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The Weighted Power Quasi Lindley Distribution with Properties and Applications of Life-time Data

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Abstract

The present study deals with a new version of power quasi Lindley distribution known as weighted power quasi Lindley distribution. Its different mathematical and statistical properties like moments, order statistics, reliability analysis, Renyi entropy, bonferroni and Lorenz curves have been presented. The parameters of proposed new distribution are estimated by employing the method of maximum likelihood estimation. Finally, a new distribution has been illustrated with three real lifetime data sets from medical sciences to examine its significance and supremacy.

Keywords: Weighted distribution, Power quasi Lindley distribution, Order statistics, Entropy, Maximum likelihood estimation, Reliability Analysis.

1. Introduction

In probability and statistics, the recent studies have shown that fitting of models to classical distributions is of great importance for generalizing the existing distributions by introducing an extra parameter to it. This extra parameter brings more flexibility and reliability to a class of distribution functions and it should be very significant for data analysis purpose. This extra parameter can be added through by various techniques. One of such technique is of weighted technique. The idea of weighted distributions was suggested by Fisher (1934) to study how the methods of ascertainment can influence the form of distribution of recorded observations. Later Rao (1965) who developed this concept in a collective manner in association with modeling statistical data were the routine practice of using standard distributions was found to be inappropriate. The weighted distributions play a dominant and practical role in probability, statistics and mathematics. The weighted distributions are used as a tool in selection of appropriate models for observed data especially when samples are drawn without a proper frame. The weighted distributions arise when observations are recorded by an investigator in nature according to certain stochastic model the distribution of recorded observation will not have the original distribution unless every observation is given an equal chance of being recorded and hence they are recorded according to some weight function. The weighted distributions provide a collective access for the problem of model specification and data interpretation. The weighted distributions have been employed in various areas of research related to reliability, biomedicine, ecology, survival analysis, analysis of family data, Meta analysis, forestry and other areas for the proper development of statistical models. The weighted distributions are remarkable for efficient modeling of statistical data and prediction. The weighted distributions provide a technique for fitting models to the unknown weight function when samples can be taken both from original and developed distributions.

Many authors have developed and contributed some important weighted probability models along with their illustrations and applications in different fields. Dar, Ahmad and Reshi (2020) presented weighted Gamma-Pareto distribution and its application. Hassan, wani and Para (2018) proposed three parameter weighted quasi Lindley distribution and introduce its necessary properties and applications. Hassan, Dar, Ahmad and Para (2019) discussed on the weighted Pranav distribution with properties and its applications. Para and Jan (2018) studies three parameter weighted Pareto type II distribution with properties and its applications in medical sciences. Ganaie and Rajagopalan (2021) proposed length biased weighted new quasi Lindley distribution with its statistical properties and applications. Kilany (2016) obtained the weighted Lomax distribution with applications. Hussain and Mohammad (2022) discussed on weighted Zeghdoudi distribution with its properties and applications. Ahmad et al. (2014) executed the characterization and estimation of double weighted Rayleigh distribution. Gharaibeh (2022) studied the weighted Gharaibeh distribution with real data applications. Benchiha et al. (2021) proposed weighted generalized quasi Lindley distribution with different methods of estimation and application for covid-19 and engineering data. Khan and Mustafa (2022) obtained the weighted power hazard rate distribution with application. Ghitany et al. (2011) introduced two parameter weighted Lindley distribution with its application to survival data. Hassan, Almetwally, Khaleel and Nagy (2021) studies weighted power Lomax distribution and its length biased version with properties and estimation based on censored samples. Al-Omari et al. (2019) proposed power length biased Suja distribution with properties and its application. Recently, Almuqrin (2023) studied weighted power Maxwell distribution with statistical inference and covid-19 applications.

Power quasi Lindley distribution is a three parametric lifetime distribution proposed by Said Hofan Alkarni (2015). The proposed distribution is introduced by considering the power transformation of Quasi Lindley distribution. The different statistical properties of proposed distribution that include density and hazard rate function with their behaviour, moments, skewness, kurtosis, Renyi entropy, quantile function and moment generating function has been discussed. Its parameters have been estimated by using the maximum likelihood estimation and their asymptotic distribution and confidence intervals has also been discussed. Finally an application of proposed distribution with real data set is presented and the results shows better fit over Lindley, two parameter Lindley, weibull and power Lindley distributions.

In this paper, we introduce a new distribution with four parameters known as weighted power quasi Lindley (WPQL) distribution. We execute the new distribution with the hope that it will provide a better result and will be reliable and flexible in comparison with other distributions. On applying the weighted version, the fourth parameter in this distribution makes it more flexible to describe different types of real data than its sub-models.

2. Weighted Power Quasi Lindley (WPQL) Distribution

The probability density function of power quasi Lindley distribution is given by

$$f\left(x;\theta,\beta,\ \alpha\right)=\frac{\alpha\,\theta}{\beta+1}\left(\beta+\theta x^{\alpha}\right)x^{\alpha-1}\,e^{-\theta x^{\alpha}}\ ;\ \theta,\alpha,\beta,x>0 \tag{1}$$

and the cumulative distribution function of power quasi Lindley distribution is given by

$$F(x;\theta,\beta,\ \alpha)=1-\left(1+\frac{\theta}{\beta+1}x^{\alpha}\right)e^{-\theta x^{\alpha}}\ ;\ \theta,\alpha,\beta,x>0 \tag{2}$$

Let X be the random variable following non-negative condition with probability density function $f(x)$.Let the non-negative weight function be $w(x)$, then the probability density function of weighted random variable X_w is given by

$$f_w(x)=\frac{w(x)f(x)}{E(w(x))},\quad x>0.$$

Where $w(x)$ be the non - negative weight function and $E(w(x)) = \int w(x)f(x)dx < \infty$.

Depending upon the various choices of weighted function $w(x)$, weighted models are of various forms obviously when $w(x) = x^c$, the resulting distribution is known as weighted distribution. In this paper, we have to study the weighted version of power quasi Lindley distribution known as weighted power quasi Lindley distribution, so consequently the weight function at $w(x) = x^c$, resulting distribution is known as weighted distribution with probability density function given by

$$f_w(x;\theta,\beta,\alpha,c)=\frac{x^c f(x;\theta,\beta,\alpha)}{E(x^c)} \tag{3}$$

Where $E(x^c) = \int_0^\infty x^c f(x;\theta,\beta,\alpha)dx$

$$E(x^c) = \frac{\beta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}}{\alpha\theta^{\frac{c}{\alpha}}(\beta+1)} \tag{4}$$

By using the equations (1) and (4) in equation (3), we will obtain the probability density function of weighted power quasi Lindley distribution.

$$f_w(x;\theta,\beta,\alpha,c) = \frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\beta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}} x^{\alpha+c-1}\left(\beta+\theta x^{\alpha}\right)e^{-\theta x^{\alpha}} \tag{5}$$

and the cumulative distribution function of weighted power quasi Lindley distribution can be obtained as

$$F_w(x;\theta,\beta,\alpha,c) = \int_0^x f_w(x;\theta,\beta,\alpha,c)dx$$

$$F_w(x;\theta,\beta,\alpha,c) = \int_0^x \frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\beta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}} x^{\alpha+c-1}\left(\beta+\theta x^{\alpha}\right)e^{-\theta x^{\alpha}}dx$$

$$F_w(x;\theta,\beta,\alpha,c) = \frac{1}{\beta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}} \int_0^x \alpha\theta^{\frac{\alpha+c}{\alpha}} x^{\alpha+c-1}\left(\beta+\theta x^{\alpha}\right)e^{-\theta x^{\alpha}}dx$$

$$F_w(x;\theta,\beta,\alpha,c) = \frac{1}{\beta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}} \left(\alpha\beta\theta^{\frac{\alpha+c}{\alpha}} \int_0^x x^{\alpha+c-1}e^{-\theta x^{\alpha}}dx + \alpha\theta^{\frac{2\alpha+c}{\alpha}} \int_0^x x^{2\alpha+c-1}e^{-\theta x^{\alpha}}dx\right) \tag{6}$$

