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A New Lifetime Parametric Model for the Survival and Relief Times with Copulas and Properties

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Abstract
In this article, we introduce a new generalization of the Exponentiated Exponential distribution. Various structural mathematical properties are derived. Numerical analysis for mean, variance, skewness and kurtosis and the dispersion index are performed. The new density can be right skewed and symmetric with "unimodal" and "bimodal" shapes. The new hazard function can be "constant", "monotonically decreasing", " monotonically increasing", "increasing-constant", "upside-down-constant", "decreasing-constant". Many bivariate and multivariate type model have been also derived. We assess the performance of the maximum likelihood method graphically via the biases and mean squared errors. The applicability of the new life distribution is illustrated by means of two real data sets.

Key Words: Exponentiated Exponential; Morgenstern family; Clayton Copula; Real Data Modeling; Hazard Function.

Mathematical Subject Classification: 62N01; 62N02; 62E10.

1. Introduction

A random variable (RV) Z is said to have the Exponentiated Exponential (EE) distribution (see Gupta et al. (1998)) if its probability density function (PDF) and cumulative distribution function (CDF) are given by

$$h_{a,b}(z) = abe^{-bz}\left(1-e^{-bz}\right)^{a-1}|_{(z>0,a>0\text{ and }b>0)}, \quad (1)$$

and

$$H_{a,b}(z) = \left(1-e^{-bz}\right)^{a}|_{(z>0,a>0\text{ and }b>0)}, \quad (2)$$

respectively. We write $Z \sim$ EE (a,b). For $a=1$, the EE reduces to the standard exponential distribution. If $1>a$, the $h_{a,b}(z)$ decreases monotonically with z. If $a>1$, the $h_{a,b}(z)$ attains a mode at $z=\frac{1}{b}log(a)$. The properties

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of the EE distribution have been studied by many authors. The original paper, Gupta and Kundu (1999), provided expressions for survival function (SF), hazard rate function (HRF), shapes of PDF and HRF, moment generating function (MGF), stochastic orders, ordinary moments (OMs), the distribution of a sum of RVs following (1) and distribution of extreme values (EVs). Many other authors have derived and studied the EE model, see Gupta and Kundu ((2001a), (2001b), (2007)), Zheng (2002), Zheng and Park (2004), Kundu and Pradhan (2009), Aslam et al. (2010), Aryal et al. (2017), Brito et al. (2017), Alizadeh et al. (2017), Cordeiro et al. (2018), Yousof et al. (2018 and 2019), Korkmaz et al. (2018), and Ibrahim et al. (2020) among others. Cordeiro et al. (2016) investigated a new flexible class of continuous distributions called the generalized odd log-logistic-G (GOLL-G) family with only two extra shape parameters. In the work, we introduce a new version of the EE model using the GOLL-G family called the generalized odd log-logistic EE (GOLLEE). For an arbitrary baseline CDF $H_{\underline{\xi}}(w)$, the CDF of the GOLL-G family is given by

$$F_{\underline{\xi}}(w) = H_{\underline{\xi}}(\boldsymbol{z})^{\alpha\beta}\left\{H_{\underline{\xi}}(\boldsymbol{z})^{\alpha\beta} + \left[1 - H_{\underline{\xi}}(\boldsymbol{z})^{\beta}\right]^{\alpha}\right\}^{-1}, \quad (3)$$

where $\underline{\xi} = (\alpha, \beta)$ refers to the parameter vector of the base line model. For $\beta = 1$ we get the OLL-G family (Gleaton and Lynch (2006)). For $\alpha = 1$ we get the proportional reversed hazard rate G (PRHR-G) family (Gupta and Gupta (2007)). The PDF corresponding to (3) is

$$f_{\underline{\xi}}(w)|_{\underline{\xi}=(\alpha,\beta)} = \frac{\alpha\beta h_{\underline{\xi}}(\boldsymbol{z}) H_{\underline{\xi}}(\boldsymbol{z})^{\alpha\beta-1}\left[1 - H_{\underline{\xi}}(\boldsymbol{z})^{\beta}\right]^{\alpha-1}}{\left\{H_{\underline{\xi}}(\boldsymbol{z})^{\alpha\beta} + \left[1 - H_{\underline{\xi}}(\boldsymbol{z})^{\beta}\right]^{\alpha}\right\}^{2}}. \quad (4)$$

Then, the CDF of the GOLLEE is given by

$$F_{\underline{\boldsymbol{w}}}(\boldsymbol{z}) = \frac{\boldsymbol{\varrho}_{b,\boldsymbol{z}}^{\alpha\alpha\beta}}{\boldsymbol{\varrho}_{b,\boldsymbol{z}}^{\alpha\alpha\beta} + \left(1 - \boldsymbol{\varrho}_{b,\boldsymbol{z}}^{a\beta}\right)^{\alpha}}|_{(\varrho_{b,\boldsymbol{z}}=1-e^{-b\boldsymbol{z}})}, \quad (5)$$

where $\underline{\boldsymbol{w}} = (\alpha, \beta, a, b)$. The PDF corresponding to (5) is given by

$$f_{\underline{\boldsymbol{w}}}(\boldsymbol{z}) = \frac{\alpha\beta a b e^{-b\boldsymbol{z}} \boldsymbol{\varrho}_{b,\boldsymbol{z}}^{\alpha a\beta-1}\left(1 - \boldsymbol{\varrho}_{b,\boldsymbol{z}}^{a\beta}\right)^{\alpha-1}}{\left[\boldsymbol{\varrho}_{b,\boldsymbol{z}}^{\alpha a\beta} + \left(1 - \boldsymbol{\varrho}_{b,\boldsymbol{z}}^{a\beta}\right)^{\alpha}\right]^{2}}. \quad (6)$$

The asymptotics of the CDF, PDF and HRF as $\boldsymbol{z} \to \boldsymbol{t}$ are given by

$$F_{\underline{\boldsymbol{w}}}(\boldsymbol{z})|_{(\boldsymbol{z}\to\boldsymbol{t})} \sim \boldsymbol{\varrho}_{b,\boldsymbol{z}}^{\alpha a\beta}, f_{\underline{\boldsymbol{w}}}(\boldsymbol{z})|_{(\boldsymbol{z}\to\boldsymbol{t})} \sim \alpha\beta a b e^{-b\boldsymbol{z}}\boldsymbol{\varrho}_{b,\boldsymbol{z}}^{\alpha a\beta-1}$$

and

$$\boldsymbol{t}_{\underline{\boldsymbol{w}}}(\boldsymbol{z})|_{(\boldsymbol{z}\to\boldsymbol{t})} \sim \alpha\beta a b e^{-b\boldsymbol{z}}\boldsymbol{\varrho}_{b,\boldsymbol{z}}^{\alpha a\beta-1}.$$

The asymptotics of CDF, PDF and HRF as $\boldsymbol{z} \to \infty$ are given by

$$1 - F_{\underline{\boldsymbol{w}}}(\boldsymbol{z})|_{(\boldsymbol{z}\to\infty)} \sim \left[\beta\left(1 - \boldsymbol{\varrho}_{b,\boldsymbol{z}}^{a}\right)\right]^{\alpha}, f_{\underline{\boldsymbol{w}}}(\boldsymbol{z})|_{(\boldsymbol{z}\to\infty)} \sim \alpha\beta^{\alpha} a b e^{-b\boldsymbol{z}}\boldsymbol{\varrho}_{b,\boldsymbol{z}}^{a-1}\left(1 - \boldsymbol{\varrho}_{b,\boldsymbol{z}}^{a}\right)^{\alpha-1}$$

and

$$\boldsymbol{t}_{\underline{\boldsymbol{w}}}(\boldsymbol{z})|_{(\boldsymbol{z}\to\infty)} \sim \alpha a b e^{-b\boldsymbol{z}}\boldsymbol{\varrho}_{b,\boldsymbol{z}}^{a-1}\left(1 - \boldsymbol{\varrho}_{b,\boldsymbol{z}}^{a}\right)^{-1}.$$

The HRF for the new model can be obtained from $f_{\underline{\boldsymbol{w}}}(\boldsymbol{z})/\left[1 - F_{\underline{\boldsymbol{w}}}(\boldsymbol{z})\right]$. Figure 1 gives some PDF plots for some selected parameters value. Figure 2 gives some HRF plots for some selected parameters value. Based on Figure 1 the GOLLEE density can be right skewed and symmetric with unimodal and bimodal PDFs. Based on Figure 2 the GOLLEE HRF can be "constant" ($\alpha = 1, \beta = 1, a = 1, b = 1$), "decreasing" ($\alpha = 2.5, \beta = 0.75, a = 1, b = 1$), "increasing" ($\alpha = 2, \beta = 1, a = 1, b = 1$), "increasing-constant" ($\alpha = 1, \beta = 2.5, a = 1, b = 2$), "upside-down-constant" ($\alpha = 2, \beta = 1, a = 1, b = 1$), "decreasing-constant" ($\alpha = 0.5, \beta = 0.05, a = 1, b = 1$).

