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Addressing the Autocorrelation Problem in the Poisson Regression Model: Theory and Numerical Illustrations

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Abstract
The Poisson regression model (PRM) is usually applied in the situations where the dependent variable is in the form of count data. The purpose of this study is to compare methods of estimation for the Poisson Regression Model's first-order autocorrelation (AR(1)). The Kibria and Lukman Estimator Method (KL), Generalized Least Square Estimator Method (GLS), the Liu Estimator Method (LE), and the Reduction Liu Estimator Method (RLE) were employed. Monte Carlo simulations are used to compare these methods. The data generated follows Poisson Regression Model, however because of sample size and autocorrelation levels among other things, to create first-order autocorrelation among random errors. The Mean square Error (MSE) criterion was used for comparison. The methods are also evaluated on actual data, Moreover, the findings demonstrated that the KL approach is superior to the other estimation techniques in terms of its performance.

Key Words: Poisson Regression Model ،Autocorrelation Problem, KL Estimator, GLS Estimator, RLE Estimator, LE Estimator.

1. Introduction

The Poisson distribution is considered one of the important discrete probability distributions in many statistical applications, and it is sometimes called the distribution of rare events such as ship collisions, plane crashes and other examples that are classified as rare. The Poisson distribution represents an approximate case of the binomial distribution, as shown by the French mathematician and physicist (Simeon Poisson), and after whom the distribution is named (Nouri and Abdul Latif (2019)).

Alkhateeb and Algamal (2020) proposed a Jackknifed Liu-type Poisson estimator (JPLTE) in the Poisson regression model with the presence of multicollinearity problem. and the researchers compared the proposed estimator with the maximum likelihood estimator (MLE), the Liu estimator LE, and the Liu-type estimator (LTE). Through studying the simulation and applying it to a set of real data, the researchers concluded that the JPLTE estimator outperforms both the LE estimator and the (MLE) estimator in terms of predictive performance. In 2021, Amin et al propose a new adjusted Poisson Liu estimator (APLE) for the Poisson regression model PRM with the multicollinearity problem. The researchers compared the proposed estimator with the maximum likelihood estimator (MLE), Liu estimator LE, and ridge regression estimator (RR). From the findings of simulation study and two empirical applications, the researchers concluded that the proposed estimator is superior based on the mean square error criterion. In the same year, (Lukman et al) proposed the KL estimator with some biasing parameters to estimate the regression coefficients for the PRM when there is multicollinearity problem. Two methods were also proposed to estimate the estimator parameter (C). The estimator was compared with the maximum likelihood estimator (LM), the ridge regression estimator (RR) and the Liu estimator LE. Using the simulation method and application to aircraft damage data, the researchers reached the superiority of the proposed estimator based on the mean square error criterion. In the year (2022), (Açar) compared the (Kibria-LukmanKL) estimator with the (Liu) estimator, the (Ridge) estimator, and the GLS estimator in the presence AR(2) in the linear regression model. The researcher concluded, through a simulation study and application on two real examples, that the KL estimator is superior based on the mean square error criterion. In the same year, (Lukman et al) proposed a modified ridge-type estimator to address the problem of multicollinearity in PRM. The proposed estimator was compared with Liu's LE estimator, the ridge slope estimator (RR) and the maximum likelihood estimator LE. The researchers concluded, using the simulation method and application to aircraft damage data, that the proposed estimator is superior based on the MSE criterion. Ibrahim and Alheety (2023) proposed the Almost Unbiased Kibria-Lukman estimator in the Poisson regression model with multicollinearity problem. The researchers compared the proposed estimator with the Kibria-Lukman estimator and ridge regression estimator. and The bias and variance matrices of the proposed estimator are derived and compared to other estimators. At the end a real data set has been used to investigate the performance of the proposed estimator, the researchers concluded that the new estimator is most effective. In 2024, Abdelwahab et al proposed a Modified Two-Parameter Liu Estimator (MTPLE) for addressing the multicollinearity problem in the Poisson regression model PRM. The proposed estimator was compared with the Adjusted Liu Estimator (ALE), Liu estimator, Ridge Regression Estimator (RRE), and maximum likelihood estimator (MLE). Through simulation study and two empirical applications, the researchers concluded that the proposed estimator is superior based on the mean square error criterion.

Assuming that Y_i represents a discrete random variable that represents the number of times a certain event occurs during a certain time period, then(Y_i) follows a Poisson distribution with a parameter of (μ), and the probability density function for this distribution is

$$f(y_i) = \begin{cases} \dfrac{e^{-\mu_i}{\mu_i}^{y_i}}{y_i!} & Y_i = 0,1,2,\dots \ , i = 1,2,\dots,n \\ 0 & otherwise \end{cases} \quad (1)$$

where(μ) represents the distribution parameter and has a positive value $(\mu > 0)$, for more details see Rashad and Algamal (2019) and Lukman et al. (2021).

Regression models are divided into two sections: linear regression models and non-linear regression models. The Poisson Regression Model PRM is considered one of the types of linear-logarithmic regression models. By taking the natural logarithm of the distribution formula, it is converted into a linear formula. Random errors in the model follow the Poisson distribution with a parameter of (μ), and then the dependent variable (Y_i) is distributed according to the same distribution. The Poisson Regression Model is considered the method through which the response variable is modeled when the values of that variable are in the form of counted data or in the form of rates. In addition to being the appropriate model for analyzing rare events, that is, it is the appropriate tool for analyzing rare events with data that have non-negative values. PRM can be expressed according to the following formula (see Winkelmann, 2008 and Davis et al, 2000)

$$\underline{Y} = e^{X\underline{\beta}+\underline{\varepsilon}} \quad (2)$$

Where $\underline{Y}$ represents the vector of the dependent variable with degree $(n \times 1)$, X represents the matrix of explanatory variables with degree $n \times (p+1)$, $\underline{\beta}$ represents the vector of model parameters with degree $(p+1) \times 1$, $\underline{\varepsilon}$ represents the vector of random errors with degree $(n \times 1)$, n represents the sample size, and (p) represents the number of explanatory variables.

Due to Kazem and Muslim, 2002, The ordinary least squares method depends on several basic assumptions, including the assumption of the absence of autocorrelation between random errors in the sample under investigation, or in other words

$$E(\varepsilon_i \, \varepsilon_j) = 0 \qquad \forall \; i \neq j \qquad i,j = 1,2,\dots,n$$

In the case where the phenomenon under study includes the existence of an autocorrelation between errors, the hypothesis becomes as follows:

$$E(\varepsilon_i \, \varepsilon_j) \neq 0 \qquad \forall \; i \neq j$$

This means that the value of the random error in period i is not independent from the value in period j. In the presence of the problem of autocorrelation, the application of the ordinary least squares (OLS) method will not be efficient has many effects on the characteristics of the estimated regression coefficients, as it loses its efficiency, the results of the (F, T) tests are less accurate and cannot be relied upon, the inaccuracy of confidence intervals, the value of the coefficient of determination is higher than its true value, the standard errors of the regression model coefficients are less than their true values, and thus the model predictions become inaccurate. Autocorrelation can be defined as the association of successive observations of the same variable in time series data or cross-sectional data, or the lack of independence of the value of the random variable (ε_i)in a specific time from its value in a previous time. (Imran, 2003) (Tali, 2022) (Anono and Osagie, 2021)

The study of the Poisson regression model, in which random errors follow the Poisson distribution and are used in the analysis of rare events in terms of estimating parameters when that model suffers from various problems, has recently begun to accept a wide space in modern statistical studies. Accordingly, this research addresses the Poisson regression model when there is a problem of first-order autocorrelation AR(1) between random errors.

