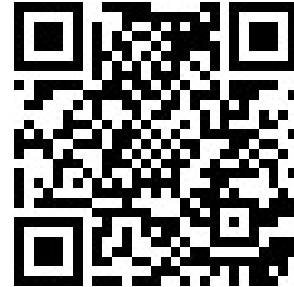## Pakistan Journal of Statistics and Operation Research

Alpha Power Exponentiated New Weibull-Pareto Distribution: Its Properties and Applications

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Abstract

This paper introduces a novel alpha power exponentiated Weibull-Pareto distribution based on the alpha power transformation. We derive several properties of the new distribution, including moments, quantile function, mean residual life, mean waiting time, and order statistics. Estimating model parameters is performed using the method of maximum likelihood. Then, for the purpose of evaluating the effectiveness of the estimates, we conduct some simulation studies. Finally, we demonstrate the superiority of this new model by analyzing three real-life data sets.

Key Words: Alpha Power Transformation; Exponentiated Method; T-X Family; New Weibull-Pareto Distribution; Maximum Likelihood Estimation.

1. Introduction

Probability distributions are required for many theoretical and practical statistical methods, including inference, modeling, survival analysis, and reliability analysis. Statistical distributions are useful in many fields. In engineering, for instance, they can be used to model the life cycle of a machine. In medical sciences, statistical distributions have been used to study the duration to recurrence of some types of cancer after surgical removal and survival times of patients after surgery. The probability distribution therefore provides essential information about statistical inference and data analysis. Such information may be useful for making some critical decisions. Accordingly, selecting the appropriate distribution to employ when modeling the data is extremely important. Identifying a suitable distribution for a set of data significantly enhances the accuracy of the statistical analysis. Frequently, data may exhibit certain characteristics that cannot be adequately explained by a classical distribution. It is therefore crucial to develop new methods for modifying existing distributions in order to improve the goodness of fit.

In recent years, there has been an increased interest in extending existing classical distributions to obtain greater flexibility in modeling data from different fields of study. Most of these extensions are obtained through developing methods for generating new classes of distributions that extend the existing standard models. Various techniques have been proposed in the literature to generate new distributions by adding one or more additional shape parameter(s). Examples of such well-known generators include the beta-G family proposed by Eugene et al. (2002), Gamma-G (type 1) by Zografos and Balakrishnan (2009), the Kw-G

by Jones (2009), McDonald-G by Alexander et al. (2012), Gamma-G (type 2) by Ristić and Balakrishnan (2012), Gamma-G (type 3) by Torabi and Hedesh (2012), among others.

Additionally, Alzaatreh et al. (2013) developed a general technique that allows the use of any baseline distribution as a generator. This innovative technique is described as the transformed transformer $(T - X)$ family of distributions.

Based on the $T - X$ technique, Nasiru and Luguterah (2015) suggested the new Weibull-Pareto (NWP) distribution. Following that, Al-Omari et al. (2019) combined the two methods of the exponentiated class and the $T - X$ family in order to introduce the exponentiated new Weibull-Pareto (ENWP) distribution. Thus, the cumulative distribution function (CDF) of the ENWP distribution is expressed as

$$F(x) = \left[1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} \right]^{\omega}. \tag{1}$$

The corresponds probability distribution function (PDF) is expressed as

$$f(x) = \frac{\omega\delta\beta}{\theta^{\beta}} x^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} \left[1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} \right]^{\omega-1}, \tag{2}$$

where $\beta, \theta, \omega, \delta > 0$ and $x > 0$.

Recently, Mahdavi and Kundu (2017) have proposed a new technique called the alpha power transformation (APT) for adding an extra parameter to a family of distributions. This parameter provides more flexibility to the CDF and PDF of an APT family which can be expressed as

$$F_{APT}(x) = \begin{cases} \frac{\alpha^{F(x)} - 1}{\alpha - 1} & \text{if } \alpha > 0, \alpha \neq 1 \\ \\ F(x) & \text{if } \alpha = 1, \end{cases} \tag{3}$$

$$f_{APT}(x) = \begin{cases} \frac{\log \alpha}{\alpha - 1} f(x) \alpha^{F(x)} & \text{if } \alpha > 0, \alpha \neq 1 \\ \\ f(x) & \text{if } \alpha = 1, \end{cases} \tag{4}$$

where $F(x)$ and $f(x)$ represent the CDF and PDF of any continuous distribution.

This family of distributions has been applied by many authors, include Mahdavi and Kundu (2017) who applied the APT technique to the exponential distribution to obtain the alpha power exponential (APE) distribution, Nassar et al. (2017) proposed a three-parameter distribution, known as the alpha power Weibull distribution, Dey et al. (2018) introduced the alpha power transformed Lindley distribution, Ihtisham et al. (2019) introduced the alpha power Pareto distribution, Dey et al. (2019) proposed the alpha power transformed inverse Lindley distribution and Eghwerido et al. (2020) presented the alpha power Gompertz distribution.

The main aim of this article is to introduce a new probability distribution, called the alpha power exponentiated new Weibull-Pareto distribution (APENWPD), based on a new technique. Particularly, this technique combines the three approaches of $T - X$, exponentiated, and APT in order to increase the flexibility of modelling real data.

This article is planned as follows: in Section 2, we define the APENWPD and provide some plots for its PDF and hazard rate function. We derive in Section 3 some fundamental statistical properties of the APENWPD. In Section 4, we discuss the estimation of the unknown model parameters using the maximum likelihood method and Section 5 provides some simulation studies that evaluate these estimates. In Section 6, we consider three applications that show the efficiency of the introduced distribution. Finally, in Section 7, we offer some concluding remarks.

2.   The alpha power exponentiated new Weibull-Pareto distribution

In this section, we present the five-parameter alpha power exponentiated new Weibull-Pareto distribution (APENWPD). We obtain the CDF and PDF of APENWPD via inserting (1) and (2) in (3) and (4) as follows

$$F(x) = \begin{cases} \dfrac{\alpha^{\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\omega}} - 1}{\alpha - 1} & \text{if } \alpha > 0; \alpha \neq 1 \\[4mm] \left[1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\omega} & \text{if } \alpha = 1, \end{cases} \tag{5}$$

$$f(x) = \begin{cases} \dfrac{\log\alpha}{\alpha-1}\dfrac{\omega\delta\beta}{\theta^{\beta}} x^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} \left[1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\omega-1} \alpha^{\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\omega}} & \text{if } \alpha > 0; \alpha \neq 1 \\[4mm] \dfrac{\omega\delta\beta}{\theta^{\beta}} x^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} \left[1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\omega-1} & \text{if } \alpha = 1, \end{cases} \tag{6}$$

where $\alpha, \beta, \theta, \omega, \delta > 0$ and $x \geq 0$.

The survival function, $S(x) = 1 - F(x)$, of the APENWPD is expressed as

$$S(x) = \begin{cases} \dfrac{\alpha}{\alpha-1}\left(1 - \alpha^{\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\omega} - 1}\right) & \text{if } \alpha > 0; \alpha \neq 1 \\[4mm] 1 - \left[1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\omega} & \text{if } \alpha = 1. \end{cases} \tag{7}$$

The hazard rate function, $h(x) = \dfrac{f(x)}{S(x)}$, of the APENWPD is expressed as

$$h(x) = \begin{cases} \dfrac{\frac{\omega\delta\beta}{\theta^{\beta}} x^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} \left[1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\omega-1} \alpha^{\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\omega} - 1}}{\left(1-\alpha^{\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\omega} - 1}\right)} \log\alpha & \text{if } \alpha > 0; \alpha \neq 1 \\[6mm] \dfrac{\frac{\omega\delta\beta}{\theta^{\beta}} x^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} \left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\omega-1}}{1 - \left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{\omega}} & \text{if } \alpha = 1. \end{cases} \tag{8}$$

2.1.   Special cases of the APENWPD

- The APENWPD reduces to the ENWP distribution at $\alpha = 1$.