Put $\theta x^{\alpha} = t \Rightarrow x^{\alpha} = \frac{t}{\theta} \Rightarrow x = \left(\frac{t}{\theta}\right)^{\frac{1}{\alpha}}$

Also $\alpha\theta x^{\alpha-1}dx = dt \Rightarrow dx = \frac{dt}{\alpha\theta x^{\alpha-1}} \Rightarrow dx = \frac{dt}{\alpha\theta\left(\frac{t}{\theta}\right)^{\frac{\alpha-1}{\alpha}}}$

After the simplification of equation (6), we will have obtained the cumulative distribution function of weighted power quasi Lindley distribution as

$$F_w(x;\theta,\beta,\alpha,c) = \frac{1}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\left(\beta\gamma\left(\frac{(\alpha+c)}{\alpha},\theta x^{\alpha}\right)+\gamma\left(\frac{(2\alpha+c)}{\alpha},\theta x^{\alpha}\right)\right) \tag{7}$$

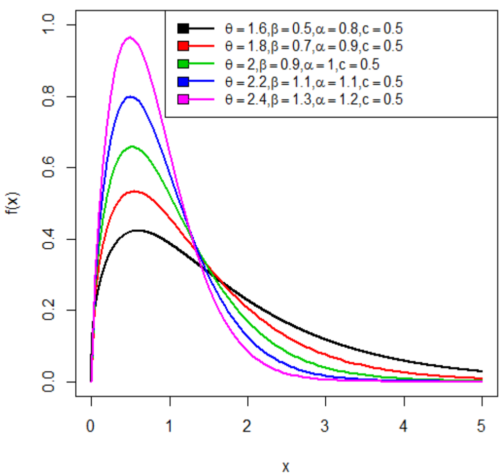

Fig.1:Pdf plot of Weighted Power Quasi Lindley Distribution

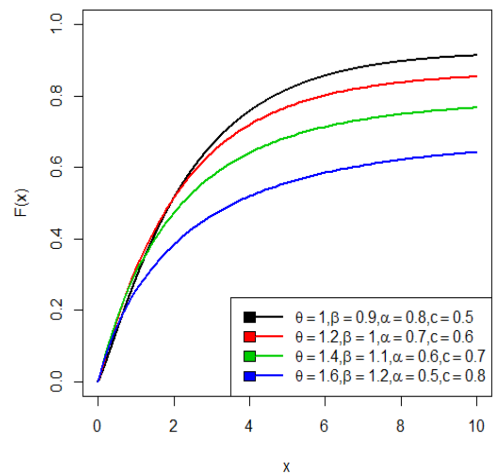

Fig.2 Cdf plot of Weighted Power Quasi Lindley Distribution

3. Reliability Analysis

In this section, we will have discussed about the reliability function, hazard rate and reverse hazard rate functions of the proposed weighted power quasi Lindley distribution.

The reliability function or survival function of weighted power quasi Lindley distribution is given by

$$R(x) = 1 - F_w(x;\theta,\beta,\alpha,c)$$

$$R(x) = 1 - \frac{1}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\left(\beta\gamma\left(\frac{(\alpha+c)}{\alpha},\theta x^{\alpha}\right)+\gamma\left(\frac{(2\alpha+c)}{\alpha},\theta x^{\alpha}\right)\right)$$

The hazard function is also termed as hazard rate, instantaneous failure rate or force of mortality is given by

$$h(x) = \frac{f_w(x;\theta,\beta,\alpha,c)}{1-F_w(x;\theta,\beta,\alpha,c)}$$

$$h(x) = \frac{\alpha\theta^{\frac{\alpha+c}{\alpha}} x^{\alpha+c-1}\left(\beta+\theta x^{\alpha}\right)e^{-\theta x^{\alpha}}}{\left(\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)-\left(\beta\gamma\left(\frac{(\alpha+c)}{\alpha},\theta x^{\alpha}\right)+\gamma\left(\frac{(2\alpha+c)}{\alpha},\theta x^{\alpha}\right)\right)}$$

The reverse hazard rate function of weighted power quasi Lindley distribution is given by

$$h_r(x) = \frac{f_w(x;\theta,\beta,\alpha,c)}{F_w(x;\theta,\beta,\alpha,c)}$$

$$h_r(x) = \frac{\alpha\theta^{\frac{\alpha+c}{\alpha}} x^{\alpha+c-1}\left(\beta+\theta x^{\alpha}\right)e^{-\theta x^{\alpha}}}{\left(\beta\gamma\left(\frac{(\alpha+c)}{\alpha},\theta x^{\alpha}\right)+\gamma\left(\frac{(2\alpha+c)}{\alpha},\theta x^{\alpha}\right)\right)}$$

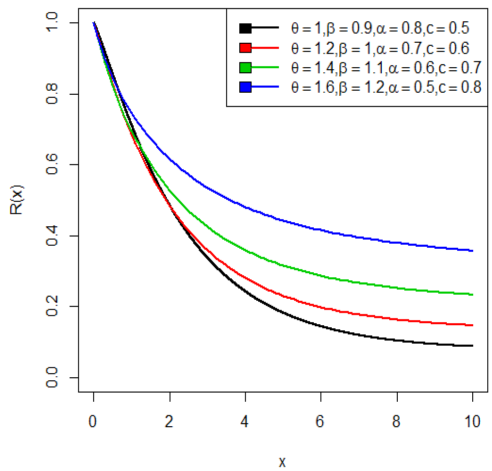

Fig 3 Reliability plot of Weighted Power Quasi Lindley Distribution

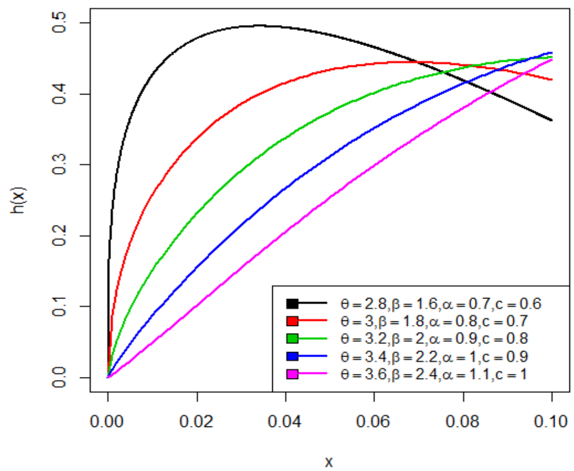

Fig 4 Hazard Plot of Weighted Power Quasi Lindley Distribution

4. Structural Properties

In this section, we will have obtained the different statistical properties of weighted power quasi Lindley distribution that include moments, harmonic mean, moment generating function and characteristic function.