2. Methodology

2.1 Liu Estimator Method

Liu's method (1993) was proposed for the linear regression model in the presence of the multicollinearity problem, as this method addresses the problem of variance inflation of the estimated model parameters. In this research, this method was used in the presence of the first-order autocorrelation problem AR(1) in the Poisson regression model (for more details see (Alheety and Kibria,2009 and Amin et al, 2021, Naya et al, 2008). By taking the natural logarithm of the model formula given in Equation (2) to convert it to the linear formula, we get:

$$\log(\underline{Y}) = \log\{e^{X\underline{\beta}+\underline{\varepsilon}}\}$$
$$\underline{Y}^* = X\underline{\beta} + \underline{\varepsilon} \tag{3}$$

Where $\underline{Y}^* = \log(\underline{Y})$, and if the error of the above model follows the first-order autocorrelation AR(1), that is

$$\underline{\varepsilon_i} = \rho\underline{\varepsilon_{i-1}} + \underline{e_i} \tag{4}$$

Where ρ represents the autocorrelation parameter$|\rho| \leq 1$, and $\underline{e_i}$ represents the first-order autoregressive error. Adopting the constraint set by (Liu), we get

$$\underline{\varepsilon}'\underline{\varepsilon} = \left(d\underline{\hat{\beta}}_{GLS} - \underline{\hat{\beta}}_{LE}\right)'\left(d\underline{\hat{\beta}}_{GLS} - \underline{\hat{\beta}}_{LE}\right) \tag{5}$$

Where ($\underline{\varepsilon}'\underline{\varepsilon}$)represents the amount of increase in the mean square error if the vector of parameters estimated using the general least squares GLS method is replaced by the vector of parameters estimated using the Liu method, $\left(\underline{\hat{\beta}}_{LE},\right)$represents the vector of parameters estimated using the LE method, ($\underline{\hat{\beta}}_{GLS}$)GLS represents the vector of parameters estimated using the General Least Squares GLS method, and d represents the bias parameter, which is a small positive quantity with a value of $(0 < d < 1)$. After the process of neutralizing the model and placing the constraint, the square of the errors for the model under study is taken with the addition of the constraint, and then the result is derived with respect to the ($\underline{\hat{\beta}}_{LE}$)vector as follows:-

$$\underline{\varepsilon}'\Omega^{-1}\underline{\varepsilon} = (\underline{Y}^* - X\underline{\hat{\beta}}_{LE})'\Omega^{-1}\left(\underline{Y}^* - X\underline{\hat{\beta}}_{LE}\right) + (d\underline{\hat{\beta}}_{GLS} - \underline{\hat{\beta}}_{LE})'(d\underline{\hat{\beta}}_{GLS} - \underline{\hat{\beta}}_{LE})$$

Which can expressed as

$$\underline{Y}^{*\prime}\Omega^{-1}\underline{Y}^* - 2\underline{\hat{\beta}}_{LE}X'\Omega^{-1}\underline{Y}^* + \underline{\hat{\beta}}_{LE}'X'\Omega^{-1}X\underline{\hat{\beta}}_{LE} + d'\underline{\hat{\beta}}_{GLS}'\underline{\hat{\beta}}_{GLS}d - 2d\underline{\hat{\beta}}_{GLS}\underline{\hat{\beta}}_{LE} + \underline{\hat{\beta}}_{LE}'\underline{\hat{\beta}}_{LE} \tag{6}$$

Where (Ω^{-1})represents the correlation matrix of degree $n \times n$ and it is as follows:

$$\frac{\partial\underline{\varepsilon}'\Omega^{-1}\underline{\varepsilon}}{\partial\underline{\hat{\beta}}_{LE}} = -2X'\Omega^{-1}\underline{Y}^* + 2X'\Omega^{-1}X\underline{\hat{\beta}}_{LE} - 2d\underline{\hat{\beta}}_{GLS} + 2\underline{\hat{\beta}}_{LE} \tag{7}$$

By equating the above equation to zero, we obtain the (Liu) estimators of the parameters

$$\underline{\hat{\beta}}_{LE} = (X'\Omega^{-1}X + I)^{-1}\left(X'\Omega^{-1}\underline{Y}^* + d\underline{\hat{\beta}}_{GLS}\right) \tag{8}$$
$$\underline{\hat{\beta}}_{GLS} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\underline{Y}^* \tag{9}$$
$$X'\Omega^{-1}\underline{Y}^* = (X'\Omega^{-1}X)\underline{\hat{\beta}}_{GLS} \tag{10}$$

By substituting equation (10) in equation (8), we obtain another form for the (Liu) estimators of the parameters as follows: (Ahmed et al, 2020) (Alkhateeb and Algamal, 2020) (AÇAR, 2022) (Mansson et al, 2012)

$$\underline{\hat{\beta}}_{LE} = (X'\Omega^{-1}X + I)^{-1}(X'\Omega^{-1}X + dI)\underline{\hat{\beta}}_{GLS} \tag{11}$$

Liu estimators are biased when $(d > 0)$ and the bias is: (Amin et al, 2021)

$$Bias\left(\underline{\hat{\beta}}_{LE}\right) = E\left(\underline{\hat{\beta}}_{LE}\right) - \underline{\beta} = (d-1)(X'\Omega^{-1}X + I)^{-1}\beta \tag{12}$$

The covariance matrix of Liu's estimators is as follows: (Abdelwahab et al, 2024)

$$Cov\left(\underline{\hat{\beta}}_{LE}\right) = \sigma^2(X'\Omega^{-1}X + I)^{-1}(X'\Omega^{-1}X + dI)(X'\Omega^{-1}X)^{-1}(X'\Omega^{-1}X + dI)(X'\Omega^{-1}X + I)^{-1} \tag{13}$$

Due to Algamal (2018), the mean square error MSE of the estimated Poisson regression model parameters using the Liu estimator is as follows:

$$MSE\left(\underline{\hat{\beta}}_{LE}\right) = E(\underline{\hat{\beta}}_{LE} - \beta)'\left(\underline{\hat{\beta}}_{LE} - \beta\right)$$
$$= \sigma^2\sum_{j=1}^{p}\frac{(\lambda_j + d)^2}{\lambda_j(\lambda_j + 1)^2} + (d-1)^2\sum_{j=1}^{p}\frac{\alpha_j^2}{(\lambda_j + 1)^2} \tag{14}$$

2.1.1 Calculating Liu Parameter

The Liu estimator (Perveen and Suhail (2023) and Mansson et al. (2011) is biased due to adding the value of d, and this value ranges between $(0 < d < 1)$, and to find the proposed estimators for d, the optimal value for it must be found, by deriving formula (14) for (d) as follows

$$\frac{\partial MSE(\underline{\hat{\beta}}_{LE})}{\partial d} = 2\sum_{j=1}^{p} \frac{\lambda_j + d}{\lambda_j(\lambda_j+1)^2} + 2(d-1)\sum_{j=1}^{p}\frac{\alpha_j^2}{(\lambda_j+1)^2} \quad (15)$$

By setting the derivative value above equal to zero, we get:

$$\frac{\partial}{\partial d} MSE(\underline{\hat{\beta}}_{LE}) = 2\sum_{j=1}^{p} \frac{\lambda_j + \hat{d}}{\lambda_j(\lambda_j+1)^2} + 2(\hat{d}-1)\sum_{j=1}^{p}\frac{\alpha_j^2}{(\lambda_j+1)^2} = 0 \quad (16)$$

By performing some algebraic operations, we get:

$$\hat{d} = \frac{\alpha_j^2 - 1}{\frac{1}{\lambda_j} + \alpha_j^2} \quad (17)$$

The above value represents the optimal value for d, and this value is negative if the value of (α_j^2) is less than one, and is positive if the value of α_j^2 is greater than one, see Alkhateeb and Algamal (2020) and Mansson et al. (2012). The estimation of the value of the bias parameter (d) is not limited to a specific rule, as a single value for this parameter can be found using several formulas. In this research, the following formula was used: (Qasim et al, 2020)

$$D_{13} = max\left(0, min\left(\frac{\hat{\alpha}_j^2 - 1}{\max\left(\frac{1}{\lambda_j}\right) + \max\left(\hat{\alpha}_j^2\right)}\right)\right) \quad (18)$$

The LE estimator of the Poisson regression model parameters is found by substituting the above formula into (11).

