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The Type II Quasi Lambert G Family of Probability Distributions

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Abstract

Probability distributions and their families play an effective role in statistical modeling and statistical analysis. Recently, researchers have been increasingly interested in generating new families with high flexibility and low number of milestones. We propose and study a new family of continuous distributions. Relevant properties are presented. Many bivariate versions of the new family are derived under the Farlie-Gumbel-Morgenstern copula, modified Farlie-Gumbel-Morgenstern copula, Clayton copula, entropy copula and Ali-Mikhail-Haq copula. We present two characterizations of the new family. Different estimation methods such as the maximum likelihood estimation, maximum product spacing estimation, least squares estimation, weighted least squares estimation, Anderson-Darling estimation and the Cramer-von Mises estimation methods are considered. Simulation studies for comparing estimation methods are performed based on the baseline Lindley model. Two real data sets are analyzed for comparing the competitive models.

Key Words: Farlie-Gumbel-Morgenstern copula; Ali-Mikhail-Haq Copula; Maximum Product Spacing Estimation; Characterizations; Maximum Likelihood Estimation; Clayton copula; Anderson-Darling Estimation.

Mathematical Subject Classification: 62N01; 62N02; 62E10.

1.Introduction

In the last few decades, there have been an increased interest in defining and studing new families of univariate continuous distributions by adding one (or more) shape parameter(s) to certain baseline model. The use of the new generators of continuous distributions from classic ones has become very common in recent years. The procedure of expanding a class of distributions by adding new shape parameter(s) is well-known in the statistical literature. In many applied sciences such as medicine, engineering and finance, among others, modeling and analyzing lifetime data are crucial. Several lifetime distributions have been adopted to model different types of survival data. The quality of the procedures used in a statistical analysis depends heavily on the generated distribution. Further, the statistical modeling of the phenomenon, the applications or the validity of data is impossible without choosing the proper probability distribution (the mathematical form of the model). Thus, considerable effort has been devoted to explore new statistical methodologies. Also the computational and analytical facilities available in programming softwares like R, Maple and Mathematica can easily tackle the problems involved in computing special functions in the new extended families. These facilities encourage several statisticians to develop new extended models. However, there still remain many important problems involving real data, which do not follow any of the classical statistical models. The chief motivation in generalizing distributions for modeling lifetime data lies in the flexibility to model both monotonic and

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non-monotonic failure rates even though the baseline failure rate may be monotonic. The role of the extra shape parameter(s) is to introduce skewness and to vary tail weights. Furthermore, various classes of distributions have been constructed by extending common families of lifetime distributions and analyze them with respect to different characteristics. The art of proposing generalized families of distributions has attracted theoretical and applied statisticians due to their flexible properties. There are many new families of distributions, that become precious for applied statisticians, proposed in the last two decades. Many wellknown generators can be cited such as beta-G (B-G) family (Eugene et al. (2002)), transmuted exponentiated generalized-G (TEG-G) family (Yousof et al. (2015)), generalized odd generalized exponential-G (GOGE-G) family by Alizadeh et al. (2017), exponentiated generalized-G Poisson (EG-GP) family (Aryal and Yousof (2017)), transmuted Topp-Leone-G (TTL-G) family (Yousof et al. (2017a)), beta Weibull-G (BW-G) family (Yousof et al. (2017b)), Topp-Leone odd log-logistic-G family (Brito et al. (2017)) (TLOLL-G), Burr XII system of densities (Cordeiro et al. (2018)) (BXII-G), transmuted Weibull-G (TW-G) family (Alizadeh et al. (2018)), generalized odd Weibull generated-G (GOWG-G)family (Korkmaz et al. (2018a)), exponential Lindley odd log-logistic-G (ELOLL-G) family (Korkmaz et al. (2018b)), Marshall-Olkin generalized-G Poisson (MOG-GP) family (Korkmaz et al. (2018c)), the Odd Power Lindley Generator (OPLG) (Korkmaz et al. (2019)), odd Nadarajah-Haghighi-G (NH-G) family (Nascimento et al. (2019)), generalized transmuted Poisson-G (GTP-G) family (Yousof et al. (2018a)), MarshallOlkin generalized-G family (Yousof et al. (2018b)) (MOG-G), Burr-Hatke-G family (Yousof et al. (2018c)) (BH-G), Type I general exponential class of distributions (TIGE-G) (Hamedani et al. (2017)), new extended-G (NE-G) family (Hamedani et al. (2018)), Type II general exponential class of distributions (TIIGE-G) (Hamedani et al. (2019)), Weibull generalized G (WG-G) family (Yousof et al. (2018d)), Weibull-G Poisson (W-GP) family (Yousof et al. (2020), The type II quasi Lambert G family of probability distributions (Hamedani et al. (2021)) and Weibull Topp-Leone generated-G (WTL-G) family (Karamikabir et al. (2020)). Many authors have used the T-X family of distributions (Alzaatreh et al. (2013)), however, in this paper we propose a new method for generating new G families.

In mathematics and statistics, the "Lambert function", or the "omega function", is a multivalued function, namely the branches of the inverse relation of the function $f(\mathbf{L})=\mathbf{L}(\cdot)\exp[\mathbf{L}(\cdot)]$ where $\mathbf{L}(\cdot)$ is any complex number and $\exp[\mathbf{L}(\cdot)]$ is the exponential function. In this work and folowoing Hamedani et al. (2021), we define and study a new G family called the type II quasi Lambert (TIIQL) family depending on the concept of the Lambert function . The cumulative distribution function (CDF) of the TIIQL family can be expressed as

$$F_{\alpha,\underline{\boldsymbol{\xi}}}(x)=\mathbf{L}_{\alpha,\underline{\boldsymbol{\xi}}}(x)\exp\left[\overline{\mathbf{L}}_{\alpha,\underline{\boldsymbol{\xi}}}(x)\right]|_{x\in\mathbb{R}}, \tag{1}$$

where $\overline{\mathbf{L}}_{\alpha,\underline{\boldsymbol{\xi}}}(x)=1-\mathbf{L}_{\alpha,\underline{\boldsymbol{\xi}}}(x)$, $\alpha>0$ is a shape parameter and $\mathbf{L}_{\alpha,\underline{\boldsymbol{\xi}}}(x)=\frac{2-\boldsymbol{\Pi}_{\underline{\boldsymbol{\xi}}}^{\alpha}(x)}{\boldsymbol{\Pi}_{\underline{\boldsymbol{\xi}}}^{\alpha}(x)}|_{\alpha>0,x\in\mathbb{R}}$. The function $\boldsymbol{\Pi}_{\underline{\boldsymbol{\xi}}}(x)$ is the CDF of any baseline model and $\underline{\boldsymbol{\xi}}$ refers to the parameter vector. For $\alpha=1$, the TIIQL family reduces to the reduced TIIQL (RTIIQL) family. The corresponding probability density function (PDF) can be expressed as

$$f_{\alpha,\underline{\boldsymbol{\xi}}}(x)=2\alpha\frac{\pi_{\underline{\boldsymbol{\xi}}}(x)}{\boldsymbol{\Pi}_{\underline{\boldsymbol{\xi}}}^{\alpha+1}(x)}\left[\mathbf{L}_{\alpha,\underline{\boldsymbol{\xi}}}(x)-1\right]\exp\left[\overline{\mathbf{L}}_{\alpha,\underline{\boldsymbol{\xi}}}(x)\right]|_{\alpha>0\text{ and }x\in\mathbb{R}}, \tag{2}$$

where $\pi_{\underline{\boldsymbol{\xi}}}(x)=d\boldsymbol{\Pi}_{\underline{\boldsymbol{\xi}}}(x)/dx$ is the PDF of the baseline model. Using the power series, the CDF in (1) can be written as

$$F_{\alpha,\underline{\boldsymbol{\xi}}}(x)=\mathbf{L}_{\alpha,\underline{\boldsymbol{\xi}}}(x)\sum_{j_0=0}^{\infty}\frac{\overline{\mathbf{L}}_{\alpha,\underline{\boldsymbol{\xi}}}^{j_0}(x)}{\Gamma(1+j_1)}. \tag{3}$$

If $\left|\frac{\vartheta_1}{\vartheta_2}\right|<1$ and $\vartheta_3>0$ is a real non-integer, the following power series holds

$$\left(1-\frac{\vartheta_1}{\vartheta_2}\right)^{\vartheta_3-1}=\sum_{j_1=0}^{\infty}\frac{\Gamma(\vartheta_3)(-\vartheta_1/\vartheta_2)^{j_1}}{\Gamma(1+j_1)!\,\Gamma(\vartheta_3-j_1)}. \tag{4}$$

Applying (4) to (3) we have

$$F_{\alpha,\underline{\boldsymbol{\xi}}}(x)=\sum_{j_0,j_1=0}^{\infty}\frac{(-1)^{j_1}2^{1+j_1}\Gamma(1+j_0)}{j_0!\,j_1!\,\Gamma(1+j_0+j_1)}\frac{\left[1-\frac{1}{2}\boldsymbol{\Pi}_{\underline{\boldsymbol{\xi}}}^{\alpha}(x)\right]^{1+j_1}}{\boldsymbol{\Pi}_{\underline{\boldsymbol{\xi}}}^{\alpha(1+j_1)}(x)}. \tag{5}$$

Applying (4) again to the term $\left[1-\frac{1}{2}\boldsymbol{\Pi}_{\underline{\boldsymbol{\xi}}}^{\alpha}(x)\right]^{1+j_1}$, Equation (5) becomes

$$F_{\alpha,\underline{\boldsymbol{\xi}}}(x)=\sum_{j_1,j_2=0}^{\infty}c_{j_1,j_2}\quad H_{\boldsymbol{\Delta}}\left(x;\underline{\boldsymbol{\xi}}\right)|_{\boldsymbol{\Delta}=\alpha j_2-\alpha(1+j_1)}, \tag{6}$$

where $c_{j_1,j_2}=\sum_{j_0=0}^{\infty}\frac{(-1)^{j_1+j_2}2^{1+j_1-j_2}\Gamma(1+j_0)\Gamma(2+j_1)}{j_0!j_1!j_2!\,\Gamma(1+j_0+j_1)\Gamma(2+j_1-j_2)}$ and $H_{\mathbf{\Delta}}\left(x;\underline{\boldsymbol{\xi}}\right)$ is the CDF of the exp-G family with power parameter $\mathbf{\Delta}>0$. Similarly, the PDF of the TIIQL family can also be expressed as a mixture of exp-G PDFs as

$$f_{\alpha,\underline{\boldsymbol{\xi}}}(x)=\sum_{j_1,j_2=0}^{\infty}c_{j_1,j_2}\;\;h_{\mathbf{\Delta}}\left(x;\underline{\boldsymbol{\xi}}\right),\tag{7}$$

where $\boldsymbol{\pi}_{\mathbf{\Delta}}\left(x;\underline{\boldsymbol{\xi}}\right)=d\mathbf{\Pi}_{\mathbf{\Delta}}\left(x;\underline{\boldsymbol{\xi}}\right)/dx$ s the PDF of the exp-G family with power parameter $\mathbf{\Delta}>0$.

2. Properties

2.1 Moments

Let $Y_{\mathbf{\Delta}}$ be a r.v. with density $h_{\mathbf{\Delta}}(x;\underline{\boldsymbol{\xi}})$. The rth ordinary moment of X, say μ_r', follows from (7) as

$$\mu_r'=\mathbf{E}(X^r)=\sum_{j_1,j_2=0}^{\infty}c_{j_1,j_2}\mathbf{E}(Y_{\mathbf{\Delta}}^r),\tag{8}$$

where $\mathbf{E}(Y_{\vartheta}^r)=\vartheta\;\int_{-\infty}^{\infty}x^r\pi_{\underline{\boldsymbol{\xi}}}(x)\,\mathbf{\Pi}_{\underline{\boldsymbol{\xi}}}(x)^{\vartheta-1}\;dx$,can be evaluated numerically in terms of the baseline qf $Q_{\mathbf{\Pi}}(u)=\mathbf{\Pi}^{-1}(u)$ as $\mathbf{E}(Y_{\vartheta}^r)=\vartheta\;\int_0^1\;\;v^{\vartheta-1}[Q_{\mathbf{\Pi}}(u)]^r du$. Setting $r=1$ in (8) gives the mean of X.