- The APENWPD reduces to the new Weibull Pareto distribution at $\alpha = \omega = 1$.

2.2.   Expansion for the PDF.

This subsection provides an expansion for the APENWPD PDF in (6). Particularly, using the following series representation

$$\alpha^{z} = \sum_{g=0}^{\infty} \frac{(\log\alpha)^{g}}{g!} (z)^{g}, \tag{9}$$

we obtain the PDF as follows

$$f(x) = \frac{\log \alpha}{\alpha - 1} \frac{\omega \delta \beta}{\theta^\beta} x^{\beta-1} e^{-\delta \left(\frac{x}{\theta}\right)^\beta} \sum_{g=0}^{\infty} \frac{(\log \alpha)^g}{g!} \left[1 - e^{-\delta \left(\frac{x}{\theta}\right)^\beta}\right]^{\omega-1+\omega g}. \tag{10}$$

Furthermore, applying the following binomial series expansion

$$(1-z)^{a-1} = \sum_{d=0}^{\infty} (-1)^d \binom{a-1}{d} (z)^d, \tag{11}$$

to $\left[1 - e^{-\delta \left(\frac{x}{\theta}\right)^\beta}\right]^{\omega-1+\omega g}$ in (10), we can rewrite the PDF of APENWPD as follows

$$f(x) = \frac{1}{\alpha - 1} \frac{\omega \delta \beta}{\theta^\beta} x^{\beta-1} \sum_{g=0}^{\infty} \sum_{d=0}^{\infty} (-1)^d \frac{(\log \alpha)^{g+1}}{g!} \binom{\omega-1+\omega g}{d} \left(e^{-\delta \left(\frac{x}{\theta}\right)^\beta}\right)^{(d+1)}$$

$$f(x) = \sum_{g=0}^{\infty} \sum_{d=0}^{\infty} \tau \, x^{\beta-1} e^{-(d+1)\delta \left(\frac{x}{\theta}\right)^\beta}, \tag{12}$$

where

$$\tau = \frac{1}{\alpha - 1} \frac{\omega \delta \beta}{\theta^\beta} (-1)^d \frac{(\log \alpha)^{g+1}}{g!} \binom{\omega(g+1)-1}{d}. \tag{13}$$

Figure 1 and Figure 2 illustrate some various shapes of the PDF and $h(x)$ of the APENWPD for some particular parameters. Several shapes such as symmetric, near symmetric, inverted J-shaped, right-skewed, and left-skewed shape are observed for the density of the APENWPD in Figure 1. Additionally, Figure 2 indicates that the hazard rate function for the APENWPD features a wide variety of asymmetrical shapes. This indicate to the flexibility of the APENWPD for modeling data set with various shapes.

3.   Properties of the APENWPD

In this section, we will discuss some distributional properties of the APENWPD. These properties are discussed in the following subsections.

3.1.   Quantile function

The quantile function for the APENWPD can be derived by inverting the distribution function in (5) as

$$x_p = \theta \left[-\frac{1}{\delta} \left(\log \left(1 - \left(\frac{\log \left(P\left(\alpha-1\right)+1\right)}{\log \alpha} \right)^{\frac{1}{\omega}} \right) \right) \right]^{\frac{1}{\beta}}, \tag{14}$$

where $0 < p < 1$. The median of the APENWPD can be then obtained as

$$x_{0.50} = \theta \left[-\frac{1}{\delta} \left(\log \left(1 - \left(\frac{-\log(2)+\log(\alpha+1)}{\log \alpha} \right)^{\frac{1}{\omega}} \right) \right) \right]^{\frac{1}{\beta}}.$$

In addition, the 25th percentile and the 75th percentile of the APENWPD are given as

$$x_{0.25} = \theta \left[-\frac{1}{\delta} \left(\log \left(1 - \left(\frac{-\log(4)+\log(\alpha+3)}{\log \alpha} \right)^{\frac{1}{\omega}} \right) \right) \right]^{\frac{1}{\beta}}.$$
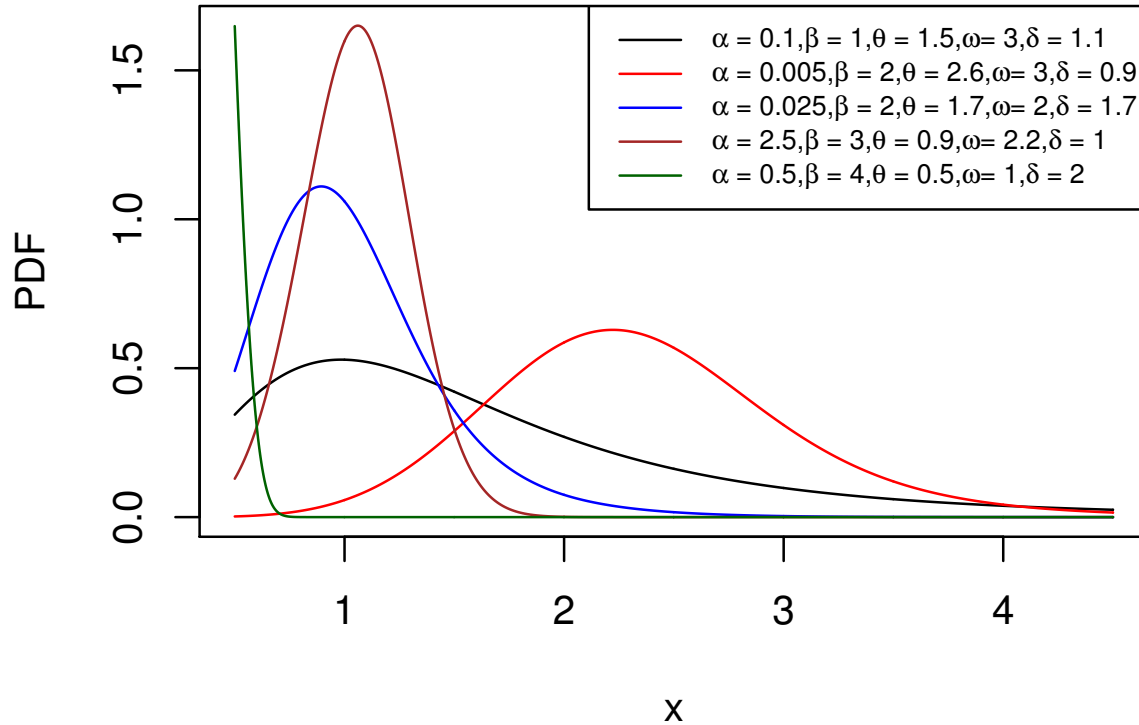
Figure 1: Plots of the PDF of the APENWPD.

$$x_{0.75} = \theta \left[-\frac{1}{\delta} \left(\log \left(1 - \left(\frac{-\log(4) + \log(3\alpha + 1)}{\log \alpha} \right)^{\frac{1}{\omega}} \right) \right) \right]^{\frac{1}{\beta}}.$$