2.2 Reduced Liu Estimator Method

Based on the constraint adopted by Liu and defined in formula (5), but the explanatory variables will be replaced by the following abbreviated form

$$Z = XT_r \quad (19)$$

Where Z represents a new vector of reduced variables, T represents an orthogonal matrix and $T_r = (T_1, T_2, T_3, \cdots, T_r)$

$$Z'\Omega^{-1}Z = T_r'X'\Omega^{-1}XT_r = \Delta_r = diag(\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_r)$$

And Δ_r represents a diagonal matrix of the eigenvalues of the matrix$(X'\Omega^{-1}X)$, and the characteristic roots (λ_i) of the matrix($X'\Omega^{-1}X$) are arranged as

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_p > 0$$

Therefore, the constraint that Liu put for the new reduced variables is as follows:

$$\underline{\varepsilon}'\underline{\varepsilon} = \left(dT_r\underline{\hat{\beta}}_{GLS} - \underline{\hat{\beta}}_{RLE}\right)'\left(dT_r\underline{\hat{\beta}}_{GLS} - \underline{\hat{\beta}}_{RLE}\right) \quad (20)$$

After the model neutralization process and placing the constraint, the square of the errors of the is taken with the addition of the constraint and then the model is derived with respect to the vector of parameters estimated according to the RLE method as follows

$$\underline{\varepsilon}'\Omega^{-1}\underline{\varepsilon} = \left(\underline{Y}^* - Z\underline{\hat{\beta}}_{RLE}\right)'\Omega^{-1}\left(\underline{Y}^* - Z\underline{\hat{\beta}}_{RLE}\right) + \left(dT_r\underline{\hat{\beta}}_{Gls} - \underline{\hat{\beta}}_{RLE}\right)'\left(dT_r\underline{\hat{\beta}}_{Gls} - \underline{\hat{\beta}}_{RLE}\right)$$

and

$$\underline{\varepsilon}'\Omega^{-1}\underline{\varepsilon} = \underline{Y}^{*\prime}\Omega^{-1}\underline{Y}^* - 2\underline{\hat{\beta}}_{RLE}Z'\Omega^{-1}\underline{Y}^* + \underline{\hat{\beta}}_{RLE}'Z'\Omega^{-1}Z\underline{\hat{\beta}}_{RLE} + dd'\underline{\hat{\beta}}_{GLS}'T_r'T_r\underline{\hat{\beta}}_{Gls} - 2d\underline{\hat{\beta}}_{RLE}'T_r\underline{\hat{\beta}}_{Gls} + \underline{\hat{\beta}}_{RLE}'\underline{\hat{\beta}}_{RLE} \quad (21)$$

By deriving the above equation with respect to the parameter vector $(\underline{\hat{\beta}}_{RLE})$ we get:

$$\frac{\partial}{\partial\underline{\hat{\beta}}_{RLE}}\underline{\varepsilon}'\Omega^{-1}\underline{\varepsilon} = -2Z'\Omega^{-1}\underline{Y}^* + 2Z'\Omega^{-1}Z\underline{\hat{\beta}}_{RLE} - 2dT_r\underline{\hat{\beta}}_{Gls} + 2\underline{\hat{\beta}}_{RLE} \quad (22)$$

By equating the above equation to zero, we obtain the RLE estimators of the PRM parameter as follows:

$$\underline{\hat{\beta}}_{RLE} = (Z'\Omega^{-1}Z + I)^{-1}\left(Z'\Omega^{-1}\underline{Y}^* + dT_r\underline{\hat{\beta}}_{Gls}\right) \quad (23)$$

$$Z'\Omega^{-1}\underline{Y}^* = (Z'\Omega^{-1}Z)T_r\underline{\hat{\beta}}_{Gls} \quad (24)$$

By substituting equation (24) in equation (23), we obtain another form for the RLE estimators of the parameters, as follows

$$\underline{\hat{\beta}}_{RLE} = (Z'\Omega^{-1}Z + I)^{-1}(Z'\Omega^{-1}Z + dI)T_r\underline{\hat{\beta}}_{Gls} \quad (25)$$

Where ($\underline{\hat{\beta}}_{RLE}$) represents the vector of parameters estimated according to the RLE method. RLE estimators are biased when $(d > 0)$ and the bias is:

$$Bias\left(\underline{\hat{\beta}}_{RLE}\right) = E\left(\underline{\hat{\beta}}_{RLE}\right) - \beta = U\underline{\beta} - \underline{\beta} = (U - I)\underline{\beta} \tag{26}$$

Where:

$$U = (Z'\Omega^{-1}Z + I)^{-1}(Z'\Omega^{-1}Z + dI)T_r \tag{27}$$

The covariance matrix of the RLE is as follows:

$$Cov\left(\underline{\hat{\beta}}_{RLE}\right) = U\,Var - Cov\left(\underline{\hat{\beta}}_{RLE}\right)U' = \sigma^2 U(Z'\Omega^{-1}Z)^{-1}U' \tag{28}$$

The mean square error of the PRM parameters estimated using the RLE estimator is as follows

$$MSE(\beta_{RLE}) = \sum_{j=1}^{p}\frac{(\lambda_j + d)^2}{\lambda_j(\lambda_j + 1)^2} + (d-1)^2 \sum_{j=1}^{p}\frac{\alpha_j^2}{(\lambda_j + 1)^2} \tag{29}$$

3.2 Kibria and Lukman Estimator Method KL

The (Kibria and Lukman) method was proposed in (2020) by (Kibria and Lukman) to address the problem of multicollinearity in the linear regression model. In (2021) (Zubair and Adenomon) proposed a new combined estimator that addresses the problem of autocorrelation in the linear regression model by merging (The Two Stage Prais Winsten Estimator) with (Kibria and Lukman Estimator) and obtaining a new estimator which can address the problem of autocorrelation and multicollinearity together. In (2022) the researcher (AÇAR) used (Kibria-Lukman) estimator to address the problem of second-order autocorrelation in the general linear regression model and proved its superiority over the general least squares estimator. In (2023), (Ibrahim and Alheety) used the (Kibria and Lukman Estimator) to address the problem of multicollinearity in the Poisson regression model, but the KL estimator was not addressed in the presence of the first-order autocorrelation problem AR(1) in the Poisson regression model, and this is what will be addressed in this research as follows: (Zubair and Adenomon, 2021) .

We multiply formula No. (3) by matrix (S) to purify data from the effect of autocorrelation, so we get (Anono and Osagie, 2021) (Ibrahim and Alheety, 2023) (AÇAR, 2022)

$$\underline{Y}^* = X\underline{\beta} + \underline{\varepsilon}$$

$$S\underline{Y}^* = SX\underline{\beta} + S\underline{\varepsilon} \tag{30}$$

$$\underline{y} = X^*\underline{\beta} + \varepsilon^* \tag{31}$$

where

$$\underline{y} = S\underline{Y}^* \quad ,$$

$$X^* = SX \quad , \quad \underline{\varepsilon}^* = S\underline{\varepsilon}$$

If the matrix (S) can be determined such that$(S\Omega S' = I \;,\; S'S = \Omega^{-1})$ then the general least squares GLS estimates of the transformed variables$(S\underline{Y}^*)$ and (SX) ,defined in the above equation, have all the optimal properties of the GLS method and thus the usual conclusions can be valid.

