MFGM family can be used for obtaining the following new versions:

Type-I MFGM type

Consider the functional forms for $W_m = m\omega(m)$ and $K_v = vC(v)$, the BGWNH-MFGM (Type-I) can be derived from

$$C_\Upsilon(m,v) = \left(1-exp\left\{-\left[\nabla_{a_1}^{-1}(m)-1\right]^{b_1}\right\}\right)^{\gamma_1}\left(1-exp\left\{-\left[\nabla_{a_2}^{-1}(v)-1\right]^{b_2}\right\}\right)^{\gamma_2}$$
$$+\Upsilon\left\{\begin{array}{c}\times\left(1-exp\left\{-\left[\nabla_{a_1}^{-1}(m)-1\right]^{b_1}\right\}\right)^{\gamma_1}\\ \times\left(1-exp\left\{-\left[\nabla_{a_2}^{-1}(v)-1\right]^{b_2}\right\}\right)^{\gamma_2}\\ \times\left[1-\left(1-exp\left\{-\left[\nabla_{a_1}^{-1}(m)-1\right]^{b_1}\right\}\right)^{\gamma_1}\right]\\ \times\left[1-\left(1-exp\left\{-\left[\nabla_{a_2}^{-1}(v)-1\right]^{b_2}\right\}\right)^{\gamma_2}\right]\end{array}\right\}|_{\Upsilon\in(-1,1)}.$$

Type-II MFGM type

Let $\omega(m)$ and $C(v)$ be two functional forms satisfying all the conditions stated earlier where $\omega(m)^{\cdot}|_{(\Upsilon_1>0)} = m^{\Upsilon_1}(1-m)^{1-\Upsilon_1}$ and $C(v)^{\cdot}|_{(\Upsilon_2>0)} = v^{\Upsilon_2}(1-v)^{1-\Upsilon_2}$. Then, the corresponding BGWNH-MFGM **(**Type-**II)** can be derived from $C_{\Upsilon,\Upsilon_1,\Upsilon_2}(m,v) = mv[1+\Upsilon\omega(m)^{\cdot}\,C(v)^{\cdot}]$. Thus

$$C_{\Upsilon,\Upsilon_1,\Upsilon_2}(m,v) = \left(1-exp\left\{-\left[\nabla_{a_1}^{-1}(m)-1\right]^{b_1}\right\}\right)^{\gamma_1}\left(1-exp\left\{-\left[\nabla_{a_2}^{-1}(v)-1\right]^{b_2}\right\}\right)^{\gamma_2}$$
$$\times\left(1+\Upsilon\left\{\begin{array}{c}\times\left(1-exp\left\{-\left[\nabla_{a_1}^{-1}(m)-1\right]^{b_1}\right\}\right)^{\gamma_1\Upsilon_1}\\ \times\left(1-exp\left\{-\left[\nabla_{a_2}^{-1}(v)-1\right]^{b_2}\right\}\right)^{\gamma_2\Upsilon_2}\\ \times\left[1-\left(1-exp\left\{-\left[\nabla_{a_1}^{-1}(m)-1\right]^{b_1}\right\}\right)^{\gamma_1}\right]^{1-\Upsilon_1}\\ \times\left[1-\left(1-exp\left\{-\left[\nabla_{a_2}^{-1}(v)-1\right]^{b_2}\right\}\right)^{\gamma_2}\right]^{1-\Upsilon_2}\end{array}\right\}\right).$$

Type-III MFGM type

Let $W(m)=m[log(1+m^{\cdot})]$ and $K(v)=v[log(1+v^{\cdot})]$ for all $\omega(m)$ and $C(v)$ which satisfy all the conditions stated earlier. In this case, one can also derive a closed form expression for the associated CDF of the BGWNH-MFGM (Type-**III**) from $C_{\Upsilon}(m,v)=mv\big(1+\Upsilon W(m)\,K(v)\big)$. Then

$$C_{\Upsilon}(m,v)=\left(1-exp\left\{-\left[\nabla_{a_1}^{-1}(m)-1\right]^{b_1}\right\}\right)^{\gamma_1}\left(1-exp\left\{-\left[\nabla_{a_2}^{-1}(v)-1\right]^{b_2}\right\}\right)^{\gamma_2}$$
$$\times\left(1+\Upsilon\left\{\begin{array}{c}\times\left(1-exp\left\{-\left[\nabla_{a_1}^{-1}(m)-1\right]^{b_1}\right\}\right)^{\gamma_1}\\ \times\left(1-exp\left\{-\left[\nabla_{a_2}^{-1}(v)-1\right]^{b_2}\right\}\right)^{\gamma_2}\\ \times\left[log\left(1+\left\{1-\left(1-exp\left\{-\left[\nabla_{a_1}^{-1}(m)-1\right]^{b_1}\right\}\right)^{\gamma_1}\right\}\right)\right]\\ \times\left[log\left(1+\left\{1-\left(1-exp\left\{-\left[\nabla_{a_2}^{-1}(v)-1\right]^{b_2}\right\}\right)^{\gamma_2}\right\}\right)\right]\end{array}\right\}\right).$$

2.3 Ali-Mikhail-Haq (A-M-H) copula

Following Ali et al. (1987), the A-M-H copula can be expressed as

$$C(m,v)=vm/(1-\Upsilon\overline{vm})|_{\Upsilon\in[-1,1]}.$$

Then for $m,v\in(0,1)$, the J-CDF of the BGWNH can be written as

$$C(m,v)=\frac{\left(1-exp\left\{-\left[\nabla_{a_1}^{-1}(m)-1\right]^{b_1}\right\}\right)^{\gamma_1}\left(1-exp\left\{-\left[\nabla_{a_2}^{-1}(v)-1\right]^{b_2}\right\}\right)^{\gamma_2}}{1-\Upsilon\left\{\begin{array}{l}\left[1-\left(1-exp\left\{-\left[\nabla_{a_1}^{-1}(m)-1\right]^{b_1}\right\}\right)^{\gamma_1}\right]\\ \times\left[1-\left(1-exp\left\{-\left[\nabla_{a_2}^{-1}(v)-1\right]^{b_2}\right\}\right)^{\gamma_2}\right]\end{array}\right\}}|_{\Upsilon\in[-1,1]}.$$

2.4 BGWNH and MGWNH type via Clayton Copula

The Clayton Copula can be considered as $C(v_1,v_2)=[(1/v_1)^{\Upsilon}+(1/v_2)^{\Upsilon}-1]^{-\Upsilon^{-1}}|_{\Upsilon\in(0,\infty)}$. Setting $v_1=F_{\underline{\delta}_1}(m)$ and $v_2=F_{\underline{\Phi}_2}(m)$, the BGWNH type can be derived as

$$C(v_1,v_2)=\left\{\begin{array}{c}\left(1-exp\left\{-\left[\nabla_{a_1}^{-1}(m)-1\right]^{b_1}\right\}\right)^{-\Upsilon\gamma_1}\\ +\left(1-exp\left\{-\left[\nabla_{a_2}^{-1}(v)-1\right]^{b_2}\right\}\right)^{-\Upsilon\gamma_2}-1\end{array}\right\}^{-\Upsilon^{-1}}|_{\Upsilon\in(0,\infty)}$$

Similarly, the MGWNH can be derived from

$$C(v_i)=\left[\sum_{i=1}^{d}\left(1-exp\left\{-\left[\nabla_{a_i}^{-1}(v_i)-1\right]^{b_i}\right\}\right)^{-\Upsilon\gamma_i}+1-d\right]^{-\Upsilon^{-1}}.$$

2.5 BGWNH type via Renyi's entropy copula

Due to Pougaza and Djafari (2011), the BGWNH type via Renyi's entropy copula can be derived using $C(m,v)=m_2m+m_1v-m_1m_2$. Then, the associated BGWNH can be expressed as

$$C(m_1,m_2)=m_2\left(1-exp\left\{-\left[\nabla_{a_1}^{-1}(m_1)-1\right]^{b_1}\right\}\right)^{\gamma_1}$$
$$+m_1(1-exp\{-[\nabla_{a2}^{-1}(m_2)-1]^{b_2}\})^{\gamma_2}-m_1m_2.$$

3. Mathematical properties

3.1 Linear representation

In this section, we provide a useful linear representation for the GWNH density function. If $|\phi|<1$ and $b>0$ is a real non-integer, the power series holds

$$\left(1-\frac{\phi_1}{\phi_2}\right)^{\phi_3}=\sum_{\phi_4=0}^{\infty}(-1)^{\phi_4}\frac{\Gamma(1+\phi_3)}{\phi_4!\,\Gamma(\phi_3-\phi_4+1)}\left(\frac{\phi_1}{\phi_2}\right)^{\phi_4}.$$

Applying (5) to the last term in (4) gives

$$f(w)=b\gamma a(1+w)^{a-1}\nabla_a(w)\frac{[1-\nabla_a(w)]^{b-1}}{\nabla_a^{b+1}(w)}$$

$$\times\sum_{\ell_1=0}^{\infty}\frac{(-1)^{\ell_1}\Gamma(\gamma)}{\ell_1!\,\Gamma(\gamma-\ell_1)}\underbrace{exp\left(-(\ell_1+1)\left\{\frac{[1-\nabla_a(w)]}{\nabla_a(w)}\right\}^b\right)}_{A(w)}.$$

Expanding the quantity $A(w)$ in power series, we can write

$$A(w)=\sum_{k_1=0}^{\infty}\frac{(-1)^{k_1}(\ell_1+1)^{k_1}}{k_1!\,\nabla_a^{k_1 b}(w)}[1-\nabla_a(w)]^{k_1 b}.$$

Inserting the above expression of $A(w)$ in (6), the GWNH density reduces to

$$f(w)=\ a(1+w)^{a-1}\nabla_a(w)\sum_{\ell_1,k_1=0}^{\infty}\frac{(-1)^{k_1+\ell_1}\gamma b\Gamma(\gamma)(\ell_1+1)^{k_1}[1-\nabla_a(w)]^{(1+k_1)b-1}}{\ell_1!\,k_1!\,\Gamma(\gamma-\ell_1)\nabla_a^{(1+k_1)b+1}(w)}.$$

Using the generalized binomial expansion to $\nabla_a^{-[(1+k_1)b+1]}(w)$ the inserting result in (7), the GWNH density can be expressed as an infinite linear combination of ENH density functions

$$f(w)=\sum_{k_1,k_2=0}^{\infty}C_{(k_1,k_2)}\ \pi_\theta(w;a),$$

where $\theta=(1+k_1)b+k_2$ and

$$\pi_\theta(w;a)=\theta\,\underbrace{a(1+w)^{a-1}\nabla_a(w)}_{h_a(w)}\underbrace{[1-\nabla_a(w)]^{\theta-1}}_{[H_a(w)]^{\theta-1}}$$

is the ENH PDF with power parameter θ and

$$C_{(k_1,k_2)}=\sum_{\ell_1=0}^{\infty}\frac{(-1)^{k_1+\ell_1}\gamma b(\ell_1+1)^{k_1}\Gamma(\gamma)\Gamma(\theta+1)}{\ell_1!\,k_1!\,k_2!\,\theta\Gamma(\gamma-\ell_1)\Gamma([1+k_1]b+1)}.$$

Equation (8) reveals that the density of W can be expressed as a linear combination of ENH densities. So, several mathematical properties of the new family can be obtained by knowing those of the eENH distribution. Similarly, the CDF of the GWNH model can also be expressed as a linear combination of ENH CDFs given by

$$F(w)=\sum_{k_1,k_2=0}^{\infty}C_{(k_1,k_2)}\ \Pi_\theta(w;a),$$

where $\Pi_\theta(w;a)=[H_a(w)]^\theta$ is the ENH CDF with power parameter θ .

