A New Flexible Probability Model: Theory, Estimation and Modeling Bimodal Left Skewed Data

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Abstract
In this work, we introduced a new three-parameter Nadarajah-Haghighi model. We derived explicit expressions for some of its statistical properties. The Farlie Gumbel Morgenstern, modified Farlie Gumbel Morgenstern, Clayton, Renyi and Ali-Mikhail-Haq copulas are used for deriving some bivariate type extensions. We consider maximum likelihood, Cramér-vonMises, ordinary least squares, weighted least squares, Anderson Darling, right tail Anderson Darling and left tail Anderson Darling estimation procedures to estimate the unknown model parameters. Simulation study for comparing estimation methods is performed. An application for comparing methods as also presented. The maximum likelihood estimation method is the best method. However, the other methods performed well. Another application for comparing the competitive models is investigated.

Key Words: Nadarajah Haghighi model; Farlie Gumbel Morgenstern, Anderson Darling, Maximum Likelihood Estimation, Ordinary Least Squares, Generating Function, Moments.

Mathematical Subject Classification: 62N01; 62N02; 62E10.

1. Introduction
Among several parametric probability models, the exponential (Exp) distribution is the most widely applied statistical model. One of the reasons for its importance is that the Exp distribution has constant hazard rate function (HRF). A random variable (RV) \( T \) is said to have the Exp distribution if its cumulative distribution function (CDF) is given by

\[
G_{\lambda}(t) = 1 - \exp(-\lambda t) \quad |(t>0).
\]

The Exp distribution is the probability distribution of the time between events in a Poisson point process in which events occur continuously and independently at a constant average rate. It is a particular case of the well-known gamma (Gam) distribution. The \( r^{th} \) ordinary and incomplete moments, the moment generating function (MGF), Quantile function (QF) and several other properties of the Exp distribution can be expressed in terms of elementary functions; see, for example Balakrishnan and Basu (1995) and Marshall and Olkin (2007). Recently, a new useful generalization of the Exp distribution as an alternative to the Exp, Weibull (W) Gam, and exponentiated exponential (ExpExp) distributions was proposed by Nadarajah and Haghighi (2011). A RV \( W \) is said to have the Weibull Nadarajah Haghighi (WNH) distribution if its PDF is given as

\[
h_{\alpha}(w) = \alpha (1 + w)^{\alpha - 1} P_{\alpha}(w) \quad |(w>0),
\]

where \( P_{\alpha}(w) = \exp[1 - (1 + w)^{\alpha}] \) refers to the reliability function of the NH distribution. Nadarajah and Haghighi (2011) proved that the PDF in (2) always has the zero mode. A RV \( W \) is said to have the GWNH distribution if its CDF is given as
\( F_\delta(w) = (1 - \exp[-V_\alpha^{-1}(w) - 1]^b])^\gamma, \) 

\( f_\delta(w) = \frac{\gamma - a}{(1 + w)^{1-a} \exp[V_\alpha^{-1}(w) - 1]^b} (1 - \exp[-V_\alpha^{-1}(w) - 1]^b])^{\gamma-1}. \)

The additional shape parameters \( \gamma \) and \( b \) can allow us to study the tail behavior of the PDF (4) with greater flexibility (see Figure 1 and Figure 2). Further, the GWNH due to its wide flexibility in accommodating all shapes of the HRF (monotonically increasing, monotonically decreasing, J-shape, upside-down-bathtub, bathtub and constant) as given in Figure 2.
Figure 1: Plots of the GWNH PDF for selected parameter values.

Figure 2: Plots of the GWNH HRF for selected parameter values.
Several extensions of the NH model can be found in the statistical literature such as the exponentiated NH model (ENH) model (Lemonte (2013)), gamma NH (GNH) (Ortega et al. (2015)), Poisson gamma NH (PGNH) (Ortega et al. (2015)), transmuted NH (TNH) (Ahmed et al. (2015)), Kumaraswamy NH (KNH) (Lima (2015)), The modified NH (MNH) (El Damcese and Ramadan (2015)), Marshall-Olkin NH (MNONH) (Lemonte et al. (2016)), The beta NH distribution (Cicero et al. (2016)), Topp-Leone NH (TLNH) (Yousof and Korkmaz (2017)), Lindley NH (LNH) (Yousof el al. (2017)), extended exponentiated NH (ExENH) (Alizadeh et al. (2018)), beta NH (BNH) (Dias (2016)), inverted NH (I-NH) (Tahir et al. (2018)), the Burr X NH (BXNH) (Elsayed and Yousof (2019)), the odd NH-G (NHG) family (Nascimento et al. (2019)), the Topp-Leone exponentiated NH model for modeling extreme values (Almazah et al. (2021)) and the Lindley exponentiated NH (Shehata and Yousof (2021)). In this paper, we present a new three parameter NH version called the generalized Weibull Nadarajah Haghighi (GWNH) distribution.

Table 1 gives some special cases of the GWNH model. The HRF of the GWNH model can be derived from \( h_\gamma(w) = f_\gamma(w) / [1 - F_\gamma(w)] \) and henceforth the RV \( W \) having CDF as in (3) is denoted by \( W \sim \text{GWNH}(\delta) \).

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2. Copula and related results

We derive some new bivariate GWNH (BGWNH) type distributions using Farlie Gumbel Morgenstern (FGM) Copula (see Gumbel (1958), Gumbel (1960 and 1961), Johnson and Kotz (1975) and Johnson and Kotz (1977)), modified FGM Copula (see Rodriguez-Lallena and Ubeda-Flores (2004)), Clayton Copula and Renyi’s entropy (Pougaza and Djafari (2011)) and Ali-Mikhail-Haq copula (see Ali et al (1978)). For other copulas and related derivations see Ali et al. (2020a,b), Elgohari et al. (2021), Elgohari and Yousof (2020a,b and 2021), Shehata and Yousof (2021a,b), Shehata et al. (2021) and Aboraya et al. (2022). The Multivariate GWNH (MGWNH) type is also presented. However, future works may be allocated to the study of these new models. First, we consider the joint CDF (JCDF) of the FGM family where

\[
C(m, v) = m v (1 + Y m v) | m = 1 - m v = 1 - v
\]

and the marginal function \( m = F_1, v = F_2, Y \in (-1, 1) \) is a dependence parameter and for every \( m, v \in (0, 1) \), \( C(m, 0) = C(0, v) = 0 \) which is “grounded minimum” and \( C(m, 1) = m \) and \( C(1, v) = v \) which is “grounded maximum”, \( C(m_1, v_1) + C(m_2, v_2) - C(m_1, v_2) - C(m_2, v_1) \geq 0 \).