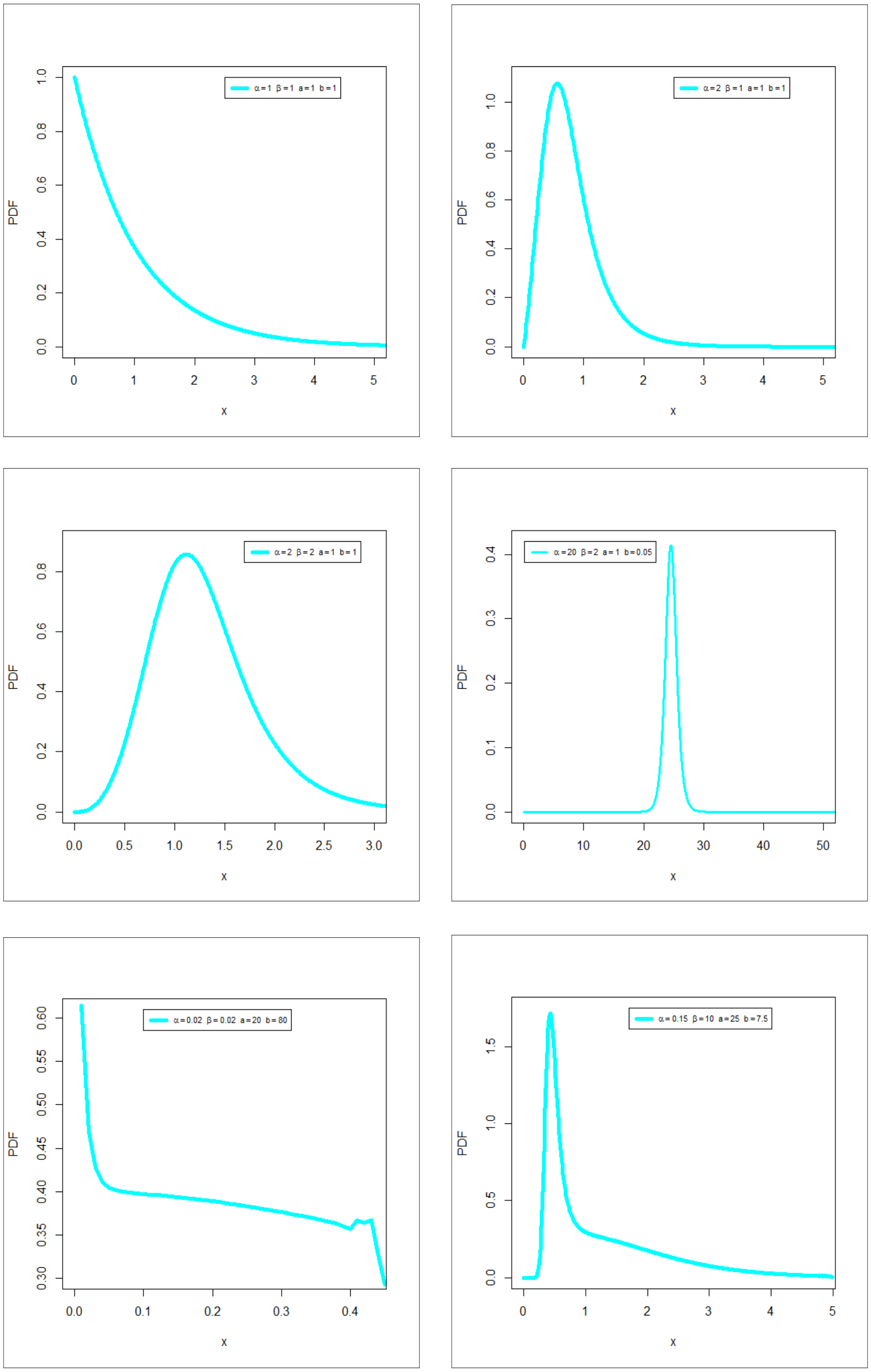


Figure 1: PDF plots for some selected parameters value.

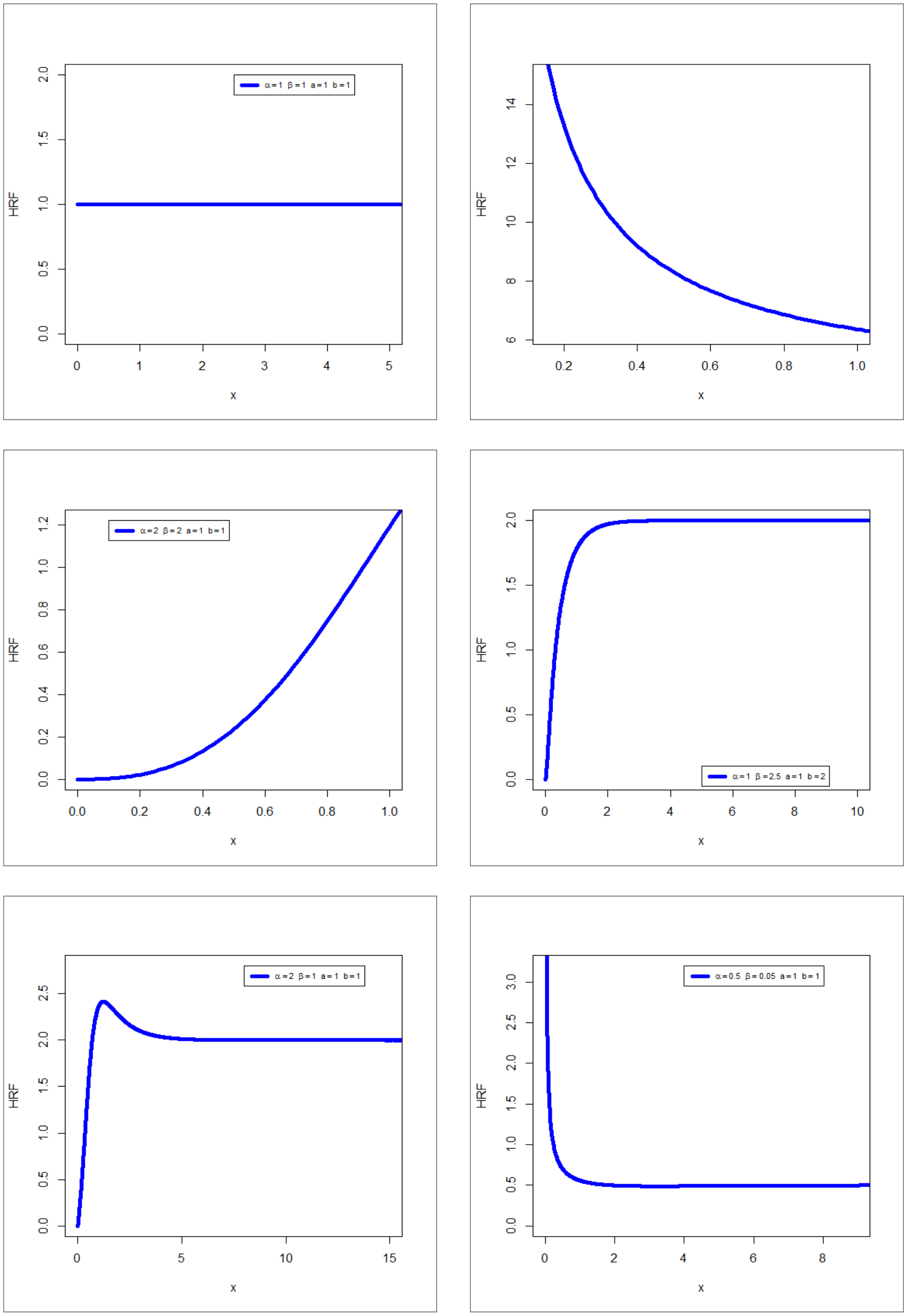


Figure 2: HRF plots for some selected parameters value.

After some algebraic manipulations, the PDF in (6) can be rewritten as

$$f_{\underline{W}}(z) = \sum_{\varepsilon=0}^{\infty} \nabla_{\varepsilon}\, \pi_{(a^{*},b)}(z)|_{(a^{*}=a(1+\varepsilon))}, \tag{7}$$

where

$$\nabla_{\varepsilon} = \frac{\alpha\beta}{1+\varepsilon}\sum_{j_1,j_2=0}^{\infty}\sum_{\varepsilon=0}^{j_3}(-1)^{j_2+\varepsilon+l}\binom{[(j_1+1)\alpha+j_2]\beta-1}{j_3}\binom{-(j_1+1)\alpha}{j_2}\binom{-2}{j_1}\binom{j_3}{\varepsilon},$$

and $\pi_{(a^*,b)}(z)$ is the PDF of the EW model with parameters a^* and b. By Integrating (7), the CDF of the GOLLEE model is given as

$$F_{\underline{w}}(z) = \sum_{\varepsilon=0}^{\infty}\nabla_{\varepsilon} H_{(a^*,b)}(z),$$

where $H(a^*,b)(z)$ is the CDF of the EW model with parameters a^* and b.

2. Properties

2.1 Moments

The r^{th} OM of Z is given by

$$\mu'_{r,Z} = E(Z^r) = \int_{-\infty}^{\infty} z^r \, dF_{\underline{w}}(z),$$

then we obtain

$$\mu'_{r,Z}|_{(r>-1)} = b^r\Gamma(1+r)\sum_{\varepsilon,h=0}^{\infty}\nabla_{\varepsilon,h}^{(r,a^*)} \quad (8)$$

where

$$\nabla_{\varepsilon,h}^{(r,a^*)} = \nabla_{\varepsilon}\frac{a^*(-1)^h}{(h+1)^{1+r}}\binom{a^*-1}{h}$$

and

$$\Gamma(1+r)|_{[r\in R^+]} = r! = \prod_{h=0}^{r-1}(r-h).$$

The variance (V (Z)), skewness (S (Z)) and kurtosis (K (Z)) can be easily derived from the well-known relationships.

2.2 Incomplete moment, MGF and dispersion index

The r^{th} incomplete moment, say $I_{r,Z}(t)$, of Z can be expressed from (7) as

$$I_{r,Z}(t)|_{(r>-1)} = \sum_{\varepsilon=0}^{\infty}\nabla_{\varepsilon}\int_{-\infty}^{t} z^r \, dF_{\underline{w}}(z),$$

then

$$I_{r,Z}(t)|_{(r>-1)} = b^r\gamma(1+r,bt)\sum_{\varepsilon,h=0}^{\infty}\nabla_{\varepsilon,h}^{(r,a^*)}, \quad (9)$$

where $\gamma(\Delta_1,\Delta_2)$ is the incomplete gamma function such that

$$\gamma(\Delta_1,\Delta_2) = \int_0^{\Delta_2} z^{\Delta_1-1}\,exp(-z)\,dz = \frac{\Delta_2^{\Delta_1}}{\Delta_1}\{F_{1,1}[\Delta_1;\Delta_1+1;-\Delta_2]\} = \sum_{d=0}^{\infty}\frac{(-1)^d}{d!\,(\Delta_1+d)}\Delta_2^{\Delta_1+d},$$

and $F_{1,1}[\cdot,\cdot,\cdot]$ refers to the confluent hypergeometric function. The first incomplete moment given by (9) with $r = 1$ as

$$I_{1,Z}(z) = b\gamma(1+,t^{-1})\sum_{\varepsilon,h=0}^{\infty}\nabla_{\varepsilon,h}^{(1,a^*)}.$$

The MGF can be derived via (7) as

$$M_Z(\boldsymbol{t}) = \Gamma(1+\mathcal{r}) \sum_{\varepsilon,\hbar,\mathcal{r}=0}^{\infty} b^{\mathcal{r}} \frac{\boldsymbol{t}^{\mathcal{r}}}{\mathcal{r}!} \nabla_{\varepsilon,\hbar}^{(\mathcal{r},a^*)} |_{(\mathcal{r}>-1)},$$

Based on Gupta and Kundu (2001a) and Gupta and Kundu (2007), another formula for the MGF can be derived as

$$M_Z(\boldsymbol{t}) = \sum_{\varepsilon=0}^{\infty} \nabla_\varepsilon \, a^* B\left(1 - \frac{\boldsymbol{t}}{b}, a^*\right),$$

Where $B(\Delta_1, \Delta_2) = \int_0^1 \boldsymbol{z}^{\Delta_1 - 1} (1-\boldsymbol{z})^{\Delta_2 - 1} d\boldsymbol{z}$ is the beta function. So, the characteristic function (CF) of Z can be easily written as

$$\varphi_Z(\boldsymbol{t}) = \sum_{\varepsilon=0}^{\infty} \nabla_\varepsilon \, a^* B\left(1 - \frac{i\boldsymbol{t}}{b}, a^*\right) |_{(i=\sqrt{-1})}$$

The dispersion index (DisIx) or the variance to the mean ratio of the GOLLEE model can derived as

$$\mathrm{DisIx}(Z) = b\left[3 \frac{\sum_{\varepsilon,\hbar=0}^{\infty} \nabla_{\varepsilon,\hbar}^{(2,a^*)}}{\sum_{\varepsilon,\hbar=0}^{\infty} \nabla_{\varepsilon,\hbar}^{(1,a^*)}} - \sum_{\varepsilon,\hbar=0}^{\infty} \nabla_{\varepsilon,\hbar}^{(1,a^*)}\right].$$

2.3 Numerical analysis

Table 1 gives Numerical analysis for the mean, V (Z), S(Z), K(Z) and DisIx(Z). Based on Table 1, we note that: 1-The skewness of the GOLLEE distribution can range in the interval $(-2.7792, 8.2978)$. 2-The spread for the GOLLEE kurtosis is much larger ranging from -46.275 to 35.526. 3-The DisIx (Z) can be "between 0 and 1 " or "equal 1 " or more than 1.