3.2.   Moments

If $X \sim APENWPD(\alpha, \beta, \theta, \omega, \delta)$, then the rth moment of X is given by the following

$$\mu_r = E(x^r) = \frac{\omega \theta^r \delta^{-\frac{r}{\beta}}}{\alpha - 1} \sum_{g=0}^{\infty} \sum_{d=0}^{\infty} (-1)^d \frac{(\log \alpha)^{g+1}}{g!} \binom{\omega(g+1)-1}{d} \left(\frac{1}{d+1} \right)^{\frac{r}{\beta}+1} \Gamma \left(\frac{r}{\beta} + 1 \right).$$

Proof. The rth moment of the random variable (RV) X with PDF $f(x)$ is defined by

$$\mu_r = E(x^r) = \int_0^{\infty} x^r f(x) dx. \tag{15}$$

Substituting by (12), we obtain

$$E(x^r) = \sum_{g=0}^{\infty} \sum_{d=0}^{\infty} \tau \int_0^{\infty} x^{r+\beta-1} e^{-(d+1)\delta \left(\frac{x}{\theta} \right)^{\beta}} dx. \tag{16}$$
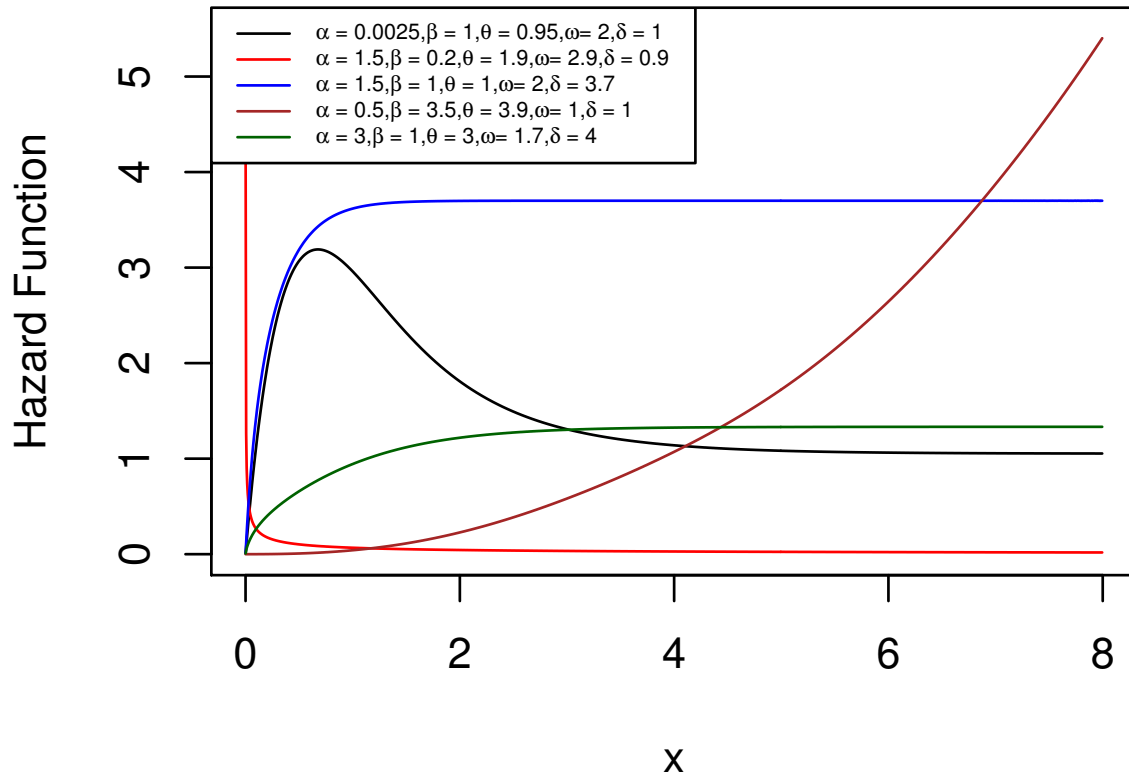
Figure 2: Plots of the hazard rate function of the APENWPD.

Applying the substitution $y = (d+1)\delta \left(\frac{x}{\theta}\right)^{\beta}$, in the integral, then we have

$$E(x^r) = \sum_{g=0}^{\infty}\sum_{d=0}^{\infty} \tau \; \frac{\theta}{\beta} \left(\frac{1}{(d+1)\delta}\right)^{\frac{r+\beta-1}{\beta}+\frac{1}{\beta}} (\theta)^{r+\beta-1} \int_{0}^{\infty} (y)^{\frac{r}{\beta}} e^{-y} dy,$$

after integrating, we obtain

$$E(x^r) = \sum_{g=0}^{\infty}\sum_{d=0}^{\infty} \tau \; \frac{\theta}{\beta} \left(\frac{1}{(d+1)\delta}\right)^{\frac{r+\beta-1}{\beta}+\frac{1}{\beta}} (\theta)^{r+\beta-1}\Gamma\left(\frac{r}{\beta}+1\right), \tag{17}$$

where τ is given by (13), on using τ in the above integral, we have

$$\mu_r = \frac{\omega\theta^r\delta^{-\frac{r}{\beta}}}{\alpha-1} \sum_{g=0}^{\infty}\sum_{d=0}^{\infty}(-1)^d \frac{(\log\alpha)^{g+1}}{g!}\binom{\omega(g+1)-1}{d}\left(\frac{1}{d+1}\right)^{\frac{r}{\beta}+1}\Gamma\left(\frac{r}{\beta}+1\right). \tag{18}$$

$\square$

Therefore, the mean of the APENWPD is easily expressed as

$$\mu_{=}E(x) = \frac{\omega\theta\delta^{-\frac{1}{\beta}}}{\alpha-1}\sum_{g=0}^{\infty}\sum_{d=0}^{\infty}(-1)^d\frac{(\log\alpha)^{g+1}}{g!}\binom{\omega(g+1)-1}{d}\left(\frac{1}{d+1}\right)^{\frac{1}{\beta}+1}\Gamma\left(\frac{1}{\beta}+1\right). \tag{19}$$

Additionally, from (18) and (19), the variance for the APENWPD can be given by

$$\begin{aligned}\sigma^2 &=E(x^2)-\mu^2\\ &=\left(\frac{\omega\theta^2\delta^{-\frac{2}{\beta}}}{\alpha-1}\sum_{g=0}^{\infty}\sum_{d=0}^{\infty}(-1)^d\frac{(\log\alpha)^{g+1}}{g!}\binom{\omega(g+1)-1}{d}\left(\frac{1}{d+1}\right)^{\frac{2}{\beta}+1}\Gamma\left(\frac{2}{\beta}+1\right)\right)-\mu^2.\end{aligned} \tag{20}$$

3.3. Moment generating function

If X follows APENWPD, then the moment generating function (MGF) $(M_x(t))$ can be derived as

$$M_x(t) = \sum_{g=0}^{\infty}\sum_{d=0}^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}\frac{\omega\theta^r\delta^{-\frac{r}{\beta}}}{\alpha-1}(-1)^d\frac{(\log\alpha)^{g+1}}{g!}\binom{\omega(g+1)-1}{d}\left(\frac{1}{d+1}\right)^{\frac{r}{\beta}+1}\Gamma\left(\frac{r}{\beta}+1\right). \tag{21}$$