3.2 Moments

The r^{th} moment of w , say μ_r' , follows from equation (10) as

$$\mu_r'=E(w^r)=\sum_{k_1,k_2,\ell_1=0}^{\infty}\sum_{\ell_2=0}^{r}C_{(k_1,k_2)}\,\xi_{\ell_1,\ell_2}^{\{\theta,r\}}\Gamma\left(1+\frac{\ell_2}{a},1+\ell_1\right),$$

where

$$\xi_{\ell_1,\ell_2}^{(\theta,r)}=\theta\lambda^{-r}exp(1+\ell_1)(-1)^{r+\ell_1-\ell_2}(1+\ell_1)^{-\left(\frac{\ell_2}{a}+1\right)}\binom{\theta-1}{\ell_1}\binom{r}{\ell_2}$$

or

$$\mu_r'=\sum_{k_1,k_2=0}^{\infty}\sum_{\ell_1=0}^{\theta-1}\sum_{\ell_2=0}^{r}C_{(k_1,k_2)}\,\xi_{\ell_1,\ell_2}^{\{\theta,r\}}\Gamma\left(1+\frac{\ell_2}{a},1+\ell_1\right)|_{(\theta>0\text{ and integer})},$$

The variance ($\mathbf{V}\,(W)$), skewness ($\beta_{(1)}(W)$) and kurtosis ($\beta_{(2)}(W)$) measures can be calculated from the ordinary moments using well-known relationships. Table 2 gives a comprehensive numerical analysis for the $E(W)$, $\mathbf{V}\,(W)$, $\beta_{(1)}(X)$ and $\beta_{(2)}(X)$ for the GWNH distribution. Based on Table 2 we note that $\beta_{(1)}(W)$ of the GWNH distribution is always positive. $\beta_{(2)}(X)$ of the GWNH distribution can be only greater than three. Figure 3 displays some three-dimensional skewness plots for some selected parameter values. Figure 4 gines some three-dimensional kurtosis plots for some selected parameter values.

Table 2: $E(W)$, $\mathbf{V}(W)$, $\beta_{(1)}(W)$ and $\beta_{(1)}(W)$ of the GWNH distribution.

γ	b	a	$E(W)$	$\mathbf{V}\,(W)$	$\beta_{(1)}(W)$	$\beta_{(2)}(W)$

0.0001	0.5	0.0001	1453839429.0	7.155913×10^{23}	784.1044	693448.7
0.001			14500802391	7.137271×10^{24}	248.2717	69523.23
0.01			141301884123	6.953540×10^{25}	79.51745	7133.25
0.1			1.090706×10^{12}	5.360817×10^{25}	28.57712	922.6559
0.5			1.725651×10^{12}	8.496843×10^{26}	22.68522	581.7479
0.001	0.1	0.001	2112861321	1.040022×10^{24}	650.4194	477144.3
	0.2		4727026176	2.326208×10^{24}	434.8631	213298.4
	0.5		14016807165	6.898551×10^{24}	252.5255	71926.77
	1		30233101685	1.488669×10^{25}	171.9247	33337.48
	5		151763639081	7.473551×10^{25}	76.71699	6638.997
0.0001	0.25	0.0001	662007318	3.257589×10^{23}	1162.043	1523102
		0.001	617083324	3.036617×10^{23}	1203.595	1633966
		0.01	680691480	3.361784×10^{23}	1145.266	1478548
0.0001	0.25	0.0001	662007318	3.257589×10^{23}	1162.043	1523102
		0.001	617083324	3.036617×10^{23}	1203.595	1633966
		0.01	680691480	3.361784×10^{23}	1145.266	1478548

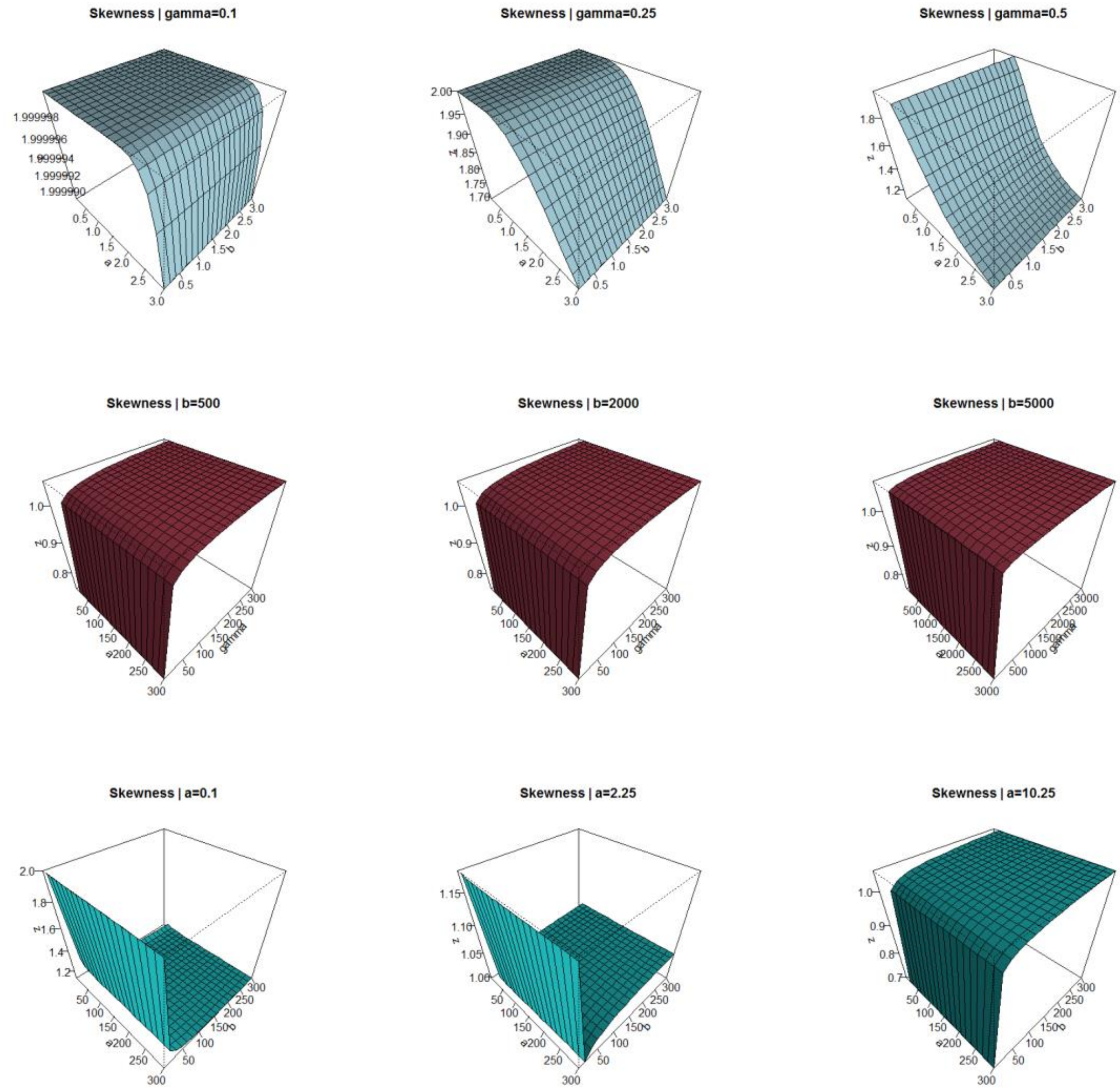

Figure 3: Three-dimensional skewness plots for some selected parameter values.

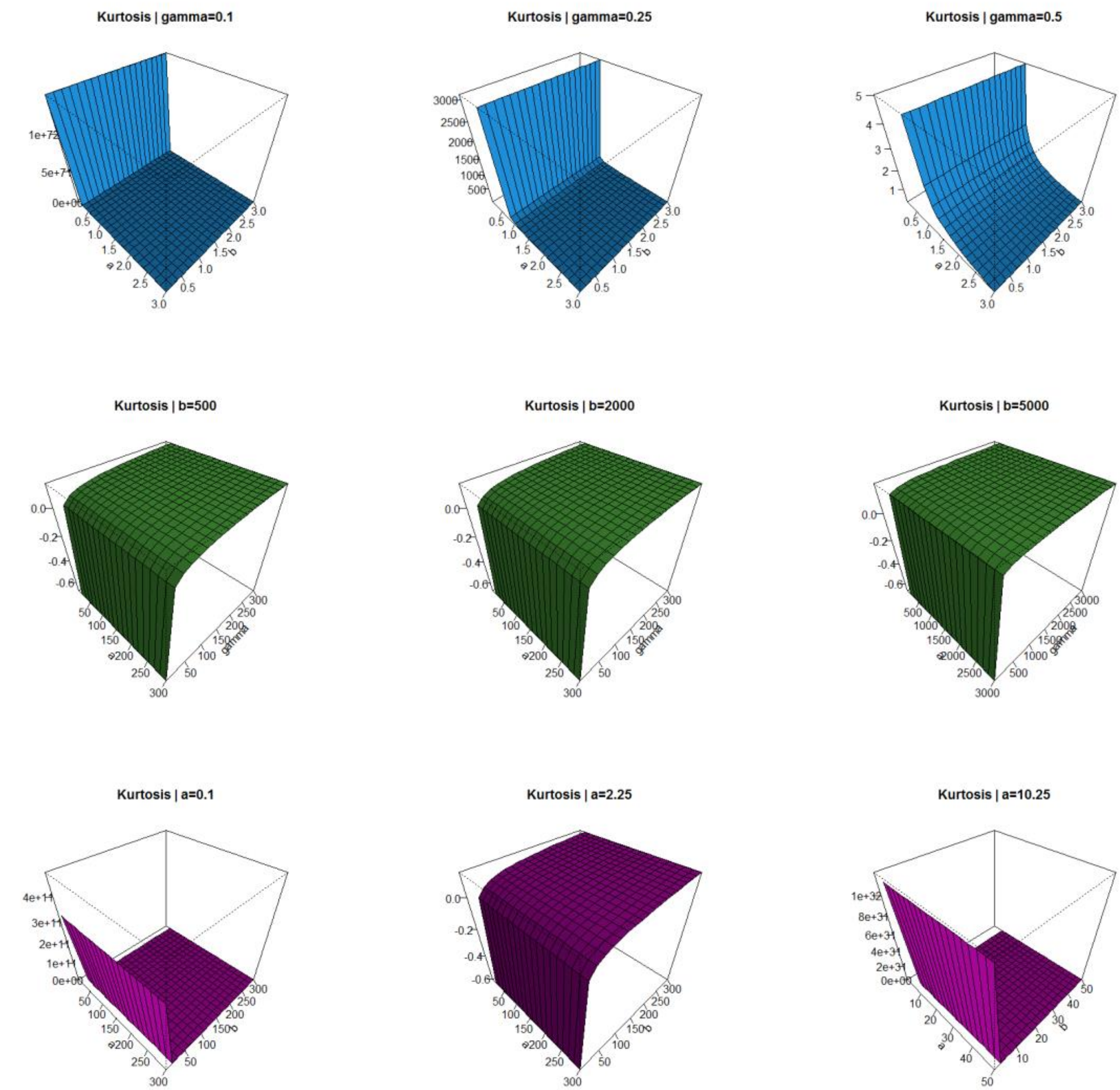

Figure 4: Three-dimensional kurtosis plots for some selected parameter values.