Table 1: Mean, variance, skewness and kurtosis.

α	β	a	b	E(Z)	V(Z)	S(Z)	K(Z)	DisIx(Z)
1	1	1	1	1	1	2	9	1
5				0.709245	0.03367585	0.811776	5.306741	0.04748127
10				0.697236	0.00827721	0.427330	4.524585	0.01187146
50				0.693312	0.00032907	0.086995	4.213821	0.00047464
1	2	1.5	1.5	1.222222	0.60493830	1.463557	6.48105	0.4949495
5				1.060796	0.02409643	0.506570	4.568689	0.0227154
10				1.054440	0.00601534	0.263991	4.305228	0.0057048
50				1.052371	0.00024048	0.0535771	4.204421	0.0002285
100				1.052305	6.1401×10^{-5}	-2.7791700	380.8274	5.8348×10^{-5}
150				1.052294	2.6725×10^{-5}	-0.0221445	12.34413	2.5397×10^{-5}
300				1.052287	6.6799×10^{-6}	0.0088806	4.222261	6.3480×10^{-5}
5	1	0.75	2.5	0.209060	0.00422373	0.9779566	5.87453	0.0202034
	5			0.716636	0.00908887	0.4779385	4.519582	0.0126827
	20			1.243380	0.01044063	0.3853573	4.819704	0.0083969
	100			1.879628	0.01083027	0.3722241	4.367961	0.0057619
	500			2.521891	0.01090973	0.3678802	4.362733	0.0043260
	1000			2.798961	0.01091954	0.3685512	4.297662	0.0039013
	5000			3.442585	0.01092768	0.3669034	4.361590	0.0031743
1.5	2	0.1	15	0.0102539	0.000458083	4.608957	35.52639	0.0446739
		0.5		0.0565069	0.001861421	1.784471	8.381003	0.0329415
		1		0.0908589	0.002442455	1.369496	6.466796	0.0268818
		10		0.2324249	0.003205821	0.971558	5.162603	0.0137929
		50		0.3385982	0.003288293	0.934989	5.069589	0.0097115
		100		0.3846670	0.003298794	0.930406	5.058231	0.0085757
		500		0.4918499	0.003307225	0.926737	5.049208	0.0067241

1.25	3	2	0.1	23.74879	96.142040	1.183845	5.664692	4.0482910
			1	2.374879	0.9614204	1.183845	5.664692	0.4048291
			30	0.079163	0.0010682	1.183845	5.665166	0.0134943
			50	0.047498	0.0003846	1.183843	5.380101	0.0080966
			100	0.023749	9.6142×10^{-5}	1.255683	7.545873	0.0040483
			200	0.011874	2.4036×10^{-5}	4.172638	−25.25049	0.0020242
			250	0.009499	1.5382×10^{-5}	8.297841	−46.27513	0.0016192
			300	0.007916	1.0829×10^{-5}	6.624094	−21.67089	0.0013679
			1000	0.002375	6.7918×10^{-5}	-1.985067	2.809855	0.0028599
			2000	0.001187	1.5772×10^{-6}	-0.453349	1.054604	0.0013282

2.4 Moments of residual life (MRL)

The n^{th} MRL is

$$\pi_{n,Z}(\boldsymbol{t}) = E(Z-\boldsymbol{t})^n \ |_{(Z>\boldsymbol{t},n=1,2,\dots)},$$

The n^{th} moment of the residual life of Z is given by

$$\pi_{n,Z}(\boldsymbol{t}) = \frac{\int_{\boldsymbol{t}}^{\infty}(Z-\boldsymbol{t})^n\, dF_{\underline{\boldsymbol{w}}}(\boldsymbol{t})}{1-F_{\underline{\boldsymbol{w}}}(\boldsymbol{t})}.$$

Therefore,

$$\pi_{n,Z}(\boldsymbol{t}) = \frac{b^n\gamma(n+1,b\boldsymbol{t})}{1-F_{\underline{\boldsymbol{w}}}(\boldsymbol{t})}\sum_{\varepsilon,\hbar=0}^{\infty}\nabla_{\varepsilon,\hbar}^{(n,a^*)*} \ |_{(n>-1)},$$

where

$$\nabla_{\varepsilon,\hbar}^{(n,a^*)*} = \nabla_{\varepsilon}\sum_{r=0}^{n}(-\boldsymbol{t})^{n-r}\binom{n}{r}\frac{a^*(-1)^{\hbar}}{(\hbar+1)^{n+1}}\binom{a^*-1}{\hbar}.$$

2.5 Moments of the reversed residual life (MRRL)

The n^{th} MRRL is

$$\Pi_{n,Z}(\boldsymbol{t}) = E(\boldsymbol{t}-Z)^n \ |_{(Z\leq\boldsymbol{t},\boldsymbol{t}>0,n=1,2,\dots)},$$

hence

$$\Pi_{,Zn}(\boldsymbol{t}) = \frac{\int_0^{\boldsymbol{t}}(\boldsymbol{t}-Z)^n\, dF_{\underline{\boldsymbol{w}}}(\boldsymbol{t})}{F_{\underline{\boldsymbol{w}}}(\boldsymbol{t})}.$$

Then, the n^{th} MRRL of $\boldsymbol{t}$ becomes

$$\Pi_{n,Z}(\boldsymbol{t}) = \frac{b^n\gamma(n+1,b\boldsymbol{t})}{F_{\underline{\boldsymbol{w}}}(\boldsymbol{t})}\sum_{\varepsilon,\hbar=0}^{\infty}\nabla_{\varepsilon,\hbar}^{(n,a^*)**} \ |_{(n>-1)},$$

where

$$\nabla_{\varepsilon,\hbar}^{(n,a^*)**} = \nabla_{\varepsilon}\sum_{r=0}^{n}(-1)^{r}\frac{a^*\boldsymbol{t}^{n-r}(-1)^{\hbar}}{(\hbar+1)^{n+1}}\binom{a^*-1}{\hbar}\binom{n}{r}.$$

3. Maximum likelihood method

Let $\boldsymbol{z}_1,\dots,\boldsymbol{z}_n$ be a random sample from the GOLLEE distribution with parameters a,b,α and β. Let $\underline{\boldsymbol{\mathcal{W}}}$ be the 4×1 parameter vector. For getting the maximum likelihood estimates (MLE) of $\underline{\boldsymbol{\mathcal{W}}}$, we have the log-likelihood (ℓ) function

$$\ell = \ell(\underline{\boldsymbol{\mathcal{W}}}) = n\, log\, \alpha + n\, log\, \beta + n\, log\, a + log\, b + -b\sum_{i=1}^{n} \boldsymbol{z}_i + (a\alpha\beta - 1)\sum_{i=1}^{n} log\boldsymbol{\varrho}_{b,\boldsymbol{z}}$$
$$-2\sum_{i=1}^{n} \quad log\left[\boldsymbol{\varrho}_{b,\boldsymbol{z}}^{a\alpha\beta} + \left(1-\boldsymbol{\varrho}_{b,\boldsymbol{z}}^{a\beta}\right)^{\alpha}\right] + (\alpha - 1)\sum_{i=1}^{n} log\left(1-\boldsymbol{\varrho}_{b,\boldsymbol{z}}^{a\beta}\right),$$

where

$$\boldsymbol{\varrho}_{b,\boldsymbol{z}_i} = 1 - e^{-b\boldsymbol{z}_i}.$$

The score vector components are available if needed. We can compute the maximum values of the unrestricted and restricted log-likelihoods to obtain likelihood ratio (LR) statistics for testing some sub-models of the GOLLEE distribution.

The maximum likelihood estimators and the Bayesian estimators are equivalent asymptotically (Ibragimov (1962) and Chao (1970)), that can be expressed as

$$\sqrt{n}\left(\underline{\widehat{\boldsymbol{\mathcal{W}}}}_{\text{Bayesian}} - \underline{\widehat{\boldsymbol{\mathcal{W}}}}_{\text{MLE}}\right) \overset{a.s}{\rightarrow} 0.$$

A direct result of the above theorem is that all asymptotic properties of the MLEs also hold for the Bayesian estimators. Also, since the determination of the MLE is independent of the loss function and the prior measure, the asymptotic properties of Bayesian estimators hold for all priors and loss functions in a certain class.

4. Simple type copula-based construction

4.1 Via Farlie Gumbel Morgenstern (FGM) family

First, we start with CDF for FGM family of two RVs (Z_1, Z_2) (see Farlie (1960), Gumbel (1960 and 1961), Morgenstern (1956), Johnson and Kotz (1977 & 1975)) which has the following form

$$F_\lambda(\boldsymbol{z}_1, \boldsymbol{z}_2)|_{(|\lambda|\le 1)} = F_1(\boldsymbol{z}_1)F_2(\boldsymbol{z}_2)\{1 + \lambda[1 - F_1(\boldsymbol{z}_1)][1 - F_2(\boldsymbol{z}_2)]\},$$

setting

$$F_1(\boldsymbol{z}_1) = F_{\underline{\boldsymbol{w}}_1}(\boldsymbol{z}) = \frac{\boldsymbol{\varrho}_{b_1,\boldsymbol{z}_1}^{a_1\alpha_1\beta_1}}{\boldsymbol{\varrho}_{b_1,\boldsymbol{z}_1}^{a_1\alpha_1\beta_1} + \left(1-\boldsymbol{\varrho}_{b_1,\boldsymbol{z}_1}^{a_1\beta_1}\right)^{\alpha_1}},$$

where $\underline{\boldsymbol{\mathcal{W}}}_1 = \left(\alpha_1, \beta_1, a_1, b_1\right)$, $\boldsymbol{\varrho}_{b_1,\boldsymbol{z}_1} = 1 - e^{-b_1\boldsymbol{z}_1}$ and

$$F_2(\boldsymbol{z}_2) = F_{\underline{\boldsymbol{w}}_2}(\boldsymbol{z}) = \frac{\boldsymbol{\varrho}_{b_2,\boldsymbol{z}_2}^{a_2\alpha_2\beta_2}}{\boldsymbol{\varrho}_{b_2,\boldsymbol{z}_2}^{a_2\alpha_2\beta_2} + \left(1-\boldsymbol{\varrho}_{b_2,\boldsymbol{z}_2}^{a_2\beta_2}\right)^{\alpha_2}},$$