Moments

Let X be the random variable following weighted power quasi Lindley distribution with parameters θ, β, α and c, then the r^{th} order moment $E(X^r)$ of executed distribution can be obtained as

$$E(X^r)=\mu_r'=\int_0^\infty x^r f_w(x;\theta,\beta,\alpha,c)dx$$

$$E(X^r)=\mu_r'=\int_0^\infty \frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}} x^{\alpha+c+r-1}\left(\beta+\theta x^{\alpha}\right)e^{-\theta x^{\alpha}}dx$$

$$E(X^r)=\mu_r'=\frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\int_0^\infty x^{\alpha+c+r-1}\left(\beta+\theta x^{\alpha}\right)e^{-\theta x^{\alpha}}dx$$

$$E(X^r)=\mu_r'=\frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\left(\beta\int_0^\infty x^{\alpha+c+r-1}e^{-\theta x^{\alpha}}dx+\theta\int_0^\infty x^{2\alpha+c+r-1}e^{-\theta x^{\alpha}}dx\right) \tag{8}$$

Put $x^{\alpha}=t \;\Rightarrow x=t^{\frac{1}{\alpha}}$

Also $\alpha x^{\alpha-1}dx=dt \;\Rightarrow dx=\dfrac{dt}{\alpha x^{\alpha-1}}=\dfrac{dt}{\alpha t^{\frac{\alpha-1}{\alpha}}}$

After simplifying, the equation (8) becomes

$$E(X^r) = \mu_r' = \frac{\beta\Gamma\frac{(\alpha+c+r)}{\alpha} + \Gamma\frac{(2\alpha+c+r)}{\alpha}}{\alpha\theta^{\frac{r}{\alpha}}\left(\beta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}\right)} \tag{9}$$

Now putting r = 1, 2, 3 and 4 in equation (9), we will obtain the first four moments of weighted power quasi Lindley distribution.

$$E(X) = \mu_1' = \frac{\beta\Gamma\frac{(\alpha+c+1)}{\alpha} + \Gamma\frac{(2\alpha+c+1)}{\alpha}}{\alpha\theta^{\frac{1}{\alpha}}\left(\beta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}\right)}$$

$$E(X^2) = \mu_2' = \frac{\beta\Gamma\frac{(\alpha+c+2)}{\alpha} + \Gamma\frac{(2\alpha+c+2)}{\alpha}}{\alpha\theta^{\frac{2}{\alpha}}\left(\beta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}\right)}$$

$$E(X^3) = \mu_3' = \frac{\beta\Gamma\frac{(\alpha+c+3)}{\alpha} + \Gamma\frac{(2\alpha+c+3)}{\alpha}}{\alpha\theta^{\frac{3}{\alpha}}\left(\beta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}\right)}$$

$$E(X^4) = \mu_4' = \frac{\beta\Gamma\frac{(\alpha+c+4)}{\alpha} + \Gamma\frac{(2\alpha+c+4)}{\alpha}}{\alpha\theta^{\frac{4}{\alpha}}\left(\beta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}\right)}$$

$$\text{Variance} = \frac{\beta\Gamma\frac{(\alpha+c+2)}{\alpha} + \Gamma\frac{(2\alpha+c+2)}{\alpha}}{\alpha\theta^{\frac{2}{\alpha}}\left(\beta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}\right)} - \left(\frac{\beta\Gamma\frac{(\alpha+c+1)}{\alpha} + \Gamma\frac{(2\alpha+c+1)}{\alpha}}{\alpha\theta^{\frac{1}{\alpha}}\left(\beta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}\right)}\right)^2$$

$$S.D(\sigma) = \sqrt{\left(\frac{\beta\Gamma\frac{(\alpha+c+2)}{\alpha} + \Gamma\frac{(2\alpha+c+2)}{\alpha}}{\alpha\theta^{\frac{2}{\alpha}}\left(\beta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}\right)} - \left(\frac{\beta\Gamma\frac{(\alpha+c+1)}{\alpha} + \Gamma\frac{(2\alpha+c+1)}{\alpha}}{\alpha\theta^{\frac{1}{\alpha}}\left(\beta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}\right)}\right)^2\right)}$$

Harmonic mean

The harmonic mean for proposed weighted power quasi Lindley distribution can be obtained as

$$H.M = E\left(\frac{1}{x}\right) = \int_0^\infty \frac{1}{x} f_w(x;\theta,\beta,\alpha,c)dx$$

$$H.M = \int_0^\infty \frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}} x^{\alpha+c-2}\left(\beta+\theta x^{\alpha}\right)e^{-\theta x^{\alpha}}\,dx$$

$$H.M = \frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}} \int_0^\infty x^{\alpha+c-2}\left(\beta+\theta x^{\alpha}\right)e^{-\theta x^{\alpha}}\,dx$$

$$H.M = \frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\left(\beta\int_0^\infty x^{\alpha+c-2}\;e^{-\theta x^{\alpha}}\,dx+\;\theta\int_0^\infty x^{2\alpha+c-2}\;e^{-\theta x^{\alpha}}\,dx\right) \tag{10}$$

Put $x^{\alpha}=t \;\Rightarrow\; x=t^{\frac{1}{\alpha}}$

Also $\alpha\, x^{\alpha-1}dx=dt \Rightarrow dx=\dfrac{dt}{\alpha\, x^{\alpha-1}} \;\Rightarrow dx=\dfrac{dt}{\alpha\, t^{\frac{\alpha-1}{\alpha}}}$

After the simplification of equation (10), we obtain the harmonic mean of WPQL distribution

$$H.M = \frac{\beta\Gamma\frac{(\alpha+c-1)}{\alpha}+\Gamma\frac{(2\alpha+c-1)}{\alpha}}{\alpha\theta^{\alpha}\left(\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)}$$

Moment generating function and characteristic function

Let X be the random variable follows weighted power quasi Lindley distribution with parameters θ, β, α and c, then the moment generating function of proposed distribution can be obtained as

$$M_X(t)=E\left(e^{tx}\right)=\int_0^\infty e^{tx} f_w(x\,;\theta,\beta,\alpha,c)dx$$

Using Taylor series, we obtain

$$=\int_0^\infty\left(1+tx+\frac{(tx)^2}{2!}+....\right)f_w(x;\theta,\beta,\alpha,c)dx$$

$$=\int_0^\infty\sum_{j=0}^\infty\frac{t^j}{j!}x^j f_w(x;\theta,\beta,\alpha,c)dx$$

$$=\sum_{j=0}^\infty\frac{t^j}{j!}\mu_j{}'$$

$$= \sum_{j=0}^{\infty} \frac{t^j}{j!} \left(\frac{\beta \Gamma \frac{(\alpha + c + j)}{\alpha} + \Gamma \frac{(2\alpha + c + j)}{\alpha}}{\alpha \theta^{\frac{j}{\alpha}} \left(\beta \Gamma \frac{(\alpha + c)}{\alpha} + \Gamma \frac{(2\alpha + c)}{\alpha} \right)} \right)$$

$$M_X(t) = \frac{1}{\alpha \left(\beta \Gamma \frac{(\alpha + c)}{\alpha} + \Gamma \frac{(2\alpha + c)}{\alpha} \right)} \sum_{j=0}^{\infty} \frac{t^j}{j! \theta^{\frac{j}{\alpha}}} \left(\beta \Gamma \frac{(\alpha + c + j)}{\alpha} + \Gamma \frac{(2\alpha + c + j)}{\alpha} \right)$$

Similarly, the characteristic function of weighted power quasi Lindley distribution is given by

$$\varphi_x(t) = M_X(it)$$

$$M_X(it) = \frac{1}{\alpha \left(\beta \Gamma \frac{(\alpha + c)}{\alpha} + \Gamma \frac{(2\alpha + c)}{\alpha} \right)} \sum_{j=0}^{\infty} \frac{it^j}{j! \theta^{\frac{j}{\alpha}}} \left(\beta \Gamma \frac{(\alpha + c + j)}{\alpha} + \Gamma \frac{(2\alpha + c + j)}{\alpha} \right)$$