Where S is the non-singular error correlation matrix of degree $(n \times n)$ and it is as follows

$$S = \begin{pmatrix} \sqrt{1-\rho^2} & 0 & 0 & \cdots & 0 & 0 \\ -\rho & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\rho & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\rho & 1 \end{pmatrix}$$

By taking the square of the errors for the model under study, we get:

$$\underline{\varepsilon}^{*\prime}\underline{\varepsilon}^* = (Y^* - X\underline{\beta})'S'S\left(Y^* - X\underline{\beta}\right) + C\{(\underline{\beta} + \underline{\hat{\beta}}_{GLS})'\left(\underline{\beta} + \underline{\hat{\beta}}_{GLS}\right) - R\}$$

$$= \underline{Y}^{*\prime}S'S\underline{Y}^* - 2\underline{\beta}'X'S'S\underline{Y}^* + \underline{\beta}'X'S'SX\underline{\beta} + C\underline{\beta}^2 + 2C\underline{\beta}'\underline{\hat{\beta}}_{GLS} + C\underline{\hat{\beta}}_{GLS}^2 + CR,$$

Which can be simplified as

$$\underline{Y}^{*\prime}\Omega^{-1}\underline{Y}^* - 2\underline{\beta}'X'\Omega^{-1}\underline{Y}^* + \underline{\beta}'X'\Omega^{-1}X\underline{\beta} + C\underline{\beta}^2 + 2C\underline{\beta}'\underline{\hat{\beta}}_{GLS} + C\underline{\hat{\beta}}_{GLS}^2 + CR \tag{32}$$

By deriving the above equation with respect to the parameter vector $(\underline{\beta}')$, we get

$$\frac{\partial \underline{\varepsilon}^{*\prime}\underline{\varepsilon}^*}{\partial \underline{\beta}'} = -2X'\Omega^{-1}\underline{Y}^* + 2X'\Omega^{-1}X\underline{\beta} + 2C\underline{\beta} + 2C\underline{\hat{\beta}}_{GLS} \tag{33}$$

By equating the above derivative to zero, we get

$$(X'\Omega^{-1}X - CI)\underline{\hat{\beta}}_{GLS} = (X'\Omega^{-1}X + CI)\underline{\beta} \tag{34}$$

After finding the GLS estimator, the KL estimator can be obtained to address the problem of autocorrelation in the , which is as follows (Zubair and Adenomon, 2021) (Aladeitan et al, 2021)

$$\underline{\hat{\beta}}_{KL} = (X'\Omega^{-1}X + CI)^{-1}(X'\Omega^{-1}X - CI)\underline{\hat{\beta}}_{GLS} \quad , k > 0 \tag{35}$$

To obtain the KL estimator, the value of (*C*) must be estimated, and a new value for (*C*) will be proposed to improve the KL estimator, which is as follows

$$C_{proposed} = \frac{1}{men\ (\alpha_j^2)}\hat{\theta}, \tag{36}$$

where

$$\hat{\theta} = \frac{1}{n-p}\sum_{i=1}^{n}(Y_i^* - \mu_i)^2 \tag{37}$$

$$\alpha_j^2 = \gamma\underline{\hat{\beta}}_{GLS} \tag{38}$$

Where ($men\ (\alpha_j^2)$ represents the mean squared value of the proposed parameter, γ represents the Eigen vector of the matrix ($X'\Omega^{-1}X$).

The KL estimator is a biased estimator unless (C = 0), and its properties can be explained as follows: (Kibria and Lukman, 2020) (Aladeitan et al, 2021) (Owolabi et al, 2022) (Oladapo et al, 2023)

Bias:

$$Bias\left(\underline{\hat{\beta}}_{KL}\right) = [W(C)\ M(C) - I]\ \beta, \tag{39}$$

where

$$S = (X'\Omega^{-1}X)$$
$$W(C) = [I + CS^{-1}]^{-1}$$
$$M(C) = [I - CS^{-1}]$$

The covariance matrix of the KL is as follows:

$$Cov(\underline{\hat{\beta}}_{KL}) = \sigma^2 W(C)M(C)S^{-1}M'(C)\ W'(C) \tag{40}$$

Mean square error: (Shewa and Ugwuowo, 2023)

$$MSE\left(\underline{\hat{\beta}}_{KL}\right) = \sigma^2 W(C)M(C)S^{-1}M'(C)\ W'(C) + [W(C)M(C) - I]\beta\beta'[W(C)M(C) - I]' \tag{41}$$

$$MSE\left(\underline{\hat{\beta}}_{KL}\right) = \sigma^2\sum_{j=1}^{p}\frac{(\lambda_j - C)^2}{\lambda_j(\lambda_j + C)^2} + 4C^2\sum_{j=1}^{p}\frac{\alpha_j^2}{(\lambda_j + C)^2} \tag{42}$$

3. Simulation Study

The simulation method was used to generate random errors according to the Poisson distribution with distribution parameter (μ) in the Poisson regression model, but it includes the problem of first-order autocorrelation AR(1), according to the following model

$\varepsilon_i = \rho\varepsilon_{i-1} + e_i$ ، $i = 1,2,\cdots,n$

that describes the dependence of an observation (i) on (i-1). Concerning calculating the values of the dependent variable (Y_i)in the Poisson regression model, they will be calculated as follows

$$Y_i = EXP(\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \varepsilon_i) \qquad i = 1,2,\cdots,n \tag{43}$$

The values of the model parameters shown in formula (43) are as follows:

$\beta_0 = -0.286,\ \beta_1 = -0.023,\ \beta_2 = 0.011,\ \beta_3 = 0.023$

Five levels of autocorrelation coefficient were also used, as follows

$\rho = 0.1$, 0.4 , 0.7 , 0.9 , 0.99

we consider the Mean Square Error MSE criterion as a criterion for conducting the comparison between the methods, which is calculated as:

$$MSE(\hat{\beta}) = \frac{1}{2000}\sum_{i=1}^{2000}(\hat{\beta} - \beta)'(\hat{\beta} - \beta) \tag{44}$$

As for the sample size, five sample sizes will be used, which are (25, 50, 100, 150, 200) to study the comparison between the estimation methods used according to different sample sizes (large, medium, small), and with a repetition of 2000 times. To apply the simulation concept, the Monte-Carlo method was used in the simulation and varied factors were taken to show the comparison between the methods in different conditions. To compare the performance of the estimation methods used in this research, a simulation experiment was conducted using the statistical programming language (R), and several results were obtained, which are as follows:

Table 1: The results of comparing methods at a correlation ($\rho = 0.1, 0.4$).

		$\rho = 0.1$				$\rho = 0.4$			
N	μ	GLS	LE	KL	RLE	GLS	LE	KL	RLE
25	1.50	0.02716442	0.02522395	0.01926391	0.02264004	0.14008706	0.05263845	0.04187486	0.03942814
	2.067	0.13733417	0.05064428	0.02981316	0.03896657	0.12434094	0.04898556	0.03294854	0.03748053
	3.50	0.12997512	0.04631001	0.04081594	0.03500578	0.12107959	0.04527569	0.02653790	0.03446225
50	1.50	0.07567518	0.04346096	0.02404229	0.03287831	0.06896125	0.04157136	0.02289645	0.03174058
	2.067	0.06644880	0.03936728	0.02298014	0.03066190	0.06345897	0.03863950	0.02220074	0.03007654
	3.50	0.06120028	0.03619131	0.02150558	0.02829898	0.05710299	0.03511353	0.02145838	0.02800919
100	1.50	0.04560545	0.03504482	0.02054886	0.02770095	0.06026827	0.03413921	0.02079236	0.02713958
	2.067	0.04064149	0.03209366	0.01985986	0.02642830	0.04043644	0.03186490	0.01936121	0.02584733
	3.50	0.03623049	0.02941142	0.01946030	0.02484749	0.03644270	0.02937245	0.01920434	0.02456332
150	1.50	0.03798216	0.03212123	0.01971129	0.02600206	0.03645282	0.03121724	0.01889092	0.02541026
	2.067	0.03504585	0.02998266	0.01934959	0.02477589	0.03313406	0.02901476	0.01875672	0.02449481
	3.50	0.03149074	0.02738742	0.01934137	0.02360692	0.03012712	0.02678961	0.01889195	0.02335200
200	1.50	0.03412662	0.03021542	0.01931253	0.02485760	0.03278470	0.02943667	0.01868319	0.02440896
	2.067	0.03192568	0.02860706	0.01917475	0.02407415	0.03062233	0.02778277	0.01872705	0.02367035
	3.50	0.02853530	0.02615709	0.01915722	0.02298021	0.02750096	0.02551329	0.01896084	0.02273987

It is noted from Table (1) that the value of the mean square error standard begins to decrease with the increase in value (μ) and that the efficiency of the KL method increases with the increase in sample size.

Table 2: The results of comparing methods at a correlation ($\rho = 0.7, 0.9$).