where $\underline{\boldsymbol{\mathcal{W}}}_2 = \left(\alpha_2, \beta_2, a_2, b_2\right)$ and $\boldsymbol{\varrho}_{b_2,\boldsymbol{z}_2} = 1 - e^{-b_2,\boldsymbol{z}_2}$ then we have

$$F_\lambda(\boldsymbol{z}_1, \boldsymbol{z}_2) = \frac{\boldsymbol{\varrho}_{b_1,\boldsymbol{z}_1}^{a_1\alpha_1\beta_1}}{\boldsymbol{\varrho}_{b_1,\boldsymbol{z}_1}^{a_1\alpha_1\beta_1} + \left(1-\boldsymbol{\varrho}_{b_1,\boldsymbol{z}_1}^{a_1\beta_1}\right)^{\alpha_1}} \frac{\boldsymbol{\varrho}_{b_2,\boldsymbol{z}_2}^{a_2\alpha_2\beta_2}}{\boldsymbol{\varrho}_{b_2,\boldsymbol{z}_2}^{a_2\alpha_2\beta_2} + \left(1-\boldsymbol{\varrho}_{b_2,\boldsymbol{z}_2}^{a_2\beta_2}\right)^{\alpha_2}}$$
$$\times\left(1 + \lambda\left\{\begin{array}{l}\left[1 - \dfrac{\boldsymbol{\varrho}_{b_1,\boldsymbol{z}_1}^{a_1\alpha_1\beta_1}}{\boldsymbol{\varrho}_{b_1,\boldsymbol{z}_1}^{a_1\alpha_1\beta_1} + \left(1-\boldsymbol{\varrho}_{b_1,\boldsymbol{z}_1}^{a_1\beta_1}\right)^{\alpha_1}}\right] \\ \left[1 - \dfrac{\boldsymbol{\varrho}_{b_2,\boldsymbol{z}_2}^{a_2\alpha_2\beta_2}}{\boldsymbol{\varrho}_{b_2,\boldsymbol{z}_2}^{a_2\alpha_2\beta_2} + \left(1-\boldsymbol{\varrho}_{b_2,\boldsymbol{z}_2}^{a_2\beta_2}\right)^{\alpha_2}}\right]\end{array}\right\}\right).$$

4.2 Bivariate GOLLEE type via modified FGM Copula

Due to Rodriguez-Lallena and Ubeda-Flores (2004), the modified version of the FGM copula defined by

$$\mathcal{C}_\Delta(u, \mathcal{Z}) = u\mathcal{Z} + \Delta\widetilde{\boldsymbol{\psi}(\mathcal{Z})}\widetilde{\boldsymbol{\phi}(u)},$$

where $u\boldsymbol{\phi}(u) = \widetilde{\boldsymbol{\phi}(u)}$, and $\mathcal{Z}\overline{\boldsymbol{\psi}(\mathcal{Z})} = \widetilde{\boldsymbol{\psi}(\mathcal{Z})}$. Where the functions $\boldsymbol{\phi}(u)$ and $\boldsymbol{\psi}(\mathcal{Z})$ are two absolutely continuous functions on (0,1). Let

$$\boldsymbol{\alpha} = inf\left\{\frac{1}{\partial u}\partial\widetilde{\boldsymbol{\phi}(u)}: \mathcal{C}_1(u)\right\} < 0, \boldsymbol{\beta} = sup\left\{\frac{1}{\partial u}\partial\widetilde{\boldsymbol{\phi}(u)}: \mathcal{C}_1(u)\right\} < 0,$$

$$\boldsymbol{\xi} = inf\left\{\frac{1}{\partial \mathcal{Z}}\partial\widetilde{\boldsymbol{\psi}(\mathcal{Z})}: \mathcal{C}_2(\mathcal{Z})\right\} > 0, \boldsymbol{\eta} = sup\left\{\frac{1}{\partial \mathcal{Z}}\partial\widetilde{\boldsymbol{\psi}(\mathcal{Z})}: \mathcal{C}_2(\mathcal{Z})\right\} > 0.$$

Then,

$$1 \leq min(\boldsymbol{\beta\alpha}, \boldsymbol{\eta\xi}),$$

where

$$\frac{1}{\partial u}\partial\widetilde{\boldsymbol{\phi}(u)} = u\frac{1}{\partial u}\partial\boldsymbol{\phi}(u) + \boldsymbol{\phi}(u)$$

$$\mathcal{C}_1(u) = \left\{u: u \in (0,1) | \frac{1}{\partial u}\partial\widetilde{\boldsymbol{\phi}(u)} \text{ exists}\right\},$$

and

$$\mathcal{C}_2(\mathcal{Z}) = \left\{\mathcal{Z}: \mathcal{Z} \in (0,1) | \frac{1}{\partial \mathcal{Z}}\partial\widetilde{\boldsymbol{\psi}(\mathcal{Z})} \text{ exists}\right\}.$$

4.2.1 Bivariate GOLLEE-FGM (Type I) model

Consider $\boldsymbol{\phi}(u)$ and $\boldsymbol{\psi}(\mathcal{Z})$ under the previous conditions. Then,

$$\mathcal{C}_\Delta(u, v) = \Delta\left[\widetilde{\boldsymbol{\phi}(u)}\widetilde{\boldsymbol{\psi}(\mathcal{Z})}\right] + \frac{\boldsymbol{\varrho}_{b_1,u}^{a_1\alpha_1\beta_1}\boldsymbol{\varrho}_{b_1,v}^{a_1\alpha_1\beta_1}}{\left[\boldsymbol{\varrho}_{b_1,u}^{a_1\alpha_1\beta_1} + \left(1 - \boldsymbol{\varrho}_{b_1,u}^{a_1\beta_1}\right)^{\alpha_1}\right]\left[\boldsymbol{\varrho}_{b_1,v}^{a_1\alpha_1\beta_1} + \left(1 - \boldsymbol{\varrho}_{b_1,v}^{a_1\beta_1}\right)^{\alpha_1}\right]},$$

where

$$\widetilde{\boldsymbol{\phi}(u)} = u\left[1 - \frac{\boldsymbol{\varrho}_{b_1,u}^{a_1\alpha_1\beta_1}}{\boldsymbol{\varrho}_{b_1,u}^{a_1\alpha_1\beta_1} + \left(1 - \boldsymbol{\varrho}_{b_1,u}^{a_1\beta_1}\right)^{\alpha_1}}\right]|_{\underline{\boldsymbol{w}}_1>0},$$

and

$$\widetilde{\boldsymbol{\psi}(\mathcal{Z})} = \mathcal{Z}\left[1 - \frac{\boldsymbol{\varrho}_{b_1,\mathcal{Z}}^{a_1\alpha_1\beta_1}}{\boldsymbol{\varrho}_{b_1,\mathcal{Z}}^{a_1\alpha_1\beta_1} + \left(1 - \boldsymbol{\varrho}_{b_1,\mathcal{Z}}^{a_1\beta_1}\right)^{\alpha_1}}\right]|_{\underline{\boldsymbol{w}}_2>0}.$$

4.2.2 Bivariate GOLLEE-FGM (Type II) model:

Consider the following functional form for both $\boldsymbol{\phi}(u)$ and $\boldsymbol{\psi}(\mathcal{Z})$ which satisfy all the conditions stated earlier where

$$\boldsymbol{\phi}(u)|_{(\Delta_1>0)} = u^{\Delta_1}(1-u)^{1-\Delta_1} \text{ and } \boldsymbol{\psi}(\mathcal{Z})|_{(\Delta_2>0)} = \mathcal{Z}^{\Delta_2}(1-\mathcal{Z})^{1-\Delta_2}.$$

The corresponding bivariate copula can then be derived from

$$\mathcal{C}_{\Delta,\Delta_1,\Delta_2}(u, \mathcal{Z}) = u\mathcal{Z}\left[1 + \Delta u^{\Delta_1}\mathcal{Z}^{\Delta_2}(1-u)^{1-\Delta_1}(1-\mathcal{Z})^{1-\Delta_2}\right].$$

4.2.3 Bivariate GOLLEE -FGM (Type III) model:

due to Ghosh and Ray (2016) the CDF of the Bivariate GOLLEE -FGM (Type **III**) model can be derived from

$$\mathcal{C}(u, \mathcal{Z}) = uF^{-1}(\mathcal{Z}) + \mathcal{Z}F^{-1}(u) - F^{-1}(u)F^{-1}(\mathcal{Z}),$$

where

$$F^{-1}(u)=-\frac{1}{b_1}ln\left[1-\left(\frac{\widetilde{u}^{\frac{1}{\alpha_1}}}{1+\widetilde{u}^{\frac{1}{\alpha_1}}}\right)^{\frac{1}{\alpha_1\beta_1}}\right]|\widetilde{u}=\frac{u}{1-u},$$

and

$$F^{-1}(z)=-\frac{1}{b_2}ln\left[1-\left(\frac{\widetilde{z}^{\frac{1}{\alpha_2}}}{1+\widetilde{z}^{\frac{1}{\alpha_2}}}\right)^{\frac{1}{\alpha_2\beta_2}}\right]|\widetilde{z}=\frac{z}{1-z}.$$

4.3 Via Clayton copula

4.3.1 The bivariate GOLLEE extension

The bivariate extension via Clayton copula can be considered as a weighted version of the Clayton copula, which is of the form

$$C(u,z)|_{[(\eta_1+\eta_2)\geq 0]}=\left[u^{-(\eta_1+\eta_2)}+z^{-(\eta_1+\eta_2)}-1\right]^{-\frac{1}{\eta_1+\eta_2}}.$$

Assume that $Z\sim$ GOLLEE $\left(\underline{w}_1\right)$ and $Y\sim$ GOLLEE $(\underline{w}_2)$. Then, setting

$$u=u_z=\frac{\varrho_{b_1,z}^{a_1\alpha_1\beta_1}}{\varrho_{b_1,z}^{a_1\alpha_1\beta_1}+\left(1-\varrho_{b_1,z}^{a_1\beta_1}\right)^{\alpha_1}},$$

where $\varrho_{b_2,z}=1-e^{-b_2 z}$ and

$$v=v_y=\frac{\varrho_{b_2,y}^{a_2\alpha_2\beta_2}}{\varrho_{b_2,y}^{a_2\alpha_2\beta_2}+\left(1-\varrho_{b_2,y}^{a_2\beta_2}\right)^{\alpha_2}},$$

where $\varrho_{b_2,y}=1-e^{-b_2 y}$. Then, the associated CDF bivariate GOLLEE type distribution will be

$$H(z,y)=\left\{\begin{matrix}\left[\frac{\varrho_{b_1,z}^{a_1\alpha_1\beta_1}}{\varrho_{b_1,z}^{a_1\alpha_1\beta_1}+\left(1-\varrho_{b_1,z}^{a_1\beta_1}\right)^{\alpha_1}}\right]^{-(\eta_1+\eta_2)}\\+\left[\frac{\varrho_{b_2,y}^{a_2\alpha_2\beta_2}}{\varrho_{b_2,y}^{a_2\alpha_2\beta_2}+\left(1-\varrho_{b_2,y}^{a_2\beta_2}\right)^{\alpha_2}}\right]^{-(\eta_1+\eta_2)}-1\end{matrix}\right\}^{-\frac{1}{\eta_1+\eta_2}}.$$

4.3.2 The Multivariate GOLLEE extension

A straightforward d-dimensional extension from the above will be

$$H(z_1,z_2,\cdots,z_d)=\left\{\begin{matrix}\sum_{i=1}^{d}\left[\frac{\varrho_{b_i,z_i}^{a_i\alpha_i\beta_i}}{\varrho_{b_i,z_i}^{a_i\alpha_i\beta_i}+\left(1-\varrho_{b_i,z_i}^{a_i\beta_i}\right)^{\alpha_i}}\right]^{-(\eta_1+\eta_2)}\\+1-d\end{matrix}\right\}^{-\frac{1}{\eta_1+\eta_2}},$$

where $\varrho_{b_i,z_i}=1-e^{-b_i,z_i}$ .Further future works could be allocated for studying the bivariate and the multivariate extensions of the GOLLEE model.