2.2 Incomplete moments

The r[th] incomplete moment of X is defined by $m_{r,X}(y)=\int_{-\infty}^{y}x^r\,f_{\alpha,\underline{\boldsymbol{\xi}}}(x)dx$. We can write from (7)

$$m_{r,X}(y)=\sum_{j_1,j_2=0}^{\infty}c_{j_1,j_2}m_{r,\mathbf{\Delta}}(y),\tag{9}$$

where

$$m_{r,\alpha}(y)=\int_0^{\mathbf{\Pi}(y)}v^{\alpha-1}\;[Q_{\mathbf{\Pi}}(u)]^r dv.\tag{10}$$

where $Q_{\mathbf{\Pi}}(u)$ refers to the quantile function of the the TIIQL family. The integral $m_{r,\alpha}(y)$ can be determined analytically for special models with closed-form expressions for $Q_{\mathbf{\Pi}}(u)$ or computed at least numerically for most baseline distributions. Two important applications of the first incomplete moment are related to the mean deviations about the mean and median and to the Bonferroni and Lorenz curves.

2.3 Moment generating functions

The moment generating function (MGF) of X, say $M(t)=\boldsymbol{E}(\exp(t\,X))$, is obtained from (7) as

$$M(t)=\sum_{j_1,j_2=0}^{\infty}c_{j_1,j_2}M_{\mathbf{\Delta}}(t),\tag{11}$$

where $M_{\vartheta}(t)$ is the generating function of Y_{ϑ} given by

$$M_{\vartheta}(t)=\vartheta\;\int_{-\infty}^{\infty}\exp(t\,x)\pi_{\underline{\boldsymbol{\xi}}}(x)\left[\mathbf{\Pi}_{\underline{\boldsymbol{\xi}}}(x)\right]^{\vartheta-1}dx=\vartheta\;\int_0^1\exp[t\;Q_{\mathbf{\Pi}}(u;\alpha\,)]\;u^{\vartheta-1}dv.\tag{12}$$

The last two integrals can be computed numerically for most parent distributions.

3. Bivariate versions via copula

In probability theory, a copula is a multivariate CDF for which the marginal probability distribution of each variable is uniform on the interval $[0,1]$. copulas are used to describe the dependence between random variables. In this Section, we derive some new bivariate TIIQL (Biv-TIIQL) type distributions using Farlie Gumbel Morgenstern (FGM) copula (see Morgenstern (1956), and Kotz (1977)), modified FGM copula (see Rodriguez-Lallena and Ubeda-Flores (2004)), Clayton copula, Renyi's entropy (Pougaza and Djafari (2011)) and Ali–Mikhail–Haq copula (Ali et al. (1987)). The Multivariate TIIQL (M-TIIQL) type is also presented. However, future works may be allocated to the study of these new models (see Elgohari and Yousof (2021a,b and 2021), Elgohari et al. (2021), Shehata and Yousof (2021a,b), Shehata et al. (2022)). First, we consider the joint CDF of the FGM family, where

$$\mathbf{H}_{\vartheta}(\upsilon,\nu)=\upsilon\nu(1+\vartheta\upsilon^{\bullet}\nu^{\bullet})|_{\upsilon^{\bullet}=1-\upsilon,\nu^{\bullet}=1-\nu},$$

and the marginal function $\upsilon=F_1$, $\nu=F_2$, $\vartheta\in(-1,1)$ is a dependence parameter and for every $\upsilon,\nu\in(0,1)$, $\mathbf{H}(\upsilon,0)=\mathbf{H}(0,\nu)=0$ which is "grounded minimum" and $\mathbf{H}(\upsilon,1)=\upsilon$ and $\mathbf{H}(1,\nu)=\nu$ which is "grounded maximum", where

$$\mathbf{H}(\upsilon_1,\nu_1)+\mathbf{H}(\upsilon_2,\nu_2)-\mathbf{H}(\upsilon_1,\nu_2)-\mathbf{H}(\upsilon_2,\nu_1)\geq 0.$$

3.1 Via FGM copula

A copula is continuous in υ and ν; actually, it satisfies the stronger Lipschitz condition, where

$$|\mathbf{H}(\upsilon_2,\nu_2)-\mathbf{H}(\upsilon_1,\nu_1)|\le|\upsilon_2-\upsilon_1|+|\nu_2-\nu_1|.$$

For $0\le\upsilon_1\le\upsilon_2\le1$ and $0\le\nu_1\le\nu_2\le1$, we have

$$\Pr(\upsilon_1\le\upsilon\le\upsilon_2,\nu_1\le\nu\le\nu_2)=\mathbf{H}(\upsilon_1,\nu_1)+\mathbf{H}(\upsilon_2,\nu_2)-\mathbf{H}(\upsilon_1,\nu_2)-\mathbf{H}(\upsilon_2,\nu_1)\ge0.$$

Then, setting

$$\upsilon^\bullet=1-F_{\alpha_1,\underline{\xi}}(x_1)|_{[\upsilon^\bullet=(1-\upsilon)\in(0,1)]}\text{ and }\nu^\bullet=1-F_{\alpha_2,\underline{\xi}}(x_2)|_{[\nu^\bullet=(1-\nu)\in(0,1)]},$$

we can easily obtain the joint CDF of the TIIQL using the FGM family ($\forall\ \underline{\mathbf{\Psi}}_i=(\alpha_i,\underline{\boldsymbol{\xi}})|_{\alpha_i>0,i=1,2.}$)

$$\mathbf{H}_\vartheta(\upsilon,\nu)=\mathbf{L}_{\alpha_1,\underline{\xi}}(\upsilon)\mathbf{L}_{\alpha_2,\underline{\xi}}(\nu)\exp\left[\overline{\mathbf{L}}_{\alpha_1,\underline{\xi}}(\upsilon)+\overline{\mathbf{L}}_{\alpha_2,\underline{\xi}}(\nu)\right]\left[1+\vartheta\begin{pmatrix}\left\{1-\mathbf{L}_{\alpha_1,\underline{\xi}}(\upsilon)\exp\left[\overline{\mathbf{L}}_{\alpha_1,\underline{\xi}}(\upsilon)\right]\right\}\\\times\left\{1-\mathbf{L}_{\alpha_2,\underline{\xi}}(\nu)\exp\left[\overline{\mathbf{L}}_{\alpha_2,\underline{\xi}}(\nu)\right]\right\}\end{pmatrix}\right].$$

The joint PDF can then be derived from $c_\vartheta(\upsilon,\nu)=1+\vartheta\upsilon^\bullet\nu^\bullet|_{(\upsilon^\bullet=1-2\upsilon\text{and }\nu^\bullet=1-2\nu)}$ or from $c_\vartheta(\upsilon,\nu)=f(x_1,x_2)=\mathbf{H}(F_1,F_2)f_1f_2$.

3.2 Via modified FGM copula

The modified FGM copula is defined as

$$\mathbf{H}_\vartheta(\upsilon,\nu)=\upsilon\nu[1+\vartheta D(\upsilon)C(\nu)]|_{\vartheta\in(-1,1)}\text{ or }\mathbf{H}_\vartheta(\upsilon,\nu)=\upsilon\nu+\vartheta\widetilde{D}_\upsilon\tilde{C}_\nu|_{\vartheta\in(-1,1)},$$

where $\widetilde{D}_\upsilon=\upsilon D(\upsilon)$, and $\tilde{C}_\nu=\nu C(\nu)$ and $D(\upsilon)$ and $C(\nu)$ are two continuous functions on (0,1) with $D(0)=D(1)=C(0)=C(1)=0$. Let

$$c_1(\widetilde{D}_\upsilon)=\inf\left\{\widetilde{D}_\upsilon:\frac{\partial}{\partial\upsilon}\widetilde{D}_\upsilon\right\}|_{\boldsymbol{\zeta}_{1,\upsilon}}<0,c_2(\widetilde{D}_\upsilon)=\sup\left\{\widetilde{D}_\upsilon:\frac{\partial}{\partial\upsilon}\widetilde{D}_\upsilon\right\}|_{\boldsymbol{\zeta}_{1,\upsilon}}<0,$$

$$d_1(\tilde{C}_\nu)=\inf\left\{\tilde{C}_\nu:\frac{\partial}{\partial\nu}\tilde{C}_\nu\right\}|_{\boldsymbol{\zeta}_{2,\nu}}>0,d_2(\tilde{C}_\nu)=\sup\left\{\tilde{C}_\nu:\frac{\partial}{\partial\nu}\tilde{C}_\nu\right\}|_{\boldsymbol{\zeta}_{2,\nu}}>0.$$

Then, $1\le\min\{c_1(\widetilde{D}_\upsilon)c_2(\widetilde{D}_\upsilon),d_1(\tilde{C}_\nu)d_2(\tilde{C}_\nu)\}<\infty$, whereb $\upsilon\frac{\partial}{\partial\upsilon}D(\upsilon)=\frac{\partial}{\partial\upsilon}\widetilde{D}_\upsilon-D(\upsilon)$,

$$\boldsymbol{\zeta}_{1,\upsilon}=\left\{\upsilon:\upsilon\in(0,1)|_{\frac{\partial}{\partial\upsilon}\widetilde{D}_\upsilon\ \text{ exists}}\right\},\boldsymbol{\zeta}_{2,\nu}=\left\{\nu:\nu\in(0,1)|_{\frac{\partial}{\partial\nu}\tilde{C}_\nu\ \text{ exists}}\right\}.$$

3.2.1 Type-I

Consider the following functional form for both $D(\upsilon)$ and $C(\nu)$. Then, the Biv-TIIQL-FGM (Type-I) can be derived from

$$\begin{aligned}\mathbf{H}_\vartheta(\upsilon,\nu)=&\ \mathbf{L}_{\alpha_1,\underline{\xi}}(\upsilon)\mathbf{L}_{\alpha_2,\underline{\xi}}(\nu)\exp\left[\overline{\mathbf{L}}_{\alpha_1,\underline{\xi}}(\upsilon)+\overline{\mathbf{L}}_{\alpha_2,\underline{\xi}}(\nu)\right]\\&+\vartheta\begin{pmatrix}\mathbf{L}_{\alpha_1,\underline{\xi}}(\upsilon)\exp\left[\overline{\mathbf{L}}_{\alpha_1,\underline{\xi}}(\upsilon)\right]\left\{1-\mathbf{L}_{\alpha_1,\underline{\xi}}(\upsilon)\exp\left[\overline{\mathbf{L}}_{\alpha_1,\underline{\xi}}(\upsilon)\right]\right\}\\\times\mathbf{L}_{\alpha_2,\underline{\xi}}(\nu)\exp\left[\overline{\mathbf{L}}_{\alpha_2,\underline{\xi}}(\nu)\right]\left\{1-\mathbf{L}_{\alpha_2,\underline{\xi}}(\nu)\exp\left[\overline{\mathbf{L}}_{\alpha_2,\underline{\xi}}(\nu)\right]\right\}\end{pmatrix}|_{\vartheta\in(-1,1)}.\end{aligned}$$

3.2.2 Type-II

Let $D(\upsilon)$ and $C(\nu)$ be two functional form satisfying all the conditions stated earlier where $D(\upsilon)^\bullet|_{(\vartheta_1>0)}=\upsilon^{\vartheta_1}(1-\upsilon)^{1-\vartheta_1}$ and $C(\nu)^\bullet|_{(\vartheta_2>0)}=\nu^{\vartheta_2}(1-\nu)^{1-\vartheta_2}$. Then, the corresponding Biv-TIIQL-FGM **(**Type-**II)** can be derived from $\mathbf{H}_{\vartheta,\vartheta_1,\vartheta_2}(\upsilon,\nu)=\upsilon\nu[1+\vartheta D(\upsilon)^\bullet C(\nu)^\bullet]$. Thus

$$\begin{aligned}\mathbf{H}_{\vartheta,\vartheta_1,\vartheta_2}(\upsilon,\nu)=&\ \mathbf{L}_{\alpha_1,\underline{\xi}}(\upsilon)\mathbf{L}_{\alpha_2,\underline{\xi}}(\nu)\exp\left[\overline{\mathbf{L}}_{\alpha_1,\underline{\xi}}(\upsilon)+\overline{\mathbf{L}}_{\alpha_2,\underline{\xi}}(\nu)\right]\\&\times\left[1+\vartheta\begin{pmatrix}\left\{\mathbf{L}_{\alpha_1,\underline{\xi}}(\upsilon)\exp\left[\overline{\mathbf{L}}_{\alpha_1,\underline{\xi}}(\upsilon)\right]\right\}^{\vartheta_1}\\\times\left\{\mathbf{L}_{\alpha_2,\underline{\xi}}(\nu)\exp\left[\overline{\mathbf{L}}_{\alpha_2,\underline{\xi}}(\nu)\right]\right\}^{\vartheta_2}\\\times\left(1-\mathbf{L}_{\alpha_1,\underline{\xi}}(\upsilon)\exp\left[\overline{\mathbf{L}}_{\alpha_1,\underline{\xi}}(\upsilon)\right]\right)^{1-\vartheta_1}\\\times\left(1-\mathbf{L}_{\alpha_2,\underline{\xi}}(\nu)\exp\left[\overline{\mathbf{L}}_{\alpha_2,\underline{\xi}}(\nu)\right]\right)^{1-\vartheta_2}\end{pmatrix}\right]\end{aligned}$$