3.3 QF and MGF

The QF of W is determined by inverting (3). We have

$$\phi(u) = \left[1 - log\left(1 - \left\{1 + \left[-log\left(1 - u^{1/\gamma}\right)\right]^{-1/b}\right\}^{-1}\right)\right]^{\frac{1}{a}} - 1, 0 < u < 1.$$

The MGF can follow from equation (8) as

$$M_w(t) = E(e^{tw}) = \sum_{k_1,k_2,\ell_1,r=0}^{\infty} \sum_{\ell_2=0}^{r} \frac{t^r}{r!} C_{(k_1,k_2)} \xi_{\ell_1,\ell_2}^{\{\theta,r\}} \Gamma\left(1 + \frac{\ell_2}{a}, 1 + \ell_1\right),$$

or

$$M_w(t) = \sum_{k_1,k_2,r=0}^{\infty} \sum_{\ell_1=0}^{\theta-1} \sum_{\ell_2=0}^{r} \frac{t^r}{r!} C_{(k_1,k_2)} \xi_{\ell_1,\ell_2}^{\{\theta,r\}} \Gamma\left(1 + \frac{\ell_2}{a}, 1 + \ell_1\right) |_{(\theta>0 \text{ and integer})}.$$

3.4 Incomplete moments

The main applications of the first incomplete moment are related to the mean deviations and Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The s^{th} incomplete moment, say $I_s(t)$, of W can be expressed from (8) as

$$I_s(t)=\int_{-\infty}^{t} w^s f(w)dw=\sum_{k_1,k_2,\ell_1=0}^{\infty}\sum_{\ell_2=0}^{s} C_{(k_1,k_2)}\xi_{\ell_1,\ell_2}^{\{\theta,s\}}\begin{bmatrix}\Gamma\left(1+\frac{\ell_2}{a},1+\ell_1\right)\\ -\Gamma\left(1+\frac{\ell_2}{a},(1+\ell_1)(1+t)^a\right)\end{bmatrix},$$

or

$$I_s(t)=\sum_{k_1,k_2=0}^{\infty}\sum_{\ell_1=0}^{\theta-1}\sum_{\ell_2=0}^{s} C_{(k_1,k_2)}\xi_{\ell_1,\ell_2}^{\{\theta,s\}}\begin{bmatrix}\Gamma\left(1+\frac{\ell_2}{a},1+\ell_1\right)\\ -\Gamma\left(1+\frac{\ell_2}{a},(1+\ell_1)(1+t)^a\right)\end{bmatrix}\Big|_{(\theta>0\text{ and integer})}$$

The mean deviations about the mean $[\boldsymbol{\delta}_1=E(|w-\mu_1'|)]$ and about the median $\left[\boldsymbol{\delta}_2=E\left(\left|w-\phi\left(\frac{1}{2}\right)\right|\right)\right]$ of W are given by $\boldsymbol{\delta}_1=2\mu_1'F(\mu_1')-2I_1(\mu_1')$ and $\boldsymbol{\delta}_2=\mu_1'-2I_1\left(\phi\left(\frac{1}{2}\right)\right)$, respectively, where $\mu_1'=E(w)$, $\phi\left(\frac{1}{2}\right)$ is the median, $F(\mu_1')$ is easily evaluated from (3) and $I_1(t)$ is the first incomplete moment given by (9) (or by (10)) with $s=1$. The $I_1(t)$ can be obtained as

$$I_1(t)=\sum_{k_1,k_2,\ell_1=0}^{\infty}\sum_{\ell_2=0}^{1} C_{(k_1,k_2)}\xi_{\ell_1,\ell_2}^{\{\theta,1\}}\begin{bmatrix}\Gamma\left(1+\frac{\ell_2}{a},1+\ell_1\right)\\ -\Gamma\left(1+\frac{\ell_2}{a},(1+\ell_1)(1+t)^a\right)\end{bmatrix},$$

or

$$I_1(t)=\sum_{k_1,k_2=0}^{\infty}\sum_{\ell_1=0}^{\theta-1}\sum_{\ell_2=0}^{1} C_{(k_1,k_2)}\xi_{\ell_1,\ell_2}^{\{\theta,1\}}\begin{bmatrix}\Gamma\left(1+\frac{\ell_2}{a},1+\ell_1\right)\\ -\Gamma\left(1+\frac{\ell_2}{a},(1+\ell_1)(1+t)^a\right)\end{bmatrix}\Big|_{(\theta>0\text{ and integer})}.$$

3.5 Lorenz and Bonferroni curves

The Bonferroni $[B_{(\text{curve})}]$ and Lorenz $[L_{(\text{curve})}]$ curves have many applications especially in economics, demography, insurance, reliability, medicine etc. The Bonferroni and Lorenz curves can be derived as

$$L_{(\text{curve})}=\frac{1}{E(W)}\int_0^x tf(t)dt,$$

and

$$B_{(\text{curve})}=\frac{1}{E(W)F(w)}\int_0^x tf(t)dt=\frac{L_{(\text{curve})}}{F(w)}.$$

Then

$$L_{(\text{curve})}=\frac{\sum_{k_1,k_2,\ell_1=0}^{\infty}\sum_{\ell_2=0}^{s} C_{(k_1,k_2)}\xi_{\ell_1,\ell_2}^{\{\theta,s\}}\begin{bmatrix}\Gamma\left(1+\frac{\ell_2}{a},1+\ell_1\right)\\ -\Gamma\left(1+\frac{\ell_2}{a},(1+\ell_1)(1+t)^a\right)\end{bmatrix}}{\sum_{k_1,k_2,\ell_1=0}^{\infty}\sum_{\ell_2=0}^{r} C_{(k_1,k_2)}\xi_{\ell_1,\ell_2}^{\{\theta,r\}}\Gamma\left(1+\frac{\ell_2}{a},1+\ell_1\right)}$$

or

$$L_{(\text{curve})}=\frac{\sum_{k_1,k_2=0}^{\infty}\sum_{\ell_1=0}^{\theta-1}\sum_{\ell_2=0}^{1} C_{(k_1,k_2)}\xi_{\ell_1,\ell_2}^{\{\theta,1\}}\begin{bmatrix}\Gamma\left(1+\frac{\ell_2}{a},1+\ell_1\right)\\ -\Gamma\left(1+\frac{\ell_2}{a},(1+\ell_1)(1+t)^a\right)\end{bmatrix}}{\sum_{k_1,k_2=0}^{\infty}\sum_{\ell_1=0}^{\theta-1}\sum_{\ell_2=0}^{r} C_{(k_1,k_2)}\xi_{\ell_1,\ell_2}^{\{\theta,r\}}\Gamma\left(1+\frac{\ell_2}{a},1+\ell_1\right)}\Big|_{(\theta>0\text{ and integer})}$$

and

$$B_{(\text{curve})}=[1-exp(-\{\nabla_a^{-1}(w)-1\}^b)]^{-\gamma}$$

$$\times \frac{\sum_{k_1,k_2,\ell_1=0}^{\infty} \sum_{\ell_2=0}^{s} \quad C_{(k_1,k_2)} \xi_{\ell_1,\ell_2}^{\{\theta,s\}} \begin{bmatrix} \Gamma\left(1+\frac{\ell_2}{a}, 1+\ell_1\right) \\ -\Gamma\left(1+\frac{\ell_2}{a}, (1+\ell_1)(1+t)^a\right) \end{bmatrix}}{\sum_{k_1,k_2,\ell_1=0}^{\infty} \sum_{\ell_2=0}^{r} \quad C_{(k_1,k_2)} \xi_{\ell_1,\ell_2}^{\{\theta,r\}} \Gamma\left(1+\frac{\ell_2}{a}, 1+\ell_1\right)},$$

or

$$B_{(\text{curve})} = [1 - exp(-\{\nabla_a^{-1}(w) - 1\}^b)]^{-\gamma}$$
$$\times \frac{\sum_{k_1,k_2=0}^{\infty} \sum_{\ell_1=0}^{\theta-1} \sum_{\ell_2=0}^{1} \quad C_{(k_1,k_2)} \xi_{\ell_1,\ell_2}^{\{\theta,1\}} \begin{bmatrix} \Gamma\left(1+\frac{\ell_2}{a}, 1+\ell_1\right) \\ -\Gamma\left(1+\frac{\ell_2}{a}, (1+\ell_1)(1+t)^a\right) \end{bmatrix}}{\sum_{k_1,k_2=0}^{\infty} \sum_{\ell_1=0}^{\theta-1} \sum_{\ell_2=0}^{1} \quad C_{(k_1,k_2)} \xi_{\ell_1,\ell_2}^{\{\theta,1\}} \Gamma\left(1+\frac{\ell_2}{a}, 1+\ell_1\right)} \Big|_{(\theta>0 \text{ and integer})}.$$