5. Simulations

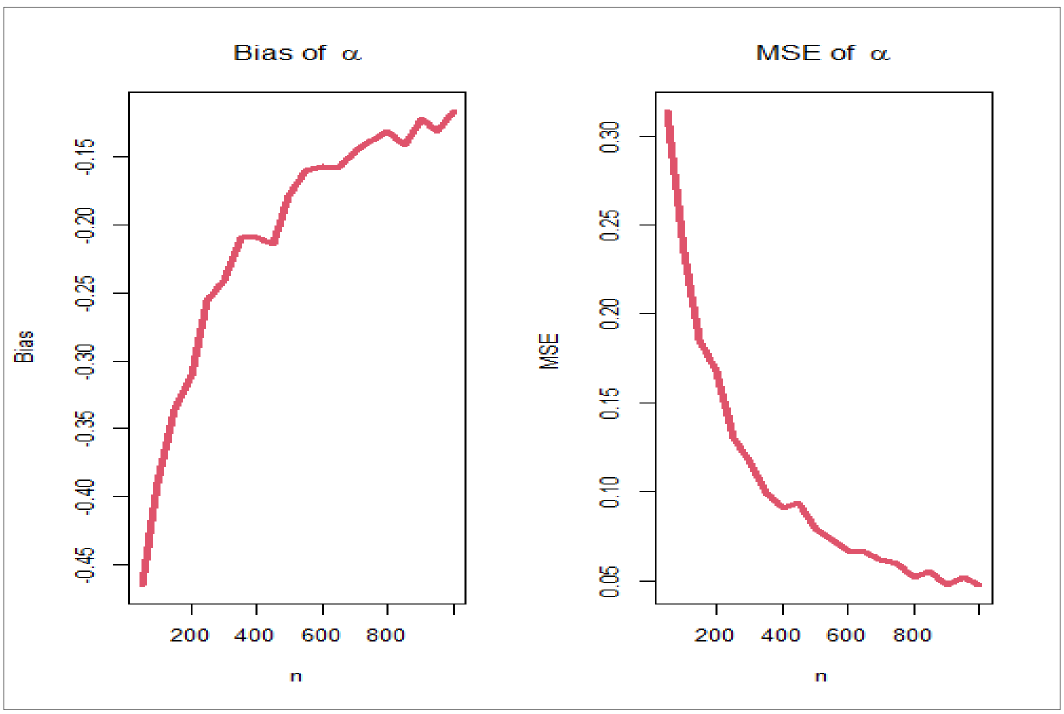
In this Section, we assess the performance of the maximum likelihood (ML) method. The assessment can be performed numerically or graphically. Graphically, we can perform the simulation experiments to assess of the finite sample behavior of the ML estimators (MLEs) via the biases and mean squared errors (MSEs). The following algorithm is considered for the assessment:

1) Using the inversion method, we generate N=1000 samples of size n from the GOLLEE distribution using

$$z_u = -\frac{1}{b} ln\left(1 - \left[\frac{\tilde{u}^{\frac{1}{\alpha}}}{1 + \tilde{u}^{\frac{1}{\alpha}}}\right]^{\frac{1}{a\beta}}\right) |\tilde{u} = \frac{u}{1-u}$$

2) Compute the MLEs for the 1000 samples.
3) Compute the standard errors (SEs) of the MLEs for the 1000 samples. The SEs were computed by inverting the observed information matrix.
4) Compute the biases and MSEs given for $\underline{\boldsymbol{W}} = (\alpha, \beta, a, b)$. We repeated these steps for $n = 50,100,150 \ldots,1000$ with $\alpha = \beta = a = b = 1$, so computing biases $\left(\text{Bias}_{\underline{\boldsymbol{W}}}(n)\right)$, $MSEs$ $\left(MSE_{\underline{\boldsymbol{W}}}(n)\right)$ for α, β, a, b $\forall$ $n = 50,100,150, \ldots,1000$.

Figures 3, 4, 5 and 6 gives the biases (left panel) and MSEs (right panel) for the parameters α, β, a and b respectively. These figures (left panels) shows how the four biases vary with respect to n and also shows how the four MSEs vary with respect to n. The broken lines in Figure 5 corresponds to the biases being 0. From Figures 3, 4, 5 and 6, the biases for each parameter are generally negative and decrease to zero as $n \to \infty$, the MSEs for each parameter decrease to zero as $n \to \infty$.



Figures 3: Biases (left panel) and MSEs (right panel) for the parameter ☺

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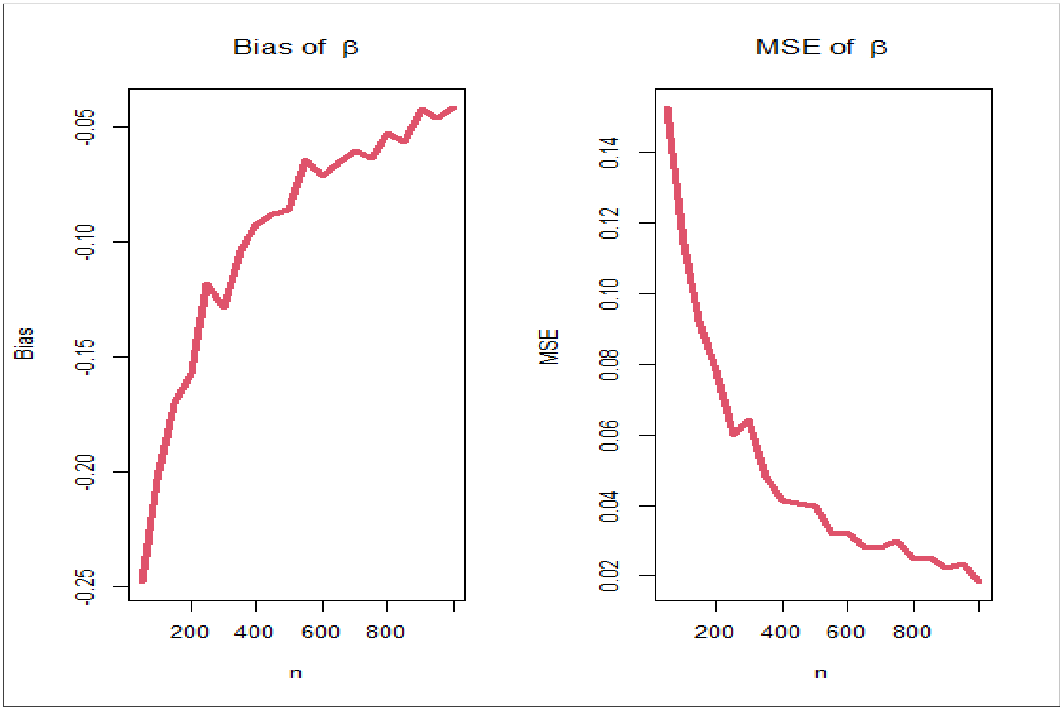
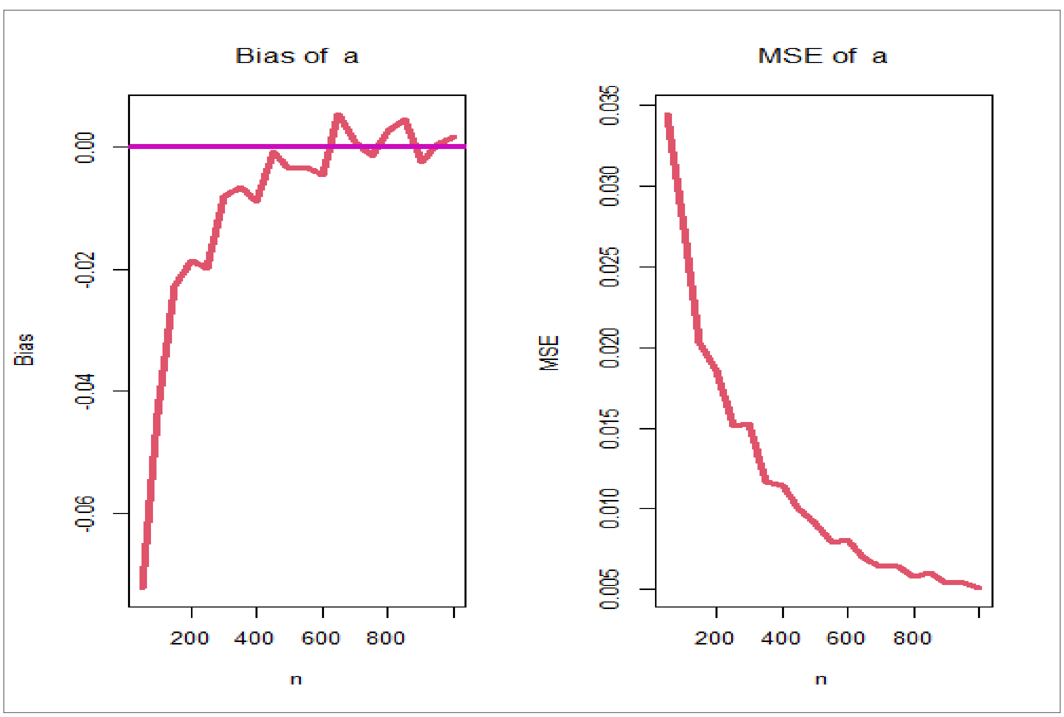
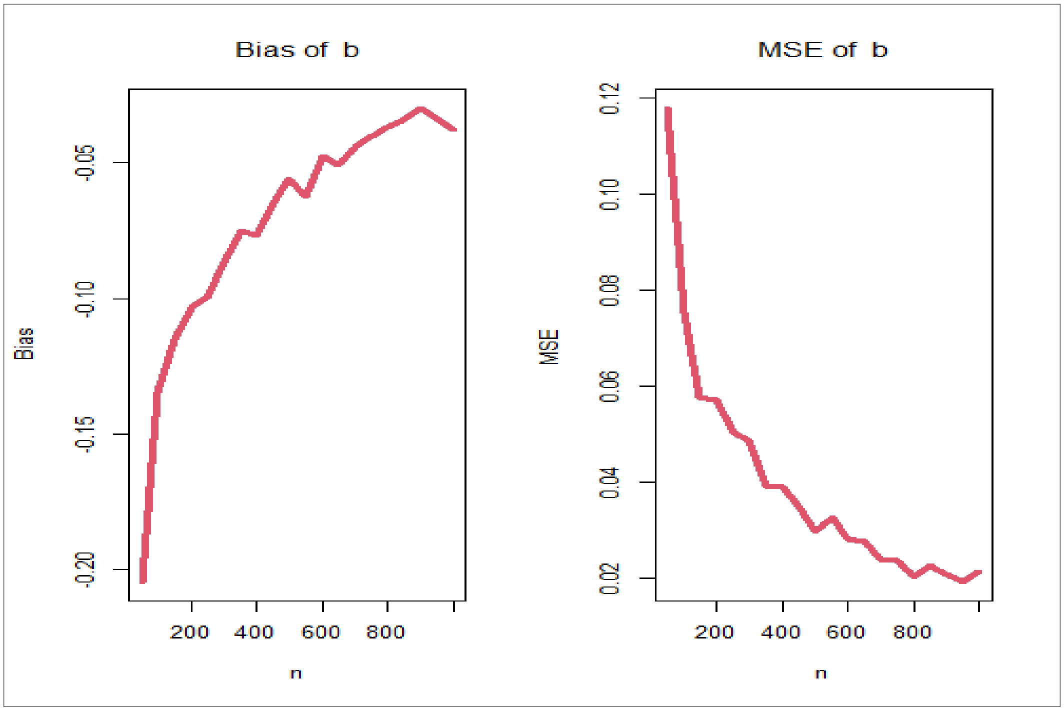