2.1 Via FGM family

A Copula is continuous in \( m \) and \( v \); actually, it satisfies the stronger Lipschitz condition, where

\[
|C(m_2, v_2) - C(m_1, v_1)| \leq |m_2 - m_1| + |v_2 - v_1|.
\]

For \( 0 \leq m_1 \leq m_2 \leq 1 \) and \( 0 \leq v_1 \leq v_2 \leq 1 \), we have

\[
Pr(m_1 \leq m \leq m_2, v_1 \leq v \leq v_2) = C(m_1, v_1) + C(m_2, v_2) - C(m_1, v_2) - C(m_2, v_1) \geq 0.
\]

Then, setting \( m' = 1 - F_{\gamma}(m_1)|m = (1 - m) \in (0, 1)| \) and \( v' = 1 - F_{\gamma}(m_2)|v = (1 - v) \in (0, 1)| \), we can easily get the JCDF of the GWNH using the FGM family as

\[
C_\gamma(m, v) = \left(1 - \exp\left[-\left[\frac{1}{m'}\right]^{b_1}\right]\right)^{Y_1} \left(1 - \exp\left[-\left[\frac{1}{v'}\right]^{b_2}\right]\right)^{Y_2} \times \left(1 + Y \left\{\left[1 - \left(1 - \exp\left[-\left[\frac{1}{m'}\right]^{b_1}\right]\right]\right]^{Y_1}\right\}\right)^{Y_1} \times \left\{1 - \left(1 - \exp\left[-\left[\frac{1}{v'}\right]^{b_2}\right]\right)\right\}^{Y_2}.
\]
2.2 Via modified (MFGM) family

The MFGM copula is defined as $C_Y(m, v) = mv[1 + Y\omega(m)C(v)]_{|Y\in(-1,1)}$ or $C_Y(m, v) = mv + YW_mK_v|_{Y\in(-1,1)}$, where $W_m = m\omega(m)$ and $K_v = vC(v)$ and $\omega(m)$ and $C(v)$ are two continuous functions on $(0,1)$ with $\omega(0) = \omega(1) = C(0) = C(1) = 0$. Let

$$a_1(W_m) = \inf \left\{ W_m: \frac{\partial}{\partial m} W_m \right\}_{|k_{1,m}} < 0, a_2(W_m) = \sup \left\{ W_m: \frac{\partial}{\partial m} W_m \right\}_{|k_{1,m}} < 0,$$

$$b_1(K_v) = \inf \left\{ K_v: \frac{\partial}{\partial v} K_v \right\}_{|k_{2,v}} > 0, b_2(K_v) = \sup \left\{ K_v: \frac{\partial}{\partial v} K_v \right\}_{|k_{2,v}} > 0.$$

Then,

$$1 \leq \min \{a_1(W_m)a_2(W_m), b_1(K_v)b_2(K_v)\} < \infty,$$

where

$$m \frac{\partial}{\partial m} \omega(m) = \frac{\partial}{\partial m} W_m - \omega(m),$$

$$\kappa_{1,m} = \left\{ m: m \in (0,1) \left| \frac{\partial}{\partial m} W_m \text{ exists} \right. \right\}$$

and

$$\kappa_{2,v} = \left\{ v: v \in (0,1) \left| \frac{\partial}{\partial v} K_v \text{ exists} \right. \right\}.$$

The MFGM family can be used for obtaining the following new versions:

**Type-I MFGM type**
Consider the functional forms for $W_m = m\omega(m)$ and $K_v = vC(v)$, the BGWNH-MFGM (Type-I) can be derived from

$$C_Y(m, v) = \left( 1 - \exp \left[ -\left[ \frac{1}{a_1} (m) - 1 \right]^{b_1} \right] \right)^{Y_1} \left( 1 - \exp \left[ -\left[ \frac{1}{a_2} (v) - 1 \right]^{b_2} \right] \right)^{Y_2} + Y \left[ \left( 1 - \exp \left[ -\left[ \frac{1}{a_1} (m) - 1 \right]^{b_1} \right] \right)^{Y_1} \times \left( 1 - \exp \left[ -\left[ \frac{1}{a_2} (v) - 1 \right]^{b_2} \right] \right)^{Y_2} \right] \left| Y \in (-1,1) \right. \right.$$

**Type-II MFGM type**
Let $\omega(m)$ and $C(v)$ be two functional forms satisfying all the conditions stated earlier where $\omega(m)\mid_{Y>0} = m^{Y_1}(1 - m)^{1-Y_1}$ and $C(v)\mid_{Y>0} = v^{Y_2}(1 - v)^{1-Y_2}$. Then, the corresponding BGWNH-MFGM (Type-II) can be derived from $C_{Y_1,Y_2}(m, v) = mv[1 + Y\omega(m)C(v)]$. Thus

$$C_{Y_1,Y_2}(m, v) = \left( 1 - \exp \left[ -\left[ \frac{1}{a_1} (m) - 1 \right]^{b_1} \right] \right)^{Y_1} \left( 1 - \exp \left[ -\left[ \frac{1}{a_2} (v) - 1 \right]^{b_2} \right] \right)^{Y_2} \times \left( 1 + Y \right) \left( \left( 1 - \exp \left[ -\left[ \frac{1}{a_1} (m) - 1 \right]^{b_1} \right] \right)^{Y_1} \times \left( 1 - \exp \left[ -\left[ \frac{1}{a_2} (v) - 1 \right]^{b_2} \right] \right)^{Y_2} \right) \left| Y \in (-1,1) \right. \right.$$


Type-III MFGM type
Let \( W(m) = m[\log(1 + m^2)] \) and \( K(v) = v[\log(1 + v^2)] \) for all \( \omega(m) \) and \( C(v) \) which satisfy all the conditions stated earlier. In this case, one can also derive a closed form expression for the associated CDF of the BGWNH-MFGM (Type-III) from \( C(m,v) = mn(1 + YW(m)K(v)) \). Then
\[
C(m,v) = \left( 1 - \exp \left\{ -\left[ \frac{1}{m} \right](m - 1)^b_1 \right\} \right)^{\gamma_1} \left( 1 - \exp \left\{ -\left[ \frac{1}{m} \right](m - 1)^b_2 \right\} \right)^{\gamma_2} \left( 1 + \sum \left\{ \frac{1}{1 - \exp \left\{ -\left[ \frac{1}{m} \right](m - 1)^b_2 \right\}} \right\} \right)}
\]

2.3 Ali-Mikhail-Ha (A-M-H) copula
Following Ali et al. (1987), the A-M-H copula can be expressed as
\[
C(m,v) = \frac{vm}{(1 - Yvm)} \text{ if } Y \in [-1,1].
\]
Then for \( m,v \in (0,1) \), the J-CDF of the BGWNH can be written as
\[
C(m,v) = \left( 1 - \exp \left\{ -\left[ \frac{1}{m} \right](m - 1)^b_1 \right\} \right)^{\gamma_1} \left( 1 - \exp \left\{ -\left[ \frac{1}{m} \right](m - 1)^b_2 \right\} \right)^{\gamma_2} \mid Y \in [-1,1].
\]