5. Order Statistics

Order statistics is useful in statistical sciences and has been largely used in reliability and life testing. Let $X_{(1)}$, $X_{(2)}$, ..., $X_{(n)}$ be the order statistics of a random sample $X_1, X_2, \ldots, X_n$ from a continuous population with probability density function $f_x(x)$ and cumulative distribution function $F_X(x)$, then the probability density function of r^{th} order statistics $X_{(r)}$ is given by

$$f_{x(r)}(x) = \frac{n!}{(r-1)!\,(n-r)!} \; f_X(x)[F_X(x)]^{r-1} \; [1 - F_X(x)]^{n-r} \tag{11}$$

By using the equations (5) and (7) in equation (11), we will obtain the probability density function of r^{th} order statistics of weighted power quasi Lindley distribution which is given by

$$f_{x(r)}(x)=\frac{n!}{(r-1)!(n-r)!}\left(\frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}x^{\alpha+c-1}\left(\beta+\theta x^{\alpha}\right)e^{-\theta x^{\alpha}}\right)$$

$$\times\left(\frac{1}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\left(\beta\gamma\left(\frac{(\alpha+c)}{\alpha},\theta x^{\alpha}\right)+\gamma\left(\frac{(2\alpha+c)}{\alpha},\theta x^{\alpha}\right)\right)\right)^{r-1}$$

$$\times\left(1-\frac{1}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\left(\beta\gamma\left(\frac{(\alpha+c)}{\alpha},\theta x^{\alpha}\right)+\gamma\left(\frac{(2\alpha+c)}{\alpha},\theta x^{\alpha}\right)\right)\right)^{n-r}$$

Therefore, the probability density function of higher order statistic $X_{(n)}$ of weighted power quasi Lindley distribution can be obtained as

$$f_{x(n)}(x)=\frac{n\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}x^{\alpha+c-1}\left(\beta+\theta x^{\alpha}\right)e^{-\theta x^{\alpha}}$$

$$\times\left(\frac{1}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\left(\beta\gamma\left(\frac{(\alpha+c)}{\alpha},\theta x^{\alpha}\right)+\gamma\left(\frac{(2\alpha+c)}{\alpha},\theta x^{\alpha}\right)\right)\right)^{n-1}$$

and the probability density function of first order statistic $X_{(1)}$ of weighted power quasi Lindley distribution can be obtained as

$$f_{x(1)}(x)=\frac{n\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}x^{\alpha+c-1}\left(\beta+\theta x^{\alpha}\right)e^{-\theta x^{\alpha}}$$

$$\times\left(1-\frac{1}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\left(\beta\gamma\left(\frac{(\alpha+c)}{\alpha},\theta x^{\alpha}\right)+\gamma\left(\frac{(2\alpha+c)}{\alpha},\theta x^{\alpha}\right)\right)\right)^{n-1}$$

6. Likelihood Ratio Test

Let the random sample $X_1, X_2, \ldots X_n$ of size n drawn from the weighted power quasi Lindley distribution or power quasi Lindley distribution. To examine its usefulness, we introduce the hypothesis for testing

$$H_o : f(x)=f(x;\theta,\beta,\alpha) \quad \text{against} \quad H_1 : f(x)=f_w(x;\theta,\beta,\alpha,c)$$

In order to analyze, whether the random sample of size *n* comes from the power quasi Lindley distribution or weighted power quasi Lindley distribution, the following test statistic procedure is applied.

$$\Delta = \frac{L_1}{L_o} = \prod_{i=1}^{n} \frac{f_w(x;\theta,\beta,\alpha,c)}{f(x;\theta,\beta,\alpha)}$$

$$\Delta = \frac{L_1}{L_o} = \prod_{i=1}^{n} \left(\frac{\alpha\theta^{\frac{c}{\alpha}} x_i^{\,c} \left(\beta+1\right)}{\beta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}} \right)$$

$$\Delta = \frac{L_1}{L_o} = \left(\frac{\alpha\theta^{\frac{c}{\alpha}} \left(\beta+1\right)}{\beta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}} \right)^n \prod_{i=1}^{n} x_i^{\,c}$$

We should refuse to retain the null hypothesis, if

$$\Delta = \left(\frac{\alpha\theta^{\frac{c}{\alpha}} \left(\beta+1\right)}{\beta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}} \right)^n \prod_{i=1}^{n} x_i^{\,c} > k$$

Or equivalently we should refuse to accept the null hypothesis, where

$$\Delta^* = \prod_{i=1}^{n} x_i^{\,c} > k \left(\frac{\beta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}}{\alpha\theta^{\frac{c}{\alpha}} \left(\beta+1\right)} \right)^n$$

$$\Delta^* = \prod_{i=1}^{n} x_i^{\,c} > k^*, \text{Where } k^* = k \left(\frac{\beta\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}}{\alpha\theta^{\frac{c}{\alpha}} \left(\beta+1\right)} \right)^n$$

Whether, for the large sample of size *n*, *2log Δ* is distributed as chi-square distribution with one degree of freedom and also the chi-square distribution is applied for getting the *p* value. Thus, we should refuse to accept the null hypothesis, when the probability value is given by

$p\left(\Delta^* > \gamma^*\right)$, Where $\gamma^* = \prod_{i=1}^{n} x_i^{\,c}$ is lower than a specified level of significance and $\prod_{i=1}^{n} x_i^{\,c}$ is the observed value of the statistic Δ^*.

7. Income Distribution Curves

The income distribution curves also known as classical or bonferroni and Lorenz curves are used to measure the distribution of poverty or inequality in income. The bonferroni and Lorenz curves are given by

$$B(p) = \frac{1}{p\mu_1'} \int_0^q x\, f_w(x;\theta,\beta,\alpha,c)dx$$

and $$L(p) = pB(p) = \frac{1}{\mu_1'} \int_0^q x\, f_w(x;\theta,\beta,\alpha,c)dx$$

Where $\mu_1' = \dfrac{\beta\Gamma\dfrac{(\alpha+c+1)}{\alpha}+\Gamma\dfrac{(2\alpha+c+1)}{\alpha}}{\alpha\theta^{\frac{1}{\alpha}}\left(\beta\Gamma\dfrac{(\alpha+c)}{\alpha}+\Gamma\dfrac{(2\alpha+c)}{\alpha}\right)}$ and $q = F^{-1}(p)$

$$B(p)=\frac{\alpha\theta^{\frac{1}{\alpha}}\left(\beta\Gamma\dfrac{(\alpha+c)}{\alpha}+\Gamma\dfrac{(2\alpha+c)}{\alpha}\right)}{p\left(\beta\Gamma\dfrac{(\alpha+c+1)}{\alpha}+\Gamma\dfrac{(2\alpha+c+1)}{\alpha}\right)}\int_0^q\frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\beta\Gamma\dfrac{(\alpha+c)}{\alpha}+\Gamma\dfrac{(2\alpha+c)}{\alpha}}x^{\alpha+c}\left(\beta+\theta x^{\alpha}\right)e^{-\theta x^{\alpha}}dx$$

$$B(p)=\frac{\alpha\theta^{\frac{\alpha+c+1}{\alpha}}}{p\left(\beta\Gamma\dfrac{(\alpha+c+1)}{\alpha}+\Gamma\dfrac{(2\alpha+c+1)}{\alpha}\right)}\int_0^q x^{\alpha+c}\left(\beta+\theta x^{\alpha}\right)e^{-\theta x^{\alpha}}dx$$

$$B(p)=\frac{\alpha\theta^{\frac{\alpha+c+1}{\alpha}}}{p\left(\beta\Gamma\dfrac{(\alpha+c+1)}{\alpha}+\Gamma\dfrac{(2\alpha+c+1)}{\alpha}\right)}\left(\beta\int_0^q x^{\alpha+c}e^{-\theta x^{\alpha}}dx+\theta\int_0^q x^{2\alpha+c}e^{-\theta x^{\alpha}}dx\right) \tag{12}$$