Figures 4: Biases (left panel) and MSEs (right panel) for the parameter [illegible]



Figures 5: Biases (left panel) and MSEs (right panel) for the parameter *a*.



Figures 6: Biases (left panel) and MSEs (right panel) for the parameter b.

6. Real data applications

In this Section, we shall compare the fits of the GOLLEE distribution with those of other competitive models, namely: exponential (E), odd-Lindley exponential (OLE), Marshall-Olkin exponential (MOE), Moment exponential (ME), The Burr-Hatke exponential (BRHE), Generalized Marshall-Olkin exponential (GzMOE), Beta exponential (BE), Marshall-Olkin Kumaraswamy exponential (MOKrE), Kumaraswamy exponential (KrE), the Burr X exponential (BrXE) and Kumaraswamy Marshall-Olkin exponential (KwMOE). Some details related to these competitive models are available in Aboraya, M. (2019a,b), Aboraya and Butt (2019) Ibrahim et al. (2020), Al-Babtain et al. (2020) Aboraya, M. (2021a,b) with some other useful real data sets. For comparing models, we consider the Akaike Information Criteria ($\mathcal{C}_1$), the Consistent Information Criteria ($\mathcal{C}_2$), the Bayesian Information Criteria ($\mathcal{C}_3$) and Hannan-Quinn Information Criteria ($\mathcal{C}_4$), the Cramér-Von Mises ($\mathcal{C}^*$) and the Anderson-Darling ($\mathcal{A}^*$). Additionally, the Kolmogorov–Smirnov (K.S) test and its corresponding p-value is also performed.

6.1 Modeling failure (relief) times

The first data set called the failure or relief times (in minutes). This data represents the lifetime data relating to relief times of patients receiving an analgesic (see Gross and Clark (1975)). This data was recently analyzed by Ibrahim et al. (2020) and Al-Babtain et al. (2020). Table 2 lists the MLEs, SEs confidence intervals (C.I.s). Table 3 lists the $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ $\mathcal{A}^*$, $\mathcal{C}^*$, K.S. and p-value. Figure 7 gives the total time in test test (TTT) plot for the relief times data along with the corresponding box plot. Based on Figure 7, the HRF of the relief times is "increasing HRF" and this data has only one EV observation. Figure 8 gives the estimated PDF, estimated CDF, estimated HRF and P-P plot for relief times data. Figure 9 below gives Kaplan-Meier survival plot for relief times data. Based on Table 3, we conclude that the proposed lifetime GOLLEE model is much better than the “Exponential”, “Odd Lindley-E”, “Marshall-Olkin-E”, “Moment-E”, “Burr-Hatke-E”, “generalized Marshall-Olkin-E”, “Beta-E”, “Marshall-Olkin-Kumaraswamy-E”, “Kumaraswamy-E”, “Burr type X-E” and “Kumaraswamy-Marshall-Olkin-E” models with $\mathcal{C}_1 = 39.62$, $\mathcal{C}_2 = 43.60$, $\mathcal{C}_3 = 42.28$, $\mathcal{C}_4 = 40.39$, $\mathcal{A}^* = 0.255$, $\mathcal{C}^* = 0.045$, K.S=0.12233 and p-value=0.9257 so the new lifetime model is a good alternative to these models in modeling relief times data set. According to Figures 8 and 9, the GOLLEE distribution provides adequate fits to the empirical functions.

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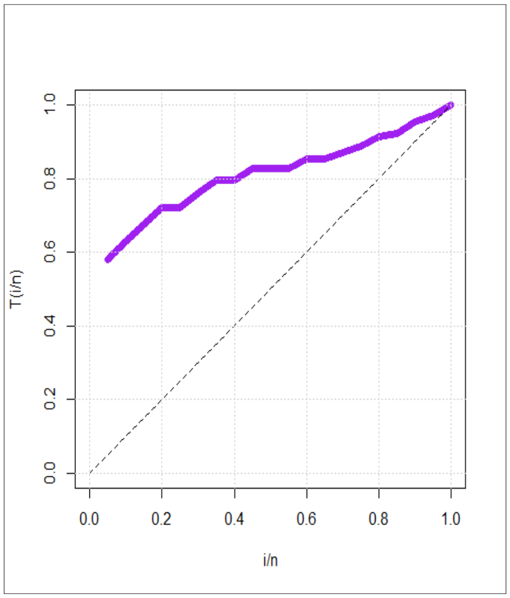
Table 2: MLEs, SEs, C.I.s (in parentheses) values for the relief times data.

Models		Estimates
	MLEs	0.52612
E(b)	SEs	(0.1172)
	C.I.s	(0.3, 0.8)
	MLEs	0.9499
ME(b)	SEs	(0.150)
	C.I.s	(0.7, 1.2)
	MLEs	0.52634
BRHE(b)	SEs	(0.1182)
	C.I.s	(0.4, 0.6)
	MLEs	0.6044
OLE(b)	SEs	(0.0535)
	C.I.s	(0.5, 0.7)
	MLEs	54.47, 2.32
MOE(α,b)	SEs	(35.58), (0.37)
	C.I.s	(0, 124.2), (1.58, 3.0)
	MLEs	1.1635, 0.3207
BrXE(a,b)	SEs	(0.33), (0.03)
	C.I.s	(0.5, 1.82), (0.26,0.38)
	MLEs	83.756, 0.568, 3.330
KrE(α,β,b)	SEs	(42.361), (0.327), (1.189)
	C.I.s	(0.7, 167), (0, 1.2), (1.00, 5.9)
	MLEs	8.868, 34.826, 0.299, 4.899
KwMOE(α,β,λ,b)	SEs	(9.15), (22.31), (0.24), (3.19)
	C.I.s	(10.9, 46.8), (0, 78.6), (0, 0.76), (0, 11)
	MLEs	0.51912, 89.4622, 3.1693
GzMOE(λ,α,b)	SEs	(0.256), (66.278), (0.77)
	C.I.s	(0.02, 1.02), (0, 219.4), (1.66, 4.9)
	MLEs	0.133, 33.232, 0.571, 1.669
MOKrE(α,β,λ,b)	SEs	(0.332), (57.84), (0.72), (1.81)
	C.I.s	(0, 0.8), (0, 146.6), (0, 2), (0, 5.3)
	MLEs	81.633, 0.5424, 3.5142
BE(α,β,b)	SEs	(120.43), (0.327), (1.411)
	C.I.s	(0, 317.65), (0, 1.19), (0.75, 6.4)
	MLEs	0.584, 13.798, 20.40, 3.415
GOLLEE(α,β,a,b)	SEs	(0.28), (193.88), (286.648), (1.29)
	C.I.s	(0.24, 1.144), (0, 402), (0, 592), (0.82, 6)

Table 3: Statistics for the relief times data.

Models	(p-value), K.S.	C_1, C_2, C_3, C_4	$\mathcal{A}^*$	$\mathcal{C}^*$
E	(0.004)0.4	68.0, 68.7, 67.9, 68.0	4.60	0.96
BrXE	(0.17)0.25	48.1, 50.1, 49.0, 48.5	1.34	0.24
OLE	(<0.1%)0.9	49.1, 50.1, 49.3, 49.3	1.30	0.22
KMOE	(0.86)0.15	43.0, 46.8, 45.6, 43.6	1.08	0.19
ME	(0.07)0.32	54.3, 55.3, 54.5, 54.5	2.76	0.53
MOKE	(0.87)0.14	41.6, 45.5, 44.3, 42.3	0.60	0.13
BRHE	(<0.1%)0.4	67.7, 68.7, 67.9, 67.8	0.62	0.12

BE	(0.80)0.16	43.5, 46.5, 44.9, 44.0	0.70	0.14
MOE	(0.55)0.18	43.5, 45.5, 44.2, 43.9	0.80	0.15
KrE	(0.86)0.14	42.0, 44.8, 43.3, 42.3	0.45	0.07
GzMOE	(0.78)0.15	42.8, 45.7, 44.3, 43.3	0.51	0.08
GOLLEE	**(0.93)0.122**	**39.6, 43.6, 42.3, 40.3**	**0.25**	**0.05**



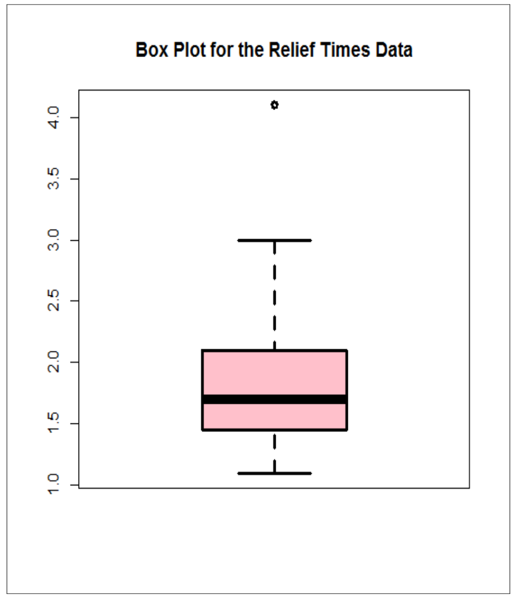


Figure 7: The TTT plot (left panel) and box plot (right panel) for the relief times data