3.6 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Let $w_1, w_2, \ldots, w_n$ be a random sample (RS) from the GWNH model of distributions. The PDF of the i^{th} order statistic, say $W_{i:n}$, can be expressed as

$$f_{i:n}(w) = \frac{f(w)}{B(i, n-i+1)} \sum_{k_2=0}^{n-i} \quad (-1)^{\ell_1} \binom{n-i}{k_2} F^{k_2+i-1}(w),$$

where $B(\cdot,\cdot)$ is the beta function. Based on equations (5) and (6), we have

$$f(w)F^{k_2+i-1}(w) = \gamma b a (1+w)^{a-1} \nabla_a(w) \frac{\{1-\nabla_a(w)\}^{b-1}}{\{\nabla_a(w)\}^{b+1}}$$
$$\times exp\left\{-\left[\frac{1-\nabla_a(w)}{\nabla_a(w)}\right]^b\right\} \left[1 - exp\left(-\left\{\frac{1-\nabla_a(w)}{\nabla_a(w)}\right\}^b\right)\right]^{\gamma(k_2+i)-1}.$$

Following the same steps of the linear representation (8), we obtain

$$f(w)F^{k_2+i-1}(w) = a(1+w)^{a-1}\nabla_a(w) \sum_{l,k_1=0}^{\infty} \quad \frac{(-1)^{l+k_1}\gamma b(1+l)^{k_1}\Gamma([k_2+i]\gamma)}{l!\,k_1!\,\Gamma([k_2+i]\gamma - l)} \frac{\{1-\nabla_a(w)\}^{(1+k_1)b-1}}{\{\nabla_a(w)\}^{(1+k_1)b+1}}.$$

Then

$$f(w)F^{k_2+i-1}(w) = \sum_{k_1,k_2=0}^{\infty} \quad q_{(k_1,k_2)} \pi_\theta(w; a),$$

where $\pi_\theta(w; a)$ is the ENH density with power parameter θ and

$$q_{(k_1,k_2)} = \sum_{l=0}^{\infty} \quad \frac{(-1)^{l+k_1}\gamma b(1+l)^{k_1}\Gamma([k_2+i]\gamma)\Gamma([1+k_1]b+1+k_2)}{l!\,k_1!\,k_2!\,[(1+k_1)b+k_2]\Gamma([k_2+i]\gamma - l)\Gamma([1+k_1]b+1)}.$$

Substituting (12) in equation (11), the PDF of $W_{i:n}$ can be expressed as

$$f_{i:n}(w) = \frac{1}{B(i, n-i+1)} \sum_{k_1,k_2=0}^{\infty} \quad q^*_{(k_1,k_2)} \pi_\theta(w; a),$$

where $q^*_{(k_1,k_2)} = \sum_{k_2=0}^{n-i} \quad (-1)^{k_2} \binom{n-i}{k_2} q_{(k_1,k_2)}$. The PDF of the GWNH order statistics is a linear combination of ENH densities. The moments of $W_{i:n}$ are given by

$$E(w_{i:n}^s) = \sum_{k_1,k_2,\ell_1=0}^{\infty} \quad \sum_{\ell_2=0}^{r} \quad \frac{q^*_{(k_1,k_2)} \xi_{\ell_1,\ell_2}^{\{\theta,s\}}}{B(i, n-i+1)} \Gamma\left(1+\frac{\ell_2}{a}, 1+\ell_1\right),$$

or

$$E(w_{i:n}^s) = \sum_{k_1,k_2=0}^{\infty} \quad \sum_{\ell_1=0}^{\theta-1} \quad \sum_{\ell_2=0}^{r} \quad \frac{q_{(k_1,k_2)} \xi_{\ell_1,\ell_2}^{\{\theta,s\}}}{B(i, n-i+1)} \Gamma\left(1+\frac{\ell_2}{a}, 1+\ell_1\right) \Big|_{(\theta>0 \text{ and integer})}.$$

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3.7 Probability weighted moments

The PWMs are expectations of certain functions of a random variable and can be defined for any random variable whose ordinary moments exist. The $(s,r)^{\text{th}}$ PWM of the GWNH distribution, say $\rho_{s,r}$, can be formally defined by $\rho_{s,r}=E\{w^s F(w)^r\}$. From equations (5) and (6), we can write

$$f(w)F(w)^r=\sum_{k_1,k_2=0}^{\infty} a_{(k_1,k_2)}^{(r)}\pi_\theta(w;a),$$

where

$$a_{(k_1,k_2)}^{(r)}=\sum_{\ell_1=0}^{\infty}\frac{(-1)^{k_1+\ell_1}\gamma b(\ell_1+1)^{k_1}\Gamma([1+r]\gamma)\Gamma(\theta+1)}{\ell_1!\,k_1!\,k_2!\,\theta\Gamma([1+r]\gamma-\ell_1)\Gamma([1+k_1]b+1)}.$$

Then, $\rho_{s,r}$ can be expressed as

$$\rho_{s,r}=\sum_{k_1,k_2=0}^{\infty} a_{(k_1,k_2)}^{(r)}\int_{-\infty}^{\infty} w^s\pi_\theta(w;a)dw.$$

Finally, the $(s,r)^{\text{th}}$ PWM of W can be obtained from an infinite linear combination of ENH moments given by

$$\rho_{s,r}=\sum_{k_1,k_2,\ell_1=0}^{\infty}\sum_{\ell_2=0}^{s} a_{(k_1,k_2)}^{(r)}\xi_{\ell_1,\ell_2}^{\{\theta,s\}}\Gamma\left(1+\frac{\ell_2}{a},1+\ell_1\right),$$

or

$$\rho_{s,r}=\sum_{k_1,k_2=0}^{\infty}\sum_{\ell_1=0}^{\theta-1}\sum_{\ell_2=0}^{s} a_{(k_1,k_2)}^{(r)}\xi_{\ell_1,\ell_2}^{\{\theta,s\}}\Gamma\left(1+\frac{\ell_2}{a},1+\ell_1\right)|_{(\theta>0\text{ and integer})}.$$

4. Estimation

Consider the following classical estimation methods:
1-Maximum likelihood (ML) method.
2-Cramér-von-Mises (CVM) method.
3-Ordinary least squares (OLS) method.
4-Whighted least squares (WLS) method.
5-Anderson Darling (AD) method.
6-Right Tail-Anderson Darling (ADERT) method.
7-Left Tail-Anderson Darling (ADELT) method.
All these methods are discussed in the statistical literature with more details. In this work, we may ignore many some of its derivation details for avoiding the replication.

4.1 The ML method

The maximum likelihood estimates (MLEs) enjoy desirable properties and can be used when constructing confidence intervals. Let $w_1,w_2,\dots,w_n$ be a RS from this distribution with parameter vector $\underline{\boldsymbol{\delta}}=(\gamma,b,a)^{\mathrm{T}}$. The log-likelihood function for $\underline{\boldsymbol{\delta}}$, say $\ell(\underline{\boldsymbol{\delta}})$, is given by

$$\begin{aligned}\ell(\underline{\boldsymbol{\delta}})&=n\,log\,b+n\,log\,\gamma+n\,log\,a-\sum_{i=0}^{n}\left[\nabla_a^{-1}(w_{[i:n]})-1\right]^b\\&+(b-1)\sum_{i=0}^{n} log\{1-exp[1-(1+w_{[i:n]})^a]\}\\&+(-1+a)\sum_{i=0}^{n} log(1+w_{[i:n]})-b\sum_{i=0}^{n}\left[1-(1+w_{[i:n]})^a\right]\\&+(\gamma-1)\sum_{i=0}^{n} log\left[1-exp\left(-\{\nabla_a^{-1}(w_{[i:n]})-1\}^b\right)\right],\end{aligned}$$

which can be maximized either using the statistical programs or by solving the nonlinear system obtained from $\ell(\underline{\boldsymbol{\delta}})$ by differentiation. The score vector, $U(\underline{\boldsymbol{\delta}})=\left(\frac{\partial\ell(\boldsymbol{\delta})}{\partial\gamma},\frac{\partial\ell(\boldsymbol{\delta})}{\partial b},\frac{\partial\ell(\boldsymbol{\delta})}{\partial a}\right)^{\mathrm{T}}$, are easy to derive.

4.2 The CVM method

The Cramér-von Mises estimates (CVME) of the parameters γ, b and a are obtained via minimizing the following expression with respect to the parameters γ, b and a respectively, where

$$CVM_{(\underline{\delta})} = \frac{1}{12}n^{-1} + \sum_{i=1}^{n} \left[F_{\underline{\delta}}(w_{[i:n]}) - \varepsilon_{(i,n)}\right]^2,$$

where $\varepsilon_{(i,n)} = \frac{2i-1}{2n}$ and

$$CVM_{(\underline{\delta})} = \sum_{i=1}^{n} \left[\left(1 - exp\left\{-\left[\nabla_a^{-1}\left(w_{[i:n]}\right) - 1\right]^b\right\}\right)^\gamma - \varepsilon_{(i,n)}\right]^2.$$

Then, CVME of the parameters γ , b and a are obtained by solving the two following non-linear equations

$$\sum_{i=1}^{n} \left[\left(1 - exp\left\{-\left[\nabla_a^{-1}\left(w_{[i:n]}\right) - 1\right]^b\right\}\right)^\gamma - \varepsilon_{(i,n)}\right] \varsigma_{(\gamma)}(w_{[i:n]}, \underline{\boldsymbol{\delta}}) = 0,$$

$$\sum_{i=1}^{n} \left[\left(1 - exp\left\{-\left[\nabla_a^{-1}\left(w_{[i:n]}\right) - 1\right]^b\right\}\right)^\gamma - \varepsilon_{(i,n)}\right] \varsigma_{(b)}(w_{[i:n]}, \underline{\boldsymbol{\delta}}) = 0,$$

and

$$\sum_{i=1}^{n} \left[\left(1 - exp\left\{-\left[\nabla_a^{-1}\left(w_{[i:n]}\right) - 1\right]^b\right\}\right)^\gamma - \varepsilon_{(i,n)}\right] \varsigma_{(a)}(w_{[i:n]}, \underline{\boldsymbol{\delta}}) = 0,$$

where $\varsigma_{(\gamma)}(w_{[i:n]}, \underline{\boldsymbol{\delta}})$, $\varsigma_{(b)}(w_{[i:n]}, \underline{\boldsymbol{\delta}})$ and $\varsigma_{(a)}(w_{[i:n]}, \underline{\boldsymbol{\delta}})$ are the first derivatives of the CDF of GWNH distribution with respect to γ, b and a respectively.