3.4. Characteristic function

Let $X \sim APENWPD(\alpha,\beta,\theta,\omega,\delta)$, then the characteristic function, $\phi_x(t)$, of X can be obtained as

$$\phi_x(t) = \sum_{g=0}^{\infty}\sum_{d=0}^{\infty}\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\frac{\omega\theta^r\delta^{-\frac{r}{\beta}}}{\alpha-1}(-1)^d\frac{(\log\alpha)^{g+1}}{g!}\binom{\omega(g+1)-1}{d}\left(\frac{1}{d+1}\right)^{\frac{r}{\beta}+1}\Gamma\left(\frac{r}{\beta}+1\right).$$

Proof. In order to determine the characteristic function of the APENWPD, we apply

$$\phi_x(t) = E(e^{itx}) = \sum_{d=0}^{\infty}\tau\int_0^{\infty}e^{itx}\,x^{\beta-1}e^{-(d+1)\delta\left(\frac{x}{\theta}\right)^{\beta}}dx.$$

Then, we have

$$e^{itx} = \sum_{r=0}^{\infty}\frac{(itx)^r}{r!}. \tag{22}$$

Applying the series representation of e^{itx} given in (22), we obtain

$$E(e^{itx}) = \sum_{d=0}^{\infty}\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\tau\int_0^{\infty}x^{r+\beta-1}\,e^{-(d+1)\delta\left(\frac{x}{\theta}\right)^{\beta}}dx$$

$$E(e^{itx}) = \sum_{r=0}^{\infty}\frac{(it)^r}{r!}\mu_r,$$

where μ_r is computed based on (18). Thus, the characteristic function of the APENWPD can be described as follows

$$\phi_x(t) = \sum_{g=0}^{\infty}\sum_{d=0}^{\infty}\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\frac{\omega\theta^r\delta^{-\frac{r}{\beta}}}{\alpha-1}(-1)^d\frac{(\log\alpha)^{g+1}}{g!}\binom{\omega(g+1)-1}{d}\left(\frac{1}{d+1}\right)^{\frac{r}{\beta}+1}\Gamma\left(\frac{r}{\beta}+1\right). \tag{23}$$

$\square$

3.5. Mean residual life and mean waiting time

The mean residual life (MRL) function of the APENWPD, say $\mu(t)$, with $f(x)$ given by (6), is obtained from

$$\mu(t) = \frac{(E(t) - I)}{S(t)} - t, \tag{24}$$

where $S(t)$ is the survival function, $E(t)$ is the mean and $I = \int_0^t x f(x) dx$.

$$I = \int_0^t \sum_{d=0}^{\infty} \tau \, x^{\beta} e^{-(d+1)\delta\left(\frac{x}{\theta}\right)^{\beta}} dx$$

$$I = \frac{\omega\theta\delta^{-\frac{1}{\beta}}}{\alpha - 1} \sum_{g=0}^{\infty} \sum_{d=0}^{\infty} (-1)^d \frac{(\log \alpha)^{g+1}}{g!} \binom{\omega(g+1)-1}{d} \left(\frac{1}{d+1}\right)^{\frac{1}{\beta}+1} \gamma\left((d+1)\delta\left(\frac{t}{\theta}\right)^{\beta}, \frac{1}{\beta}+1\right), \tag{25}$$

where $\gamma(a,b) = \int_0^a x^{b-1} e^{-x} dx$ denotes the lower incomplete gamma function. Thus, the MRL of APENWPD is obtained by substituting (7), (19), and (25) into (24), as follows

$$\mu(t) = \frac{(\alpha-1)}{\alpha\left(1 - \alpha^{\left[1 - e^{-\delta\left(\frac{t}{\theta}\right)^{\beta}}\right]^{\omega}} - 1\right)} \left(\frac{\omega\theta\delta^{-\frac{1}{\beta}}}{\alpha-1} \sum_{g=0}^{\infty} \sum_{d=0}^{\infty} (-1)^d \frac{(\log \alpha)^{g+1}}{g!} \binom{\omega(g+1)-1}{d}\right.$$

$$\left. \left(\frac{1}{d+1}\right)^{\frac{1}{\beta}+1} \left[\Gamma\left(\frac{1}{\beta}+1\right) - \gamma\left((d+1)\delta\left(\frac{t}{\theta}\right)^{\beta}, \frac{1}{\beta}+1\right)\right]\right) - t.$$

Similarly, if X has the CDF (5), then its mean waiting time (MWT), $\bar{\mu}(t)$ can be defined as follows

$$\bar{\mu}(t) = t - \frac{1}{F(t)} \int_0^t x f(x) dx, \tag{26}$$

where $I = \int_0^t x f(x) dx$ denotes the first incomplete moment given by (25). Thus, the MWT of the APENWPD can be derived by substituting the equations (5) and (25) in equation (26) as follows

$$\bar{\mu}(t) = t - \frac{(\alpha-1)\left(\frac{\omega\theta\delta^{-\frac{1}{\beta}}}{\alpha-1} \sum_{g=0}^{\infty} \sum_{d=0}^{\infty} (-1)^d \frac{(\log \alpha)^{g+1}}{g!} \binom{\omega(g+1)-1}{d} \left(\frac{1}{d+1}\right)^{\frac{1}{\beta}+1} \gamma\left((d+1)\delta\left(\frac{t}{\theta}\right)^{\beta}, \frac{1}{\beta}+1\right)\right)}{\alpha^{\left[1 - e^{-\delta\left(\frac{t}{\theta}\right)^{\beta}}\right]^{\omega}} - 1}. \tag{27}$$

3.6. Shannon and Rényi entropies

Entropy is a measure of variation or uncertainty of the RV X. The Shannon entropy, SE_X, of an RV X with PDF $f(x)$ is defined as follows

$$SE_X = E\left[-\log f(x)\right] = -\int_0^{\infty} \log(f(x)) f(x) dx. \tag{28}$$

Using the PDF specified in (6), SE_X can be derived as

$$
\begin{aligned}
SE_X = {} & \log\left(\frac{\alpha-1}{\log\alpha}\frac{\theta^\beta}{\omega\delta\beta}\right) + \frac{\omega}{\alpha-1}\sum_{g=0}^{\infty}\sum_{d=0}^{\infty}(-1)^d\frac{(\log\alpha)^{g+1}}{g!}\left\{\binom{\omega(g+1)-1}{d}\left[\frac{\beta-1}{\beta}\right.\right. \\
& \left(\left(\frac{(k+\log(d+1))}{d+1}\right)+\log\left(\frac{\theta^\beta}{\delta}\right)\left(\frac{1}{d+1}\right)\right)-\frac{1}{(d+1)^2}+(\omega-1)\sum_{s=1}^{\infty}\frac{1}{s}\frac{1}{(d+1+s)}\Bigg]- \\
& \binom{\omega(g+2)-1}{d}\frac{\log\alpha}{(d+1)}\Bigg\},
\end{aligned}
\tag{29}
$$

where k is the Euler constant.