2.4 BGWNH and MGWNH type via Clayton Copula
The Clayton Copula can be considered as \( C(u_1,u_2) = [(1/u_1)^{\gamma_1} + (1/u_2)^{\gamma_2} - 1]^{-1/\gamma_1} \mid y \in (0,\infty) \). Setting \( u_1 = F_{\frac{m_1}{m}}(m) \) and \( u_2 = F_{\frac{m_2}{m}}(m) \), the BGWNH type can be derived as
\[
C(u_1,u_2) = \left\{ \left( 1 - \exp \left\{ -\left[ \frac{1}{m} \right](m - 1)^b_1 \right\} \right)^{-\gamma_1} + \left( 1 - \exp \left\{ -\left[ \frac{1}{m} \right](m - 1)^b_2 \right\} \right)^{-\gamma_2} - 1 \right\} \mid Y \in (0,\infty)
\]
Similarly, the MGWNH can be derived from
\[
C(u_1) = \sum_{i=1}^{d} \left( 1 - \exp \left\{ -\left[ \frac{1}{m} \right](m - 1)^b_1 \right\} \right)^{-\gamma_i} + 1 - d \right\} \mid Y \in (0,\infty)
\]

2.5 BGWNH type via Renyi's entropy copula
Due to Pougaza and Djafari (2011), the BGWNH type via Renyi's entropy copula can be derived using \( C(m,v) = m_2 m + m_1 v - m_1 m_2 \). Then, the associated BGWNH can be expressed as
\[
C(m_1,m_2) = m_2 \left( 1 - \exp \left\{ -\left[ \frac{1}{m} \right](m - 1)^b_1 \right\} \right)^{\gamma_1} + m_1 \left( 1 - \exp \left\{ -\left[ \frac{1}{m} \right](m - 1)^b_2 \right\} \right)^{\gamma_2} - m_1 m_2.
\]

3. Mathematical properties
3.1 Linear representation
In this section, we provide a useful linear representation for the GWNH density function. If \( |\phi| < 1 \) and \( b > 0 \) is a real non-integer, the power series holds
\[
\left( 1 - \frac{\phi_1}{\phi_2} \right)^{\phi_3} = \sum_{\phi_4=0}^{\infty} (-1)^{\phi_4} \frac{\Gamma(\phi_3)}{\phi_4! \Gamma(\phi_3 - \phi_4 + 1)} \left( \frac{\phi_1}{\phi_2} \right)^{\phi_4}.
\]
Applying (5) to the last term in (4) gives
\[
f(w) = b\gamma a(1 + w)^{a-1}v_a(w)^{b-1} \frac{1 - v_a(w)\gamma^1 - \gamma^2}{v_a^{d+1}(w)}
\]
Expanding the quantity \( A(w) \) in power series, we can write

\[
A(w) = \sum_{k=0}^{\infty} \frac{(-1)^k k! |(\gamma + 1) + 1|!\gamma!}{k! |(\gamma + 1) + 1|!\gamma!} [1 - \varpi_a(w)]^{k+1}.
\]

Inserting the above expression of \( A(w) \) in (6), the GWNH density reduces to

\[
f(w) = a (1 + w)^{a-1} \varpi_a(w) \sum_{\ell_1=0}^{\infty} \frac{(-1)^k k! (\ell_1 + 1) k!}{k! |(\gamma + 1) + 1|!\gamma!} \varpi_a(w) [1 - \varpi_a(w)]^{(1+k_1)\ell_1 + 1}(w).
\]

Using the generalized binomial expansion to \( \varpi_a(w) \) the inserting result in (7), the GWNH density can be expressed as an infinite linear combination of ENH density functions

\[
f(w) = \sum_{k_1,k_2=0}^{\infty} C_{(k_1,k_2)} \pi_\theta(w, a),
\]

where \( \theta = (1 + k_1) + k_2 \) and

\[
\pi_\theta(w, a) = \theta a (1 + w)^{a-1} \varpi_a(w) [1 - \varpi_a(w)]^{\theta-1}
\]

is the ENH PDF with power parameter \( \theta \) and

\[
C_{(k_1,k_2)} = \sum_{\ell_1=0}^{\infty} \frac{(-1)^k k! (\ell_1 + 1) k!}{k! |(\gamma + 1) + 1|!\gamma!} \varpi_a(w) [1 - \varpi_a(w)]^{(1+k_1)\ell_1 + 1}(w).
\]

Equation (8) reveals that the density of \( W \) can be expressed as a linear combination of ENH densities. So, several mathematical properties of the new family can be obtained by knowing those of the eENH distribution. Similarly, the CDF of the GWNH model can also be expressed as a linear combination of ENH CDFs given by

\[
F(w) = \sum_{k_1,k_2=0}^{\infty} C_{(k_1,k_2)} \Pi_\theta(w, a),
\]

where \( \Pi_\theta(w, a) = [H_a(w)]^\theta \) is the ENH CDF with power parameter \( \theta \).

### 3.2 Moments

The \( r \)th moment of \( w \), say \( \mu'_r \), follows from equation (10) as

\[
\mu'_r = E(w^r) = \sum_{k_1,k_2=0}^{\infty} \sum_{\ell_1=0}^{\infty} C_{(k_1,k_2)} \xi^{(\theta, r)}_{(\ell_1, \ell_2)} \gamma \left( 1 + \frac{\ell_2}{a} \right),
\]

where

\[
\xi^{(\theta, r)}_{(\ell_1, \ell_2)} = \theta \lambda^{-r} \exp\left( 1 + \ell_1 \right) (-1)^{r-\ell_1-\ell_2} (1 + \ell_1)^{-(\ell_2+1)(\theta-1)} (\ell_1)^r.
\]

The variance (\( V(W) \)), skewness (\( \beta_1(W) \)) and kurtosis (\( \beta_2(W) \)) measures can be calculated from the ordinary moments using well-known relationships. Table 2 gives a comprehensive numerical analysis for the \( E(W), V(W), \beta_1(W) \) and \( \beta_2(W) \) for the GWNH distribution. Based on Table 2 we note that \( \beta_1(W) \) of the GWNH distribution is always positive. \( \beta_2(X) \) of the GWNH distribution can be only greater than three. Figure 3 displays some three-dimensional skewness plots for some selected parameter values. Figure 4 gives some three-dimensional kurtosis plots for some selected parameter values.