4.3 OLS method

Let $F_{\underline{\delta}}(w_{[i:n]})$ denote the CDF of GWNH model and let $w_{1,n} < w_{2,n} < \cdots < w_{n,n}$ be the n ordered RS. The ordinary least squares estimates (OLSEs) are obtained upon minimizing

$$OLSE(\underline{\boldsymbol{\delta}}) = \sum_{i=1}^{n} \left[F_{\underline{\delta}}(w_{[i:n]}) - \tau_{(i,n)}\right]^2.$$

Then, we have

$$OLSE(\underline{\boldsymbol{\delta}}) = \sum_{i=1}^{n} \left[\left(1 - exp\left\{-\left[\nabla_a^{-1}\left(w_{[i:n]}\right) - 1\right]^b\right\}\right)^\gamma - \tau_{(i,n)}\right]^2,$$

where $\tau_{(i,n)} = \frac{i}{n+1}$ .The LSEs are obtained via solving the following non-linear equations

$$0 = \sum_{i=1}^{n} \left[\left(1 - exp\left\{-\left[\nabla_a^{-1}\left(w_{[i:n]}\right) - 1\right]^b\right\}\right)^\gamma - \tau_{(i,n)}\right] \varsigma_\gamma(w_{[i:n]}, \underline{\boldsymbol{\delta}}),$$

$$0 = \sum_{i=1}^{n} \left[\left(1 - exp\left\{-\left[\nabla_a^{-1}\left(w_{[i:n]}\right) - 1\right]^b\right\}\right)^\gamma - \tau_{(i,n)}\right] \varsigma_b(w_{[i:n]}, \underline{\boldsymbol{\delta}}),$$

and

$$0 = \sum_{i=1}^{n} \left[\left(1 - exp\left\{-\left[\nabla_a^{-1}\left(w_{[i:n]}\right) - 1\right]^b\right\}\right)^\gamma - \tau_{(i,n)}\right] \varsigma_a(w_{[i:n]}, \underline{\boldsymbol{\delta}}),$$

where $\varsigma_\gamma(w_{[i:n]}, \underline{\boldsymbol{\delta}})$, $\varsigma_b(w_{[i:n]}, \underline{\boldsymbol{\delta}})$ and $\varsigma_a(w_{[i:n]}, \underline{\boldsymbol{\delta}})$ are as defined above.

4.4 WLS method

The weighted least squares estimates (WLSEs) are obtained by minimizing the function $WLSE(\underline{\boldsymbol{\delta}})$ with respect to γ, b and a where

$$WLSE(\underline{\boldsymbol{\delta}}) = \sum_{i=1}^{n} \omega_{(i,n)}\left[F_{\underline{\delta}}(w_{[i:n]}) - \tau_{(i,n)}\right]^2,$$

where

$$\omega_{(i,n)} = [(1+n)^2(2+n)]/[i(1+n-i)].$$

The WLSEs are obtained by solving

$$0=\sum_{i=1}^{n} \omega_{(i,n)}\left[\left(1-exp\left\{-\left[\nabla_a^{-1}\left(w_{[i:n]}\right)-1\right]^b\right\}\right)^{\gamma}-\tau_{(i,n)}\right]\varsigma_{\gamma}(w_{[i:n]},\underline{\boldsymbol{\delta}}),$$

$$0=\sum_{i=1}^{n} \omega_{(i,n)}\left[\left(1-exp\left\{-\left[\nabla_a^{-1}\left(w_{[i:n]}\right)-1\right]^b\right\}\right)^{\gamma}-\tau_{(i,n)}\right]\varsigma_{b}(w_{[i:n]},\underline{\boldsymbol{\delta}}),$$

and

$$0=\sum_{i=1}^{n} \omega_{(i,n)}\left[\left(1-exp\left\{-\left[\nabla_a^{-1}\left(w_{[i:n]}\right)-1\right]^b\right\}\right)^{\gamma}-\tau_{(i,n)}\right]\varsigma_{a}(w_{[i:n]},\underline{\boldsymbol{\delta}}),$$

where $\varsigma_{\gamma}(w_{[i:n]},\underline{\boldsymbol{\delta}})$, $\varsigma_{b}(w_{[i:n]},\underline{\boldsymbol{\delta}})$ and $\varsigma_{a}(w_{[i:n]},\underline{\boldsymbol{\delta}})$ are as defined above.

4.5 The AD method

The Anderson-Darling estimates (ADEs) of γ , b and a are obtained by minimizing the function

$$\mathrm{ADE}(\underline{\boldsymbol{\delta}}) = -n-n^{-1}\sum_{i=1}^{n}(2i-1)\left\{\begin{matrix} log\, F_{(\underline{\boldsymbol{\delta}})}\,(w_{[i:n]}) \\ +\, log\left[1-F_{(\underline{\boldsymbol{\delta}})}(w_{[-i+1+n:n]})\right]\end{matrix}\right\}.$$

The parameter estimates of γ , b and a follow by solving the nonlinear equations

$$\frac{\partial}{\partial\gamma}\left[\mathrm{ADE}(\underline{\boldsymbol{\delta}})\right]=0, \frac{\partial}{\partial b}\left[\mathrm{ADE}(\underline{\boldsymbol{\delta}})\right]=0,$$

and

$$\frac{\partial}{\partial a}\left[\mathrm{ADE}(\underline{\boldsymbol{\delta}})\right]=0.$$

4.6 The ADERT method

The right-tail Anderson-Darling estimates (ADERTEs) of γ , b and a are obtained by minimizing

$$\mathrm{ADERT}(\underline{\boldsymbol{\delta}}) = \frac{1}{2}n-2\sum_{i=1}^{n}F_{(\underline{\boldsymbol{\delta}})}(w_{[i:n]})-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{log\left[1-F_{(\underline{\boldsymbol{\delta}})}(w_{[-i+1+n:n]})\right]\right\}.$$

The parameter estimates of γ , b and a follow by solving the nonlinear equations

$$\frac{\partial}{\partial\gamma}\left[\mathrm{ADERT}(\underline{\boldsymbol{\delta}})\right]=0, \frac{\partial}{\partial b}\left[\mathrm{ADERT}(\underline{\boldsymbol{\delta}})\right]=0,$$

and

$$\frac{\partial}{\partial a}\left[\mathrm{ADERT}(\underline{\boldsymbol{\delta}})\right]=0.$$

4.7 The ADELT method

The left-tail Anderson-Darling estimates (LTADEs) of γ , b and a are obtained by minimizing

$$\mathrm{ADELT}(\underline{\boldsymbol{\delta}}) = -\frac{3}{2}n+2\sum_{i=1}^{n}F_{(\underline{\boldsymbol{\delta}})}(w_{[i:n]})-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\, log\, F_{(\underline{\boldsymbol{\delta}})}\,(w_{[i:n]}).$$

The parameter estimates of γ , b and a follow by solving the nonlinear equations

$$\frac{\partial}{\partial\gamma}\left[\mathrm{ADELT}(\underline{\boldsymbol{\delta}})\right]=0, \frac{\partial}{\partial \gamma}\left[\mathrm{ADELT}(\underline{\boldsymbol{\delta}})\right]=0,$$

and

$$\frac{\partial}{\partial a}\left[\mathrm{ADELT}(\underline{\boldsymbol{\delta}})\right]=0.$$

5. Simulation studies for comparing estimation methods

A numerical simulation is performed to compare the classical estimation methods. The simulation study is based on N= 1000 generated data sets from the GWNH version where $n=50,100,150$ and 300 and

	a	b	γ
I	0.2	1.5	2.0
II	0.5	3.5	0.5
III	0.8	0.8	0.8

The estimates are compared in terms of bias $\left(\mathrm{BIAS}_{\underline{\delta}}\right)$, root mean-standard error $\left(\mathrm{RMSE}_{\underline{\delta}}\right)$, the mean of the absolute difference between the theoretical and the estimates (D.abs) and the maximum absolute difference between the true parameters and estimates (D.max). From Tables 3, 4 and 5 we note that:

1-The BIAS $\underline{\boldsymbol{\delta}}$ tend to zero when n increases which means that all estimators are consistent.

2-The RMSE $\underline{\boldsymbol{\delta}}$ tend to zero when n increases which means incidence of consistency property.

Table 3: Simulation results for blend I.

	n	BIAS			RMSE			D.abs	D.max
		a	b	γ	a	b	γ		
ML	50	0.00072	0.04465	0.02272	0.00662	0.22270	0.50014	0.00682	0.01322
CVM		0.00015	0.25160	0.05732	0.00776	1.41353	0.38440	0.02786	0.05070
OLS		-0.00156	0.11434	0.13982	0.00785	1.65217	0.41653	0.02097	0.03914
WLS		-0.00107	0.13130	0.11673	0.00733	0.83280	0.40431	0.01871	0.03469
ADE		0.00605	0.53256	0.20783	0.03755	1.56995	1.23693	0.05810	0.11315
ADERT		0.00546	0.76817	0.19440	0.03872	2.34255	1.28307	0.07395	0.14418
ADELT		0.00480	1.02845	0.16264	0.04154	2.48622	1.36804	0.08971	0.17857
MLE	100	0.00034	0.02144	0.01313	0.00468	0.14870	0.35468	0.00325	0.00617
CVM		-0.00004	0.18119	0.03429	0.00560	0.97837	0.26676	0.02017	0.03682
OLS		-0.00067	0.12534	0.06272	0.00547	0.82525	0.26827	0.01468	0.02443
WLS		-0.00063	0.09886	0.06226	0.00513	0.59287	0.26913	0.01236	0.02172
ADE		0.00521	0.29812	0.18884	0.03508	0.94263	1.13413	0.04043	0.07840
ADERT		0.00446	0.32906	0.16384	0.03435	0.99592	1.11831	0.04015	0.07710
ADELT		0.00412	0.46356	0.14367	0.03785	1.28490	1.24698	0.05679	0.11129
MLE	150	0.00019	0.01216	0.00827	0.00360	0.11345	0.26483	0.00183	0.00341
CVM		0.00021	0.11086	0.01028	0.00437	0.5152	0.20369	0.01313	0.02533
OLS		-0.00032	0.06696	0.03486	0.00438	0.46514	0.20862	0.00806	0.01332
WLS		-0.00032	0.04650	0.03592	0.00408	0.33081	0.20851	0.00635	0.01152
ADE		0.00347	0.16210	0.13527	0.03283	0.70864	1.06473	0.03029	0.05867
ADERT		0.00214	0.22576	0.10082	0.03334	0.85142	1.08310	0.03538	0.06875
ADELT		0.00377	0.35411	0.14352	0.03615	1.05010	1.18921	0.04759	0.09346
MLE	300	-0.00001	0.00286	0.01203	0.00255	0.07821	0.19130	0.00139	0.00238
CVM		0.00006	0.04546	0.00828	0.00321	0.26897	0.14836	0.00541	0.01009
OLS		-0.00030	0.01511	0.02406	0.00315	0.23486	0.14878	0.00378	0.00672
WLS		-0.00029	0.01034	0.02506	0.00293	0.18177	0.14850	0.00404	0.00660
ADE		-0.00209	0.07854	0.09715	0.03003	0.53451	0.96874	0.02275	0.04403
ADERT		0.00114	0.08782	0.08134	0.02937	0.56100	0.95343	0.02236	0.04309
ADELT		0.00067	0.17408	0.01935	0.03202	0.68573	1.04719	0.03076	0.06013