Put $x^{\alpha}=t \;\Rightarrow\; x=t^{\frac{1}{\alpha}}$

Also $\alpha x^{\alpha-1}dx=dt \quad\Rightarrow\quad dx=\dfrac{dt}{\alpha x^{\alpha-1}} \quad\Rightarrow dx=\dfrac{dt}{\alpha t^{\frac{\alpha-1}{\alpha}}}$

After the simplification of equation (12), we obtain

$$B(p)=\frac{\theta^{\frac{\alpha+c+1}{\alpha}}}{p\left(\beta\Gamma\dfrac{(\alpha+c+1)}{\alpha}+\Gamma\dfrac{(2\alpha+c+1)}{\alpha}\right)}\left(\beta\gamma\left(\frac{(\alpha+c+1)}{\alpha},\theta q\right)+\theta\gamma\left(\frac{(2\alpha+c+1)}{\alpha},\theta q\right)\right)$$

$$L(p)=\frac{\theta^{\frac{\alpha+c+1}{\alpha}}}{\left(\beta\Gamma\dfrac{(\alpha+c+1)}{\alpha}+\Gamma\dfrac{(2\alpha+c+1)}{\alpha}\right)}\left(\beta\gamma\left(\frac{(\alpha+c+1)}{\alpha},\theta q\right)+\theta\gamma\left(\frac{(2\alpha+c+1)}{\alpha},\theta q\right)\right)$$

8. Maximum Likelihood Estimation and Fisher's Information Matrix

In this section, we will discuss the method of maximum likelihood estimation to estimate the parameters of weighted power quasi Lindley distribution and also obtained its Fisher's Information matrix. Consider $X_1, X_2,\ldots,X_n$ be a random sample of size n from the weighted power quasi Lindley distribution, then the likelihood function can be defined as.

$$L(x)=\prod_{i=1}^{n} f_w(x;\theta,\beta,\alpha,c)$$

$$L(x)=\prod_{i=1}^{n}\left(\frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}x_i^{\alpha+c-1}\left(\beta+\theta x_i^{\alpha}\right)e^{-\theta x_i^{\alpha}}\right)$$

$$L(x)=\left(\frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\right)^{n}\prod_{i=1}^{n}\left(x_i^{\alpha+c-1}\left(\beta+\theta x_i^{\alpha}\right)e^{-\theta x_i^{\alpha}}\right)$$

$$L(x)=\frac{\alpha\theta^{n\left(\frac{\alpha+c}{\alpha}\right)}}{\left(\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)^{n}}\prod_{i=1}^{n}\left(x_i^{\alpha+c-1}\left(\beta+\theta x_i^{\alpha}\right)e^{-\theta x_i^{\alpha}}\right)$$

The log likelihood function is given by

$$\log L=n\left(\frac{\alpha+c}{\alpha}\right)\log\alpha\theta-n\log\left(\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)$$

$$+(\alpha+c-1)\sum_{i=1}^{n}\log x_i+\sum_{i=1}^{n}\log\left(\beta+\theta x_i^{\alpha}\right)-\theta\sum_{i=1}^{n}x_i^{\alpha} \tag{13}$$

By differentiating the log likelihood equation (13) with respect to *θ, β, α* and *c*. we must satisfy the following normal

equations

$$\frac{\partial\log L}{\partial\theta}=\frac{n\alpha}{\alpha\theta}\left(\frac{\alpha+c}{\alpha}\right)+\sum_{i=1}^{n}\left(\frac{x_i^{\alpha}}{\left(\beta+\theta x_i^{\alpha}\right)}\right)-\sum_{i=1}^{n}x_i^{\alpha}=0$$

$$\frac{\partial\log L}{\partial\alpha}=-\frac{n\theta c}{\alpha^{3}\theta}-n\psi\left(\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)+\sum_{i=1}^{n}\log x_i+\sum_{i=1}^{n}\frac{\theta x_i^{\alpha}\log x_i}{\left(\beta+\theta x_i^{\alpha}\right)}-\theta\sum_{i=1}^{n}x_i^{\alpha}\log x_i=0$$

$$\frac{\partial\log L}{\partial\beta}=-n\left(\frac{\Gamma\frac{(\alpha+c)}{\alpha}}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\right)+\sum_{i=1}^{n}\left(\frac{1}{\left(\beta+\theta x_i^{\alpha}\right)}\right)=0$$

$$\frac{\partial\log L}{\partial c}=\frac{n}{\alpha}\log\alpha\theta-n\psi\left(\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}\right)+\sum_{i=1}^{n}\log x_i=0$$

Where ψ (.) is the digamma function.

The above likelihood equations are too complicated to solve it algebraically. Therefore we use the numerical technique like Newton- Raphson method for estimating the required parameters of the proposed distribution.

In order to use the asymptotic normality results for obtaining the confidence interval. We have that if $\hat{\lambda} = \left(\hat{\theta}, \hat{\beta}, \hat{\alpha}, \hat{c}\right)$ denotes the MLE of $\lambda = \left(\theta, \beta, \alpha, c\right)$, we can determine the results as

$$\sqrt{n}(\hat{\lambda} - \lambda) \rightarrow N_4\left(0, I^{-1}(\lambda)\right)$$

Where $I(\lambda)$ is the Fisher's Information matrix

$$I(\lambda) = -\frac{1}{n}\begin{pmatrix} E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \theta \partial \beta}\right) & E\left(\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial \theta \partial c}\right) \\ E\left(\frac{\partial^2 \log L}{\partial \beta \partial \theta}\right) & E\left(\frac{\partial^2 \log L}{\partial \beta^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \beta \partial \alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial \beta \partial c}\right) \\ E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \theta}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \beta}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha \partial c}\right) \\ E\left(\frac{\partial^2 \log L}{\partial c \partial \theta}\right) & E\left(\frac{\partial^2 \log L}{\partial c \partial \beta}\right) & E\left(\frac{\partial^2 \log L}{\partial c \partial \alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial c^2}\right) \end{pmatrix}$$

Here, we define

$$E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) = -\frac{n\alpha^2}{(\alpha\theta)^2}\left(\frac{\alpha + c}{\alpha}\right) - \sum_{i=1}^{n}\left(\frac{\left(x_i^{\alpha}\right)^2}{\left(\beta + \theta x_i^{\alpha}\right)^2}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) = \frac{3n\alpha^2\theta^2 c}{\left(\alpha^3\theta\right)^2} - n\psi'\left(\beta\Gamma\frac{(\alpha + c)}{\alpha} + \Gamma\frac{(2\alpha + c)}{\alpha}\right) + \sum_{i=1}^{n}\frac{\left(\beta + \theta x_i^{\alpha}\right)\theta x_i^{\alpha}\left(\log x_i\right)^2 - \left(\theta x_i^{\alpha}\log x_i\right)^2}{\left(\beta + \theta x_i^{\alpha}\right)^2}$$