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<th>( b )</th>
<th>( a )</th>
<th>( E(W) )</th>
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Figure 3: Three-dimensional skewness plots for some selected parameter values.
3.3 QF and MGF

The QF of $W$ is determined by inverting (3). We have

$$
\phi(u) = \left[1 - \log \left(1 - \left\{1 + \left[- \log(1 - u^1/\gamma)\right]^{-1/\delta}\right\}^{-1}\right)\right]^{1/\alpha} - 1, \ 0 < u < 1.
$$

The MGF can follow from equation (8) as

$$
M_w(t) = E(e^{tw}) = \sum_{k_1,k_2,\ell_1=0}^{\infty} \sum_{r=0}^{\ell_1} \frac{t^r}{r!} C(k_1,k_2) \xi_{\ell_1,\ell_2}^{\theta,r} \Gamma\left(1 + \frac{\ell_2}{\alpha}, 1 + \ell_1\right),
$$

or

$$
M_w(t) = \sum_{k_1,k_2,\ell_1=0}^{\infty} \sum_{r=0}^{\ell_1} \sum_{\ell_2=0}^{r} \frac{t^r}{r!} C(k_1,k_2) \xi_{\ell_1,\ell_2}^{\theta,r} \Gamma\left(1 + \frac{\ell_2}{\alpha}, 1 + \ell_1\right) \left|\theta > 0 \text{ and integer}\right|.
$$
3.4 Incomplete moments
The main applications of the first incomplete moment are related to the mean deviations and Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The \( s \)-th incomplete moment, say \( I_s(t) \), of \( W \) can be expressed from (8) as

\[
I_s(t) = \int_{-\infty}^{t} w^s f(w) dw = \sum_{k_1,k_2=t}^{\infty} \sum_{\ell_1=0}^{s} c_{(k_1,k_2)} \xi_{\ell_1,\ell_2} \left[ \Gamma \left( 1 + \frac{\ell_2}{a}, 1 + \ell_1 \right) - \Gamma \left( 1 + \frac{\ell_2}{a}, (1 + \ell_1)(1 + t)^a \right) \right],
\]

or

\[
I_s(t) = \sum_{k_1,k_2=0}^{\infty} \sum_{\ell_1=0}^{s} \sum_{\ell_2=0}^{\theta-1} c_{(k_1,k_2)} \xi_{\ell_1,\ell_2} \left[ \Gamma \left( 1 + \frac{\ell_2}{a}, 1 + \ell_1 \right) - \Gamma \left( 1 + \frac{\ell_2}{a}, (1 + \ell_1)(1 + t)^a \right) \right] \quad \text{for} \theta > 0 \text{ and integer}.
\]

The mean deviations about the mean \( \delta_1 = E(|w - \mu_1|) \) and about the median \( \delta_2 = E \left( \left| w - \phi \left( \frac{1}{2} \right) \right| \right) \) of \( W \) are given by \( \delta_1 = 2 \mu_1 F(\mu_1) - 2I_1(\mu_1) \) and \( \delta_2 = \mu_1 - 2I_1 \left( \phi \left( \frac{1}{2} \right) \right) \), respectively, where \( \mu_1 = E(w) \), \( \phi \left( \frac{1}{2} \right) \) is the median, \( F(\mu_1) \) is easily evaluated from (3) and \( I_1(t) \) is the first incomplete moment given by (9) or (10) with \( s = 1 \). The \( I_1(t) \) can be obtained as

\[
I_1(t) = \sum_{k_1,k_2=0}^{\infty} \sum_{\ell_1=0}^{\theta-1} \sum_{\ell_2=0}^{1} c_{(k_1,k_2)} \xi_{\ell_1,\ell_2} \left[ \Gamma \left( 1 + \frac{\ell_2}{a}, 1 + \ell_1 \right) - \Gamma \left( 1 + \frac{\ell_2}{a}, (1 + \ell_1)(1 + t)^a \right) \right],
\]

or

\[
I_1(t) = \sum_{k_1,k_2=0}^{\infty} \sum_{\ell_1=0}^{\theta-1} \sum_{\ell_2=0}^{1} c_{(k_1,k_2)} \xi_{\ell_1,\ell_2} \left[ \Gamma \left( 1 + \frac{\ell_2}{a}, 1 + \ell_1 \right) - \Gamma \left( 1 + \frac{\ell_2}{a}, (1 + \ell_1)(1 + t)^a \right) \right] \quad \text{for} \theta > 0 \text{ and integer}.
\]

3.5 Lorenz and Bonferroni curves
The Bonferroni \( B_{\text{curve}} \) and Lorenz \( L_{\text{curve}} \) curves have many applications especially in economics, demography, insurance, reliability, medicine etc. The Bonferroni and Lorenz curves can be derived as

\[
L_{\text{curve}} = \frac{1}{E(W)} \int_{0}^{x} tf(t) dt,
\]

and

\[
B_{\text{curve}} = \frac{1}{E(W)F(w)} \int_{0}^{x} tf(t) dt = \frac{L_{\text{curve}}}{F(w)}.
\]

Then

\[
L_{\text{curve}} = \sum_{k_1,k_2=0}^{\infty} \sum_{\ell_1=0}^{\theta-1} \sum_{\ell_2=0}^{1} c_{(k_1,k_2)} \xi_{\ell_1,\ell_2} \left[ \Gamma \left( 1 + \frac{\ell_2}{a}, 1 + \ell_1 \right) - \Gamma \left( 1 + \frac{\ell_2}{a}, (1 + \ell_1)(1 + t)^a \right) \right],
\]

or

\[
L_{\text{curve}} = \sum_{k_1,k_2=0}^{\infty} \sum_{\ell_1=0}^{\theta-1} \sum_{\ell_2=0}^{1} c_{(k_1,k_2)} \xi_{\ell_1,\ell_2} \left[ \Gamma \left( 1 + \frac{\ell_2}{a}, 1 + \ell_1 \right) - \Gamma \left( 1 + \frac{\ell_2}{a}, (1 + \ell_1)(1 + t)^a \right) \right] \quad \text{for} \theta > 0 \text{ and integer}.
\]

and

\[
B_{\text{curve}} = \left[ 1 - \exp \left( -\left\{ -\left[ \frac{1}{a} (w) - 1 \right] b \right\} \right) \right]^{-\gamma}
\]
\[
\sum_{k_1,k_2=0}^{\infty} \sum_{\ell_1=0}^{\infty} \frac{C(k_1,k_2)E_{\ell_1,\ell_2}}{(k_1+1)!(k_2+1)!} \left[ \Gamma\left(1+\frac{\ell_2}{\alpha},1+\ell_1\right) - \Gamma\left(1+\frac{\ell_2}{\alpha},(1+\ell_1)(1+t)^{\alpha}\right) \right], \\
\sum_{k_1,k_2,\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} \frac{C(k_1,k_2)E_{\ell_1,\ell_2}}{(k_1+1)!(k_2+1)!} \left[ \Gamma\left(1+\frac{\ell_2}{\alpha},1+\ell_1\right) - \Gamma\left(1+\frac{\ell_2}{\alpha},(1+\ell_1)(1+t)^{\alpha}\right) \right]
\]

or

\[
B_{\text{curve}} = \left[1 - \exp\left(-\left(\frac{\gamma}{\alpha} - 1\right)\right)\right]^{\gamma}
\]

\[
\sum_{k_1,k_2=0}^{\infty} \sum_{\ell_1=0}^{\infty} \frac{C(k_1,k_2)E_{\ell_1,\ell_2}}{(k_1+1)!(k_2+1)!} \left[ \Gamma\left(1+\frac{\ell_2}{\alpha},1+\ell_1\right) - \Gamma\left(1+\frac{\ell_2}{\alpha},(1+\ell_1)(1+t)^{\alpha}\right) \right].
\]