		$\rho = 0.7$				$\rho = 0.9$			
N	μ	GLS	LE	KL	RLE	GLS	LE	KL	RLE
25	1.50	0.13371048	0.05213846	0.04624208	0.03808483	5.02875094	0.25770063	24.89581145	0.03477259
	2.067	0.24100992	0.04867551	0.63019828	0.03599439	76.66714246	0.14172048	78.92787921	0.03306281
	3.50	0.18136643	0.04450375	0.50622048	0.03371144	2.49674581	0.04452236	99.05438915	0.03149519
50	1.50	0.06618041	0.03992266	0.26755205	0.02998731	0.08032774	0.03893448	0.59880169	0.02819380
	2.067	0.07209304	0.03749930	0.03562050	0.02857513	0.37395299	0.03569649	0.55952479	0.02712977
	3.50	0.06398656	0.03421446	0.04713790	0.02726250	0.12112470	0.03394633	0.06960764	0.02646101
100	1.50	0.04626860	0.03259479	0.02660144	0.02570551	0.08505299	0.03096600	0.02070141	0.02454526
	2.067	0.05163655	0.03094968	0.02407489	0.02491916	0.06852444	0.02959023	0.02352819	0.02409900
	3.50	0.03504847	0.02833382	0.01910705	0.02404729	0.06178955	0.02775285	0.02418030	0.02350282
150	1.50	0.03454936	0.02932382	0.01828127	0.02431147	0.04546152	0.02816251	0.02318199	0.02329264
	2.067	0.03424981	0.02814099	0.01903554	0.02370939	0.04062729	0.02678571	1.52680961	0.02291253
	3.50	0.03755953	0.02605941	0.02336733	0.02292522	0.04216260	0.02542606	0.02647705	0.02258643
200	1.50	0.03325353	0.02812873	0.01814513	0.02359584	0.05974900	0.02699247	0.04299795	0.02274373
	2.067	0.03020236	0.02658364	0.01905859	0.02302053	0.05619583	0.02557283	0.02133675	0.02243759
	3.50	0.02896571	0.02512453	0.01892068	0.02239249	0.03629669	0.02420393	0.02450318	0.02209213

According to Table (1), it is observed that the value of the mean square error standard starts to drop as the value (μ) increases. Additionally, it is observed that the efficiency of the KL method increases as the sample size increases.

Table 3: The results of comparing methods at a correlation ($\rho = 0.99$).

$\rho = 0.99$					
N	μ	GLS	LE	KL	RLE
25	1.50	1.03534586	0.05701777	17.64682318	0.03314750
	2.067	0.86007673	0.05179190	17.50258779	0.02936058
	3.50	0.896484	0.633542	9.405264	0.084325
50	1.50	2.40325485	0.11784287	4.92607040	0.02677095
	2.067	9.97989459	0.05343473	144.83518892	0.04263943
	3.50	4.35078152	0.03504626	5.38482914	0.02471862
100	1.50	1.73366348	0.03565916	24.77068972	0.02299823
	2.067	3.17623198	0.02828416	15.53549393	0.02196793

	3.50	2.62472399	0.02719102	1.41264977	0.02183624
150	1.50	1.08754918	0.02873258	0.22223820	0.02207040
	2.067	3.83523791	0.02584652	0.97130496	0.02179629
	3.50	1.86616407	0.02478391	0.67189448	0.02169379
200	1.50	0.98348207	0.02591362	0.21266974	0.02184556
	2.067	3.18634388	0.02508047	7.41056321	0.02169571
	3.50	1.42616407	0.02466271	0.33258614	0.02170131

that It is noted from the table above that the best way to deal with the problem of first-degree autocorrelation in the Poisson regression model is RLE, and it is also noted that the efficiency of the KL method declines in the presence of high autocorrelation between random errors.

4. Application

The experiment was applied to a set of real data that represent Number of cases of repeating the IVF process until pregnancy is achieved as a dependent variable (Y), namely: Wife's age, Husband's age, Wife's weight as independent variables (X_1, X_2, X_3) respectively, where the data was initially tested according to the Durbin-Watson test to test whether the data suffers from the presence of autocorrelation or not, as follows.

Table 4: Durbin-Watson test for real data.

Model Summary					
Model	R	R Square	Adjusted R Square	Std. Error of the Estimate	Durbin-Watson
1	.228[a]	.052	.001	1.03882	2.821
a. Predictors: (Constant), Wife's weight , Husband's age, Wife's age					
b. Dependent Variable: Number of repetitions until success					

After conducting the Durbin-Watson test for real data and proving that data suffers from the problem of negative autocorrelation, estimation methods were applied and several results were obtained, which are as follows:

Table 5: The results of the parameter values and the MSE criterion values and R square criterion values of the methods for real data.

parameters	GLS	LE	KL	RLE	Sample size
β_0	0.2405	0.1007	-0.2443	0.0005	60
β_1	-0.0492	-0.0503	-0.0617	-0.0078	
β_2	0.0082	0.01	-0.0296	-0.0191	
β_3	0.0509	0.0527	0.0842	0.0491	
MSE	1.0108	1.0100	1.0965	1.0329	
R squared	0.0484	0.0491	0.0323	0.0276	

It is noted from the above table that the best method used to address the problem of first-degree autocorrelation according to the criterion of the mean square error is the LE method. It is also noted that the RLE method outperforms the LE method according to R square criterion.

5. Conclusions

After applying the simulation experiment and practical application and the results presented, the researcher concluded that the best method to address the problem of first-order autocorrelation AR(1) in the Poisson regression model for different sample sizes, variances, and different values of the autocorrelation coefficient (ρ) is the KL method. The worst method used is the GLS method because it gave the largest values for the Mean Square Error MSE criterion. And for that we recommend using KL method to estimate the parameters of AR(1) model in the presence of first order autocorrelation.

Autocorrelation in Poisson regression models, often arising in time-series or spatial data, challenges the independence assumption and necessitates advanced strategies for accurate modeling. Future research should focus on incorporating temporal or spatial correlation through methods like generalized estimating equations (GEE), hierarchical models, or Bayesian frameworks to account for dependencies. Residual analysis and the development of diagnostic tools specific to Poisson regression can further aid in identifying and addressing autocorrelation. Additionally, employing copula-based methods, generalized additive models (GAMs), or introducing smooth terms for time or spatial effects could mitigate autocorrelation impacts. Extending the current literature on statistical distributions and regression models to explicitly handle autocorrelation is a promising direction. For example, various statistical families such as Burr XII (Afify et al., 2018; Cordeiro et al., 2018; Yousof et al., 2018; Ibrahim et al., 2020), Lomax (Ibrahim & Yousof, 2020), Weibull (Aryal et al., 2017; Cordeiro et al., 2018; Rasekhi et al., 2022), and other flexible models (Altun et al., 2021, 2022a; Korkmaz et al., 2022) have shown potential applications in regression modeling, though their adaptation to account for autocorrelation remains underexplored. Studies on odd log-logistic families (Alizadeh et al., 2021; Korkmaz et al., 2019) and extended distributions (Altun et al., 2018; Yousof et al., 2018, 2019) highlight the need for integrating autocorrelation handling into model design. Simulation studies evaluating robustness under autocorrelation (Cordeiro et al., 2020; Ibrahim et al., 2020) and refining model selection criteria for dependent data (Altun et al., 2018; Hamedani et al., 2018) could bridge this gap. Incorporating autocorrelation-specific adjustments into existing tools and advancing statistical software packages (Minkah et al., 2023; Yousof & Gad, 2017) will further enhance their applicability in big data and machine learning contexts, ensuring these solutions are both accessible and robust across real-life scenarios. Many future visions can be explored in insurance, economics, and risk analysis through various research contributions. These include works by Alizadeh et al. (2023, 2024, 2025), Al-Essa et al. (2023), Salem et al. (2023), Hamedani et al. (2023), and Bandar et al. (2023). Additionally, studies by Aljadani et al. (2024), Yousof et al. (2024a, 2024b), and Shehata et al. (2024) provide valuable insights. Contributions from Korkmaz et al. (2017), Alizadeh et al. (2018), Rasekhi et al. (2020), Ahmed et al. (2024), Abiad et al. (2024), Khan et al. (2024), Abonongo et al. (2025) and Das et al. (2025) further enrich this field. These works collectively shape potential advancements in these domains.