$$- \theta\sum_{i=1}^{n}\left(x_i^{\alpha}\log x_i\right)^2$$

$$E\left(\frac{\partial^2 \log L}{\partial \beta^2}\right) = n\left(\frac{\left(\Gamma\frac{(\alpha + c)}{\alpha}\right)\left(\Gamma\frac{(\alpha + c)}{\alpha}\right)}{\left(\beta\Gamma\frac{(\alpha + c)}{\alpha} + \Gamma\frac{(2\alpha + c)}{\alpha}\right)^2}\right) - \sum_{i=1}^{n}\left(\frac{1}{\left(\beta + \theta x_i^{\alpha}\right)^2}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial c^2}\right) = -n\psi'\left(\beta\Gamma\frac{(\alpha + c)}{\alpha} + \Gamma\frac{(2\alpha + c)}{\alpha}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right) = \sum_{i=1}^{n}\frac{\left(\beta + \theta x_i^{\alpha}\right)x_i^{\alpha}\log x_i - x_i^{\alpha}\left(\theta x_i^{\alpha}\log x_i\right)}{\left(\beta + \theta x_i^{\alpha}\right)^2} - \sum_{i=1}^{n}x_i^{\alpha}\log x_i$$

$$E\left(\frac{\partial^2 \log L}{\partial \beta \partial \alpha}\right) = -n\psi\left(\frac{\Gamma\frac{(\alpha + c)}{\alpha}}{\beta\Gamma\frac{(\alpha + c)}{\alpha} + \Gamma\frac{(2\alpha + c)}{\alpha}}\right) - \sum_{i=1}^{n}\frac{\theta x_i^{\alpha}\log x_i}{\left(\beta + \theta x_i^{\alpha}\right)^2}$$

$$E\left(\frac{\partial^2 \log L}{\partial\theta\,\partial c}\right) = \frac{n\alpha}{\alpha^2\theta}$$

$$E\left(\frac{\partial^2 \log L}{\partial\alpha\,\partial c}\right) = -\frac{n\theta}{\alpha^3\theta} - n\psi'\left(\beta\,\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial\beta\,\partial c}\right) = -n\psi\left(\frac{\Gamma\frac{(\alpha+c)}{\alpha}}{\beta\,\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial\theta\,\partial\beta}\right) = -\sum_{i=1}^{n}\left(\frac{E(x_i{}^{\alpha})}{\left(\beta+\theta x_i{}^{\alpha}\right)^2}\right)$$

where $\psi(.)'$ is the first order derivative of digamma function.

Since λ being unknown, we estimate $I^{-1}(\lambda)$ by $I^{-1}(\hat{\lambda})$ and this can be used to obtain asymptotic confidence intervals for θ, β, α and c.

9. Entropy

The idea of entropy is important in different areas of research such as probability and statistics, physics, communication theory and economics. The term entropy was coined by German physicist Rudolf Clausius in 1865 and it also measures quantify the diversity, uncertainty or randomness of a system. Entropy is also termed as decline into disorder. Entropy of a random variable *X* is a measure of variation of the uncertainty.

Renyi entropy

The term Renyi entropy was given by Alfred Renyi (1957). The Renyi entropy is important in ecology and statistics as index of diversity. The Renyi entropy of order γ for a random variable *X* is given by.

$$e(\gamma) = \frac{1}{1-\gamma}\log\left(\int f_w{}^{\gamma}(x)dx\right)$$

Where $\gamma > 0$ *and* $\gamma \neq 1$

$$e(\gamma) = \frac{1}{1-\gamma}\log\int_0^\infty\left(\frac{\alpha\,\theta^{\frac{\alpha+c}{\alpha}}}{\beta\,\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}}\,x^{\alpha+c-1}\left(\beta+\theta x^{\alpha}\right)e^{-\theta x^{\alpha}}\right)^{\gamma}dx$$

$$e(\gamma) = \frac{1}{1-\gamma}\log\left(\left(\frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\beta\,\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}}\right)^{\gamma}\int_0^\infty x^{\gamma(\alpha+c-1)}\,e^{-\theta\gamma\,x^{\alpha}}\left(\beta+\theta x^{\alpha}\right)^{\gamma}dx\right) \tag{14}$$

Using Binomial expansion in (14), we obtain

$$e(\gamma) = \frac{1}{1-\gamma}\log\left(\left(\frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\beta\,\Gamma\frac{(\alpha+c)}{\alpha} + \Gamma\frac{(2\alpha+c)}{\alpha}}\right)^{\gamma}\sum_{i=0}^{\infty}\binom{\gamma}{i}\beta^{\gamma-i}\left(\theta x^{\alpha}\right)^{i}\int_0^\infty x^{\gamma(\alpha+c-1)}\,e^{-\theta\gamma\,x^{\alpha}}dx\right)$$

$$e(\gamma)=\frac{1}{1-\gamma}\log\left(\left(\frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\right)^{\gamma}\sum_{i=0}^{\infty}\binom{\gamma}{i}\beta^{\gamma-i}\theta^{i}\int_{0}^{\infty}x^{\gamma(\alpha+c-1)+\alpha i}e^{-\theta\gamma x^{\alpha}}dx\right) \tag{15}$$

Put $x^{\alpha}=t \Rightarrow x=t^{\frac{1}{\alpha}}$

Also $\alpha x^{\alpha-1}dx=dt \Rightarrow dx=\frac{dt}{\alpha x^{\alpha-1}}=\frac{dt}{\alpha t^{\frac{\alpha-1}{\alpha}}}$

After simplifying, the equation (15) becomes

$$e(\gamma)=\frac{1}{1-\gamma}\log\left(\left(\frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\right)^{\gamma}\sum_{i=0}^{\infty}\binom{\gamma}{i}\frac{\beta^{\gamma-i}\theta^{i}}{\alpha}\frac{\Gamma\frac{\gamma(\alpha+c-1)+\alpha i+1}{\alpha}}{(\theta\gamma)^{\frac{\gamma(\alpha+c-1)+\alpha i+1}{\alpha}}}\right)$$

Tsallis Entropy

Tsallis entropy was introduced in 1988 is the basis of so called non-extensive statistical mechanics. The generalization of B-G statistics was proposed firstly by introducing the mathematical expression of Tsallis entropy (Tsallis, 1988) for a continuous random variable is defined as follows.

$$S_{\lambda}=\frac{1}{\lambda-1}\left(1-\int_{0}^{\infty}f_{w}^{\lambda}(x)dx\right)$$

$$S_{\lambda}=\frac{1}{\lambda-1}\left(1-\int_{0}^{\infty}\left(\frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}x^{\alpha+c-1}\left(\beta+\theta x^{\alpha}\right)e^{-\theta x^{\alpha}}\right)^{\lambda}dx\right)$$

$$S_{\lambda}=\frac{1}{\lambda-1}\left(1-\left(\frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\right)^{\lambda}\int_{0}^{\infty}x^{\lambda(\alpha+c-1)}e^{-\lambda\theta x^{\alpha}}\left(\beta+\theta x^{\alpha}\right)^{\lambda}dx\right) \tag{16}$$

Using Binomial expansion in equation (16), we get

$$S_{\lambda}=\frac{1}{\lambda-1}\left(1-\left(\frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\right)^{\lambda}\sum_{i=0}^{\infty}\binom{\lambda}{i}\beta^{\lambda-i}\theta^{i}\ x^{\alpha i}\int_{0}^{\infty}x^{\lambda(\alpha+c-1)}e^{-\lambda\theta x^{\alpha}}dx\right)$$

$$S_{\lambda}=\frac{1}{\lambda-1}\left(1-\left(\frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\right)^{\lambda}\sum_{i=0}^{\infty}\binom{\lambda}{i}\beta^{\lambda-i}\theta^{i}\int_{0}^{\infty}x^{\lambda(\alpha+c-1)+\alpha i}e^{-\lambda\theta x^{\alpha}}dx\right) \tag{17}$$