3.6 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Let \( W_1, W_2, ..., W_n \) be a random sample (RS) from the GWNH model of distributions. The PDF of the \( i^{th} \) order statistic, say \( W_{i:n} \), can be expressed as

\[
f_{i:n}(w) = \frac{f(w)}{B(i,n-i+1)} \sum_{k_2=0}^{n-i} (-1)^{k_2} \binom{n-i}{k_2} F^{k_2+i-1}(w),
\]

where \( B(\cdot, \cdot) \) is the beta function. Based on equations (5) and (6), we have

\[
f(w) F^{k_2+i-1}(w) = yba(1+w)^{a-1}v_a(w) \frac{\{1-v_a(w)\}^{b-1}}{\{v_a(w)\}^{b+1}} \\
\times \exp\left(-\left[1-v_a(w)\right]\right) \left[1-\exp\left(-\left[\frac{1-v_a(w)}{v_a(w)}\right]\right)\right]^{\gamma(k_2+i)-1}.
\]

Following the same steps of the linear representation (8), we obtain

\[
f(w) F^{k_2+i-1}(w) = a(1+w)^{a-1}v_a(w) \sum_{k_1=0}^{\infty} \binom{-1+i+k_1+b}{k_1+k_2} \frac{y \Gamma([k_2+i]+1) \Gamma([1+k_2+b+1]+1)}{l! k_1! \Gamma([k_2+i]+1)} \frac{\{1-v_a(w)\}^{1+k_1+b-1}}{\{v_a(w)\}^{1+k_1+b+1}}.
\]

Then

\[
f(w) F^{k_2+i-1}(w) = \sum_{k_1,k_2=0}^{\infty} q(k_1,k_2) \pi_\theta(w; \alpha),
\]

where \( \pi_\theta(w; \alpha) \) is the ENH density with power parameter \( \theta \) and

\[
q(k_1,k_2) = \sum_{i=0}^{\infty} \frac{(-1)^{i+k_1}y b(1+i+l)^{k_1+1} \Gamma([k_2+i]+1) \Gamma([1+k_2+b+1]+1)}{l! k_1! \Gamma([k_2+i]+1)} \Gamma([1+k_2+b+1]+1).
\]

Substituting (12) in equation (11), the PDF of \( W_{i:n} \) can be expressed as

\[
f_{i:n}(w) = \frac{1}{B(i,n-i+1)} \sum_{k_1,k_2=0}^{\infty} q(k_1,k_2)^* \pi_\theta(w; \alpha),
\]

where \( q^*(k_1,k_2) = \sum_{k_2=0}^{n-i} (-1)^{k_2} \binom{n-i}{k_2} q(k_1,k_2) \). The PDF of the GWNH order statistics is a linear combination of ENH densities. The moments of \( W_{i:n} \) are given by

\[
E(W_{i:n}^r) = \sum_{k_1,k_2,\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} \frac{q(k_1,k_2)E_{\ell_1,\ell_2}}{B(i,n-i+1)} \Gamma\left(1+\frac{\ell_2}{\alpha},1+\ell_1\right),
\]

or

\[
E(W_{i:n}^r) = \sum_{k_1,k_2,\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} \frac{q(k_1,k_2)E_{\ell_1,\ell_2}}{B(i,n-i+1)} \Gamma\left(1+\frac{\ell_2}{\alpha},1+\ell_1\right) \quad (\theta>0 \text{ and integer}).
\]
3.7 Probability weighted moments
The PWMs are expectations of certain functions of a random variable and can be defined for any random variable whose ordinary moments exist. The $(s,r)^{th}$ PWM of the GWNH distribution, say $\rho_{s,r}$, can be formally defined by $\rho_{s,r} = E\{w^s F(w)^r\}$. From equations (5) and (6), we can write

$$f(w)F(w)^r = \sum_{k_1,k_2=0}^{\infty} a^{(r)}_{(k_1,k_2)} \pi_\theta (w; a),$$

where

$$a^{(r)}_{(k_1,k_2)} = \sum_{\ell_1=0}^{\infty} \frac{(-1)^{k_1+\ell_1} b(\ell_1 + 1)^k \Gamma([1 + r] \gamma) \gamma \Gamma(1 + \theta \Gamma([1 + \gamma] \gamma - \ell_1) \Gamma([1 + k_1] \gamma + 1)}{(1 + \theta \gamma)^{-1}}.$$

Then, $\rho_{s,r}$ can be expressed as

$$\rho_{s,r} = \sum_{k_1,k_2=0}^{\infty} \sum_{\ell_1=0}^{\infty} a^{(r)}_{(k_1,k_2)} \int_{-\infty}^{\infty} w^s \pi_\theta (w; a) dw.$$

Finally, the $(s,r)^{th}$ PWM of $W$ can be obtained from an infinite linear combination of ENH moments given by

$$\rho_{s,r} = \sum_{k_1,k_2=0}^{\infty} \sum_{\ell_1=0}^{\infty} a^{(r)}_{(k_1,k_2)} \xi_{(\theta),\gamma}^{(\ell_1,\ell_2)} (1 + \frac{\ell_2}{a}, 1 + \ell_1).$$

or

$$\rho_{s,r} = \sum_{k_1,k_2=0}^{\infty} \sum_{\ell_1=0}^{\ell_2} a^{(r)}_{(k_1,k_2)} \xi_{(\theta),\gamma}^{(\ell_1,\ell_2)} (1 + \frac{\ell_2}{a}, 1 + \ell_1) |(\theta > 0 \text{ and integer}).$$

4. Estimation
Consider the following classical estimation methods:
1. Maximum likelihood (ML) method.
2. Cramér-von-Mises (CVM) method.
3. Ordinary least squares (OLS) method.
4. Weighted least squares (WLS) method.
5. Anderson Darling (AD) method.
6. Right Tail-Anderson Darling (ADERT) method.
7. Left Tail-Anderson Darling (ADELT) method.

All these methods are discussed in the statistical literature with more details. In this work, we may ignore many of its derivation details for avoiding the replication.