3.2.3 Type-III

Let $\overline{D^\bullet(\upsilon)}=\upsilon[\log(1+\upsilon^\bullet)]$ and $\overline{C^\bullet(\nu)}=\nu[\log(1+\nu^\bullet)]$ for all $D(\upsilon)$ and $C(\nu)$ which satisfy all the conditions stated earlier. In this case, one can also derive a closed form expression for the associated CDF of the Biv-TIIQL-FGM (Type-**III**) from $\mathbf{H}_\vartheta(\upsilon,\nu)=\upsilon\nu\left(1+\vartheta\overline{D^\bullet(\upsilon)C^\bullet(\nu)}\right)$. Then

$$\mathbf{H}_{\vartheta}(v,\nu)=\mathbf{L}_{\alpha_1,\underline{\xi}}(v)\mathbf{L}_{\alpha_2,\underline{\xi}}(\nu)\exp\left[\overline{\mathbf{L}}_{\alpha_1,\underline{\xi}}(v)+\overline{\mathbf{L}}_{\alpha_2,\underline{\xi}}(\nu)\right]\left[1+\vartheta\begin{pmatrix}\mathbf{L}_{\alpha_1,\underline{\xi}}(v)\exp\left[\overline{\mathbf{L}}_{\alpha_1,\underline{\xi}}(v)\right]\mathbf{L}_{\alpha_2,\underline{\xi}}(\nu)\exp\left[\overline{\mathbf{L}}_{\alpha_2,\underline{\xi}}(\nu)\right]\\ \times\left[\log\left(2-\mathbf{L}_{\alpha_1,\underline{\xi}}(v)\exp\left[\overline{\mathbf{L}}_{\alpha_1,\underline{\xi}}(v)\right]\right)\right]\\ \times\left[\log\left(2-\mathbf{L}_{\alpha_2,\underline{\xi}}(\nu)\exp\left[\overline{\mathbf{L}}_{\alpha_2,\underline{\xi}}(\nu)\right]\right)\right]\end{pmatrix}\right].$$

3.3 Via Clayton copula

The Clayton copula can be considered as $\mathbf{H}(\nu_1,\nu_2)=\left[(1/\nu_1)^{\vartheta}+(1/\nu_2)^{\vartheta}-1\right]^{-\vartheta^{-1}}|_{\vartheta\in(0,\infty)}$. Setting $\nu_1=F_{\alpha_1,\underline{\xi}}(v)$ and $\nu_2=F_{\alpha_2,\underline{\xi}}(x)$, the Biv-TIIQL type can be derived from $\mathbf{H}(\nu_1,\nu_2)=\mathbf{H}(F_{\alpha_1,\underline{\xi}}(\nu_1),F_{\alpha_1,\underline{\xi}}(\nu_2))$. Then

$$\mathbf{H}(\nu_1,\nu_2)=\begin{Bmatrix}\mathbf{L}^{-\vartheta}_{\alpha_1,\underline{\xi}}(\nu_1)\exp\left[-\vartheta\overline{\mathbf{L}}_{\alpha_1,\underline{\xi}}(\nu_1)\right]\\ +\mathbf{L}^{-\vartheta}_{\alpha_2,\underline{\xi}}(\nu_2)\exp\left[-\vartheta\overline{\mathbf{L}}_{\alpha_2,\underline{\xi}}(\nu_2)\right]-1\end{Bmatrix}^{-\vartheta^{-1}}|_{\vartheta\in(0,\infty)}$$

Similarly, the M-TIIQL can be derived from $\mathbf{H}(\nu_k)=\left(\sum_{k=1}^{d}\nu_k^{-\vartheta}+1-d\right)^{-\vartheta^{-1}}$.

3.4 Via Renyi's entropy copula

Using the theorem of Pougaza and Djafari (2011) where

$$\mathbf{H}(v,\nu)=x_2 v+x_1\nu-x_1x_2,$$

the associated Biv-TIIQL can be derived from

$$\mathbf{H}(v,\nu)=x_2\mathbf{L}_{\alpha_1,\underline{\xi}}(x_1)\exp\left[\overline{\mathbf{L}}_{\alpha_1,\underline{\xi}}(x_1)\right]+x_1\mathbf{L}_{\alpha_2,\underline{\xi}}(x_2)\exp\left[\overline{\mathbf{L}}_{\alpha_2,\underline{\xi}}(x_2)\right]-x_1x_2.$$

3.5 Via Ali–Mikhail–Haq copula

Under the stronger Lipschitz condition, the Archimedean Ali–Mikhail–Haq copula can expressed as

$$\mathbf{H}(v,\nu)=v\nu[1-\vartheta v^{\bullet}\nu^{\bullet}]^{-1}|_{\boldsymbol{\tau}\in(-1,1)},$$

then for any $v^{\bullet}=1-F_{\alpha_1,\underline{\xi}}(h_1)|_{[v^{\bullet}=(1-v)\in(0,1)]}$ and $\nu^{\bullet}=1-F_{\alpha_2,\underline{\xi}}(h_2)|_{[\nu^{\bullet}=(1-\nu)\in(0,1)]}$ we have

$$\mathbf{H}(h_1,h_2)=\frac{\mathbf{L}_{\alpha_1,\underline{\xi}}(h_1)\mathbf{L}_{\alpha_2,\underline{\xi}}(h_2)\exp\left[\overline{\mathbf{L}}_{\alpha_1,\underline{\xi}}(h_1)+\overline{\mathbf{L}}_{\alpha_2,\underline{\xi}}(h_2)\right]}{1-\vartheta\left\{1-\mathbf{L}_{\alpha_1,\underline{\xi}}(h_1)\exp\left[\overline{\mathbf{L}}_{\alpha_1,\underline{\xi}}(h_1)\right]\right\}\left\{1-\mathbf{L}_{\alpha_2,\underline{\xi}}(h_2)\exp\left[\overline{\mathbf{L}}_{\alpha_2,\underline{\xi}}(h_2)\right]\right\}}$$

4. Characterizations of the TIIQL Distribution

To understand the behavior of the data obtained through a given process, we need to be able to describe this behavior via its approximate probability law. This, however, requires to establish conditions which govern the required probability law. In other words we need to have certain conditions under which we may be able to recover the probability law of the data. So, characterization of a distribution is important in applied sciences, where an investigator is vitally interested to find out if their model follows the selected distribution. Therefore, the investigator relies on conditions under which their model would follow a specified distribution. A probability distribution can be characterized in different directions one of which is based on the truncated moments. This type of characterization initiated by Galambos and Kotz (1978) and followed by other authors such as Kotz and Shanbhag (1980), Glanzel et al. (1984), Glannzel (1987), Glanzel and Hamedani (2001) and Kim and Jeon (2013), to name a few. For example, Kim and Jeon (2013) proposed a credibility theory based on the truncation of the loss data to estimate conditional mean loss for a given risk function. It should also be mentioned that characterization results are mathematically challenging and elegant. In this section, we present two characterizations of the TIIQL distribution based on: (i) conditional expectation (truncated moment) of certain function of a random variable and (ii) the reversed hazard function.

4.1 Characterizations based on two truncated moments

This subsection deals with the characterizations of TIIQL distribution in terms of a simple relationship between two truncated moments. We will employ Theorem 1 of Glanzel (1987) given in the Appendix A. As shown in Glanzel (1990), this characterization is stable in the sense of weak convergence.

Proposition 4.1.1. Let $X:\Omega\to\mathbb{R}$ be a continuous random variable and let $q_1(x)=\exp\left[\overline{\mathbf{L}}_{\alpha,\underline{\xi}}(x)\right]\left[1-\mathbf{\Pi}^{\alpha}_{\underline{\xi}}(x)\right]^{-1}$ and $q_2(x)=q_1(x)\mathbf{\Pi}^{-2\alpha}_{\underline{\xi}}(x)$ for $x\in\mathbb{R}$. Then X has PDF (2) if and only if the function $\eta(x)$ defined in Theorem 1 is of the form

$$\eta(x) = \frac{1}{2}\left\{\mathbf{\Pi}_{\underline{\xi}}^{-2\alpha}(x) + 1\right\}, \quad x \in \mathbb{R}.$$

Proof. If X has PDF (2), then

$$\left[1 - F_{\alpha,\underline{\xi}}(x)\right] E[q_1(X)|X \geq x] = 2\left\{\mathbf{\Pi}_{\underline{\xi}}^{-2\alpha}(x) - 1\right\}, \; x \in \mathbb{R},$$

and

$$\left[1 - F_{\alpha,\underline{\xi}}(x)\right] E[q_2(X)|X \geq x] = \left\{\mathbf{\Pi}_{\underline{\xi}}^{-4\alpha}(x) - 1\right\}, \quad x \in \mathbb{R},$$

and hence

$$\eta(x) = \frac{1}{2}\left[\mathbf{\Pi}_{\underline{\xi}}^{-2\alpha}(x) + 1\right], \quad x \in \mathbb{R}.$$

We also have

$$\eta(x) q_1(x) - q_2(x) = \frac{1}{2} q_1(x)\left\{1 - \mathbf{\Pi}_{\underline{\xi}}^{-2\alpha}(x)\right\} < 0, \quad \text{for } x \in \mathbb{R}.$$

Conversely, if ξ is of the above form, then

$$s'(x) = \frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \frac{\alpha \pi_{\underline{\xi}}(x) \mathbf{\Pi}_{\underline{\xi}}^{-2\alpha-1}(x)}{\mathbf{\Pi}_{\underline{\xi}}^{-2\alpha}(x) - 1}, \quad x \in \mathbb{R},$$

and

$$s(x) = -\frac{1}{2}\log\left(\mathbf{\Pi}_{\underline{\xi}}^{-2\alpha}(x) - 1\right).$$

Now, according to Theorem 1, X has density (2).

Corollary 4.1.1. Suppose X is a continuous random variable. Let $q_1(x)$ be as in Proposition 4.1.1. Then X has density (2) if and only if there exist functions $q_2(x)$ and $\eta(x)$ defined in Theorem 1 for which the following first order differential equation holds

$$\frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \frac{\alpha \pi_{\underline{\xi}}(x) \mathbf{\Pi}_{\underline{\xi}}^{-2\alpha-1}(x)}{\mathbf{\Pi}_{\underline{\xi}}^{-2\alpha}(x) - 1}, \quad x \in \mathbb{R}.$$

Corollary 4.1.2. The differential equation in Corollary 4.1.1 has the following general solution

$$\eta(x) = \left\{\mathbf{\Pi}_{\underline{\xi}}^{-2\alpha}(x) - 1\right\}^{-1}\left[-\int \alpha \pi_{\underline{\xi}}(x) \mathbf{\Pi}_{\underline{\xi}}^{-2\alpha-1}(x)\left(q_1(x)\right)^{-1} q_2(x) + D\right],$$

where D is a constant. A set of functions satisfying the above differential equation is given in Proposition 4.1.1 with $D = \frac{1}{2}$. Clearly, there are other triplets $\left(q_1(x), q_2, \eta(x)\right)$ satisfying the conditions of Theorem 1.

4.2 Characterization based on reverse hazard function

The reverse hazard function, r_F, of a twice differentiable distribution function, F , is defined as

$$r_F(x) = \frac{f_{\alpha,\underline{\xi}}(x)}{F_{\alpha,\underline{\xi}}(x)}, \quad x \in \text{support of} F.$$

In this subsection we present a characterization of the TIIQL which is not of the above trivial form.