Put $x^{\alpha}=t \Rightarrow x=t^{\frac{1}{\alpha}}$

and $\alpha x^{\alpha-1}dx=dt \Rightarrow dx=\frac{dt}{\alpha x^{\alpha-1}} \Rightarrow dx=\frac{dt}{\alpha t^{\frac{\alpha-1}{\alpha}}}$

After simplifying, the equation (17) becomes

$$S_{\lambda}=\frac{1}{\lambda-1}\left(1-\left(\frac{\alpha\theta^{\frac{\alpha+c}{\alpha}}}{\beta\Gamma\frac{(\alpha+c)}{\alpha}+\Gamma\frac{(2\alpha+c)}{\alpha}}\right)^{\lambda}\sum_{i=0}^{\infty}\binom{\lambda}{i}\frac{\beta^{\lambda-i}\theta^{i}}{\alpha}\frac{\Gamma\frac{\lambda(\alpha+c-1)+\alpha i+1}{\alpha}}{(\lambda\theta)^{\frac{\lambda(\alpha+c-1)+\alpha i+1}{\alpha}}}\right)$$

10. Application

In this section, we have fitted three real lifetime data sets in weighted power quasi Lindley distribution to discuss its goodness of fit and the fit has been compared over power quasi Lindley, quasi Lindley, Lindley and exponential distributions. The three real lifetime data sets are given below as

The following first real lifetime data set represents the remission times (in months) of a random sample of 128 bladder cancer patients reported by Lee and Wang (2003) and the data set is given below as

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 6.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69

The second real life data set represents the waiting time (in minutes) of 65 dental patients, waiting before OPD (out Patient Diagnosis) at Halibet hospital, Asmara, from 25th to 29th December, 2017 available in the master thesis of Berhane Abebe and is given below as.

2(5), 6, 7, 8(3), 9, 10, 11, 12(2), 13, 14(4), 15, 16, 17(2), 18(2), 19, 20(3), 22, 23(2), 26, 27, 28, 29(2), 30(3), 31, 32, 33, 35, 36(2), 37(2), 40(2), 41(2), 42, 43, 44, 46, 47, 49, 52, 53, 55, 56, 58, 90

The following real lifetime data set reported by Efron (1988) represents the survival times of a group of patients suffering from head and neck cancer disease and should be treated using a combination of radiotherapy and chemotherapy (RT + CT) and the data set is given below as

12.2, 23.6, 23.7, 25.9, 32.0, 37.0, 41.4, 47.4, 55.5, 58.4, 63.5, 68.5, 78.3, 74.5, 81.4, 84.0, 92.0, 94.0, 110.0, 112.0, 119.0, 127.0, 130.0, 133.0, 140.0, 146.0, 155.0, 159.0, 173.0, 179.0, 194.0, 195.0, 209.0, 249.0, 281.0, 319.0, 339.0, 432.0, 469.0, 519, 633, 725, 817, 1776

To compute the model comparison criterion values along with the estimation of unknown parameters, the technique of R software is used. In order to compare the performance of weighted power quasi Lindley distribution over power quasi Lindley, quasi Lindley, Lindley and exponential distributions, we consider the criterion values like Bayesian Information criterion (*BIC*), Akaike Information Criterion (*AIC*), Akaike Information Criterion Corrected (*AICC*), Consistent Akaike Information Criterion (*CAIC*), Shannon's entropy *H(X)* and *-2logL*. The better distribution is which corresponds to the lesser values of *AIC, BIC, AICC, CAIC, H(X)* and *-2logL*. For determining the criterions like *AIC, BIC, AICC, CAIC, H(X)* and *-2logL*, given below following formulas are used

$$AIC = 2k - 2\log L, \qquad BIC = k\log n - 2\log L, \qquad AICC = AIC + \frac{2k(k+1)}{n-k-1}$$

$$CAIC = -2\log L + \frac{2kn}{n-k-1} \qquad \text{and} \qquad H(X) = -\frac{2\log L}{n}$$

Where k is the number of parameters in the statistical model, n is the sample size and $-2\log L$ is the maximized value of the log-likelihood function under the considered model.

Table 1: Shows ML Estimates and S.E estimates for the following data sets.

Data Set 1		
Distributions	**MLE**	**S.E**
WPQL	$\hat{\alpha} = 0.50111774$, $\hat{\theta} = 1.49656867$ $\hat{\beta} = 0.00100000$, $\hat{c} = 1.28832235$	$\hat{\alpha} = 0.02704925$, $\hat{\theta} = 0.28967742$ $\hat{\beta} = 0.01322245$, $\hat{c} = 0.28761569$
PQL	$\hat{\alpha} = 0.73003378$, $\hat{\theta} = 0.43096681$ $\hat{\beta} = 0.00100000$	$\hat{\alpha} = 0.07085634$, $\hat{\theta} = 0.09492552$ $\hat{\beta} = 0.10154280$
QL	$\hat{\alpha} = 651.9872757$ $\hat{\theta} = 0.1087553$	$\hat{\alpha} = 0.25576589$ $\hat{\theta} = 0.0000000$
Lindley	$\hat{\theta} = 0.1991393$	$\hat{\theta} = 0.0125326$
Exponential	$\hat{\theta} = 0.108588071$	$\hat{\theta} = 0.009597106$
Data set 2		
WPQL	$\hat{\alpha} = 0.32619907$, $\hat{\theta} = 38.96656167$ $\hat{\beta} = 0.00100000$, $\hat{c} = 13.70854469$	$\hat{\alpha} = 0.06888757$, $\hat{\theta} = 21.94062234$ $\hat{\beta} = 0.023655784$, $\hat{c} = 5.08134437$
PQL	$\hat{\alpha} = 1.3434580$, $\hat{\theta} = 1.0733575$ $\hat{\beta} = 0.0010000$	$\hat{\alpha} = 0.1259919$, $\hat{\theta} = 0.1069377$ $\hat{\beta} = 0.0211345$
OL	$\hat{\alpha} = 0.0010000$ $\hat{\theta} = 1.3226709$	$\hat{\alpha} = 0.06654378$ $\hat{\theta} = 0.1238838$
Lindley	$\hat{\theta} = 0.9934028$	$\hat{\theta} = 0.1144634$
Exponential	$\hat{\theta} = 0.6615390$	$\hat{\theta} = 0.1008835$
Data set 3		
WPQL	$\hat{\alpha} = 0.48896558$, $\hat{\theta} = 0.32135216$ $\hat{\beta} = 0.00100000$, $\hat{c} = 1.19170325$	$\hat{\alpha} = 0.04052444$, $\hat{\theta} = 0.13739106$ $\hat{\beta} = 0.0110000$, $\hat{c} = 0.45856633$

PQL	$\hat{\alpha} = 0.66575780$, $\hat{\theta} = 0.06192627$ $\hat{\beta} = 0.00100000$	$\hat{\alpha} = 0.05189612$, $\hat{\theta} = 0.01370739$ $\hat{\beta} = 0.01000000$
QL	$\hat{\alpha} = 1.363482$ $\hat{\theta} = 4.814857$	$\hat{\alpha} = 2.578744$ $\hat{\theta} = 8.324753$
Lindley	$\hat{\theta} = 0.0089115585$	$\hat{\theta} = 0.0009378505$
Exponential	$\hat{\theta} = 0.0044781396$	$\hat{\theta} = 0.0006392403$

Table 2: Shows comparison and performance of fitted distributions for data set 1.