4.1 The ML method
The maximum likelihood estimates (MLEs) enjoy desirable properties and can be used when constructing confidence intervals. Let $w_1, w_2, ... , w_n$ be a RS from this distribution with parameter vector $\theta = (\gamma, b, a)^T$. The log-likelihood function for $\theta$, say $\ell(\theta)$, is given by

$$\ell(\theta) = n \log b + n \log \gamma + n \log a - \sum_{i=0}^{n} \left[ \mathcal{W}_a^{-1}(w_{[i:n]}) - 1 \right]^b + (b - 1) \sum_{i=0}^{n} \log \left[ 1 - \exp \left[ 1 - (1 + w_{[i:n]})^a \right] \right] + (-1 + a) \sum_{i=0}^{n} \log \left[ 1 + w_{[i:n]} \right] - b \sum_{i=0}^{n} \left[ 1 - (1 + w_{[i:n]})^a \right] + (\gamma - 1) \sum_{i=0}^{n} \log \left[ 1 - \exp \left( -\mathcal{W}_a^{-1}(w_{[i:n]})^b \right) \right],$$

which can be maximized either using the statistical programs or by solving the nonlinear system obtained from $\ell(\theta)$ by differentiation. The score vector, $U(\theta) = \left( \frac{\partial \ell(\theta)}{\partial \gamma}, \frac{\partial \ell(\theta)}{\partial b}, \frac{\partial \ell(\theta)}{\partial a} \right)^T$, are easy to derive.
4.2 The CVM method
The Cramér-von Mises estimates (CVME) of the parameters $\gamma$, $b$ and $\alpha$ are obtained via minimizing the following expression with respect to the parameters $\gamma$, $b$ and $\alpha$ respectively, where

$$CVM(\delta) = \frac{1}{12}n^{-1} + \sum_{i=1}^{n} \left[ F_{\hat{G}}(w_{[i:n]}) - \varepsilon_{(i:n)} \right]^2,$$

where $\varepsilon_{(i:n)} = \frac{2i-1}{2n}$ and

$$CVM(\delta) = \sum_{i=1}^{n} \left[ \left( 1 - \exp \left\{ -\left[ \frac{1}{b} \left( w_{[i:n]} - 1 \right) \right]^{\gamma} \right\} \right)^{\gamma} - \varepsilon_{(i:n)} \right]^2.$$

Then, CVME of the parameters $\gamma$, $b$ and $\alpha$ are obtained by solving the two following non-linear equations

$$\sum_{i=1}^{n} \left[ \left( 1 - \exp \left\{ -\left[ \frac{1}{b} \left( w_{[i:n]} - 1 \right) \right]^{\gamma} \right\} \right)^{\gamma} - \varepsilon_{(i:n)} \right] \zeta(\gamma)(w_{[i:n]}, \delta) = 0,$$

$$\sum_{i=1}^{n} \left[ \left( 1 - \exp \left\{ -\left[ \frac{1}{b} \left( w_{[i:n]} - 1 \right) \right]^{\gamma} \right\} \right)^{\gamma} - \varepsilon_{(i:n)} \right] \zeta(b)(w_{[i:n]}, \delta) = 0,$$

and

$$\sum_{i=1}^{n} \left[ \left( 1 - \exp \left\{ -\left[ \frac{1}{b} \left( w_{[i:n]} - 1 \right) \right]^{\gamma} \right\} \right)^{\gamma} - \varepsilon_{(i:n)} \right] \zeta(\alpha)(w_{[i:n]}, \delta) = 0,$$

where $\zeta(\gamma)(w_{[i:n]}, \delta)$, $\zeta(b)(w_{[i:n]}, \delta)$ and $\zeta(\alpha)(w_{[i:n]}, \delta)$ are the first derivatives of the CDF of GWNH distribution with respect to $\gamma$, $b$ and $\alpha$ respectively.

4.3 OLS method
Let $F_{\hat{G}}(w_{[i:n]})$ denote the CDF of GWNH model and let $w_{1,n} < w_{2,n} < \cdots < w_{n,n}$ be the $n$ ordered RS. The ordinary least squares estimates (OLSEs) are obtained upon minimizing

$$OLSE(\delta) = \sum_{i=1}^{n} \left[ F_{\hat{G}}(w_{[i:n]}) - \tau_{(i:n)} \right]^2.$$

Then, we have

$$OLSE(\delta) = \sum_{i=1}^{n} \left[ \left( 1 - \exp \left\{ -\left[ \frac{1}{b} \left( w_{[i:n]} - 1 \right) \right]^{\gamma} \right\} \right)^{\gamma} - \tau_{(i:n)} \right]^2,$$

where $\tau_{(i:n)} = \frac{i}{n+1}$. The LSEs are obtained via solving the following non-linear equations

$$0 = \sum_{i=1}^{n} \left[ \left( 1 - \exp \left\{ -\left[ \frac{1}{b} \left( w_{[i:n]} - 1 \right) \right]^{\gamma} \right\} \right)^{\gamma} - \tau_{(i:n)} \right] \zeta(\gamma)(w_{[i:n]}, \delta),$$

$$0 = \sum_{i=1}^{n} \left[ \left( 1 - \exp \left\{ -\left[ \frac{1}{b} \left( w_{[i:n]} - 1 \right) \right]^{\gamma} \right\} \right)^{\gamma} - \tau_{(i:n)} \right] \zeta(b)(w_{[i:n]}, \delta),$$

and

$$0 = \sum_{i=1}^{n} \left[ \left( 1 - \exp \left\{ -\left[ \frac{1}{b} \left( w_{[i:n]} - 1 \right) \right]^{\gamma} \right\} \right)^{\gamma} - \tau_{(i:n)} \right] \zeta(\alpha)(w_{[i:n]}, \delta),$$

where $\zeta(\gamma)(w_{[i:n]}, \delta)$, $\zeta(b)(w_{[i:n]}, \delta)$ and $\zeta(\alpha)(w_{[i:n]}, \delta)$ are as defined above.