On the other hand, many regression models can be presented, and financial, actuarial, economic, and other applications can be provided, for more discrete distributions see Aboraya et al. (2020), Yousof et al. (2018; 020; 2021; 2024a,b), Eliwa et al. (2022), Chesneau et al. (2022), Emam et al. (2024), and Ibrahim et al. (2021; 2025).

References

1. Abdelwahab, M. M., Abonazel, M. R., Hammad, A. T., and El-Masry, A. M., (2024). Modified Two-Parameter Liu Estimator for Addressing Multicollinearity in the Poisson Regression Model. Axioms, Vol. (13), No. (46), pp, (1-22).
2. Korkmaz, M. Ç., Yousof, H. M., & Ali, M. M. (2017). Some theoretical and computational aspects of the odd Lindley Fréchet distribution. İstatistikçiler Dergisi: İstatistik ve Aktüerya, 10(2), 129-140.
3.
4. Alizadeh, M., Lak, F., Rasekhi, M., Ramires, T. G., Yousof, H. M., & Altun, E. (2018). The odd log-logistic Topp–Leone G family of distributions: heteroscedastic regression models and applications. Computational Statistics, 33, 1217-1244.
5.
6. Yousof, H. M., Altun, E., & Hamedani, G. G. (2018). A new extension of Frechet distribution with regression models, residual analysis and characterizations. Journal of Data Science, 16(4), 743-770.
7.
8. Rasekhi, M., Saber, M. M., & Yousof, H. M. (2020). Bayesian and classical inference of reliability in multicomponent stress-strength under the generalized logistic model. Communications in Statistics-Theory and Methods, 50(21), 5114-5125.
9. Abiad, M., Alsadat, N., Abd El-Raouf, M. M., Yousof, H. M., & Kumar, A. (2025). Different copula types and reliability applications for a new fisk probability model. Alexandria Engineering Journal, 110, 512-526.
10. Abonongo, J., Abonongo, A. I. L., Aljadani, A., Mansour, M. M., & Yousof, H. M. (2025). Accelerated failure model with empirical analysis and application to colon cancer data: Testing and validation. *Alexandria Engineering Journal*, *113*, 391-408.
11. Aboraya, M., M. Yousof, H. M., Hamedani, G. G., & Ibrahim, M. (2020). A new family of discrete distributions with mathematical properties, characterizations, Bayesian and non-Bayesian estimation methods. *Mathematics*, *8*, 1648.
12. AÇAR, T. S., (2022). Kibria-Lukman Estimator for General Linear Regression Model with AR (2) Errors: A Comparative Study with Monte Carlo Simulation. Journal of New Theory, Vol. (41), pp. (1-17).
13. Afify, A. Z., Cordeiro, G. M., Ortega, E. M., Yousof, H. M., & Butt, N. S. (2018). The four-parameter Burr XII distribution: Properties, regression model, and applications. Communications in Statistics-Theory and Methods, 47(11), 2605-2624.
14. Ahmed, A. E., Ali, H. M., and Amal, H. A., (2020). Almost Unbiased Liu Principal Component Estimator in the Presence of Multicollinearity and Autocorrelation. The Egyptian Statistical Journal, V0l. (64), No. (1), pp. (21-33).
15. Ahmed, B., Hamedani, G. G., Mekiso, G. T., Tashkandy, Y. A., Bakr, M. E., Hussam, E., & Yousof, H. M. (2024). Amputated life-testing based on extended Dagum percentiles for type of group inspection plans: optimal sample sizes, termination time ratios analysis. Scientific Reports, 14(1), 24144.
16. Aladeitan, B. B., Adebimpe, O., Lukman, A. F., Oludoun, O., and Abiodun, O. E., (2021). Modified Kibria-Lukman (MKL) estimator for the Poisson Regression Model: application and simulation. F1000Research.

17. Al-Essa, L. A., Eliwa, M. S., El-Morshedy, M., Alqifari, H., & Yousof, H. M. (2023). Flexible extension of the lomax distribution for asymmetric data under different failure rate profiles: Characteristics with applications for failure modeling and service times for aircraft windshields. Processes, 11(7), 2197.
18. Algamal, Z. Y., (2018). Biased estimators in Poisson regression model in the presence of multicollinearity: A subject review. Al-Qadisiyah Journal for Administrative and Economic Sciences, Vol. (20), pp. (36-43).
19. Alheety, M. I., and Kibria, B. G., (2009). On the Liu and almost unbiased Liu estimators in the presence of multicollinearity with heteroscedastic or correlated errors. Surveys in Mathematics and its Applications, Vol. (4), pp. (155-167).
20. Alizadeh, M., Afshari, M., Contreras-Reyes, J. E., Mazarei, D., & Yousof, H. M. (2024). The Extended Gompertz Model: Applications, Mean of Order P Assessment and Statistical Threshold Risk Analysis Based on Extreme Stresses Data. IEEE Transactions on Reliability, doi: 10.1109/TR.2024.3425278.
21. Alizadeh, M., Afshari, M., Cordeiro, G. M., Ramaki, Z., Contreras-Reyes, J. E., Dirnik, F., & Yousof, H. M. (2025). A New Weighted Lindley Model with Applications to Extreme Historical Insurance Claims. Stats, 8(1), 8.
22. Alizadeh, M., Afshari, M., Ranjbar, V., Merovci, F., & Yousof, H. M. (2023). A novel XGamma extension: applications and actuarial risk analysis under the reinsurance data. São Paulo Journal of Mathematical Sciences, 1-31.
23. Alizadeh, M., Afshariy, M., Karamikabir, H. and Yousof, H. M. (2021). The Odd Log-Logistic Burr-X family of distributions: Properties and Applications, Journal of Statistical Theory and Applications, 20(2), 228 – 241.
24. Alizadeh, M., Lak, F., Rasekhi, M., Ramires, T. G., Yousof, H. M., & Altun, E. (2018). The odd log-logistic Topp–Leone G family of distributions: heteroscedastic regression models and applications. Computational Statistics, 33(3), 1217-1244.
25. Aljadani, A., Mansour, M. M., & Yousof, H. M. (2024). A Novel Model for Finance and Reliability Applications: Theory, Practices and Financial Peaks Over a Random Threshold Value-at-Risk Analysis. Pakistan Journal of Statistics and Operation Research, 20(3), 489-515. https://doi.org/10.18187/pjsor.v20i3.4439
26. Alkhateeb, A., and Algamal, Z., (2020). Jackknifed Liu-type estimator in Poisson regression model. Journal of the Iranian Statistical Society, Vol. (19), No. (1), pp. (21-37).
27. Altun, E. Alizadeh, M., Yousof, H. M., Rasekhi, M. and Hamedani, G. G. (2021). A New Type II Half Logistic-G family of Distributions with Properties, Regression Models, Reliability Systems and Applications. Applications and Applied Mathematics: An International Journal, 16(2), 823-843.
28. Altun, E., Alizadeh, M., Kadilar, G. O. and Yousof, H. M. (2022) New Odd Log-Logistic Family of Distributions: Properties, Regression Models and Applications, CRC Press, Taylor & Francis Group.
29. Altun, E., Alizadeh, M., Yousof, H. M. and Hamedani, G. G. (2022). The Gudermannian generated family of distributions with characterizations, regression models and applications, Studia Scientiarum Mathematicarum Hungarica, 59 (2), 93-115.
30. Altun, E., Yousof, H. M. and Hamedani G. G. (2018a). A new flexible extension of the generalized half-normal lifetime model with characterizations and regression modeling. Bulletin of Computational Applied Mathematics, 6(1), 83-115.
31. Altun, E., Yousof, H. M. and Hamedani G. G. (2018d). A Flexible Extension of Generalized Half-Normal Distribution: Characterizations and Regression Models. International Journal of Applied Mathematics and Statistics, 57(3), 27-49.
32. Altun, E., Yousof, H. M. and Hamedani, G. G. (2018b). A new generalization of generalized half-normal distribution: properties and regression models. Journal of Statistical Distributions and Applications, 5(1), 7.
33. Altun, E., Yousof, H. M. and Hamedani, G. G. (2018c). A new log-location regression model with influence diagnostics and residual analysis. Facta Universitatis, Series: Mathematics and Informatics, 33(3), 417-449.
34. Altun, E., Yousof, H. M., Chakraborty, S. and Handique, L. (2018). Zografos-Balakrishnan Burr XII distribution: regression modeling and applications. International Journal of Mathematics and Statistics, 19(3), 46-70.
35. Amin, M., Akram, M. N., & Kibria, B. G. (2021). A new adjusted Liu estimator for the Poisson regression model. *Concurrency and computation: Practice and experience*, *33*(20), e6340.
36. Anono, Z. M., and Osagie, A. M., (2021). Robustness Test of the Two Stage KL Estimator in Models with Multicollinear Regressors and Autocorrelated Error Term. Asian Journal of Probability and Statistics, Vol. 14, No. (3), pp. (22-33).
37. Aryal, G. R., Ortega, E. M., Hamedani, G. G., & Yousof, H. M. (2017). The Topp-Leone generated Weibull distribution: regression model, characterizations and applications. International Journal of Statistics and Probability, 6(1), 126-141.
38. Bandar, S. A., Hussein, E. A., Yousof, H. M., Afify, A. Z., & Abdellatif, A. D. (2023). A NOVEL EXTENSION OF THE REDUCED-KIES FAMILY: PROPERTIES, INFERENCE, AND APPLICATIONS TO RELIABILITY ENGINEERING DATA. Advanced Mathematical Models & Applications, 8(1).
39. Chesneau, C., Yousof, H. M., Hamedani, G., & Ibrahim, M. (2022). A New One-parameter Discrete Distribution: The Discrete Inverse Burr Distribution: Characterizations, Properties, Applications, Bayesian and Non-Bayesian Estimations. Statistics, Optimization & Information Computing, 10(2), 352-371.
40. Cordeiro, G. M., Afify, A. Z., Yousof, H. M., Cakmakyapan, S., & Ozel, G. (2018). The Lindley Weibull distribution: properties and applications. Anais da Academia Brasileira de Ciências, 90(3), 2579-2598.
41. Cordeiro, G. M., Altun, E., Korkmaz, M. C., Pescim, R. R. and Afify, A. Z. and Yousof, H. M. (2020). The xgamma Family: Censored Regression Modelling and Applications. Revstat Statistical Journal, 18(5), 593–612.
42. Cordeiro, G. M., Yousof, H. M., Ramires, T. G., & Ortega, E. M. (2018). The Burr XII system of densities: properties, regression model and applications. Journal of Statistical Computation and Simulation, 88(3), 432-456.
43. Das, J., Hazarika, P. J., Alizadeh, M., Contreras-Reyes, J. E., Mohammad, H. H., & Yousof, H. M. (2025). Economic Peaks and Value-at-Risk Analysis: A Novel Approach Using the Laplace Distribution for House Prices. Mathematical and Computational Applications, 30(1), 4.