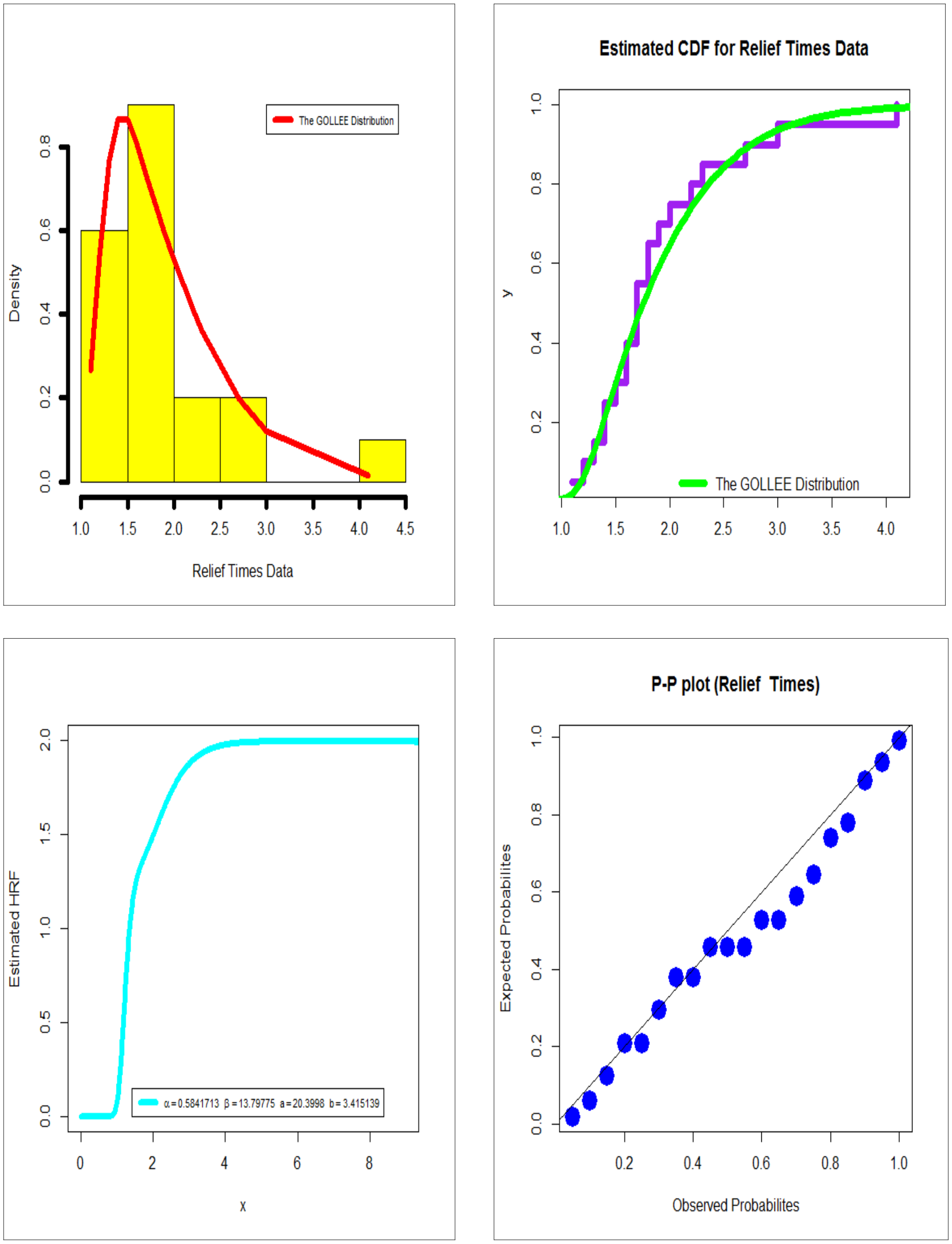


Figure 8: Estimated PDF, estimated CDF, estimated HRF and P-P plot for relief times data.

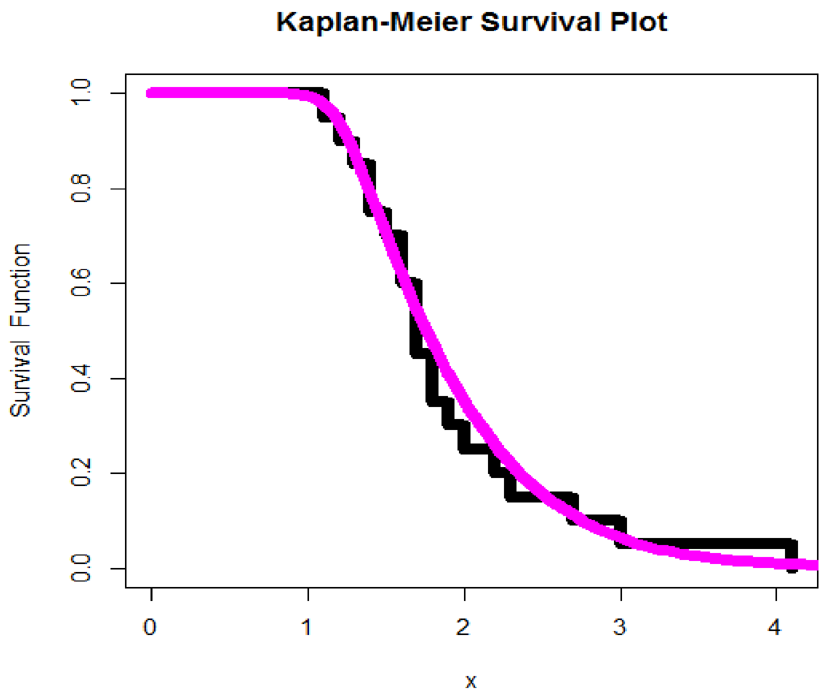


Figure 9: Kaplan-Meier survival plot for relief times data.

The proposed GOLLEE lifetime model is much better than the E, ME, MOE, GzMOE, KrE, BE, MOKE, KMOE models.

6.2 Modeling survival times

The second data set called the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). This data was recently analyzed by Ibrahim et al. (2020) and Al-Babtain et al. (2020). Table 4 lists the MLEs, SEs confidence intervals (C.I.s). Table 5 lists the $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{A}^*$, $\mathcal{C}^*$, K.S. and p-value. Figure 10 gives the TTT plot along with the corresponding box plot for the survival times data. Based on Figure 10, the HRF of the survival times is "increasing HRF" and this data has only four EV observations. Figure 11 gives the estimated PDF, estimated CDF, estimated HRF and P-P plot survival times data. Figure 12 gives the Kaplan-Meier survival plot survival times data. Based on Table 5, we conclude that the GOLLEE model is much better than the Exponential, Odd Lindley Exponential, Marshall-Olkin Exponential, Moment Exponential, The Burr-Hatke Exponential, generalized Marshall-Olkin Exponential, Beta Exponential, Marshall-Olkin Kumaraswamy Exponential, Kumaraswamy Exponential, the Burr X Exponential and Kumaraswamy Marshall-Olkin Exponential models with $\mathcal{C}_1 = 204.44$, $\mathcal{C}_2 = 213.54$, $\mathcal{C}_3 = 205.03$, $\mathcal{C}_4 = 208.06$, $\mathcal{A}^* = 0.35$, $\mathcal{C}^* = 0.058$, K.S=0.0723 and p-value=**0.846** so the new lifetime model is a good alternative to these models in modeling relief times data set. According to Figures 11 and 12, the GOLLEE distribution provides adequate fits to the empirical functions.

Table 4: MLEs, SEs, C.I.s (in parentheses) values for the survival times data.

Models		Estimates
E(b)	MLEs	0.5401
	SEs	(0.063)
	C.I.s	(0.4, 0.7)
OLE(b)	MLEs	0.38145
	SEs	(0.021)
	C.I.s	(0.3, 0.4)
ME(b)	MLEs	0.9250
	SEs	(0.0801)
	C.I.s	(0.62, 1.08)

	MLEs	0.480, 0.2060
BrXE(a,b)	SEs	(0.061), (0.012)
	C.I.s	(0.4, 0.5), (0.18, 0.25)
	MLEs	0.54221
BRHE(b)	SEs	(0.0623)
	C.I.s	(0.41, 0.68)
	MLEs	0.179, 47.635, 4.470
GzMOE(λ,α,b)	SEs	(0.07), (44.901), (1.327)
	C.I.s	(0.04, 0.32), (0, 15), (2, 8)
	MLE	8.7802, 1.3802
MOE(α,b)	SEss	(3.556), (0.193)
	C.I.s	(1.81,15.74), (1.0,1.82)
	MLEs	0.807, 3.461, 1.331
BE(a,β,b)	SEs	(0.696), (1.003), (0.856)
	C.I.s	(0, 2.17), (1.49,5.42), (0, 3.01)
	MLEs	3.3042, 1.1003, 1.0374
KrE(a,β,b)	SEs	(1.106), (0.764), (0.614)
	C.I.s	(1.13, 5.5), (0, 2.6), (0, 2.2)
	MLEs	0.37, 3.48, 3.31, 0.30
KwMOE(α,β,λ,b)	SEs	(0.14), (0.86), (0.78), (1.11)
	C.I.s	(0.11, 0.6), (1.8, 5), (1.8, 5), (0, 2.5)
	MLEs	0.008, 2.716, 1.986, 0.099
MOKE(α,β,λ,b)	SEs	(0.002), 1.316), (0.784), (0.048)
	C.I.s	(0.004,0.012), (0.14, 5.5), (0.4, 3.6), (0, 0.3)
	MLEs	7.512, 1.400, 0.140, 0.0221
GOLLEE(α,β,a,b)	SEs	(1.66), (0.000), (0.000), (0.01)
	C.I.s	(4.1, 10.9), -, -, (0, 0.047)

Table 5: Statistic for the survival times data.