Proposition 4.2.1. Suppose X is a continuous random variable. Then, X has density (2) if and only if its hazard function $r_F(x)$ satisfies the following first order differential equation

$$r_F'(x) - \frac{\pi_{\underline{\xi}}'(x)}{\pi_{\underline{\xi}}(x)} r_F(x) = 4\alpha \pi_{\underline{\xi}}(x) \frac{d}{dx}\left\{\frac{1 - \mathbf{\Pi}_{\underline{\xi}}^{\alpha}(x)}{\mathbf{\Pi}_{\underline{\xi}}^{\alpha}(x)\left[2 - \mathbf{\Pi}_{\underline{\xi}}^{\alpha}(x)\right]}\right\}, x \in \mathbb{R}.$$

Proof. Is straightforward and hence omitted.

5. Two Special TIIQL-G distributions

5.1 TIIQL-Lindley distribution

Firstly, the Lindley (L) distribution (Lindley (1958)) has been taken as baseline distribution. It is well known that the L distribution has unimodal or decreasing PDF shapes as well as it has only increasing HRF shape. To extend the shape properties of the L distribution, we define the TIIQL-Lindley (TIIQL-L) distribution. Taking

$$G(x,\beta) = 1 - \left(1 + \frac{\beta x}{1+\beta}\right)\exp(-\beta x) \text{ and } g(x,\beta) = \frac{\beta^2}{1+\beta}(1+x)\exp(-\beta x), x > 0, \beta > 0,$$

for the CDF and PDF of the L distribution, The PDF and CDF of the new distribution are respectively given by

$$f_{\alpha,\beta}(x) = \frac{4\alpha\beta^2(1+x)e^{2-\beta x}}{(1+\beta)\left[1 - \left(1 + \frac{\beta x}{1+\beta}\right)\exp(-\beta x)\right]^{\alpha+1}}$$
$$\times \left\{\left[1 - \left(1 + \frac{\beta x}{1+\beta}\right)\exp(-\beta x)\right]^{-\alpha} - 1\right\}\exp\left\{-2\left[1 - \left(1 + \frac{\beta x}{1+\beta}\right)\exp(-\beta x)\right]^{\alpha}\right\},$$

and

$$F_{\alpha,\beta}(x) = \left(2\left[1 - \left(1 + \frac{\beta x}{1+\beta}\right)\exp(-\beta x)\right]^{-\alpha} - 1\right)\exp\left\{-2\left[1 - \left(1 + \frac{\beta x}{1+\beta}\right)\exp(-\beta x)\right]^{\alpha}\right\},$$

where, $0 < x,\ \alpha, \beta > 0$. We denote with TIIQL-L$(\alpha,\beta)$. Plots of the TIIQL-L density and hazard functions for selected parameter values are displayed in Figure 1. These plots indicate that the L distribution gains excellent shape properties via the proposed family. For instance, the PDF of the TIIQL-L model have unimodal and decreasing shapes with skewed and high kurtosis. Further, the its HRF can be unimodal, increasing or decreasing.

5.2 TIIQL-Weibull distribution

Secondly, we consider the Weibull (W) distribution with CDF and PDF are

$$G(x;\beta,\theta) = 1 - \exp\left[-(\theta x)^{\beta}\right] \text{ and } g(x,\beta,\theta) = \beta\theta^{\beta}x^{\beta-1}\exp\left[-(\theta x)^{\beta}\right],\ x > 0,\ \beta, \alpha > 0,$$

respectively. The W distribution has decreasing or unimodal PDF shapes as well as its HRF has the monotone shapes. To extend its modeling ability, we propose the TIIQL-Weibull (TIIQL-W) distribution with following PDF and CDF

$$f_{\alpha,\beta,\theta}(x) = \frac{4\alpha\beta\theta^{\beta}x^{\beta-1}\exp\left[-(\theta x)^{\beta}\right]}{\{1-\exp[-(\theta x)^{\beta}]\}^{\alpha+1}}\left(\frac{1-\{1-\exp[-(\theta x)^{\beta}]\}^{\alpha}}{\{1-\exp[-(\theta x)^{\beta}]\}^{\alpha}}\right)\exp\left(\frac{2\{1-\exp[-(\theta x)^{\beta}]\}^{\alpha}-2}{(1-\exp[-(\theta x)^{\beta}])^{\alpha}}\right)$$

and

$$F_{\alpha,\beta,\theta}(x) = \left(2\{1-\exp[-(\theta x)^{\beta}]\}^{-\alpha} - 1\right)\exp\left(\frac{2\{1-\exp[-(\theta x)^{\beta}]\}^{\alpha}-2}{(1-\exp[-(\theta x)^{\beta}])^{\alpha}}\right)$$

respectively, where $0 < x,\ \alpha, \beta, \theta > 0$. We denote with TIIQL-W$(\alpha,\beta,\theta)$. For $\beta = 1$ and $\beta = 2$, we obtain the TIIQL-exponential and TIIQL-Rayleigh distributions respectively. Plots of the PDF and HRF of the TIIQL-W model for selected parameter values are displayed in Figure 2. These plots reveal that the proposed density can be unimodal, decreasing and skewed shapes. Also, the HRF can be monotonically increasing or decreasing, bathtub shaped and upside-down bathtub shaped depending basically on the parameter values.

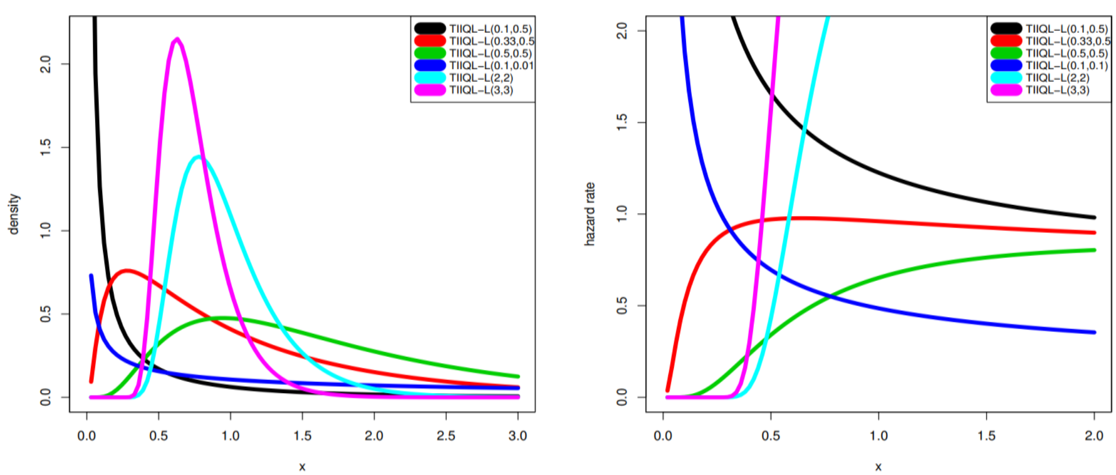

Figure 1: The possible pdf and hrf shapes of the TIIQL-L distribution

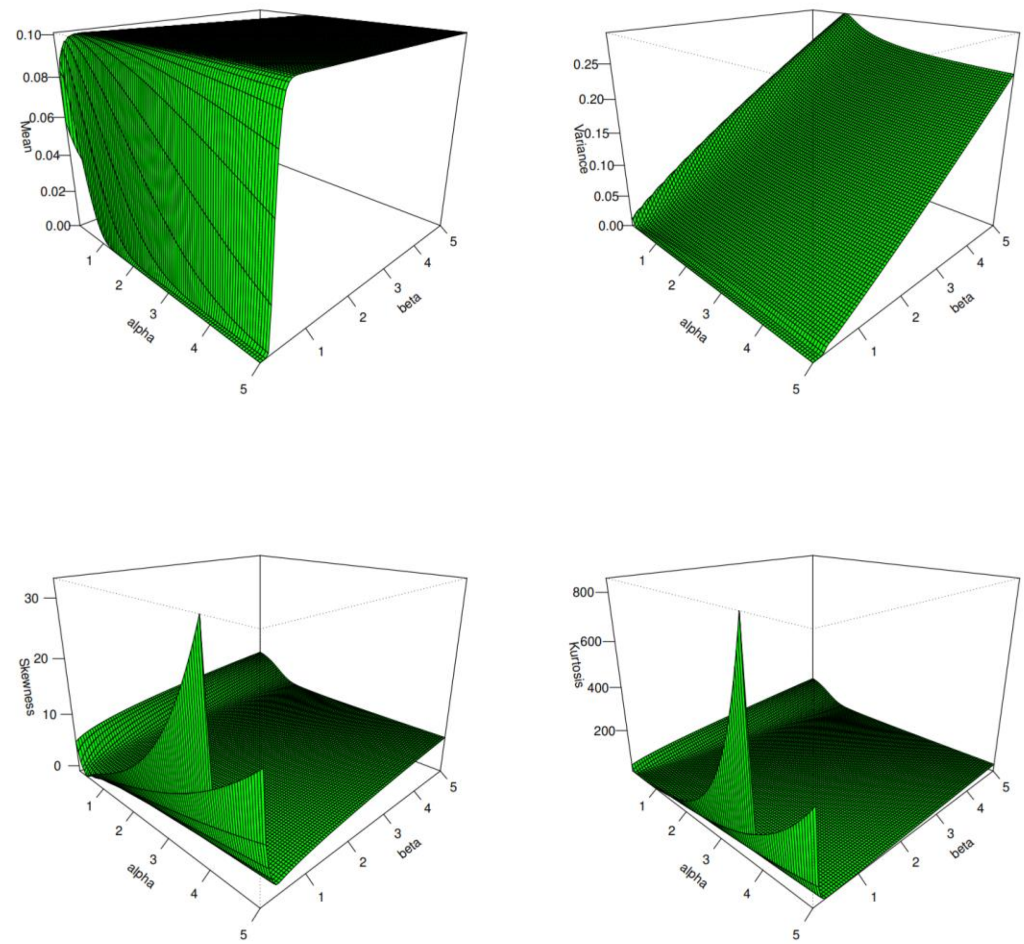

Figure 2: The mean, variance, skewness and kurtosis plots of the TIIQL-L distribution