4.4 WLS method
The weighted least squares estimates (WLSSEs) are obtained by minimizing the function $WLSSE(\delta)$ with respect to $\gamma$, $b$ and $\alpha$ where

$$WLSSE(\delta) = \sum_{i=1}^{n} \omega_{(i:n)} \left[ F_{\hat{G}}(w_{[i:n]}) - \tau_{(i:n)} \right]^2,$$
where
\[ \omega_{(\ell,n)} = \frac{(1+n)^2(2+n)}{\ell(1+n-\ell)}. \]

The WLSEs are obtained by solving
\[ 0 = \sum_{i=1}^{n} \omega_{(\ell,n)} \left( 1 - \exp \left\{ -\left[ \gamma_0^{-1}(w_{[i:n]}) - 1 \right]^b \right\} \right)^\gamma - \tau_{(\ell,n)} \] \[ \times \eta_{\gamma}(w_{[i:n]}, \delta), \]
\[ 0 = \sum_{i=1}^{n} \omega_{(\ell,n)} \left( 1 - \exp \left\{ -\left[ \gamma_0^{-1}(w_{[i:n]}) - 1 \right]^b \right\} \right)^\gamma - \tau_{(\ell,n)} \] \[ \times \eta_{\beta}(w_{[i:n]}, \delta), \] and
\[ 0 = \sum_{i=1}^{n} \omega_{(\ell,n)} \left( 1 - \exp \left\{ -\left[ \gamma_0^{-1}(w_{[i:n]}) - 1 \right]^b \right\} \right)^\gamma - \tau_{(\ell,n)} \] \[ \times \eta_{\alpha}(w_{[i:n]}, \delta), \]

where \( \eta_{\gamma}(w_{[i:n]}, \delta), \ eta_{\beta}(w_{[i:n]}, \delta) \) and \( \eta_{\alpha}(w_{[i:n]}, \delta) \) are as defined above.

4.5 The AD method

The Anderson-Darling estimates (ADEs) of \( \gamma \), \( b \) and \( a \) are obtained by minimizing the function
\[ \text{ADE}(\delta) = -n - n^{-1} \sum_{i=1}^{n} (2i-1) \left\{ -\log F(w_{[i:n]}) \log \left[ 1 - F(w_{[i+1:i+n:n]}) \right] \right\}; \]
The parameter estimates of \( \gamma \), \( b \) and \( a \) follow by solving the nonlinear equations
\[ \frac{\partial}{\partial \gamma} \left[ \text{ADE}(\delta) \right] = 0, \frac{\partial}{\partial b} \left[ \text{ADE}(\delta) \right] = 0, \]
and
\[ \frac{\partial}{\partial a} \left[ \text{ADE}(\delta) \right] = 0. \]

4.6 The ADERT method

The right-tail Anderson-Darling estimates (ADERTEs) of \( \gamma \), \( b \) and \( a \) are obtained by minimizing
\[ \text{ADERT}(\delta) = -\frac{1}{2} n - \frac{1}{2} \sum_{i=1}^{n} F(w_{[i:n]}) - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \left\{ -\log F(w_{[i+1:i+n:n]}) \right\}; \]
The parameter estimates of \( \gamma \), \( b \) and \( a \) follow by solving the nonlinear equations
\[ \frac{\partial}{\partial \gamma} \left[ \text{ADERT}(\delta) \right] = 0, \frac{\partial}{\partial b} \left[ \text{ADERT}(\delta) \right] = 0, \]
and
\[ \frac{\partial}{\partial a} \left[ \text{ADERT}(\delta) \right] = 0. \]

4.7 The ADELT method

The left-tail Anderson-Darling estimates (LTADEs) of \( \gamma \), \( b \) and \( a \) are obtained by minimizing
\[ \text{ADELT}(\delta) = -\frac{3}{2} n + \frac{3}{2} \sum_{i=1}^{n} F(w_{[i:n]}) - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \log F(w_{[i:n]}); \]
The parameter estimates of \( \gamma \), \( b \) and \( a \) follow by solving the nonlinear equations
\[ \frac{\partial}{\partial \gamma} \left[ \text{ADELT}(\delta) \right] = 0, \frac{\partial}{\partial b} \left[ \text{ADELT}(\delta) \right] = 0, \]
and
\[ \frac{\partial}{\partial a} \left[ \text{ADELT}(\delta) \right] = 0. \]

5. Simulation studies for comparing estimation methods

A numerical simulation is performed to compare the classical estimation methods. The simulation study is based on \( N = 1000 \) generated data sets from the GWNH version where \( n = 50, 100, 150 \) and \( 300 \) and
The estimates are compared in terms of bias (BIAS), root mean-standard error (RMSE), the mean of the absolute difference between the theoretical and the estimates (D.abs) and the maximum absolute difference between the true parameters and estimates (D.max). From Tables 3, 4 and 5 we note that:

1-The BIAS tend to zero when n increases which means that all estimators are consistent.

2-The RMSE tend to zero when n increases which means incidence of consistency property.

Table 3: Simulation results for blend I.

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Table 4: Simulation results for blend II.

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### Simulation results for blend II

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### Table 5: Simulation results for blend III.

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<tr>
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A New Flexible Probability Model: Theory, Estimation and Modeling Bimodal Left Skewed Data

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6. Applications
6.1 Application for comparing classical methods

For comparing classical methods, an application to real data set is introduced. We consider the Cramér-Von Mises (WC) and the Anderson-Darling (AC) statistics. The data is obtained from Section 1.15 of Klein and Moeschberger (2006). This sample is part of a larger study of psychiatric inpatients discussed by Tsuang and Woolson (1977). Data for each patient consists of age, sex, number of years of follow-up (years from admission to death or censoring) and patient status at the follow up time. The data are: 1, 1, 2, 22, 30, 28, 32, 11, 14, 36, 31, 33, 33, 37, 35, 25, 31, 22, 26, 24, 35, 34, 30, 35, 40, 39. Other real-life datasets are in Aryal and Yousof (2017), Yousof et al. (2017), Hamedani et al. (2017, 2018 and 2019), Merovci et al. (2017 and 2020), Korkmaz et al. (2018a), Korkmaz et al. (2018b), Nascimento et al. (2019), Alizadeh et al. (2020a,b), Mansour et al. (2020a-f) Korkmaz et al. (2020) and Karamikabir et al. (2020).

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The HRF of psychiatric patients’ data set is "bathtub (U-HRF)". Due to Figure 7(d), the psychiatric patient’s data set is a bimodal and left skewed data. From Table 6, the MLE method is the best method with WC =0.31136 and AC =1.90220 then ADE with WC =0.37273 and AC =2.23792. However, CVM, OLS, WLS, ADERT and ADELT performed well. Figure 5 gives probability-probability (P-P) plots for comparing classical methods. Figure 6 gives Kaplan-Meier survival plots for comparing classical methods.