Table 4: Simulation results for blend II.

	n	BIAS			RMSE			D.abs	D.max
		a	b	γ	a	b	γ		
MLE	50	0.00137	-0.30070	0.02785	0.01530	0.96570	0.15483	0.00791	0.01255
CVM		0.00200	0.42357	0.01085	0.02210	1.91896	0.09519	0.02448	0.04307
OLS		-0.00235	0.65705	0.02995	0.02100	1.92618	0.10306	0.04910	0.08272
WLS		0.00208	0.86106	0.03137	0.01899	2.34371	0.10206	0.05806	0.09912
ADE		0.03279	0.58893	0.27609	0.10492	2.71651	0.80472	0.08026	0.14804
ADERT		0.03319	0.79492	0.28981	0.10455	3.28294	0.83541	0.08937	0.16633
ADELT		0.04433	1.12668	0.30763	0.12076	3.29109	0.87957	0.09767	0.17633

MLE	100	0.00122	-0.03259	0.01030	0.01087	0.77992	0.10321	0.00131	0.00252
CVM		0.00097	0.20888	0.00598	0.01541	1.04705	0.06837	0.01270	0.02238
OLS		-0.00166	0.35143	0.01733	0.01489	1.09023	0.07042	0.02829	0.04732
WLS		-0.00177	0.47401	0.01966	0.01335	1.40497	0.07291	0.03535	0.05964
ADE		0.00602	0.34041	0.08105	0.06698	1.65934	0.45930	0.04259	0.07508
ADERT		0.01468	0.25240	0.14094	0.07402	1.82144	0.54661	0.04841	0.08808
ADELT		0.03190	0.51926	0.22071	0.09039	2.34849	0.66721	0.06732	0.12208
MLE	150	0.00083	0.03456	0.00561	0.00852	0.62830	0.08041	0.00343	0.00615
CVM		0.00057	0.11232	0.00360	0.01191	0.71733	0.05285	0.00705	0.01241
OLS		-0.00148	0.23144	0.01285	0.01197	0.76051	0.05478	0.02013	0.03325
WLS		-0.00113	0.29294	0.01251	0.01071	0.96772	0.05605	0.02245	0.03788
ADE		-0.00246	0.23528	0.02701	0.05683	1.32009	0.36683	0.02867	0.04532
ADERT		0.00326	0.18097	0.07639	0.06248	1.40593	0.43033	0.03748	0.06289
ADELT		0.02080	0.56949	0.14603	0.07333	2.16319	0.53899	0.05846	0.10681
MLE	300	-0.00001	0.03044	0.00652	0.00580	0.41832	0.05641	0.00473	0.00767
CVM		0.00001	0.06599	0.00288	0.00821	0.46159	0.03630	0.00486	0.00838
OLS		-0.00060	0.10658	0.00584	0.00843	0.49604	0.03811	0.00924	0.01531
WLS		-0.00028	0.12711	0.00498	0.00751	0.61636	0.03834	0.00944	0.01608
ADE		-0.01523	0.07299	-0.04430	0.04537	0.81859	0.22270	0.01350	0.02729
ADERT		-0.01454	0.07591	-0.03191	0.04763	0.83108	0.24286	0.01522	0.03027
ADELT		0.00895	0.26133	0.06065	0.03993	1.41856	0.29637	0.02848	0.05165
MLE	50	0.00137	-0.30070	0.02785	0.01530	0.96570	0.15483	0.00791	0.01255

Table 5: Simulation results for blend III.

	n	BIAS			RMSE			D.abs	D.max
		a	b	γ	a	b	γ		
MLE	50	0.01764	0.00619	0.01032	0.06854	0.09696	0.11911	0.00447	0.00979
CVM		0.02665	0.14129	0.01855	0.15252	0.69016	0.15203	0.03041	0.04995
OLS		-0.00779	0.21557	0.05585	0.12517	0.74504	0.16661	0.05268	0.09503
WLS		0.00022	0.23939	0.05603	0.11918	0.80326	0.16074	0.05495	0.09963
ADE		0.05914	0.18452	0.07224	0.27181	0.70441	0.48094	0.04604	0.07654
ADERT		0.09556	0.07651	0.13353	0.28474	0.63278	0.49538	0.03729	0.05889
ADELT		0.08418	0.37842	0.05434	0.31919	1.00667	0.52047	0.07152	0.11447
MLE	100	0.00860	0.00637	0.01319	0.04464	0.06444	0.08508	0.00373	0.00625
CVM		0.01627	0.04931	0.00105	0.08766	0.36906	0.09658	0.01081	0.01828
OLS		-0.00555	0.13945	0.02936	0.08559	0.52749	0.10835	0.03382	0.06121
WLS		-0.00460	0.14334	0.03170	0.06998	0.58535	0.10295	0.03486	0.06315
ADE		0.03797	0.10554	0.04452	0.21239	0.46836	0.38221	0.02837	0.04680
ADERT		0.06680	0.02484	0.10064	0.22657	0.38594	0.41014	0.02234	0.03587
ADELT		0.01484	0.36991	-0.03155	0.23310	0.88243	0.41722	0.05668	0.09521
MLE	150	0.00640	0.00409	0.00433	0.03540	0.05300	0.06796	0.00193	0.00382
CVM		0.00250	0.05136	0.01042	0.06702	0.27318	0.08141	0.01251	0.02238
OLS		-0.00101	0.07109	0.01398	0.06426	0.35363	0.08003	0.01745	0.03146
WLS		-0.00211	0.08895	0.02070	0.05539	0.38931	0.08228	0.02246	0.04055
ADE		0.03306	0.09708	0.03638	0.19898	0.41161	0.36119	0.02546	0.04191
ADERT		0.05319	0.02364	0.07417	0.20653	0.30414	0.37182	0.01790	0.02817
ADELT		0.03407	0.24868	0.00710	0.22697	0.71750	0.39549	0.04586	0.07506
MLE	300	0.00245	0.00119	0.00188	0.02455	0.03797	0.04604	0.00069	0.00137
CVM		0.00189	0.01368	0.00424	0.04616	0.09894	0.05518	0.00366	0.00647
OLS		-0.00104	0.02401	0.00838	0.04709	0.11519	0.05733	0.00704	0.01267
WLS		-0.00125	0.02385	0.01105	0.03824	0.09897	0.05618	0.00774	0.01383
ADE		0.03147	0.05189	0.04295	0.16964	0.31944	0.31392	0.01806	0.02991

ADERT	0.03743	0.02654	0.05165	0.17245	0.27484	0.31413	0.01464	0.02347
ADELT	0.01142	0.20702	-0.01753	0.19197	0.57001	0.34803	0.03506	0.05814

6. Applications

6.1 Application for comparing classical methods

For comparing classical methods, an application to real data set is introduced. We consider the Cramér-Von Mises (WC) and the Anderson-Darling (AC) statistis. The data is obtained from Section 1.15 of Klein and Moeschberger (2006). This sample is part of a larger study of psychiatric inpatients discussed by Tsuang and Woolson (1977). Data for each patient consists of age, sex, number of years of follow-up (years from admission to death or censoring) and patient status at the follow up time. The data are: 1, 1, 2, 22, 30, 28, 32, 11, 14, 36, 31, 33, 33, 37, 35, 25, 31, 22, 26, 24, 35, 34, 30, 35, 40, 39. Other real-life datasets are in Aryal and Yousof (2017), Yousof et al. (2017), Hamedani et al. (2017,2018 and 2019), Merovci et al. (2017 and 2020), Korkmaz et al. (2018a), Korkmaz et al. (2018b), Nascimento et al. (2019), Alizadeh et al. (2020a,b), Mansour et al. (2020a-f) Korkmaz et al. (2020) and Karamikabir et al. (2020).

The HRF of psychiatric patients' data set is "bathtube (**U**-HRF)". Due to Figure 7(d), the psychiatric patient's data set is a bimodal and left skewed data. From Table 6, the MLE method is the best method with WC =0.31136 and AC =1.90220 then ADE with WC =0.37273 and AC =2.23792. However, CVM, OLS, WLS, ADERT and ADELT performed well. Figure 5 gives probability-probability (P-P) plots for comparing classical methods. Figure 6 gives Kaplan-Meier survival plots for comparing classical methods.

Table 6: he values of estimators, WC and AC.

Method	γ	b	a	WC	AC
ML	0.172	15.948	0.143	0.31136	1.90220
CVM	0.176	16.516	0.138	0.41316	2.45224
OLS	0.189	16.484	0.137	0.42038	2.49014
WLS	0.131	15.109	0.136	0.42893	2.53358
ADE	0.195	16.766	0.140	0.37273	2.23792
ADERT	0.182	32.687	0.144	0.55448	3.64103
ADELT	0.144	16.399	0.134	0.46552	2.72700