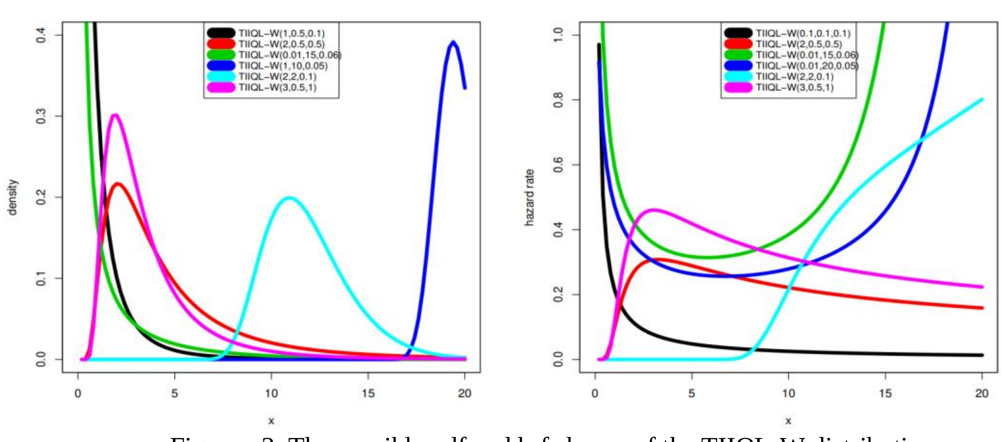

Figure 3: The possible pdf and hrf shapes of the TIIQL-W distribution

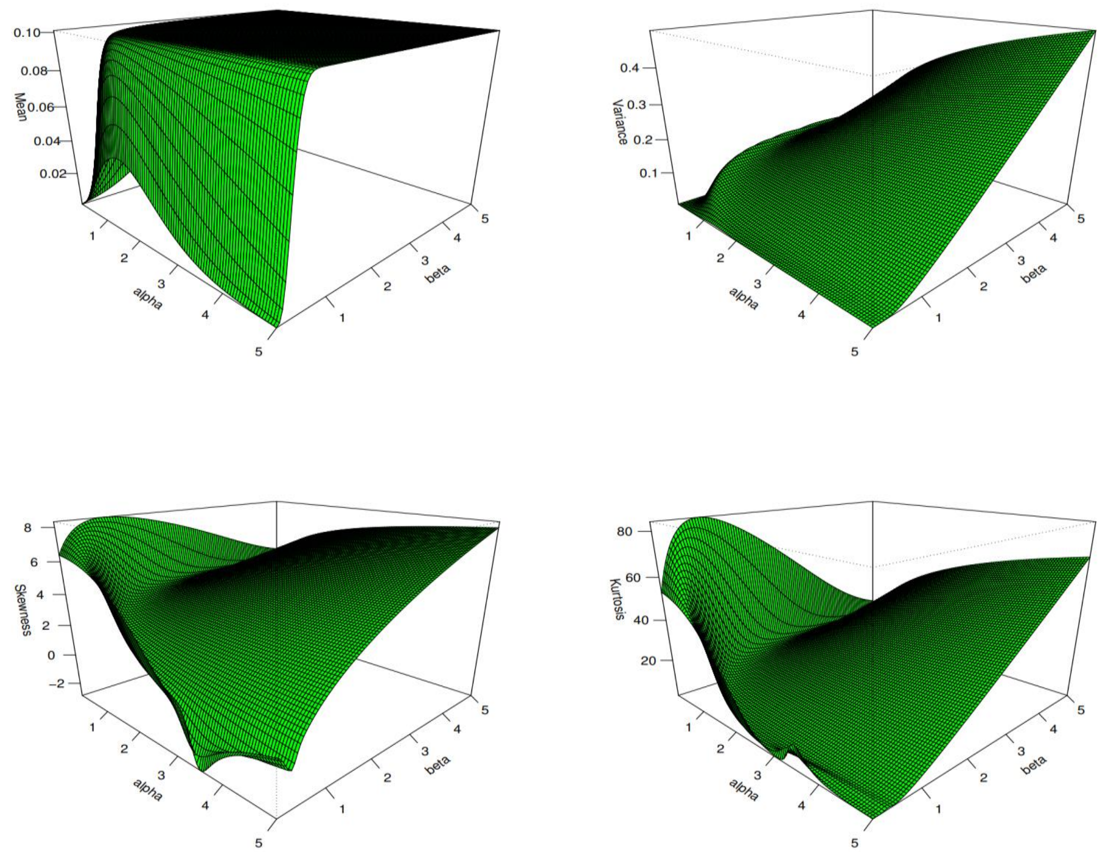

Figure 4: The mean, variance, skewness and kurtosis plots of the TIIQL-W distribution

6. Different methods of the estimation of parameters

This section is devoted to six estimations methods of the parameters of the TIIQL-G distribution. The details are given below.

6.1 Maximum likelihood estimation

Let $X_1, X_2, \dots, X_n$ be a random sample from the TIIQL-G distribution with observed values $x_1, x_2, \dots, x_n$, and $\underline{\mathbf{\Phi}} = (\alpha, \underline{\boldsymbol{\xi}})$ Then, the log-likelihood function is given by

$$\ell(\underline{\mathbf{\Phi}}) = 2n\log2 + n\log\alpha + \sum_{i=1}^{n} \log\boldsymbol{\pi}_{\underline{\boldsymbol{\xi}}}(x_i)(x_i) - (2\alpha+1)\sum_{i=1}^{n} \log\mathbf{\Pi}_{\underline{\boldsymbol{\xi}}}(x_i)$$
$$+ \sum_{i=1}^{n} \log\left[1 - \mathbf{\Pi}_{\underline{\boldsymbol{\xi}}}^{\alpha}(x_i)\right] + \sum_{i=1}^{n} \frac{2-\mathbf{\Pi}_{\underline{\boldsymbol{\xi}}}^{\alpha}(x_i)}{\mathbf{\Pi}_{\underline{\boldsymbol{\xi}}}^{\alpha}(x_i)}, \tag{13}$$

Then, the ML estimates (MLEs) of α and $\underline{\boldsymbol{\xi}}$, say $\hat{\alpha}$ and $\underline{\boldsymbol{\xi}}$, are obtained by maximizing $\ell(\underline{\mathbf{\Phi}})$ with respect to $\underline{\mathbf{\Phi}}$. Mathematically, this is equivalent to solving the following non-linear equation with respect to the parameters: $\frac{\partial}{\partial\alpha}\ell(\underline{\mathbf{\Phi}}) = 0$ and $\frac{\partial}{\partial\underline{\boldsymbol{\xi}}}\ell(\underline{\mathbf{\Phi}}) = 0$. Hence, the numerical methods are needed to obtain the MLEs. Under mild regularity conditions, one can use the multivariate normal distribution with mean $\mu = (\alpha, \underline{\boldsymbol{\xi}})$ and covariance matrix $\mathbf{I}^{-1}$, where $\mathbf{I}$ denotes the following $(p+1)\times(p+1)$ observed information matrix of real numbers to construct confidence intervals or likelihood ratio test on the parameters. The components of $\mathbf{I}$ can be requested from the authors when it is needed. Then, approximate $100(1-\upsilon)\%$ confidence intervals for α and $\underline{\boldsymbol{\xi}}$ can be determined by: $\hat{\alpha} \pm z_{\upsilon/2} s_{\hat{\alpha}}$ and

$\underline{\boldsymbol{\xi}} \pm z_{\upsilon/2} s_{\underline{\boldsymbol{\xi}}}$ where $z_{\upsilon/2}$ is the upper $(\upsilon/2)^{th}$ quantile of the standard normal distribution, $s_{\hat{\alpha}}$ is the first diagonal element of , where $\mathbf{I}^{-1}$ denotes the following and $s_{\underline{\boldsymbol{\xi}}}$ is its second diagonal element.

6.2 Maximum product spacing estimation

The maximum product spacing (MPS) method has been introduced by Cheng and Amin (1979). It is based on the idea that differences (spacings) between the values of the CDF at consecutive data points should be identically distributed. Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be the ordered statistics from the TIIQL-G distribution with sample size n, and $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ be the ordered observed values. Then, we define the MPS function by

$$MPS(\underline{\boldsymbol{\Phi}}) = \frac{1}{n+1}\sum_{i=1}^{n+1} \log\left[F(x_{(i)}, \underline{\boldsymbol{\Phi}}) - F(x_{(i-1)}, \underline{\boldsymbol{\Phi}})\right],$$

where $F(x, \underline{\boldsymbol{\Phi}}) = F_{\alpha, \underline{\boldsymbol{\xi}}}(x)$. The MPS estimates (MPSEs), say $\hat{\alpha}_{MPS}$ and $\underline{\boldsymbol{\xi}}_{MPS}$, can be obtained by minimizing $MPS(\underline{\boldsymbol{\Phi}})$ with respect to $\underline{\boldsymbol{\Phi}}$. They are also given as the simultaneous solution of the following non-linear equations:

$$\frac{\partial MPS(\underline{\boldsymbol{\Phi}})}{\partial \alpha} = \frac{1}{n+1}\sum_{i=1}^{n+1} \left[\frac{F_{\alpha}'(x_{(i)}, \underline{\boldsymbol{\Phi}}) - F_{\alpha}'(x_{(i-1)}, \underline{\boldsymbol{\Phi}})}{F(x_{(i)}, \underline{\boldsymbol{\Phi}}) - F(x_{(i-1)}, \underline{\boldsymbol{\Phi}})}\right] = 0$$

and

$$\frac{\partial MPS(\underline{\boldsymbol{\Phi}})}{\partial \underline{\boldsymbol{\xi}}} = \frac{1}{n+1}\sum_{i=1}^{n+1} \left[\frac{F_{\underline{\boldsymbol{\xi}}}'(x_{(i)}, \underline{\boldsymbol{\Phi}}) - F_{\underline{\boldsymbol{\xi}}}'(x_{(i-1)}, \underline{\boldsymbol{\Phi}})}{F(x_{(i)}, \underline{\boldsymbol{\Phi}}) - F(x_{(i-1)}, \underline{\boldsymbol{\Phi}})}\right] = 0,$$

where $F_{\alpha}'(x, \underline{\boldsymbol{\Phi}}) = \frac{\partial}{\partial \alpha} F(x, \underline{\boldsymbol{\Phi}})$ and $F_{\underline{\boldsymbol{\xi}}}'(x, \underline{\boldsymbol{\Phi}}) = \frac{\partial}{\partial \underline{\boldsymbol{\xi}}} F(x, \underline{\boldsymbol{\Phi}})$.

6.3 Least squares estimation

The least square estimates (LSEs) $\hat{\alpha}_{LSE}$ and $\underline{\boldsymbol{\xi}}_{LSE}$ of α and $\underline{\boldsymbol{\xi}}$, respectively, are obtained by minimizing the following function:

$$LSE(\underline{\boldsymbol{\Phi}}) = \sum_{i=1}^{n} \left(F(x_{(i)}, \underline{\boldsymbol{\Phi}}) - E\left[F(X_{(i)}, \underline{\boldsymbol{\Phi}})\right]\right)^2, \tag{14}$$

with respect to $\underline{\boldsymbol{\Phi}}$, where $E\left[F(X_{(i)}, \underline{\boldsymbol{\Phi}})\right] = i/(n+1)$ for $i = 1,2,\dots,n$. Then, $\hat{\alpha}_{LSE}$ and $\underline{\boldsymbol{\xi}}_{LSE}$ are solutions of the following equations:

$$\frac{\partial LSE(\underline{\boldsymbol{\Phi}})}{\partial \alpha} = 2\sum_{i=1}^{n} F_{\alpha}'(x_{(i)}, \underline{\boldsymbol{\Phi}}) \left(F(x_{(i)}, \underline{\boldsymbol{\Phi}}) - \frac{i}{n+1}\right) = 0,$$

and

$$\frac{\partial LSE(\underline{\boldsymbol{\Phi}})}{\partial \underline{\boldsymbol{\xi}}} = 2\sum_{i=1}^{n} F_{\underline{\boldsymbol{\xi}}}'(x_{(i)}, \underline{\boldsymbol{\Phi}}) \left(F\left(x_{(i)}, \underline{\boldsymbol{\Phi}}\right) - \frac{i}{n+1}\right) = 0,$$

respectively, where $F_{\alpha}'(x_{(i)}, \underline{\boldsymbol{\Phi}})$ and $F_{\underline{\boldsymbol{\xi}}}'(x_{(i)}, \underline{\boldsymbol{\Phi}})$ are mentioned before.

6.4 Weighted least squares estimation

This estimation method is a generalization of the LSE method with a weighted function. The weighted least square estimates (WLSEs) $\hat{\alpha}_{WLSE}$ and $\underline{\boldsymbol{\xi}}_{WLSE}$ of α and $\underline{\boldsymbol{\xi}}$, are obtained by minimizing the following function:

$$WLSE(\underline{\boldsymbol{\Phi}}) = \sum_{i=1}^{n} \frac{1}{V\left[F(X_{(i)}, \underline{\boldsymbol{\Phi}})\right]} \left\{F(x_{(i)}, \alpha, \beta) - E\left[F(X_{(i)}, \underline{\boldsymbol{\Phi}})\right]\right\}^2, \tag{15}$$

where, $E\left[F(X_{(i)}, \underline{\boldsymbol{\Phi}})\right] = i/(n+1)$ and $V\left[F(X_{(i)}, \underline{\boldsymbol{\Phi}})\right] = i(n-i+1)/[(n+2)(n+1)^2]$ for $i = 1,2,\dots,n$. Then, $\hat{\alpha}_{WLSE}$ and $\underline{\boldsymbol{\xi}}_{WLSE}$ are solutions of the following equations:

$$\frac{\partial WLSE(\underline{\boldsymbol{\Phi}})=}{\partial \alpha} = \sum_{i=1}^{n} \frac{(n+2)(n+1)^2}{i(n-i+1)} \frac{\partial LSE(\underline{\boldsymbol{\Phi}})}{\partial \alpha} = 0$$

and

$$\frac{\partial WLSE(\underline{\boldsymbol{\Phi}})}{\partial \underline{\boldsymbol{\xi}}} = \sum_{i=1}^{n} \frac{(n+2)(n+1)^2}{i(n-i+1)} \frac{\partial LSE(\underline{\boldsymbol{\Phi}})}{\partial \underline{\boldsymbol{\xi}}} = 0.$$