In addition, the Rényi entropy, $(RE_X(v))$ might be obtained as follows

$$
RE_X(v) = \frac{1}{1-v}\log\left(\int_0^\infty f(x)^v dx\right); v>0, v\neq 1.
\tag{30}
$$

Inserting (6) into (30) yields

$$
\begin{aligned}
RE_X(v) = {} & \frac{1}{1-v}\log\left(\left(\frac{\log\alpha}{\alpha-1}\frac{\omega\delta\beta}{\theta^\beta}\right)^v\int_0^\infty x^{v(\beta-1)}\left(e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\right)^v\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\right]^{v(\omega-1)}\right. \\
& \left(\alpha^{v\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\right]^\omega}\right)dx\Bigg).
\end{aligned}
$$

Then, after solving the integral, the Rényi entropy of the APENWPD can be written as

$$
\begin{aligned}
RE_x(v) = {} & \frac{v}{1-v}\log\left(\frac{\omega\log\alpha}{\alpha-1}\right)+\frac{1}{1-v}\log\left(\sum_{g=0}^{\infty}\sum_{d=0}^{\infty}\frac{(\log\alpha)^g}{g!}(v)^g(-1)^d\binom{v(\omega-1)+\omega g}{d}\right. \\
& \theta^{1-v}\beta^{v-1}\delta^{\frac{v-1}{\beta}}\left(\frac{1}{v+d}\right)^{\frac{v(\beta-1)}{\beta}+\frac{1}{\beta}}\Gamma\left(\frac{v(\beta-1)}{\beta}+\frac{1}{\beta}\right)\Bigg).
\end{aligned}
\tag{31}
$$

Table 1 displays the mean, variance, skewness and Kurtosis of APENWPD for various values of α, β, θ, ω and δ. For fixed β, θ, ω and δ, the values of the mean and the variance of APENWPD are increasing with the increase of α. While the skewness and Kurtosis values are decreasing as the value of α increases. Also, at fixed α, θ, ω and δ, the variance, skewness and Kurtosis decrease with increasing β.

3.7.   Order statistics

Suppose $X_1, X_2, ..., X_n$ are the observed values of a sample from the APENWPD and $X_{i:n}$ denotes the ith order statistic. The density of $X_{i:n}$ can be defined as

$$
f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}f(x)\left[F(x)\right]^{i-1}\left[1-F(x)\right]^{n-i}.
\tag{32}
$$

Table 1: The mean, variance, skewness and kurtosis of the APENWPD for some choices of parameter values.

α	β	θ	ω	δ	mean	variance	skewness	kurtosis
0.5	1	2	1	2	0.8234	0.8014	1.9769	7.3099
		3	3	4	1.2131	0.6386	1.5003	5.4793
		6	10	6	2.6972	1.3102	1.3017	4.8974
		12	20	10	4.0349	1.9485	1.2566	4.7812
1.5	1	2	1	2	1.0898	1.0744	1.6631	5.9169
		3	3	4	1.4625	0.7939	1.2892	4.7345
		6	10	6	3.0590	1.5765	1.1325	4.3594
		12	20	10	4.4771	2.3272	1.0967	4.2845
3	1	2	1	2	1.2725	1.2229	1.5051	5.3402
		3	3	4	1.6282	0.8678	1.1808	4.4337
		6	10	6	3.2965	1.6934	1.0448	4.1493
		12	20	10	4.7666	2.4810	1.0137	4.0930
0.5	2	2	1	2	1.1241	0.3839	0.8584	3.3634
		3	3	4	1.8171	0.3382	0.71137	3.3126
		6	10	6	3.9422	0.6438	0.7529	3.4692
		12	20	10	6.8648	1.2947	0.7975	3.5674
1.5	2	2	1	2	1.3246	0.4259	0.6436	2.9958
		3	3	4	2.0055	0.3662	0.5498	3.0687
		6	10	6	4.2014	0.7033	0.6058	3.2142
		12	20	10	7.2320	1.4263	0.6517	3.2932
3	2	2	1	2	1.4535	0.4333	0.5285	2.8905
		3	3	4	2.1253	0.3685	0.4623	3.0127
		6	10	6	4.3667	0.7126	0.5266	3.1473
		12	20	10	7.4666	1.4526	0.5735	3.2150

Substituting by equations (5) and (6) in (32), we have

$$f_{i:n}(x) = \frac{(-1)^{i-1}}{B(i, n-i+1)(\alpha-1)^{n-1}} f(x) \left(1 - \alpha^{\left[1 - e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\right]^\omega}\right)^{i-1}$$

$$\left(\alpha - \alpha^{\left[1 - e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\right]^\omega}\right)^{n-i}, \tag{33}$$

where $B(a,b)$ refers to the beta function. An expansion of the binomial series is given as follows

$$(x-z)^n = \sum_{y=0}^{n} (-1)^y \binom{n}{y} x^{n-y} z^y. \tag{34}$$

By applying the binomial series expansion given in equation (34), $f_{i:n}(x)$ can be expressed as follows

$$f_{i:n}(x) = \frac{\log\alpha}{B(i,n-i+1)(\alpha-1)^n} \sum_{y=0}^{i-1}\sum_{l=0}^{n-i}\binom{i-1}{y}\binom{n-i}{l}\frac{(-1)^{i-1+y+l}\omega\delta\beta\alpha^{((n-i)-l)}}{\theta^\beta}$$

$$x^{\beta-1}e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\right]^{\omega-1}\alpha^{(y+l+1)\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\right]^\omega}. \tag{35}$$

4. Maximum likelihood estimates

Assume $x_1, x_2, x_3, ..., x_n$ represent a random sample from the APENWPD, then the log-likelihood function (ℓ) is given as

$$\ell(\alpha,\beta,\theta,\omega,\delta;x) = n\log\left(\frac{\log\alpha}{\alpha-1}\right) + n\log\left(\frac{\omega\delta\beta}{\theta^\beta}\right) + (\beta-1)\sum_{i=1}^{n}\log x_i - \frac{\delta}{\theta^\beta}\sum_{i=1}^{n}x_i^\beta +$$

$$(\omega-1)\sum_{i=1}^{n}\log\left[1-e^{-\delta\left(\frac{x_i}{\theta}\right)^\beta}\right] + \log\alpha\sum_{i=1}^{n}\left[1-e^{-\delta\left(\frac{x_i}{\theta}\right)^\beta}\right]^\omega. \tag{36}$$

On taking partial derivatives of the log-likelihood in (36) with respect to the parameters and equating the results to zero, we get

$$\frac{\partial\ell}{\partial\alpha} = \frac{1}{\alpha}\left(\frac{n(\alpha-1-\alpha\log\alpha)}{(\alpha-1)\log\alpha} - \sum_{i=1}^{n}\eta_i^\omega\right) = 0, \tag{37}$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n(1-\beta\log(\theta))}{\beta} + \sum_{i=1}^{n}\log x_i - \frac{\delta}{\theta^\beta}\sum_{i=1}^{n}x_i^\beta\left\{\log x_i - \log(\theta)\right\} + \sum_{i=1}^{n}e^{-\delta\left(\frac{x_i}{\theta}\right)^\beta}$$

$$\delta\left(\frac{x_i}{\theta}\right)^\beta\log\left(\frac{x_i}{\theta}\right)\left\{\frac{(\omega-1)}{\eta_i} + \log\alpha(\omega)\eta_i^{\omega-1}\right\} = 0, \tag{38}$$

$$\frac{\partial\ell}{\partial\theta} = \frac{-n\beta}{\theta} + \sum_{i=1}^{n}x_i^\beta\delta\beta\theta^{-\beta-1} - \frac{\delta\beta}{\theta}\left(\frac{x_i}{\theta}\right)^\beta e^{-\delta\left(\frac{x_i}{\theta}\right)^\beta}\left\{\frac{(\omega-1)}{\eta_i} + \log\alpha(\omega)\eta_i^{\omega-1}\right\} = 0, \tag{39}$$

$$\frac{\partial\ell}{\partial\omega} = \frac{n}{\omega} + \sum_{i=1}^{n}\log\eta_i + \log\alpha\sum_{i=1}^{n}\eta_i^\omega\log\eta_i = 0, \tag{40}$$

$$\frac{\partial\ell}{\partial\delta} = \frac{n}{\delta} - \frac{\sum_{i=1}^{n}x_i^\beta}{\theta^\beta} + \sum_{i=1}^{n}e^{-\delta\left(\frac{x_i}{\theta}\right)^\beta}\left(\frac{x_i}{\theta}\right)^\beta\left\{\frac{(\omega-1)}{\eta_i} + \log\alpha(\omega)\eta_i^{\omega-1}\right\} = 0, \tag{41}$$

where

$$\eta_i = \left[1-e^{-\delta\left(\frac{x_i}{\theta}\right)^\beta}\right].$$

Then we can obtain the maximum likelihood estimates (MLEs) for each parameter by the solving system of equations ((37)–(41)). Also, we can find the solution to the equations analytically by using the routine "optim" in R.