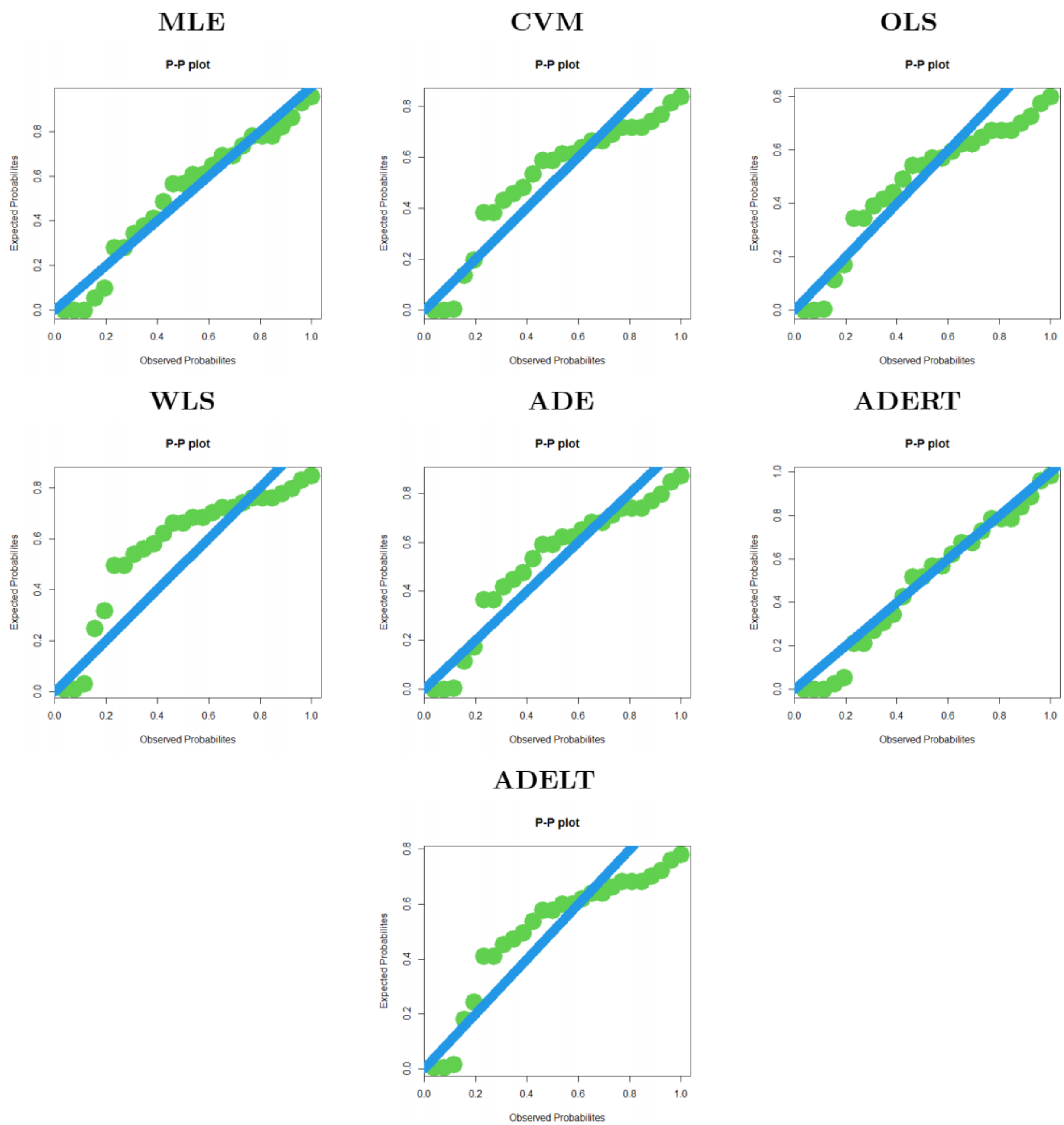

Figure 5: P-P plots for comparing classical methods.

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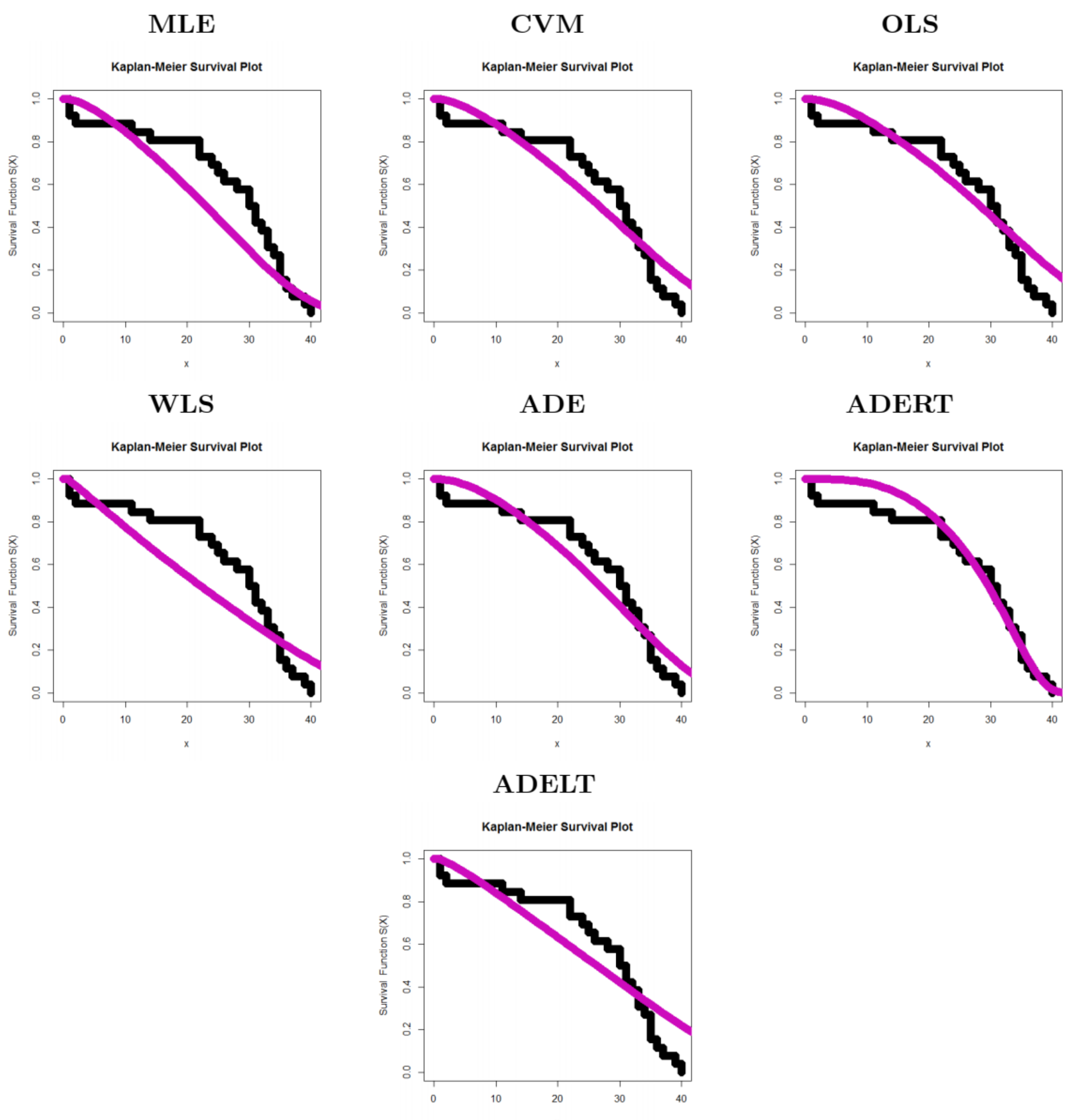


Figure 6: Kaplan-Meier survival plots for comparing classical methods.

6.2 Application for comparing models

We present an application based on a real data set related to times of death of twenty-six psychiatric patients. The data of Klein and Moeschberger (1997) is used in this part. First, we compare GWNH distribution with some well-known four and five parameter distributions with names given below

Competitive model & Abbreviation	Author
NH	Nadarajah and Haghighi (2011)
Beta NH (BNH)	Dias (2016)
Kumaraswamy NH (KumNH)	Lima (2015)
exponentiated generalized NH (EGNH)	VedoVatto (2016)
Beta Exponentiated Weibull (BEW)	Cordeiro et al. (2013)
Exponentiated Exponential (EExp)	Gupta and Kundu (2001)

For the purpose of comparisons, we shall use the following statistics:

- **I.** $-$log-likelihood function.
- **II.** Akaike information criterion (Cr(1))
- **III.** Bayesian information criterion (Cr(2)).
- **IV.** Hannan-Quinn information criterion (Cr(3)).
- **V.** Consistent Akaike information criterion (Cr(4)).

In general, the smaller values of these ceriteria show the better fit to the data sets. In all of mentioned goodness of fit test ceriteria, GWNH shows better fit to this data sets. The estimated parameters based on MLE procedure reports in Tables 1 as well as the values of goodness-of-fit statistics are given in Tables 2. Also, Table 3 includes the likelihood ratio test for sub models comparison. The total time on test (TTT) plot (a), box plot (b), quantile-quantile (Q-Q) plot (c) and nonparametric Kernel density estimation (NKDE) plot (d) are presented in Figure 7. Figure 7(a) indicates that the HRF of psychiatric patients' data set is "bathtub (**U**-HRF)". Based on Figure 7(b), the psychiatric patient's data contains some extreme values and the QQ plot (Figure 7(c)) ensure this fact. Due to the NKDE plot (Figure 7(d)), the psychiatric patients data set is a bimodal and left skewed data. Table 7 gives the parameters estimates and standard deviation in parenthesis. Table 8 gives the goodness of fit criteria. From Table 3 we note that, the new GWNH model is better that the EENH, BW, GaMW, BEW, KwNH, BNH and EGNH. Figure 8 gives the estimated CDF (ECDF), EPDF, Kaplan-Meier survival plot, P-P plot and EHRF for the times of death data. Figure 8 indicates that the new model gives a good fit to the used date set.

Table 7: Parameters estimates and standard deviation in parenthesis.

Model	Estimates				
Exp(a)	0.0379				
	(0.007)				
NH(a)	0.2483				
	(0.0309)				
EExp(b,a)	1.7970	0.0525			
	(0.4795)	(0.0109)			
GWNH(γ,b,a)	0.1723	15.948	0.1434		
	(0.035)	(0.003)	(0.0017)		
KwNH(λ,γ,b,a)	1.844	1.042	0.002	11.072	
	(0.509)	(0.901)	(0.0006)	(4.406)	
BNH(λ,γ,b,a)	3.384	0.0487	0.237	1.540	
	(0.528)	(0.009)	(0.001)	(0.002)	
EGNH(λ,γ,b,a)	0.0613	2.056	0.915	0.991	
	(0.012)	(0.574)	(0.003)	(0.002)	
BEW(λ,γ,θ,b,a)	0.1873	0.0432	5.4915	1.565	0.259
	(0.07)	(0.007)	(0.037)	(0.002)	(0.002)

Table 8: Goodness of fit criteria

Model	Goodness of fit criteria				
	-Log Likelihood	Cr(1)	Cr(2)	Cr(3)	Cr(4)
GWNH	99.180	204.360	208.134	205.450	205.446
KwNH	103.923	215.846	220.879	217.296	217.751
BEW	104.908	219.816	226.106	221.627	222.816
BNH	108.591	225.183	230.215	226.632	227.087
EGNH	108.997	225.995	231.027	227.444	227.899
EExp	108.987	221.974	224.490	222.496	222.698
Exp	111.130	224.260	225.519	224.427	224.623
NH	127.813	257.626	258.884	257.793	257.988