<table>
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<th>AC</th>
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<td>3.64103</td>
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<tr>
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<td>16.399</td>
<td>0.134</td>
<td>0.46552</td>
<td>2.72700</td>
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</table>
Figure 5: P-P plots for comparing classical methods.
6.2 Application for comparing models
We present an application based on a real data set related to times of death of twenty-six psychiatric patients. The data of Klein and Moeschberger (1997) is used in this part. First, we compare GWNH distribution with some well-known four and five parameter distributions with names given below

<table>
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<th>Competitive model &amp; Abbreviation</th>
<th>Author</th>
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<td>NH</td>
<td>Nadarajah and Haghhighi (2011)</td>
</tr>
<tr>
<td>Beta NH (BNH)</td>
<td>Dias (2016)</td>
</tr>
<tr>
<td>Kumaraswamy NH (KumNH)</td>
<td>Lima (2015)</td>
</tr>
<tr>
<td>exponentiated generalized NH (EGNH)</td>
<td>VedoVatto (2016)</td>
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<tr>
<td>Beta Exponentiated Weibull (BEW)</td>
<td>Cordeiro et al. (2013)</td>
</tr>
<tr>
<td>Exponentiated Exponential (EExp)</td>
<td>Gupta and Kundu (2001)</td>
</tr>
</tbody>
</table>

Figure 6: Kaplan-Meier survival plots for comparing classical methods.
For the purpose of comparisons, we shall use the following statistics:

I. $-\log$-likelihood function.

II. Akaike information criterion ($\text{Cr}(1)$)

III. Bayesian information criterion ($\text{Cr}(2)$).

IV. Hannan-Quinn information criterion ($\text{Cr}(3)$).

V. Consistent Akaike information criterion ($\text{Cr}(4)$).

In general, the smaller values of these criteria show the better fit to the data sets. In all of mentioned goodness of fit test criteria, GWNH shows better fit to this data sets. The estimated parameters based on MLE procedure reports in Tables 1 as well as the values of goodness-of-fit statistics are given in Tables 2. Also, Table 3 includes the likelihood ratio test for sub models comparison. The total time on test (TTT) plot (a), box plot (b), quantile-quantile (Q-Q) plot (c) and nonparametric Kernel density estimation (NKDE) plot (d) are presented in Figure 7. Figure 7(a) indicates that the HRF of psychiatric patients’ data set is "bathtub (U-HRF)". Based on Figure 7(b), the psychiatric patient’s data contains some extreme values and the Q-Q plot (Figure 7(c)) ensure this fact. Due to the NKDE plot (Figure 7(d)), the psychiatric patients data set is a bimodal and left skewed data. Table 7 gives the parameters estimates and standard deviation in parenthesis. Table 8 gives the goodness of fit criteria. From Table 3 we note that, the new GWNH model is better that the EENH, BW, GaMW, BEW, KwNH, BNH and EGNH. Figure 8 gives the estimated CDF (ECDF), EPDF, Kaplan-Meier survival plot, P-P plot and EHRF for the times of death data. Figure 8 indicates that the new model gives a good fit to the used date set.

### Table 7: Parameters estimates and standard deviation in parenthesis.

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<th>Estimates</th>
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</tr>
<tr>
<td>NH(a)</td>
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<tr>
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<td>(0.0309)</td>
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<td>EExp(b,a)</td>
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</tr>
<tr>
<td>GWNH(γ,b,a)</td>
<td>0.1723</td>
</tr>
<tr>
<td></td>
<td>(0.035)</td>
</tr>
<tr>
<td>KwNH(λ,γ,b,a)</td>
<td>1.844</td>
</tr>
<tr>
<td></td>
<td>(0.509)</td>
</tr>
<tr>
<td>BNH(λ,γ,b,a)</td>
<td>3.384</td>
</tr>
<tr>
<td></td>
<td>(0.528)</td>
</tr>
<tr>
<td>EGNH(λ,γ,b,a)</td>
<td>0.0613</td>
</tr>
<tr>
<td></td>
<td>(0.012)</td>
</tr>
<tr>
<td>BEW(λ,γ,θ,b,a)</td>
<td>0.1873</td>
</tr>
<tr>
<td></td>
<td>(0.07)</td>
</tr>
</tbody>
</table>

### Table 8: Goodness of fit criteria

<table>
<thead>
<tr>
<th>Model</th>
<th>Goodness of fit criteria</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-Log Likelihood</td>
</tr>
<tr>
<td>GWNH</td>
<td>99.180</td>
</tr>
<tr>
<td>KwNH</td>
<td>103.923</td>
</tr>
<tr>
<td>BEW</td>
<td>104.908</td>
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<tr>
<td>BNH</td>
<td>108.591</td>
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<tr>
<td>EGNH</td>
<td>108.997</td>
</tr>
<tr>
<td>EExp</td>
<td>108.987</td>
</tr>
<tr>
<td>Exp</td>
<td>111.130</td>
</tr>
<tr>
<td>NH</td>
<td>127.813</td>
</tr>
</tbody>
</table>
Figure 7: TTT plot(a), box plot(b), Q-Q plot (c) and NKDE plot(d).
A New Flexible Probability Model: Theory, Estimation and Modeling Bimodal Left Skewed Data
7. Conclusions
We introduced a new model called the generalized Weibull Nadarajah Haghighi model with three parameters. The new density can be expressed as a straightforward linear combination of the exponentiated Nadarajah Haghighi densities. The new failure rate can be monotonically increasing, monotonically decreasing, J-shape, upside-down-bathtub, bathtub and constant. We derived some of its mathematical and statistical quantities including the ordinary moments, the moment generating function, the incomplete moments, moments of residual life and reversed residual life. Some new bivariate type distributions using Farlie Gumbel Morgenstern, modified Farlie Gumbel Morgenstern, Clayton, Renyi and Ali-Mikhail-Haq copulas are investigated. The maximum likelihood, Cramér-von-Mises, ordinary least square, weighted least square, Anderson Darling, right tail Anderson Darling and left tail Anderson Darling methods to estimate the unknown model parameters. Simulation studies for comparing estimation methods were performed. Moreover, an application to real data set for comparing methods is also presented. The maximum likelihood method is the best method with $W_C=0.31136$ and $AC=1.90220$. Then, the Anderson Darling with $W_C=0.37273$ and $AC=2.23792$. However, the other methods performed well. The estimation methods are compared using probability-probability plots and Kaplan-Meier survival plots. Another real application for comparing the competitive models was also investigated.

It is suggested to present a novel discrete generalized Weibull Nadarajah Haghighi model for modeling count real-life data (see Aboraya et al. (2020), Chesneau et al. (2021), Yousof et al. (2021) and Ibrahim et al. (2022) for more details). Furthermore, applying the Nikulin–Rao-Robson and Bagdonavičius-Nikulin tests (see Ibrahim et al. (2019), Goual et al. (2019, 2020), Yadav et al. (2020 and 2022), Ibrahim et al. (2022), Goual and Yousof (2020) and Aidi et al. (2021), among others). For regression modeling under censored data sets see, some new regression models could be presented due to Altun et l. (2018a,b) and Yousof et al. (2019). Reliability estimation for the remained stress-strength model under the generalized Weibull Nadarajah Haghighi distribution could be presented (see Saber et al. (2022)) and Bayesian and classical inference for the generalized stress-strength parameter under generalized Weibull Nadarajah Haghighi model (see Saber and Yousof (2022)). Following, Ahmed and Yousof (2022) and Ahmed et al. (2022), a single acceptance sampling plan with its corresponding application in quality and risk decisions can be presented. Finally, for applications in insurance filed one can follow Mohamed et al. (2022).

References