5. Simulation study

In this section, we discuss some simulation studies to investigate the behavior of the MLEs for the unknown parameters of the APENWPD for various sample sizes and different values of the parameters. Particularly,

50, 100, 150, 200, 250 and 500 sample sizes were considered from the APENWPD with 1000 replications. Two different sets of parameters are assumed; Set 1 ($\alpha = 0.5, \beta = 0.4, \theta = 0.5, \omega = 0.5, \delta = 0.5$) and Set 2 ($\alpha = 1.5, \beta = 1.5, \theta = 1.5, \omega = 2, \delta = 2$). The average estimates and the mean squared errors (MSEs) of the MLEs are calculated for the different sample sizes. The simulation results of the average estimates

Table 2: A Simulation Study of the APENWPD.

Sample Size	Par.	Set 1		Set 2	
		MLE	MSE	MLE	MSE
50	α	1.8223	7.2409	2.8661	15.4106
	β	0.4903	0.1288	1.9816	2.1052
	θ	3.8778	21.7650	3.1326	4.9315
	ω	0.6491	0.6588	2.8277	7.5988
	δ	1.4010	1.8698	6.5020	37.8807
100	α	1.5929	5.2285	2.8559	14.1591
	β	0.4524	0.0670	1.7963	1.0122
	θ	3.3810	15.0810	3.0535	4.3191
	ω	0.5113	0.2813	2.3828	3.5758
	δ	1.2062	1.2408	6.1374	33.0459
150	α	1.4069	3.7896	2.7255	12.1431
	β	0.4338	0.0458	1.7055	0.5427
	θ	2.8847	10.6481	2.9485	3.7016
	ω	0.4937	0.1559	2.2160	2.0187
	δ	1.1226	0.9816	5.7460	27.0947
200	α	1.2731	2.74301	2.6338	10.3104
	β	0.4217	0.0375	1.6997	0.4645
	θ	2.7471	9.1424	2.8396	3.1539
	ω	0.4909	0.1490	2.1065	1.4098
	δ	1.0906	0.8509	5.4189	22.7318
250	α	1.1067	1.9696	2.3021	8.0300
	β	0.4149	0.0300	1.7221	0.4560
	θ	2.6177	8.5965	2.7985	2.8910
	ω	0.4956	0.1336	2.0346	1.0570
	δ	1.0372	0.7129	5.1597	19.6327
500	α	0.9339	0.9962	2.1483	5.5442
	β	0.4114	0.0216	1.6465	0.2463
	θ	1.8489	3.7354	2.5835	2.1862
	ω	0.4799	0.0955	2.0306	0.7557
	δ	0.8821	0.4336	4.5441	13.6413

and the MSEs are demonstrated in Table 2. We can notice that the MSE decreases as n increases and the parameters' MLEs becomes closer to the actual parameter.

6.   Applications

In this section, we consider three real-life data sets. The fit of the APENWPD is compared with some other distributions, namely the exponentiated Weibull Weibull (EWW) distribution by Elgarhy and Hassan (2019),

exponentiated new Weibull-Pareto (ENWP) distribution by Al-Omari et al. (2019), alpha power inverse Weibull (APIW) distribution by Basheer (2019), APT inverse Lomax (APTIL) distribution by ZeinEldin et al. (2020), Weibull (W) distribution and exponential (E) distribution for the three data sets.

First Data Set.

The first data set is from Meeker and Escobar (1998), this data set concerns with a large system with 30 units, in which the failure and running times are 2.75, 0.13, 1.47, 0.23, 1.81, 0.30, 0.65, 0.10, 3.00,1.73, 1.06, 3.00, 3.00, 2.12, 3.00, 3.00, 3.00, 0.02, 2.61, 2.93, 0.88, 2.47, 0.28, 1.43, 3.00, 0.23, 3.00, 0.80, 2.45, 2.66.

Second Data Set.

The second data set obtained from Nichols and Padgett (2006). This data set comprises the tensile strength of 100 observations of carbon fibers. The second data values are: 3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65.

Third Data Set.

The data represents a COVID-19 data belong to Canada of 36 days, from 10 April to 15 May 2020 see the link: `https://covid19.who.int`. The data are as follows: 3.1091, 3.3825, 3.1444, 3.2135, 2.4946, 3.5146, 4.9274, 3.3769, 6.8686, 3.0914, 4.9378, 3.1091, 3.2823, 3.8594, 4.0480, 4.1685, 3.6426, 3.2110, 2.8636, 3.2218, 2.9078, 3.6346, 2.7957, 4.2781, 4.2202, 1.5157, 2.6029, 3.3592, 2.8349, 3.1348, 2.5261, 1.5806, 2.7704, 2.1901, 2.4141, 1.9048.

The negative log-likelihood value $(-\ell)$, Cramer-von Mises (W) statistic, Anderson-Darling (A) statistic and the Kolmogorov– Smirnov (K–S) test value with its p-value are considered to compare the fit of the APEN-WPD with the other distributions. The APENWPD performance compared with other distributions for the three real data sets that shown in Tables 3, 5 and 7. In addition, the MLEs and standard errors (SEs) of the parameters for the APENWPD with the other competing distributions for the three data sets are shown in Tables 4, 6 and 8.

Tables 3, 5 and 7 demonstrate that the APENWPD has the lowest values of $-\ell$, W, A and K-S test values as well as the best p-values. This means that the APENWPD provides the best fit as compared to the other competing distributions for the three real data sets.

Table 3: Goodness-of-fit measures for the first data sets.

Distribution	Statistics				
	$-\ell$	K - S	p - value	W	A
APENWP	37.3808	0.2146	0.1261	0.2937	1.9174
EWW	37.4793	0.2364	0.07	0.3627	2.3513
ENWP	38.45037	0.2385	0.0659	0.3619	2.2925
APTIL	54.03125	0.2912	0.0124	0.3723	2.1490
APIW	56.92328	0.2719	0.0237	0.4387	2.4763
W	46.15873	0.2195	0.111	0.3317	2.1107
E	47.13504	0.2161	0.1214	0.3678	2.0022

Figures 3, 4 and 5 present the three data sets' histogram with the estimated PDF of the competing distributions. This graphical goodness-of-fit method also supports the results of Tables 3, 5 and 7. To illustrate,

Table 4: The MLEs and SE (in parenthesis) for first data set.