6.5 Anderson-Darling estimation

The Anderson-Darling minimum distance estimates (ADEs) $\hat{\alpha}_{AD}$ and $\underline{\boldsymbol{\xi}}_{AD}$ of α and $\underline{\boldsymbol{\xi}}$, respectively, are obtained by minimizing the following function:

$$AD\left(\underline{\boldsymbol{\Phi}}\right) = -n - \sum_{i=1}^{n} \frac{2i-1}{n}\left[\log F(x_{(i)}, \underline{\boldsymbol{\Phi}}) + \log\left\{1 - F(x_{(n+1-i)}, \underline{\boldsymbol{\Phi}})\right\}\right], \tag{16}$$

with respect to $\underline{\mathbf{\Phi}}$. Therefore, $\hat{\alpha}_{AD}$ and $\underline{\boldsymbol{\xi}}_{AD}$ can be obtained as the solutions of the following system of equations:

$$\frac{\partial AD(\underline{\mathbf{\Phi}})}{\partial \alpha} = -\sum_{i=1}^{n} \frac{2i-1}{n} \left[\frac{F'_{\alpha}(x_{(i)},\underline{\mathbf{\Phi}})}{F(x_{(i)}\underline{\mathbf{\Phi}})} - \frac{F'_{\alpha}(x_{(n+1-i)},\underline{\mathbf{\Phi}})}{1-F(x_{(n+1-i)},\underline{\mathbf{\Phi}})} \right] = 0$$

and

$$\frac{\partial AD(\underline{\mathbf{\Phi}})}{\partial \underline{\boldsymbol{\xi}}} = -\sum_{i=1}^{n} \frac{2i-1}{n} \left[\frac{F'_{\underline{\mathbf{\Psi}}}(x_{(i)},\underline{\mathbf{\Phi}})}{F(x_{(i)},\underline{\mathbf{\Phi}})} - \frac{F'_{\underline{\mathbf{\Psi}}}(x_{(n+1-i)},\underline{\mathbf{\Phi}})}{1-F(x_{(n+1-i)},\underline{\mathbf{\Phi}})} \right] = 0.$$

6.6 The Cramer-von Mises estimation

The Cramer-von Mises minimum distance estimates (CVMEs) $\hat{\alpha}_{CVM}$ and $\underline{\boldsymbol{\xi}}_{CVM}$ of α and $\underline{\boldsymbol{\xi}}$, respectively, are obtained by minimizing the following function:

$$CVM(\underline{\mathbf{\Phi}}) = \frac{1}{12n} + \sum_{i=1}^{n} \left[F(x_{(i)},\underline{\mathbf{\Phi}}) - \frac{2i-1}{2n} \right]^2, \tag{17}$$

with respect to $\underline{\mathbf{\Phi}}$. Therefore, the estimates $\hat{\alpha}_{CVM}$ and $\underline{\boldsymbol{\xi}}_{CVM}$ can be obtained as the solution of the following system of equations:

$$\frac{\partial CVM(\underline{\mathbf{\Phi}})}{\partial \alpha} = 2\sum_{i=1}^{n} \left(F(x_{(i)},\underline{\mathbf{\Phi}}) - \frac{2i-1}{2n} \right) F'_{\alpha}(x_{(i)},\underline{\mathbf{\Phi}}) = 0$$

and

$$\frac{\partial CVM(\underline{\mathbf{\Phi}})}{\partial \underline{\boldsymbol{\xi}}} = 2\sum_{i=1}^{n} \left(F(x_{(i)},\underline{\mathbf{\Phi}}) - \frac{2i-1}{2n} \right) F'_{\underline{\boldsymbol{\xi}}}(x_{(i)},\underline{\mathbf{\Phi}}) = 0.$$

Since all estimating equations contain non-linear functions, it is not possible to obtain explicit forms of all estimators directly. Therefore, they have to be solved via numerical methods such as the Newton-Raphson and quasi-Newton algorithms. In addition, the Equations (13), (14), (15), (16) and (17) can also be optimized directly by using the software such as R (constrOptim, optim and maxLik functions), S-Plus and Mathematica to numerically optimize the $\ell(\underline{\mathbf{\Phi}})$, $MPS(\underline{\mathbf{\Phi}})$, LSE$(\underline{\mathbf{\Phi}})$, WLSE$(\underline{\mathbf{\Phi}})$, $AD(\underline{\mathbf{\Phi}})$ and $CVM(\underline{\mathbf{\Phi}})$ functions.

7. Simulation studies for the parameters estimations

In this section, two graphical simulation studies have been pointed out to assess the performance of the above estimators of the special member of the new family with respect to varying sample size n. We take Lindley distribution (Lindley, 1958) as the baseline model. Hence, the CDF of the TIIQL-Lindley (TIIQL-L) distribution is given by

$$\begin{aligned} f_{\alpha,\beta}(x) = & \frac{4\alpha\beta^2(1+x)e^{2-\beta x}}{(1+\beta)\left[1-\left(1+\frac{\beta x}{1+\beta}\right)\exp(-\beta x)\right]^{\alpha+1}} \\ & \times \left\{ \left[1-\left(1+\frac{\beta x}{1+\beta}\right)\exp(-\beta x)\right]^{-\alpha} - 1 \right\} \\ & \times \exp\left\{ -2\left[1-\left(1+\frac{\beta x}{1+\beta}\right)\exp(-\beta x)\right]^{\alpha} \right\}, \end{aligned}$$

where, $0 < x$, $\alpha,\beta > 0$. We generate $N = 1000$ samples of size $n = 20,25,\dots,1000$ from the TIIQL-L distribution based on the actual parameter values. For the case-I and case-II, the $\alpha = 0.5$, $\beta = 0.5$ and $\alpha = 2$, $\beta = 2$ pairs have been taken as actual parameters values respectively for simulation studies. The random numbers generation is obtained by the solution of the its CDF via uniroot function in R software as well as all the estimations based on the estimation methods have been obtained by using the constrOptim function in the same software. Further, we calculate the empirical mean, bias and mean square error (MSE) of the estimators for comparisons of the estimation methods. The bias and MSE are calculated by (for $h = \alpha,\beta$)

$$Bias_h(n) = \frac{1}{N}\sum_{i=1}^{N} \left(h_i - \hat{h}_i\right),$$

and

$$MSE_h(n) = \frac{1}{N}\sum_{i=1}^{N} \left(h_i - \hat{h}_i\right)^2,$$

respectively. It is hoped that while the estimated means are close to the true values, the MSEs and biases approach 0. Figures 5 and 6 show results of the simulation studies.

For two cases, the amount of the biases and MSEs of the MPS and CVM methods are bigger than those of other methods. However, they approach each other when sample size increases. These Figures show that all the estimators are consistent since the MSE and biasedness decrease to zero with increasing sample size as expected. All estimators are asymptotically unbiased also. Therefore, all methods can be chosen as more reliable than another estimator of new distribution. Similar results can also be obtained for different parameter settings.

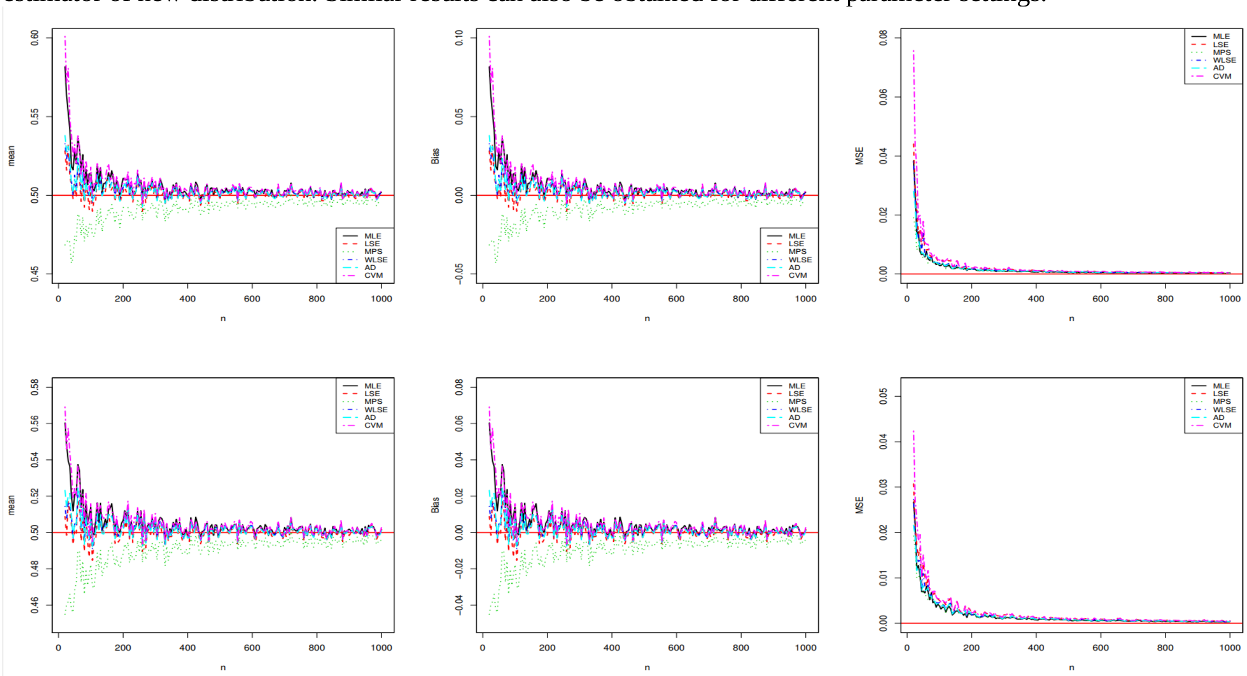

Figure 5: The simulation results of the case-I for the α (top) and β (bottom) parameters

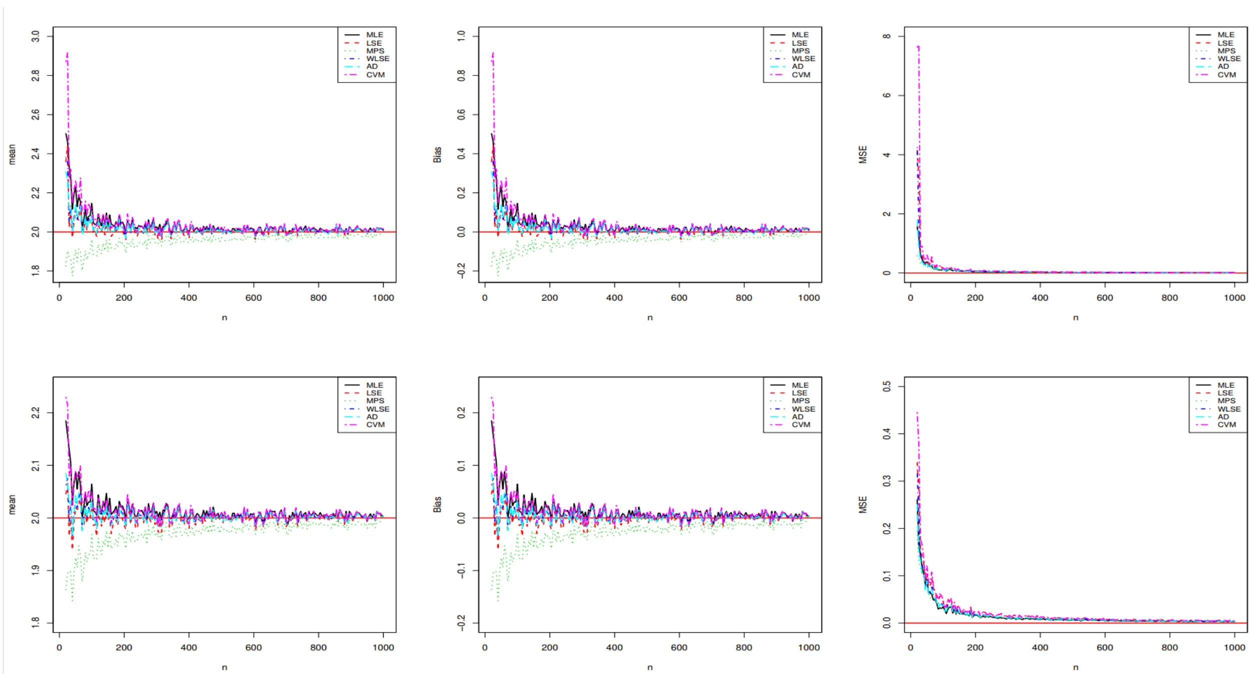

Figure 6: The simulation results of the case-II for the α (top) and β (bottom) parameters

In addition, based on the case-I and case-II simulation studies, the behaviors of the 95% confidence intervals of the MLEs. They are obtained via the empirical coverage length (CL) which is defined by

$$CL_h(n) = \frac{1}{N}\sum_{i=1}^{N} 3.9198 s_{\hat{h}_i},$$

where the $s_{\hat{h}_i} = \left(s_{\hat{\alpha}_i}, s_{\hat{\beta}_i}\right)$ are the standard errors of the MLEs which are calculated by the observed information matrix. Figure 7 displays the simulation results for the CLs. As seen from this Figure, both cases, the CLs approach

0 value when the sample size increases. The simulation results verify the consistency property of MLEs.