Data Set 1						
Distributions	**-2logL**	**AIC**	**BIC**	**AICC**	**CAIC**	**H(X)**
WPQL	705.407	713.407	724.8151	713.7322	713.7322	5.5109
PQL	818.077	824.077	832.6331	824.2705	824.2705	6.3912
QL	824.3769	828.3769	834.081	828.4729	828.4729	6.4404
Lindley	833.7925	835.7925	838.6445	835.8242	835.8242	6.5140
Exponential	824.3768	826.3768	829.2289	826.4085	826.4085	6.4404

Table 3: Shows comparison and performance of fitted distributions for data set 2.

Data Set 2						
Distributions	**-2logL**	**AIC**	**BIC**	**AICC**	**CAIC**	**H(X)**
WPQL	76.20054	84.20054	91.24534	84.8672	84.8672	1.1723
PQL	92.12017	98.12017	103.4038	98.5136	98.5136	1.4172
QL	98.82022	102.8202	106.3426	103.0137	103.0137	1.5203
Lindley	114.5096	116.5096	118.2708	116.5730	116.5730	1.7616
Exponential	121.5341	123.5341	125.2953	123.5975	123.5975	1.8697

Table 4: Shows comparison and performance of fitted distributions for data set 3.

Data Set 3						
Distributions	**-2logL**	**AIC**	**BIC**	**AICC**	**CAIC**	**H(X)**
WPQL	451.3541	459.3541	466.4909	460.3797	460.3797	10.2580
PQL	559.9548	565.9548	571.3074	566.5548	566.5548	12.7262
QL	564.1671	568.1671	571.7354	568.4597	568.4597	12.8219
Lindley	579. 1558	581.1558	582.94	581.2510	581.2510	13.1626
Exponential	564.022	566.022	567.8062	566.1172	566.1172	12.8186

From results given above in table 2, table 3 and table 4, it has been clearly observed that the weighted power quasi Lindley distribution has the lesser *AIC, BIC, AICC, CAIC, H(X)* and *-2logL* values as compared to the power quasi Lindley, quasi Lindley, Lindley and exponential distributions. Hence, it can be concluded that the weighted power quasi Lindley distribution provides a quite satisfactory results over power quasi Lindley, quasi Lindley, Lindley and exponential distributions.

11. Conclusion

In the present study, a new distribution called as weighted power quasi Lindley distribution has been introduced. The proposed new distribution is executed by using the weighted technique to the classical distribution. Its various structural properties which include moments, harmonic mean, shape of the behavior of pdf and cdf, order statistics, reliability function, hazard rate function, reverse hazard rate function, Renyi and Tsallis entropy, bonferroni and Lorenz curves have been studied. The parameters of proposed introduced distribution are estimated by employing the technique of maximum likelihood estimation. The main purpose behind this manuscript completion is to aware one so that he realize how important are the new extensions in expressing some random processes even though when we have already a number of existing standard distributions. It is also observed that the real life data sets fitted in new distribution proves a better fit rather than the classical distribution i.e. power quasi Lindley distribution. Finally, a new distribution has been demonstrated with three real lifetime data sets to illustrate its superiority and flexibility and hence it is revealed from the results that the proposed weighted power quasi Lindley distribution provides a better fit over power quasi Lindley, quasi Lindley, Lindley and exponential distributions.

References

1. Ahmad, A., Ahmad, S. P., & Ahmed, A. (2014). Characterization and estimation of double weighted Rayleigh distribution. Journal of Agriculture and Life Sciences, 1(2), 21-28.
2. Alkarni, S. H. (2015). Power quasi Lindley distribution: Properties and Application. Asian Journal of mathematics and Computer Research, 10(2), 179-195.
3. Almuqrin, M. A., Almutlak, S. A., Gemeay, A. M., Almetwally, E. M., Karakaya, K., Makumi, N., Hussam, E., & Aldallai, R. (2023). Weighted power Maxwell distribution: Statistical properties and Covid-19 applications. Plos One, 18(1) doi:10.1371/journal.pone.0278659.
4. Al-Omari, A. I., Alhyasat, K. M., Ibrahim, K., & Abu Bakar, M. A. (2019). Power length-biased Suja distribution: properties and application. Electronic Journal of Applied Statistical Analysis, 12(02), 429-452.
5. Berhane, A. (2018). Discretization of some continuous distribution and their Applications. Master thesis submitted to Department of statistics, College of Science, Eritrea Institute of technology, Asmara, Eritrea.
6. Benchiha, S. A., Al-Omari, A. I., Alotaibi, N., & Shrahili, M. (2021). Weighted generalized quasi Lindley distribution: Different methods of estimation, applications for Covid-19 and engineering data. AIMS Mathematics, 6(11), 11850-11878.
7. Dar, A. A., Ahmed, A., & Reshi, J. A. (2020). Weighted Gamma-Pareto distribution and its Application. Pak.J.Statist, 36(4), 287-304.
8. Efron, B. (1988). Logistic regression, survival analysis and the Kaplan-Meier curve. Journal of the American Statistical Association, 83, 414-425.
9. Fisher, R. A. (1934). The effects of methods of ascertainment upon the estimation of frequencies. Annals of Eugenics, 6, 13-25.
10. Ganaie, R. A., & Rajagopalan, V. (2021). Length biased weighted new quasi Lindley distribution: Statistical properties and applications. Pak.j.stat.oper.res, 17(1), 123-136.
11. Gharaibeh, M. M. (2022). Weighted Gharaibeh distribution with real data applications. Electronic Journal of Applied Statistical Analysis, 15(02), 421-437.
12. Ghitany, M. E., Alqallaf, F., Al-Mutairi, D. K., & Husain, H. A. (2011). A two-parameter weighted Lindley distribution and its applications to survival data. Mathematics and Computers in Simulation, 81(6), 1190-1201.
13. Hassan, A., Dar, M. A., Peer, B. A., & Para, B. A. (2019). A new generalization of Pranav distribution using weighted technique. International Journal of Scientific Research in Mathematical and Statistical Sciences, 6(1), 25-32.
14. Hassan, A., Wani, S. A., & Para, B. A. (2018). On three parameter weighted quasi Lindley distribution: properties and applications. International Journal of Scientific Research in Mathematical and Statistical Sciences, 5(5), 210-224.
15. Hassan, A. S., Almetwally, E. M., Khaleel, M. A., & Nagy, H. F. (2021). Weighted power Lomax distribution and its length biased version: Properties and Estimation based on censored samples. Pak.j.stat.oper.res, 17(2), 343-356.
16. Hussain, B. A., & Mohammed, R. S. (2022). Weighted Zeghdoudi distribution: Properties and Applications. Researchjet Journal of Analysis and Inventions, 3(4), 50-63.

17. Khan, M. I., & Mustafa, A. (2022). Some properties of the weighted power hazard rate distribution with application. Pak. J. Statist, 38(2), 219-234.
18. Kilany, N. M. (2016). Weighted Lomax distribution. SpringerPlus, 5(1), doi: 10.1186/s40064-016-3489-2.
19. Lee E. T., & Wang, J. W. (2003). Statistical Methods for Survival Data Analysis. 3rd ed. New York, NY, USA: John Wiley & Sons, 512.
20. Para, B. A., & Jan, T. R. (2018). On three Parameter weighted Pareto type II distribution: Properties and Applications in Medical Sciences. Applied Mathematics and Information Sciences Letters, 6(1), 13-26.
21. Rao, C. R. (1965). On discrete distributions arising out of method of ascertainment, in classical and Contagious Discrete, G.P. Patiled; Pergamum Press and Statistical publishing Society,Calcutta. 320-332.
22. R core team (2019). R version 3.5.3: A language and environment for statistical computing. R Foundation for statistical computing, Vienna, Austria.URL https:// www.R-project .org/.