Distribution	Estimated Parameters				
APENWP	2.5086	7.1883	4.2511	0.1062	8.9533
$(\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\omega}, \hat{\delta})$	(1.87036)	(0.0025)	(0.0025)	(0.0284)	(0.0025)
EWW	0.1275	13.4932	2.2933	0.0079	3.1017
$(\hat{a}, \hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\gamma})$	(0.0239)	(0.0026)	(0.0026)	(0.0012)	(0.0026)
ENWP	7.1512	4.6192	0.1283	13.2910	-
$(\hat{\beta}, \hat{\theta}, \hat{\omega}, \hat{\delta})$	(0.0026)	(0.0026)	(0.0234)	(0.0026)	
APTIL	0.9585	0.3903	18.3745	-	-
$(\hat{a}, \hat{b}, \hat{\alpha})$	(0.9273)	(0.3026)	(58.8040)		
APIW	116.3505	0.1681	0.8196	-	-
$(\hat{\alpha}, \hat{\lambda}, \hat{\beta})$	(228.7952)	(0.0774)	(0.0997)		
W	1.2650	1.8802	-	-	-
$(\hat{a}, \hat{\gamma})$	(0.2044)	(0.2821)			
E	0.5649	-	-	-	-
$(\hat{\lambda})$	(0.1031)				

Table 5: Goodness-of-fit measures for the second data sets.

Distribution	Statistics				
	$-\ell$	K - S	p - value	W	A
APENWP	140.53	0.0586	0.8821	0.0670	0.3950
EWW	140.8305	0.0653	0.7881	0.0792	0.4605
ENWP	140.7303	0.0640	0.8074	0.0754	0.4381
APTIL	207.9825	0.3468	7.181×10^{-11}	3.2218	16.6530
APIW	159.2739	0.1363	0.04878	0.4938	3.1044
W	140.9957	0.0632	0.8196	0.0701	0.4602
E	195.9886	0.3219	2.011×10^{-09}	3.4635	17.4211

Table 6: The MLEs and SE (in parenthesis) for second data set.

Distribution	Estimated Parameters				
APENWP	4.7642	1.8260	3.4698	1.6920	2.7632
$(\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\omega}, \hat{\delta})$	(10.2342)	(0.6762)	(29.8453)	(0.9437)	(43.4064)
EWW	1.7455	9.3059	0.8789	0.0107	2.2291
$(\hat{a}, \hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\gamma})$	(1.0040)	(19.0056)	(0.6620)	(0.0074)	(1.1058)
ENWP	2.3595	2.0710	1.3736	0.5740	-
$(\hat{\beta}, \hat{\theta}, \hat{\omega}, \hat{\delta})$	(0.5869)	(10.2816)	(0.6233)	(6.6881)	-
APTIL	17.4689	0.0317	53.0660	-	-
$(\hat{a}, \hat{b}, \hat{\alpha})$	(14.6067)	(0.0265)	(44.9768)		
APIW	1123.1269	0.7799	2.3420	-	-
$(\hat{\alpha}, \hat{\lambda}, \hat{\beta})$	(1944.7938)	(0.2175)	(0.1513)		
W	2.7928	2.9321	-	-	-
$(\hat{a}, \hat{\gamma})$	(0.2134)	(0.1107)			
E	0.3829	-	-	-	-
$(\hat{\lambda})$	(0.0383)				

Table 7: Goodness-of-fit measures for the third data sets.

Distribution	Statistics				
	$-\ell$	K - S	p - value	W	A
APENWP	47.4345	0.1029	0.8408	0.0654	0.3986
EWW	48.1423	0.1064	0.8101	0.0909	0.5404
ENWP	48.0936	0.1053	0.8197	0.0889	0.5274
APTIL	83.5687	0.4016	1.812×10^{-05}	1.7112	8.4869
APIW	49.7715	0.1315	0.5625	0.1289	0.8716
W	51.4743	0.1500	0.3925	0.1979	1.1424
E	78.7798	0.4097	1.128×10^{-05}	1.8685	8.9490

Table 8: The MLEs and SE (in parenthesis) for third data set.

Distribution	Estimated Parameters				
APENWP	23.4016	1.0586	0.1636	19.2673	0.1916
$(\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\omega}, \hat{\delta})$	(48.4934)	(0.4652)	(0.5582)	(36.3015)	(0.6730)
EWW	12.3997	12.2822	1.8298	0.1881	0.5991
$(\hat{a}, \hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\gamma})$	(21.2760)	(76.7683)	(4.8860)	(0.7794)	(1.5789)
ENWP	1.5117	9.7120	8.1758	13.6473	-
$(\hat{\beta}, \hat{\theta}, \hat{\omega}, \hat{\delta})$	(0.6071)	(139.0037)	(9.3442)	(294.9714)	-
APTIL	4.7407	0.1576	51.9843	-	-
$(\hat{a}, \hat{b}, \hat{\alpha})$	(3.7393)	(0.1275)	(58.6711)		
APIW	64.8470	19.3576	4.1376	-	-
$(\hat{\alpha}, \hat{\lambda}, \hat{\beta})$	(105.9448)	(11.1635)	(0.4937)		
W	3.3138	3.6372	-	-	-
$(\hat{a}, \hat{\gamma})$	(0.3789)	(0.1941)			
E	0.3047	-	-	-	-
$(\hat{\lambda})$	(0.0508)				

it is evident from Figures 3, 4 and 5 that the APENWPD fits the histogram more closely than the other competing models. That is, it is apparent that the APENWPD best fits these data sets when compared to other distributions considered here. Therefore, APENWPD has the potential to compete with other distributions used commonly in literature to fit lifetime data.

7. Conclusions

When describing and predicting real-world phenomena, statistical distributions are extremely useful. Many distributions have been developed, but there is always the opportunity to develop distributions that are more flexible or that fit specific data scenarios. This has motivated researchers to explore and develop new and more flexible distributions. In this study, a five-parameter APENWPD distribution is introduced based on a new technique for generating distributions. In this method, three techniques are combined: T-X family, exponentiated method, and APT, which provides a greater degree of adaptability to the suitability of practical data sets. Among the characteristics that are relevant to the proposed distribution is the diversity of shapes that the density and the hazard rate functions of the distribution can take. Some mathematical properties of this new distribution are provided. Estimation of the unknown parameters is discussed by employing the maximum likelihood technique. Then, to demonstrate the consistency of the estimates, various simulation studies are conducted. The results indicate that the proposed estimators demonstrate good performance. Furthermore, three real data sets are used to show the new model's flexibility against the competitive models.

The results indicate the APENWPD provides the best fit among some other competitive models. Thus, the proposed distribution possesses great potential for wider applications in statistics. In the future, it might be possible to develop regression model based on the APENWPD and compare it with some existing models.
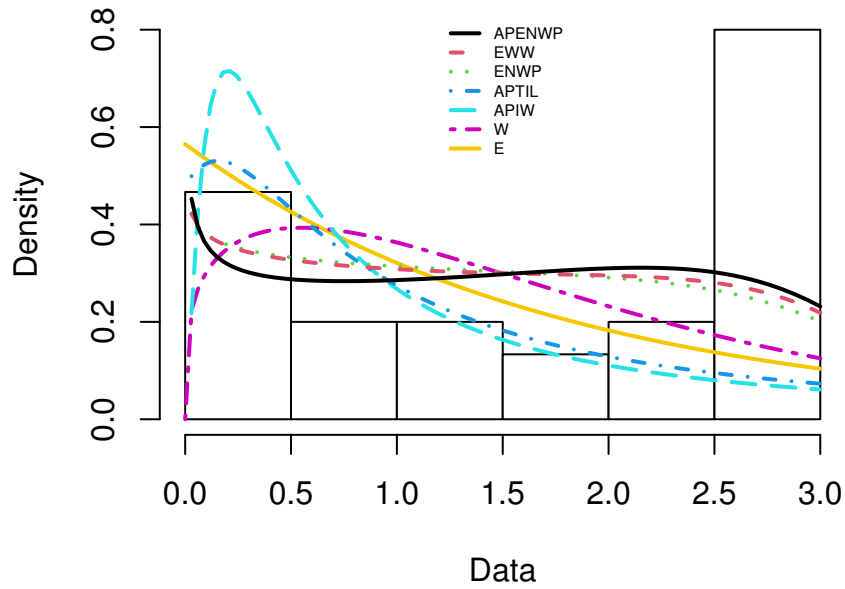