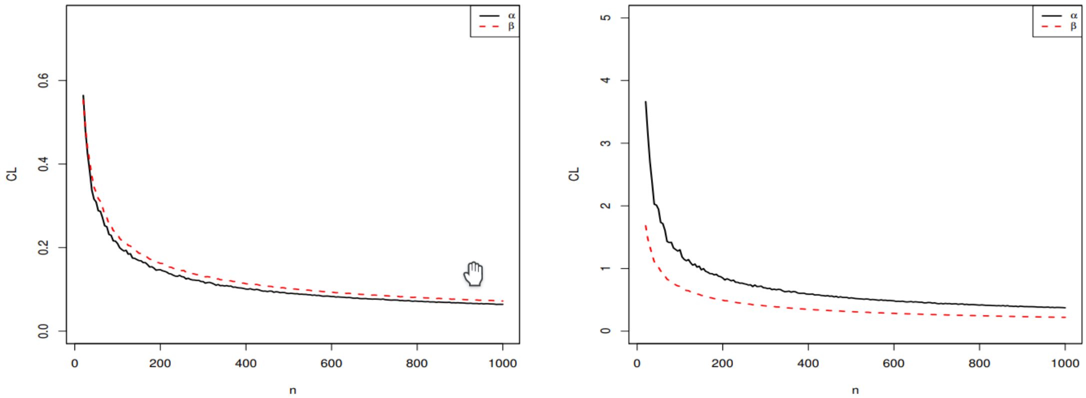

Figure 7: The empirical CLs of the case-I (left) and case-II (right)

8. Modeling data for comparing competitive models

In this section, two real data set are analyzed to prove the empirical importance and modeling ability of two special members of the TIIQL family. Based on the MLE method, we also compare these models with the well-known competitive models in the literature under the estimated log-likelihood values $\hat{\ell}$, Akaike Information Criteria (AIC), corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) and Kolmogorov-Smirnov (KS) goodness of-fit statistics for all models. We note that the AIC, CAIC, BIC and HQIC are given by $AIC = -2\hat{\ell} + 2p$, $CAIC = -2\hat{\ell} + 2pn(n-k-1)^{-1}$, $BIC = -2\hat{\ell} + p\log n$ and $HQIC = -2\hat{\ell} + p\log(\log n)$, where p is the number of the estimated model parameters and n is the sample size. Generally, it can be chosen as the proper model which has the smaller values of the AIC, CAIC, BIC, HQIC and KS statistics with larger values of $\hat{\ell}$. All computations are performed by the maxLike and goftest functions in the R software. The details are given below.

8.1 Data Set-I: The times between successive failures

The first data set is the times between successive failures (in thousands of hours) in events of secondary reactor pumps studied by Salman et al. (1999), Bebbington et al. (2007) and Lucena et al. (2015). The data are: 2.160, 0.746, 0.402, 0.954, 0.491, 6.560, 4.992, 0.347,0.150, 0.358, 0.101, 1.359, 3.465, 1.060, 0.614, 1.921,4.082 ,0.199 ,0.605 ,0.273 ,0.070, 0.062, 5.320. We compare performance of the real data fitting of the the TIIQL-L distribution under the MLE method with well know unit distribution in the literature. These competitor distributions are:

• Lindley (L) distribution:

$$f_\beta(x) = \frac{\beta^2}{1+\beta}(1+x)\exp(-\beta x),$$

where $0 < x$ and $\beta > 0$.

• Kumaraswamy Lindley (Kw-L) distribution (Cakmakyapan and Ozel (2014)):

$$f_{\alpha,\beta,\theta}(x) = \frac{\alpha\theta\beta^2}{1+\beta}(1+x)\exp(-\beta x)\left[1-\left(1+\frac{\beta x}{1+\beta}\right)\exp(-\beta x)\right]^{\alpha-1}\left\{1-\left[1-\left(1+\frac{\beta x}{1+\beta}\right)\exp(-\beta x)\right]^{\alpha}\right\}^{\theta-1},$$

where $0 < x$ and $\alpha, \beta, \theta > 0$.

• Beta Lindley (B-L) distribution (Merovci and Sharma (2014)):

$$f_{\alpha,\beta,\theta}(x) = \frac{\beta^2(1+\beta+\beta x)^{\theta-1}(1+x)e^{-\theta\beta x}}{\mathrm{B}(\alpha,\theta)(1+\beta)^\theta}\left[1-\left(1+\frac{\beta x}{1+\beta}\right)\exp(-\beta x)\right]^{\alpha-1},$$

where $0 < x$ and $\alpha, \beta, \theta > 0$ and $\mathrm{B}(\alpha,\theta)$ is the beta function.

• Odd log-logistic Lindley (OLL-L) distribution (Ozel et al. (2017)):

$$f_{\alpha,\beta}(x) = \frac{\alpha\beta^2(1+\beta+\beta x)^{\theta-1}(1+x)\exp(-\alpha\beta x)\left[1-\left(1+\frac{\beta x}{1+\beta}\right)\exp(-\beta x)\right]^{\alpha-1}}{(1+\beta)^\alpha\left\{\left[1-\left(1+\frac{\beta x}{1+\beta}\right)\exp(-\beta x)\right]^{\alpha}+\left[\left(1+\frac{\beta x}{1+\beta}\right)\exp(-\beta x)\right]^{\alpha}\right\}^2},$$

where $0 < x$ and $\alpha, \beta, \theta > 0$.

• Lindley-Lindley (L-L) distribution (Cakmakyapan and Ozel (2017)):

$$f_{\alpha,\beta}(x)=\frac{\alpha^2\beta^2(1+x)\exp(-\beta x)}{(1+\alpha)(1+\beta)}\left[\left(1+\frac{\beta x}{1+\beta}\right)\exp(-\beta x)\right]^{\alpha-1}\left\{1-\log\left[\left(1+\frac{\beta x}{1+\beta}\right)\exp(-\beta x)\right]\right\},$$

where $0<x$ and $\alpha,\beta>0$. The data analysis results are given by in Table 1. Table 1 indicates that the TIQL-L distribution has the lowest values of the K-S, AIC, CAIC, BIC and HQIC statistic with $\hat{\ell}$ and p-value among application models. It implies that the TIIQL-L model will be the best choice for the modeled data set.

Table 1: MLEs, standard errors of the estimates (in parentheses), $\hat{\ell}$ and goodness-of-fits statistics for the first data set (p value is given in $[\cdot]$)

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\ell}$	AIC	BIC	$CAIC$	$HQIC$	KS
TIIQL-L	0.2102	0.2527		-31.7099	67.4198	9.6907	68.0198	67.9909	0.1115
	(0.0365)	(0.0860)							[0.9070]
B-L	0.6388	1.2784	0.5256	-33.3950	72.7901	76.1966	74.0532	73.6468	0.1588
	(0.1919)	(1.6927)	(0.7505)						[0.5550]
Kw-L	0.6101	0.4545	1.6688	-33.4740	72.9479	76.3544	74.2111	73.8047	0.1502
	(0.1453)	(1.4025)	(5.6460)						[0.6240]
OLL-L	0.7091	1.0438		33.1707	70.3414	72.6124	70.9414	70.9125	0.1614
	(0.1283)	(0.2082)							[0.5338]
L-L	1.9567	0.6937		36.9406	77.8811	80.1521	78.4811	78.4523	0.2857
	(1.7112)	(0.5147)							[0.0373]
L		0.9575		-35.3054	72.6108	73.7463	72.8013	72.8963	0.2440
		(0.1505)							[0.1085]

Figure 8 displays the fitted PDFs and CDFs for all models. It is clear that the proposed TIQL-L model fits the data set graphically and its fitting is acceptable. Figure 9 shows that the probability-probability (PP) plot of the proposed model is closer to diagonal line than those of the other models.

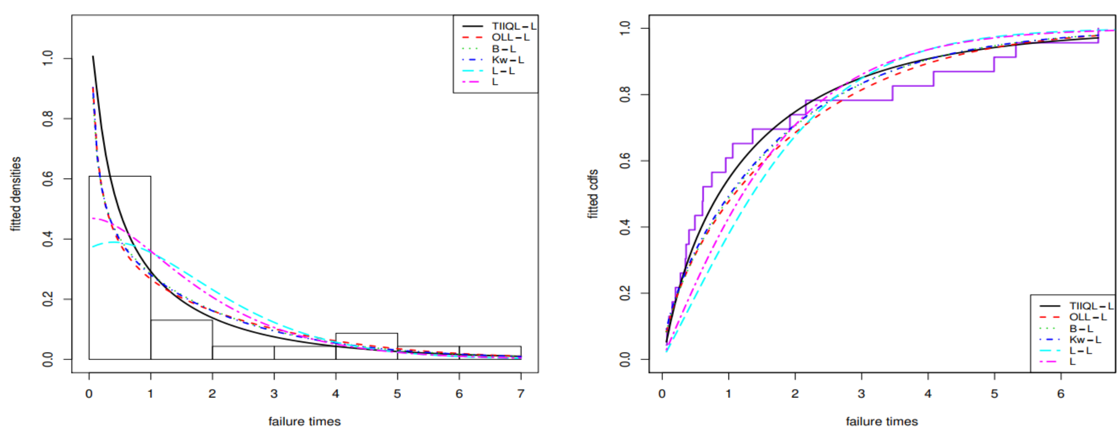

Figure 8: PP plots for the fitted models based on the first data set.

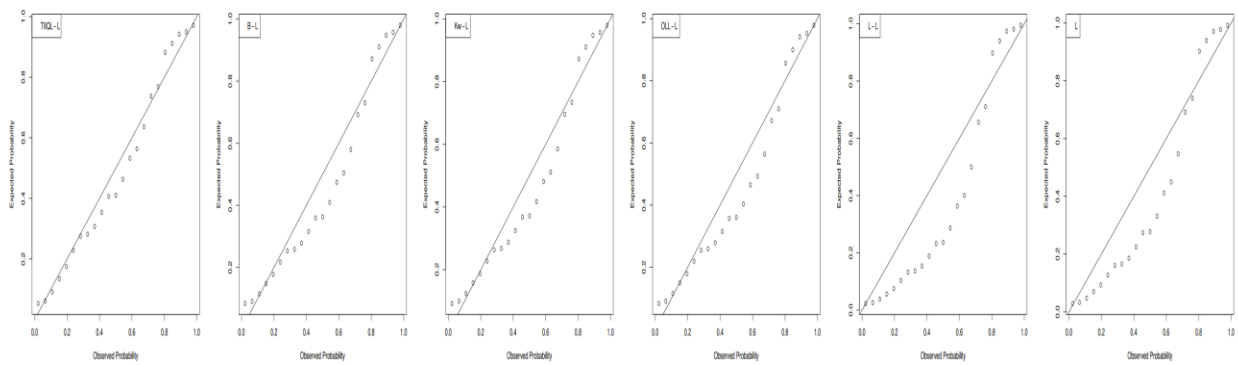

Figure 9: PP plots for the fitted models based on the first data set.