Models	(p-value), K.S.	C_1, C_2, C_3, C_4	$\mathcal{A}^*$	$\mathcal{C}^*$
E	(0.060)0.27	234.6, 236.9, 234.7, 235.5	6.53	1.25
BrXE	(0.002)0.22	235.3, 239.9, 235.5, 237.5	2.90	0.52
OLE	(<0.1%)0.49	229.1, 231.4, 229.5, 230.0	1.95	0.33
KMOE	(0.530)0.09	208.0, 217.0, 208.4, 211.4	0.61	0.14
ME	(0.130)0.14	210.4, 212.7, 210.5, 211.6	1.54	0.25
BE	(0.340)0.11	207.4, 214.2, 207.7, 210.4	0.98	0.15
BRHE	(<0.1%)0.28	235.5, 237.6, 235.0, 236.3	0.73	0.14
MOKE	(0.440)0.11	209.4, 218.6, 210.0, 213.5	0.79	0.13
MOE	(0.430)0.13	210.4, 215.0, 210.5, 212.2	1.24	0.17
KrE	(0.500)0.09	209.4, 216.4, 209.8, 212.3	0.75	0.13
GzMOE	(0.510)0.09	210.5, 217.4, 211.0, 213.2	1.05	0.17
GOLLEE	**(0.846)0.0723**	**204.4, 213.5, 205.0, 208.1**	**0.35**	**0.06**

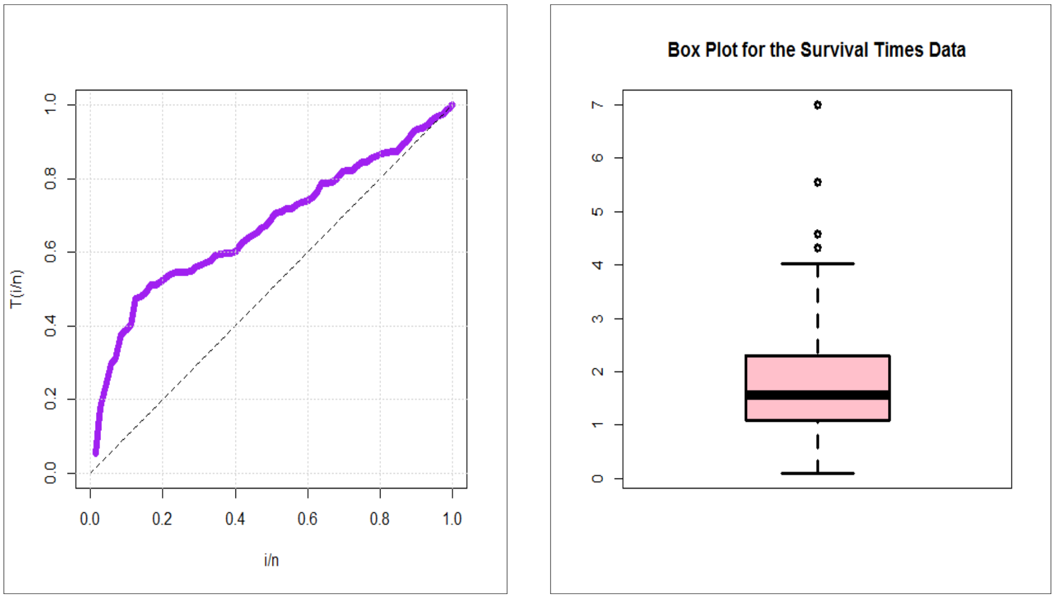


Figure 10: The TTT plot (left panel) and box plot (right panel) for the survival times data

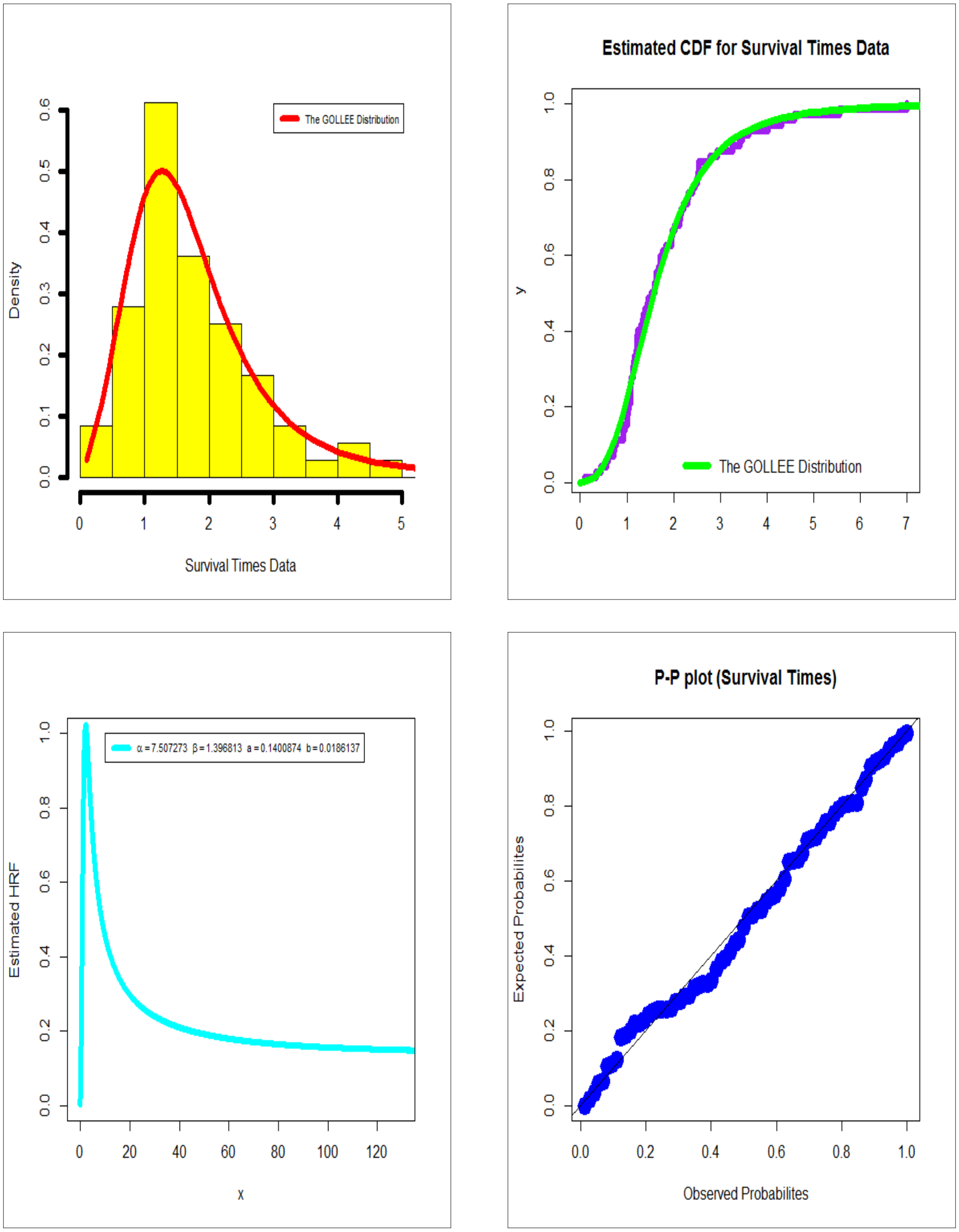


Figure 11: Estimated PDF, estimated CDF, estimated HRF and P-P plot survival times data.

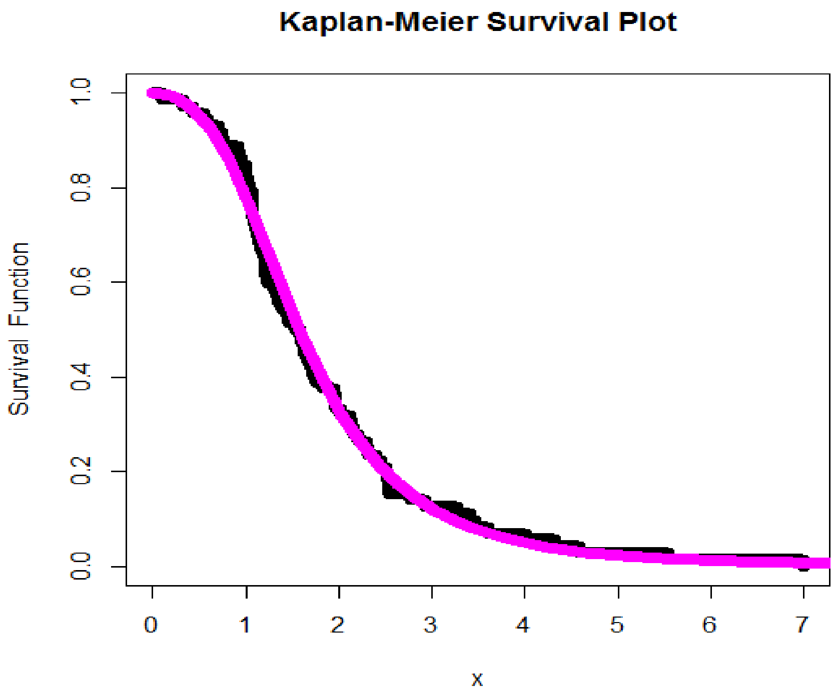


Figure 12: Kaplan-Meier survival plot survival times data.

7. Conclusions and future works

In this article, we introduced and studied a new generalization of the exponentiated exponential distribution. Various structural mathematical properties including explicit expressions for the moment generating function, the ordinary moments, incomplete moment are derived. Numerical analysis for mean, variance, skewness and kurtosis and the dispersion index is performed for illustrating the importance and flexibility of the new model. Many bivariate and multivariate type extensions have been also derived. The estimation of the model parameters is performed by maximum likelihood method. The new density can be right skewed and symmetric with unimodal and bimodal shapes. The new hazard function can be "constant", "monotonically decreasing", " monotonically increasing", "increasing-constant”, “upside-down-constant", "decreasing-constant". We assessed the performance of the maximum likelihood estimation method using a graphical simulation study via the biases and mean squared errors. The usefulness and flexibility of the new distribution is illustrated by means of two real data sets. The new model is much better than many useful models in modeling relief times and survival times data sets according to the Akaike Information Criterion, the Consistent Akaike Information Criterion, the Hannan-Quinn Information Criterion, the Bayesian Information Criterion, the Cramér-Von Mises, the Anderson-Darling statistics.

As a future related work, authors can apply many new useful goodness-of-fit tests for right censored validation such as the Nikulin-Rao-Robson goodness-of-fit test, modified Nikulin-Rao-Robson goodness-of-fit test, Bagdonavicius-Nikulin goodness-of-fit test, modified Bagdonavicius-Nikulin goodness-of-fit test, to the new BuXENH model as performed by Ibrahim et al. (2019), Goual et al. (2019, 2020), Mansour et al. (2020a,d), Yadav et al. (2020) and Goual and Yousof (2020), among others. Recently, Aidi et al. (2021) investigated and studied a novel version of the well-known goodness-of-fit test statistic for a new model alled the double Burr X distribution with many applications to right censored real medical and reliability datasets. Yousof et al. (2021a) presented new modified Chi-square type test for the right censored distributional validation with some new characterization results and various estimation methods. Yousof et al. (2021c) introduced new modified Chi-square test for the right the censoring distributional validation under a novel Nadarajah Haghighi model with some new characterization results and different estimation methods. Another new version for the right censored validity under a new Chen model with some applications in reliability and medicine is presented by Ibrahim et al. (2021).

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