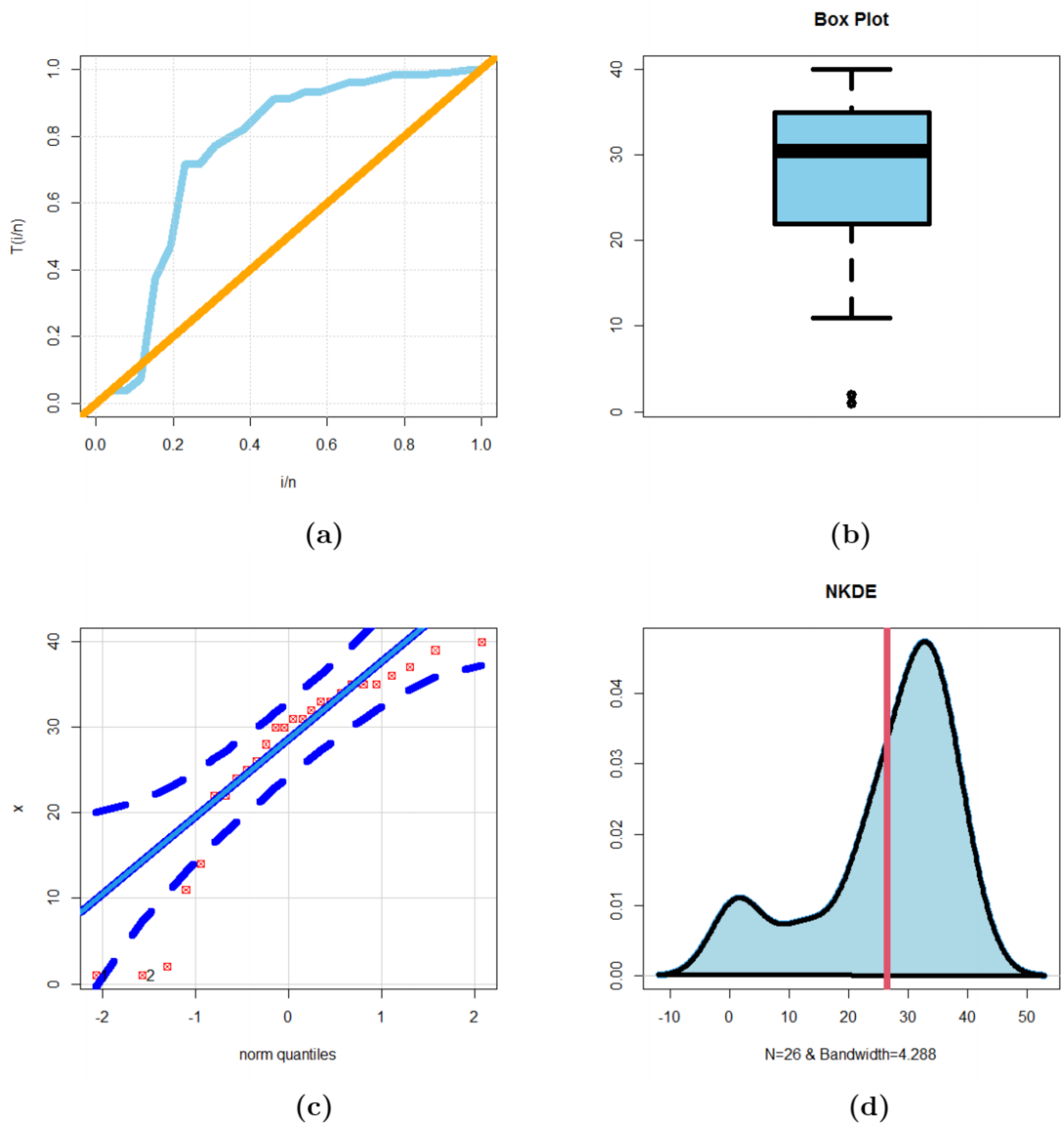
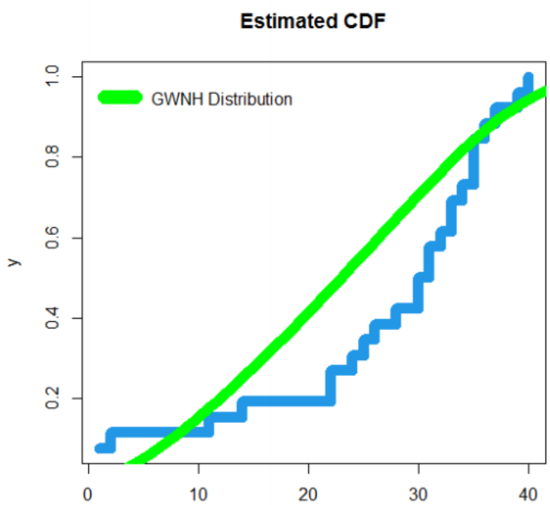
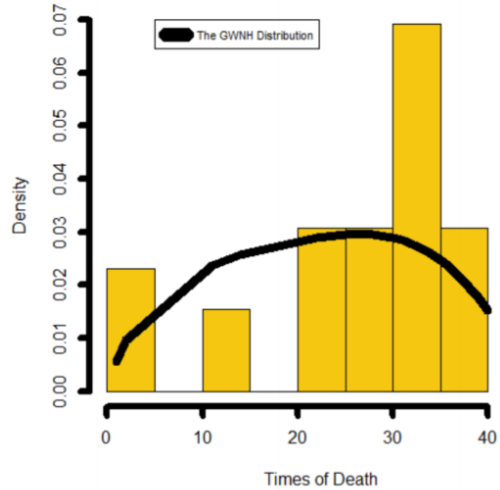
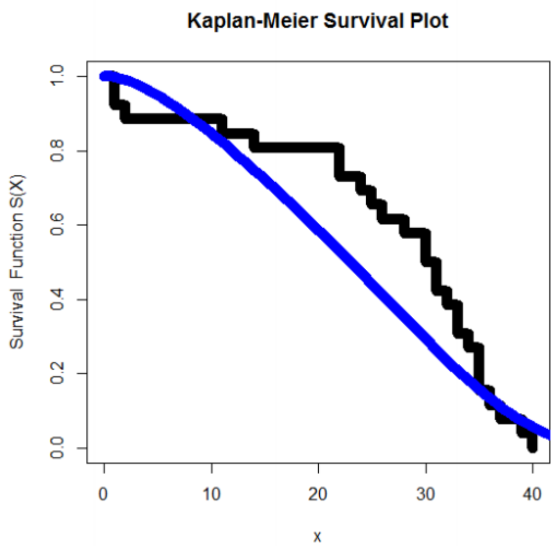
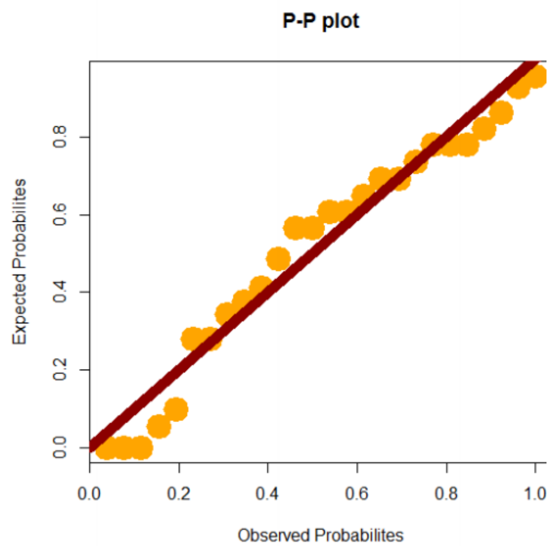


Figure 7: TTT plot(a), box plot(b), Q-Q plot (c) and NKDE plot(d).


Estimated CDF
GWNH Distribution
y
0.2 0.4 0.6 0.8 1.0
0 10 20 30 40


The GWNH Distribution
Density
0.00 0.01 0.02 0.03 0.04 0.05 0.06 0.07
0 10 20 30 40
Times of Death


Kaplan-Meier Survival Plot
Survival Function S(X)
0.0 0.2 0.4 0.6 0.8 1.0
0 10 20 30 40
x


P-P plot
Expected Probabilites
0.0 0.2 0.4 0.6 0.8
0.0 0.2 0.4 0.6 0.8 1.0
Observed Probabilites

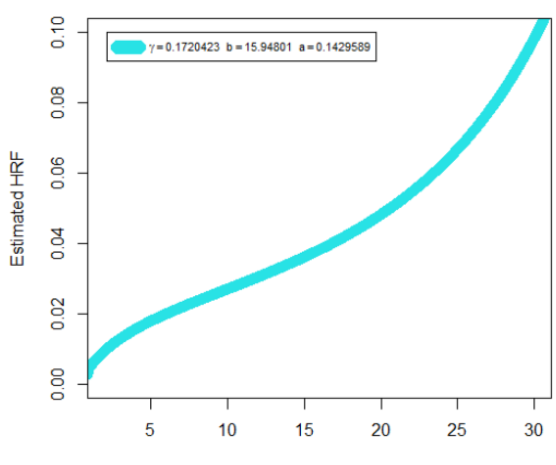

γ = 0.1720423 b = 15.94801 a = 0.1429589
Estimated HRF
0.00 0.02 0.04 0.06 0.08 0.10
5 10 15 20 25 30
x

Figure 8: ECDF, EPDF, Kaplan-Meier survival plot, P-P plot and EHRF.

7. Conclusions

We introduced a new model called the generalized Weibull Nadarajah Haghighi model with three parameters. The new density can be expressed as a straightforward linear combination of the exponentiated Nadarajah Haghighi densities. The new failure rate can be monotonically increasing, monotonically decreasing, J-shape, upside-down-bathtub, bathtub and constant. We derived some of its mathematical and statistical quantities including the ordinary moments, the moment generating function, the incomplete moments, moments of residual life and reversed residual life. Some new bivariate type distributions using Farlie Gumbel Morgenstern, modified Farlie Gumbel Morgenstern, Clayton, Renyi and Ali-Mikhail-Haq copulas are investigated. The maximum likelihood, Cramér-von-Mises, ordinary least square, whighted least square, Anderson Darling, right tail Anderson Darling and left tail Anderson Darling methods to estimate the unknown model parameters. Simulation studies for comparing estimation methods were performed. Moreover, an application to real data set for comparing methods is also presented. The maximum likelihood method is the best method with WC =0.31136 and AC =1.90220. Then, the Anderson Darling with WC =0.37273 and AC =2.23792. However, the other methods performed well. The estimation methods are compared using probability-probability plots and Kaplan-Meier survival plots. Another real application for comparing the competitive models was also investigated.

It is suggested to present a novel discrete generalized Weibull Nadarajah Haghighi model for modeling count real-life data (see Aboraya et al. (2020), Chesneau et al. (2021), Yousof et al. (2021) and Ibrahim et al. (2022) for more details). Forthermore, appling the Nikulin-Rao-Robson and Bagdonavičius-Nikulin tests (see Ibrahim et al. (2019), Goual et al. (2019, 2020), Yadav et al. (2020 and 2022), Ibrahim et al. (2022), Goual and Yousof (2020) and Aidi et al. (2021), among others). For regression modeling under censored data sets see, some new regression models could be presented due to Altun et l. (2018a,b) and Yousof et al. (2019). Reliability estimation for the remained stress-strength model under the generalized Weibull Nadarajah Haghighi distribution could be presented (see Saber et al. (2022)) and Bayesian and classical inference for the generalized stress-strength parameter under generalized Weibull Nadarajah Haghighi model (see Saber and Yousof (2022)). Following, Ahmed and Yousof (2022) and Ahmed et al. (2022), a single acceptance sampling plan with its corresponding application in quality and risk decisions can be presented. Finally, for applications in insurance filed one can follow Mohamed et al. (2022).

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