44. Davis, R. A., Dunsmuir, W. T., and Wang, Y., (2000). On autocorrelation in a Poisson regression model. Biometrika, Vol. (87), No. (3), pp. (491-505.
45. Eliwa, M. S., El-Morshedy, M. and Yousof, H. M. (2022). A Discrete Exponential Generalized-G Family of Distributions: Properties with Bayesian and Non-Bayesian Estimators to Model Medical, Engineering and Agriculture Data. Mathematics, 10, 3348. https://doi.org/10.3390/math10183348
46. Emam, W.; Tashkandy, Y.; Hamedani, G.G.; Shehab, M.A.; Ibrahim, M.; Yousof, H.M. A Novel Discrete Generator with Modeling Engineering, Agricultural and Medical Count and Zero-Inflated Real Data with Bayesian, and Non-Bayesian Inference. Mathematics 2023, 11, 1125. https://doi.org/10.3390/math11051125
47. Hamedani, G. G., Altun, E., Korkmaz, M. Ç., Yousof, H. M., & Butt, N. S. (2018). A new extended G family of continuous distributions with mathematical properties, characterizations and regression modeling. Pakistan Journal of Statistics and Operation Research, 737-758.
48. Hamedani, G. G., Goual, H., Emam, W., Tashkandy, Y., Ahmad Bhatti, F., Ibrahim, M., & Yousof, H. M. (2023). A new right-skewed one-parameter distribution with mathematical characterizations, distributional validation, and actuarial risk analysis, with applications. Symmetry, 15(7), 1297.
49. Ibrahim, M., & Yousof, H. M. (2020). A new generalized Lomax model: statistical properties and applications. Journal of Data Science, 18(1), 190-217.
50. Ibrahim, M., Ali, M. M., & Yousof, H. M. (2021). The discrete analogue of the Weibull G family: properties, different applications, Bayesian and non-Bayesian estimation methods. Annals of Data Science, 1-38.
51. Ibrahim, M., Altun, E., & Yousof, H. M. (2020). A new distribution for modeling lifetime data with different methods of estimation and censored regression modeling. Statistics, Optimization & Information Computing, 8(2), 610-630.
52. Ibrahim, M., Altun, E., Goual, H., and Yousof, H. M. (2020). Modified goodness-of-fit type test for censored validation under a new Burr type XII distribution with different methods of estimation and regression modeling. Eurasian Bulletin of Mathematics, 3(3), 162-182.
53. Ibrahim, M., Butt, N. S., Al-Nefaie, A. H., Hamedani, G. G., Yousof, H. M., & Mahmoud, A. S. (2025). An Extended Discrete Model for Actuarial Data and Value at Risk Analysis: Properties, Applications and Risk Analysis under Financial Automobile Claims Data. Statistics, Optimization & Information Computing, 13(1), 27-46.
54. Ibrahim, S. F., and Alheety, M. I., (2023). A New Estimator Biased on PKL Estimator for Poisson Regression Model. Journal of Survey in Fisheries Sciences, Vol. 10, No. (3S), pp. (4836-4843).
55. Imran, M. E. M., (2003), "Autocorrelation-Conditional Linear Regression Model: An Applied Study of the Influences on Stock Price. Journal of Accounting, Management and Insurance, Issue (61).
56. Kazem, A., H., and Muslim, B., S., (2002), "Advanced Economic Measurement: Theory and Application. Dunya Al-Amal Library, Baghdad, Iraq.
57. Kejian, L. (1993). A new class of blased estimate in linear regression. Communications in Statistics-Theory and Methods, Vol. (22), pp. (393-402).
58. Khan, M. I., Aljadani, A., Mansour, M. M., Abd Elrazik, E. M., Hamedani, G. G., Yousof, H. M., & Shehata, W. A. (2024). A New Heavy-Tailed Lomax Model With Characterizations, Applications, Peaks Over Random Threshold Value-at-Risk, and the Mean-of-Order-P Analysis. Journal of Mathematics, 2024(1), 5329529.
59. Kibria, B. M., and Lukman, A. F, (2020). A new ridge-type estimator for the linear regression model: Simulations and applications. Scientifica, 2020.
60. Korkmaz, M. C., Altun, E., Alizadeh, M., & Yousof, H. M. (2019). A new flexible lifetime model with log-location regression modeling, properties and applications. Journal of Statistics and Management Systems, 22(5), 871-891.
61. Korkmaz, M. Ç., Altun, E., Chesneau, C. and Yousof, H. M. (2022). On the unit-Chen distribution with associated quantile regression and applications. Mathematica Slovaca, 72 (2022), No. 3, 765-786.
62. Korkmaz, M. Ç., Altun, E., Yousof, H. M., & Hamedani, G. G. (2019). The odd power Lindley generator of probability distributions: properties, characterizations and regression modeling. International Journal of Statistics and Probability. 8(2), 70-89.
63. Korkmaz, M. Ç., Altun, E., Yousof, H. M., & Hamedani, G. G. (2020). The Hjorth's IDB Generator of Distributions: Properties, Characterizations, Regression Modeling and Applications. Journal of Statistical Theory and Applications, 19(1), 59-74.
64. Korkmaz, M. Ç., Cordeiro, G. M., Yousof, H. M., Pescim, R. R., Afify, A. Z., & Nadarajah, S. (2019). The Weibull Marshall–Olkin family: Regression model and application to censored data. Communications in Statistics-Theory and Methods, 48(16), 4171-4194.
65. Lukman, A. F., Adewuyi, E., Månsson, K., and Kibria, B. G., (2021). A new estimator for the multicollinear Poisson regression model: simulation and application. Scientific Reports, 11, 3732.
66. Lukman, A. F., Aladeitan, B., Ayinde, K., and Abonazel, M. R., (2022). Modified ridge-type for the Poisson regression model: simulation and application. Journal of Applied Statistics, Vol. (49), No. (8), pp. (2124-2136).
67. Mansson, K., Kibria, B. G., Sjölander, P., and Shukur, G, (2012). Improved Liu estimators for the Poisson regression model. International Journal of Statistics and Probability, Vol. (1), No. (1), pp. (2-6).
68. Mansson, K., Kibria, B. G., Sjölander, P., Shukur, G., and Sweden, V, (2011). New Liu Estimators for the Poisson regression model: Method and application. provided by research papers in economics.
69. Minkah, R., de Wet, T., Ghosh, A., & Yousof, H. M. (2023). Robust extreme quantile estimation for Pareto-type tails through an exponential regression model. Communications for Statistical Applications and Methods, 30(6), 531-550.