Figure 3: a Histogram and the fitted PDFs for first data set.
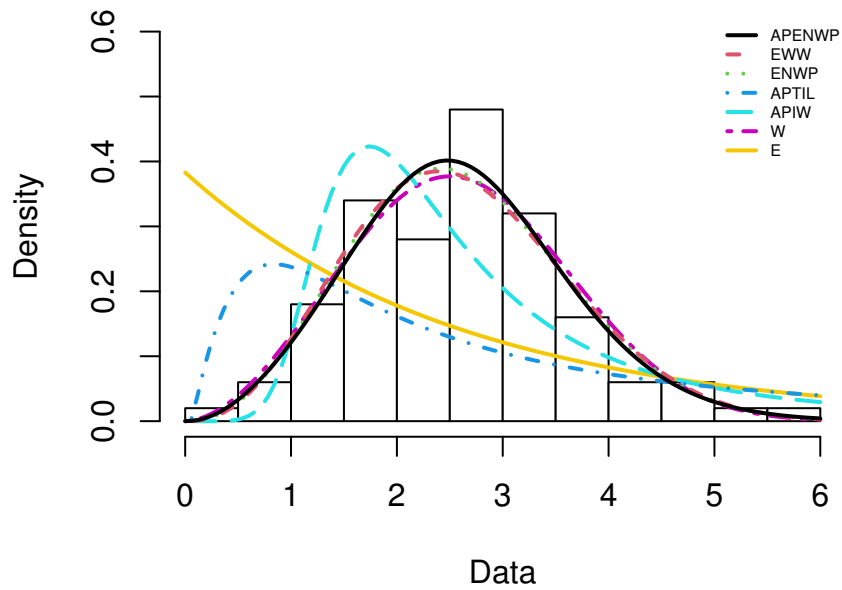
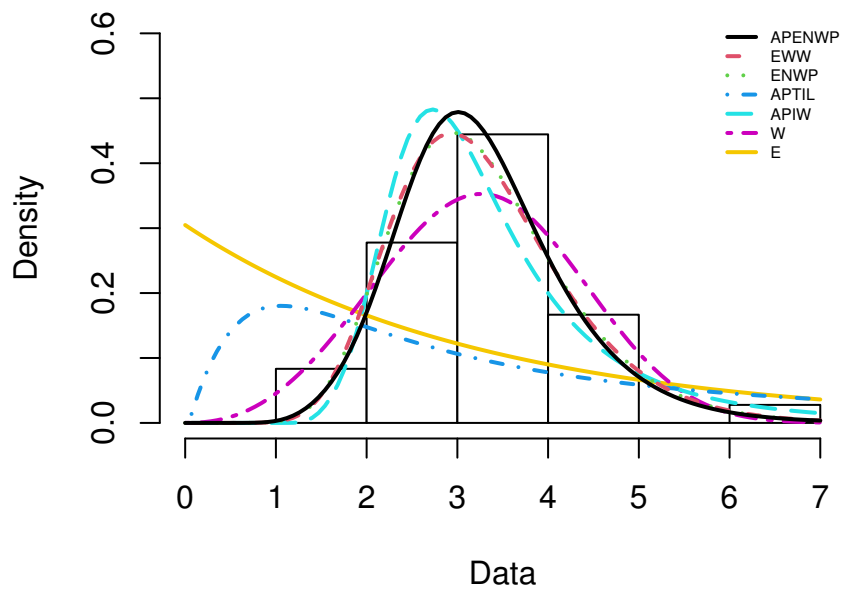Figure 4: a Histogram and the fitted PDFs for second data set.

Figure 5: a Histogram and the fitted PDFs for third data set.

References

Al-Omari, A. I., Al-khazaleh, A. M., and Al-khazaleh, M. (2019). Exponentiated new weibull-pareto distribution. *Investigación Operacional*, 40(2):165–175.

Alexander, C., Cordeiro, G. M., Ortega, E. M., and Sarabia, J. M. (2012). Generalized beta-generated distributions. *Computational Statistics & Data Analysis*, 56(6):1880–1897.

Alzaatreh, A., Lee, C., and Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, 71(1):63–79.

Basheer, A. M. (2019). Alpha power inverse weibull distribution with reliability application. *Journal of Taibah University for Science*, 13(1):423–432.

Dey, S., Ghosh, I., and Kumar, D. (2018). Alpha-power transformed lindley distribution: properties and associated inference with application to earthquake data. *Annals of Data Science*, pages 1–28.

Dey, S., Nassar, M., and Kumar, D. (2019). Alpha power transformed inverse lindley distribution: A distribution with an upside-down bathtub-shaped hazard function. *Journal of Computational and Applied Mathematics*, 348:130–145.

Eghwerido, J. T., Nzei, L. C., and Agu, F. I. (2020). The alpha power gompertz distribution: Characterization, properties, and applications. *Sankhya A*, pages 1–27.

Elgarhy, M. and Hassan, A. (2019). Exponentiated weibull weibull distribution: Statistical properties and applications. *Gazi University Journal of Science*, 32(2):616–635.

Eugene, N., Lee, C., and Famoye, F. (2002). Beta-normal distribution and its applications. *Communications in Statistics-Theory and methods*, 31(4):497–512.

Ihtisham, S., Khalil, A., Manzoor, S., Khan, S. A., and Ali, A. (2019). Alpha-power pareto distribution: Its properties and applications. *PloS one*, 14(6):e0218027.

Jones, M. (2009). Kumaraswamy's distribution: A beta-type distribution with some tractability advantages. *Statistical Methodology*, 6(1):70–81.

Mahdavi, A. and Kundu, D. (2017). A new method for generating distributions with an application to exponential distribution. *Communications in Statistics-Theory and Methods*, 46(13):6543–6557.

Meeker, W. and Escobar, L. (1998). Statistical methods for reliability data john wiley & sons new york. *New York*.

Nasiru, S. and Luguterah, A. (2015). The new weibull-pareto distribution. *Pakistan Journal of Statistics and Operation Research*, pages 103–114.

Nassar, M., Alzaatreh, A., Mead, M., and Abo-Kasem, O. (2017). Alpha power weibull distribution: Properties and applications. *Communications in Statistics-Theory and Methods*, 46(20):10236–10252.

Nichols, M. D. and Padgett, W. (2006). A bootstrap control chart for weibull percentiles. *Quality and reliability engineering international*, 22(2):141–151.

Ristić, M. M. and Balakrishnan, N. (2012). The gamma-exponentiated exponential distribution. *Journal of statistical computation and simulation*, 82(8):1191–1206.

Torabi, H. and Hedesh, N. M. (2012). The gamma-uniform distribution and its applications. *Kybernetika*, 48(1):16–30.

ZeinEldin, R. A., Ahsan ul Haq, M., Hashmi, S., and Elsehety, M. (2020). Alpha power transformed inverse lomax distribution with different methods of estimation and applications. *Complexity*, 2020.

Zografos, K. and Balakrishnan, N. (2009). On families of beta-and generalized gamma-generated distributions and associated inference. *Statistical methodology*, 6(4):344–362.