70. Naya, H., Urioste, J. I., Chang, Y. M., Rodrigues-Motta, M., Kremer, R., and Gianola, D., (2008).. A comparison between Poisson and zero-inflated Poisson regression models with an application to number of black spots in Corriedale sheep. Genetics Selection Evolution, Vol. (40), pp. (1-16).
71. Nouri, Z. Y., and Abdul Latif, A. M., (2019), "Selecting Variables in the Poisson Regression Model Using Partial Possibility Methods. Journal of Management and Economics, Issue (118).
72. Oladapo, O. J., Idowu, J. I., Owolabi, A. T., and Ayinde, K, (2023). A New Biased Two-Parameter Estimator in Linear Regression Model. EQUATIONS, Vol. (3), pp. (73-92).
73. Owolabi, A. T., Ayinde, K., Idowu, J. I., Oladapo, O. J., and Lukman, A. F, (2022). A New Two-Parameter Estimator in the Linear Regression Model with Correlated Regressors. Journal of Statistics Applications and Probability, Vol. (11), No. (2), pp. (499-512).
74. Perveen, I., and Suhail, M, (2023). Bootstrap Liu estimators for Poisson regression model. Communications in Statistics-Simulation and Computation, Vol. 52, No. (7), pp. (2811-2821).
75. Qasim, M., Kibria, B. M. G., Månsson, K., and Sjölander, P, (2020). A new Poisson Liu regression estimator: method and application. Journal of Applied Statistics, Vol. (47), No. (12), pp. (2258-2271).
76. Rasekhi, M., Altun, E., Alizadeh, M., & Yousof, H. M. (2022). The Odd Log-Logistic Weibull-G Family of Distributions with Regression and Financial Risk Models. Journal of the Operations Research Society of China, 10(1), 133-158.
77. Rashad, N. K., and Algamal, Z. Y, (2019). A new ridge estimator for the Poisson regression model. Iranian Journal of Science and Technology, Vol (43), pp (2921-2928).
78. Salem, M., Emam, W., Tashkandy, Y., Ibrahim, M., Ali, M. M., Goual, H., & Yousof, H. M. (2023). A new lomax extension: Properties, risk analysis, censored and complete goodness-of-fit validation testing under left-skewed insurance, reliability and medical data. Symmetry, 15(7), 1356.
79. Salih A.M., Abdullah M.M., 2024. Comparison between classical and Bayesian estimation with joint Jeffrey's prior to Weibull distribution parameters in the presence of large sample conditions. Statistics in Transition new series, 25(4), pp. 191-202
80. Shehata, W. A. M., Aljadani, A., Mansour, M. M., Alrweili, H., Hamed, M. S., & Yousof, H. M. (2024). A Novel Reciprocal-Weibull Model for Extreme Reliability Data: Statistical Properties, Reliability Applications, Reliability PORT-VaR and Mean of Order P Risk Analysis. Pakistan Journal of Statistics and Operation Research, 20(4), 693-718. https://doi.org/10.18187/pjsor.v20i4.4302
81. Shewa, G. A., and Ugwuowo, F. I., (2023). A new hybrid estimator for linear regression model analysis: computations and simulations. Scientific African, 19, e01441.
82. Tali, N. H., (2022), "Estimating the Linear Regression Model When There is an Autocorrelation Problem Using Lasso and Al-Lubi Regression with Application" Master's Thesis, College of Management and Economics, Al-Mustansiriya University, Iraq.
83. Winkelmann, R, (2008). Econometric analysis of count data. 5 th Edition., Springer, Verlag Berlin Heidelberg, Germany.
84. Yousof, H. M., & Gad, A. M. (2017). Bayesian Semi-Parametric Logistic Regression Model with Application to Credit Scoring Data. Journal of Data Science, 15(1), 25-39.
85. Yousof, H. M., Aljadani, A., Mansour, M. M., & Abd Elrazik, E. M. (2024a). A New Pareto Model: Risk Application, Reliability MOOP and PORT Value-at-Risk Analysis. Pakistan Journal of Statistics and Operation Research, 20(3), 383-407. https://doi.org/10.18187/pjsor.v20i3.4151
86. Yousof, H. M., Al-Nefaie, A. H., Butt, N. S., Hamedani, G., Alrweili, H., Aljadani, A., Mansour, M. M., Hamed, M. S., & Ibrahim, M. (2024b). A New Discrete Generator with Mathematical Characterization, Properties, Count Statistical Modeling and Inference with Applications to Reliability, Medicine, Agriculture, and Biology Data. Pakistan Journal of Statistics and Operation Research, 20(4), 745-770. https://doi.org/10.18187/pjsor.v20i4.4616
87. Yousof, H. M., Altun, E., & Hamedani, G. G. (2018). A new extension of Fréchet distribution with regression models, residual analysis and characterizations. Journal of Data Science, 16(4), 743-770.
88. Yousof, H. M., Altun, E., Ramires, T. G., Alizadeh, M., & Rasekhi, M. (2018). A new family of distributions with properties, regression models and applications. Journal of Statistics and Management Systems, 21(1), 163-188.
89. Yousof, H. M., Altun, E., Rasekhi, M., Alizadeh, M., Hamedani, G. G., & Ali, M. M. (2019). A new lifetime model with regression models, characterizations and applications. Communications in Statistics-Simulation and Computation, 48(1), 264-286.
90. Yousof, H. M., Chesneau, C., Hamedani, G., & Ibrahim, M. (2021). A New Discrete Distribution: Properties, Characterizations, Modeling Real Count Data, Bayesian and Non-Bayesian Estimations. Statistica, 81(2), 135-162.
91. Yousof, H. M., Hamedani, G. G. and Ibrahim, M. (2020). The Discrete Weibull Generalized Family: Characterization, Applications, Bayesian and Non-Bayesian Estimation Methods. Communications in Statistics - Theory and Method. Submitted.
92. Yousof, H. M., Jahanshahi, S. M. A., Ramires, T. G., Aryal, G. R., & Hamedani, G. G. (2018). A new distribution for extreme values: regression model, characterizations and applications. Journal of Data Science, 16(4), 677 -706.
93. Yousof, H. M., Rasekhi, M., Altun, E., & Alizadeh, M. (2018). The extended odd Fréchet family of distributions: properties, applications and regression modeling. International Journal of Applied Mathematics and Statistics, 30(1), 1-30.
94. Yousof, H. M., Saber, M. M., Al-Nefaie, A. H., Butt, N. S., Ibrahim, M., & Alkhayyat, S. L. (2024). A discrete claims-model for the inflated and over-dispersed automobile claims frequencies data: Applications and actuarial risk analysis. Pakistan Journal of Statistics and Operation Research, 20(2), 261-284.
95. Zubair, M. A., and Adenomon, M. O, (2021). Comparison of estimators efficiency for linear regressions with joint presence of autocorrelation and multicollinearity. Science World Journal, Vol. (16), No. (2), pp. (103-109).