8.2 Data Set-II: The survival times of patients suffering from acute Myelogeneous Leukaemia

The second real data set gives the survival times, in weeks, of 33 patients suffering from acute Myelogeneous Leukaemia. These data have been analyzed by Feigl and Zelen (1965) and Mead (2014). The data are: 65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43. Using Weibull (W) baseline model, we research the data modeling ability of the TIIQL-Weibull (TIIQL-W) distribution on this data set. Corresponding PDF of the TIIQL-W distribution is given by

$$f_{\alpha,\beta,\theta}(x)=\frac{4\alpha\beta\theta^{\beta}x^{\beta-1}\exp[-(\theta x)^{\beta}]}{\{1-\exp[-(\theta x)^{\beta}]\}^{\alpha+1}}\left(\frac{1-\{1-\exp[-(\theta x)^{\beta}]\}^{\alpha}}{\{1-\exp[-(\theta x)^{\beta}]\}^{\alpha}}\right)\exp\left(\frac{2\{1-\exp[-(\theta x)^{\beta}]\}^{\alpha}-2}{(1-\exp[-(\theta x)^{\beta}])^{\alpha}}\right),$$

where $0<x$ and $\alpha,\beta,\theta>0$. We compare performance of the real data fitting of the the TIQLW distribution under the MLE method with well know unit distribution in the literature. These competitor distributions are:

• W distribution (Weibull (1951)):

$$f_{\beta,\theta}(x)=\beta\theta^{\beta}x^{\beta-1}e^{-(\theta x)^{\beta}},$$

where, $0<x$ and $\beta,\theta>0$.

• Beta Weibull (B-W) distribution (Famoye et al. (2005)):

$$f_{\alpha,\beta,\theta,\gamma}(x)=\beta\theta^{\beta}\frac{x^{\beta-1}\exp[-(\theta x)^{\beta}]}{\mathrm{B}(\alpha,\gamma)}\{1-\exp[-(\theta x)^{\beta}]\}^{\alpha-1}\exp[-\gamma(\theta x)^{\beta}],$$

where, $0<x$ and $\alpha,\beta,\theta,\gamma>0$ and $\mathrm{B}(\alpha,\gamma)$ is the beta function.

• Odd log-logistic Weibull (OLL-W) distribution (Gleaton and Lynch (2006)):

$$f_{\alpha,\beta,\theta}(x)=\beta\theta^{\beta}\frac{x^{\beta-1}\exp[-\alpha(\theta x)^{\beta}]\{1-\exp[-(\theta x)^{\beta}]\}^{\alpha-1}}{\left(\{1-\exp[-(\theta x)^{\beta}]\}^{\alpha}+\exp[-(\theta x)^{\beta}]\right)^{2}},$$

where, $0<x$ and $\alpha,\beta,\theta>0$.

• Lindley-Weibull (L-W) distribution (Cakmakyapan and Ozel (2017)):

$$f_{\alpha,\beta,\theta}(x)=\frac{\alpha^{2}\beta\theta^{\beta}x^{\beta-1}}{(1+\alpha)}[1+(\theta x)^{\beta}]^{\alpha}\exp[-\alpha(\theta x)^{\beta}],$$

where $0<x$ and $\alpha,\beta,\theta>0$.

We give the data analysis results belong to other competitor models in Table 2. Table 2 shows that the proposed model has the lowest values of the K-S, AIC, CAIC, BIC and HQIC statistic with $\hat{\ell}$ and p-value among application models. It implies that the TIIQL-W model will be the best choice for the modeled data set.

Table 2: MLEs, standard errors of the estimates (in parentheses), $\hat{\ell}$ and goodness-of-fits statistics for the second data set (p value is given in $[\cdot]$)

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\ell}$	AIC	BIC	$CAIC$	$HQIC$	KS
TIQL-W	0.0160	14.1588	0.0037		-151.7055	309.4111	313.9007	310.2387	310.9217	0.1331
	(0.0071)	(6.0475)	(0.0011)							[0.6025]
L-W	5.6819	0.7687	0.0036		-153.5738	313.1476	317.6371	313.9752	314.6582	0.1366
	(19.5711)	(0.1154)	(0.0174)							[0.5685]

OLL-W	0.5547	1.2882	0.0269		-152.9761	311.9521	316.4417	312.7797	313.4627	0.1373
	(0.2564)	(0.5188)	(0.0068)							[0.5626]
B-W	2.4526	0.4639	0.0286	2.1614	-153.5415	315.0831	321.0691	316.5116	317.0972	0.1372
	(5.1603)	(0.4960)	(0.1757)	(2.8410)						[0.5638]
W		0.0282	0.7764		-153.5868	311.5737	314.1667	311.5737	312.1807	0.1367
		(0.0067)	(0.1075)							[0.5687]

Figure 10 displays the fitted PDFs and CDFs for all models. It is clear that the fitting of the TIQL-W model is acceptable as well as the Figure 11 shows that the PP plot of the proposed model is remarkable.

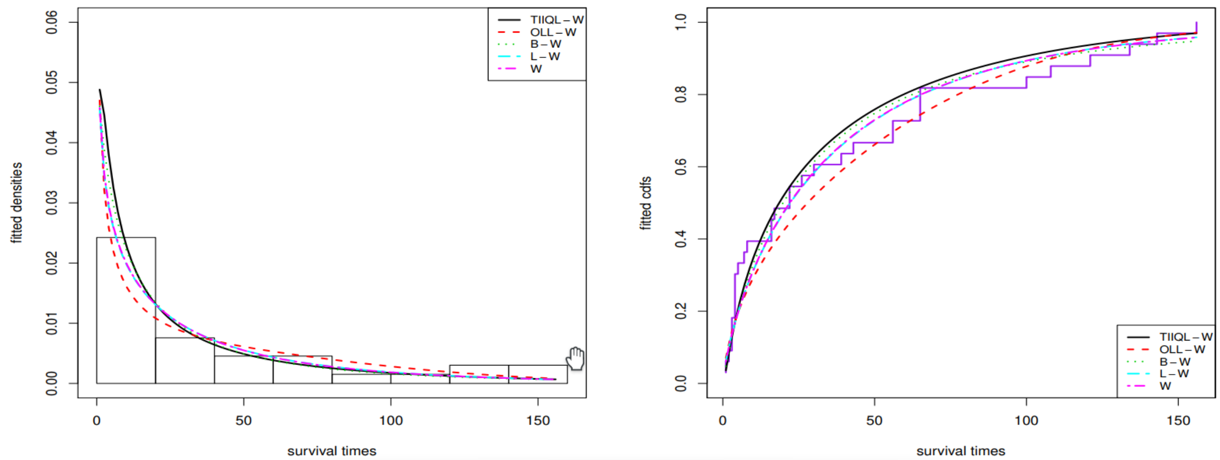

Figure 10: PP plots for the fitted models based on the second data set.

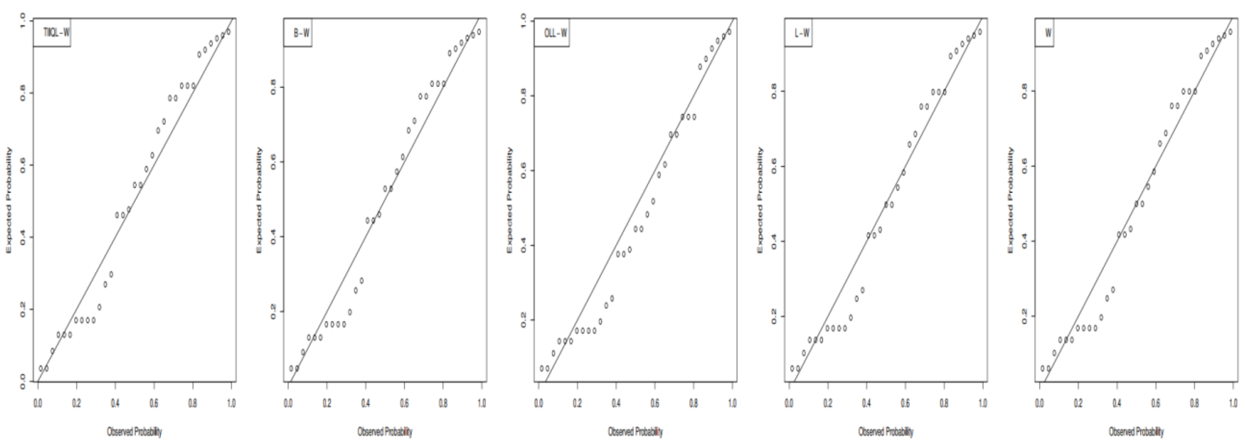

Figure 11: PP plots for the fitted models based on the second data set.

9. Conclusions

A new family of continuous distributions called the type II quasi Lambert family is proposed and studied. Relevant statistical properties are presented and analyzed. Many bivariate versions of the type II quasi Lambert family are derived via different copulas such as Farlie-Gumbel Morgenstern copula, modified Farlie-Gumbel-Morgenstern copula, Clayton copula, Renyi's entropy copula and Ali-Mikhail-Haq copula. We presented two characterizations of the type II quasi Lambert family based on the conditional expectation (truncated moment) of certain functions of a random variable and based on the reversed hazard function. Different estimation methods such as the maximum likelihood estimation, maximum product spacing estimation, least squares estimation, weighted least squares estimation, Anderson-Darling estimation and the Cramer-von Mises estimation methods are considered. Simulation studies for comparing the estimation are performed based on the baseline Lindley model. Two real data sets are analyzed for comparing competitive models.

As future potential works, we can apply many new useful goodness-of-fit tests for right censoring distributional validity such as the Nikulin-Rao-Robson goodness-of-fit test, modified Nikulin-Rao-Robson goodness-of-fit test, Bagdonavicius-Nikulin goodness-of-fit test, modified Bagdonavicius-Nikulin goodness-of-fit test to the new family as performed by Ibrahim et al. (2019), Goual et al. (2019, 2020), Mansour et al. (2020a-f) and Ibrahim et al. (2022a,b), among others. Some new acceptance sampling plans based on the type II quasi Lambert family or based on some special members can be presented in separate article (see Ahmed and Yousof (2022) and Ahmed et al. (2022a,b)). Some useful reliability studies based on multicomponent stress-strength and the remained stress-strength concepts can be presented (Rasekhi et al. (2020) and Saber et al. (2022a,b), Saber and Yousof (2022)). Relevant applications in insurance and risk analysis are available in Mohamed et al. (2022a,b,c) and Hamed et al. (2022). Following Mohamed et al. (2022a,b,c), Salem et al. (2020), and Hamed et al. (2022), the new family can be employed for establishing some risk models for actuarial data.

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Appendix A

Theorem 1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [a, b]$ be an interval for some $d < b$ $(a = -\infty, b = \infty$ mightaswellbeallowed). Let $X: \Omega \to H$ be a continuous random variable with the distribution function F and let $q_1(x)$ and q_2 be two real functions defined on H such that

$$\mathbf{E}[q_2(X)|X \geq x] = \mathbf{E}[q_1(X)|X \geq x]\eta(x), \quad x \in H,$$

is defined with some real function η. Assume that $q_1(x), q_1(x) \in C^1(H)$, $\xi \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1(x) = q_1(x)$ has no real solution in the interior of H. Then F is uniquely determined by the functions $q_1(x), q_1(x)$ and η , particularly

$$F(x) = \int_a^x C \left| \frac{\eta'(u)}{\eta(u) q_1(u) - q_2(u)} \right| \exp\bigl(-s(u)\bigr) du,$$

where the function s is a solution of the differential equation $s' = \frac{\eta' q_1(x)}{\eta q_1(x) - q_1(x)}$ and C is the normalization constant, such that $\int_H dF = 1$.

Note: The goal is to have the function $\eta(x)$ as simple as possible.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence (see, Glanzel, 1990), in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions $q_{1n(x)}$, $q_{2n}(x)$ and η_n $(n \in \mathbb{N})$ satisfy the conditions of Theorem 1 and let $q_{1n}(x) \to q_1(x)$, $q_{2n}(x) \to q_1(x)$ for some continuously differentiable real functions $q_1(x)$ and $q_1(x)$. Let, finally, X be a random variable with distribution F. Under the condition that $q_{1n}(X)$ and $q_{2n}(X)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to X in distribution if and only if $\eta_n(x)$ converges to $\eta(x)$, where

$$\eta(x) = \frac{E[q_2(X)|X \geq x]}{E[q_1(X)|X \geq x]}.$$

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions $q_1(x)$, $q_1(x)$ and η , respectively. It guarantees, for instance, the 'convergence' of characterization of the Wald distribution to that of the Lavy-Smirnov distribution if $\alpha \to \infty$. A further consequence of the stability property of Theorem 1 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions. For such purpose, the functions $q_1(x)$, $q_1(x)$ and, specially, η should be as simple as possible. Since the function triplet is not uniquely determined it is often possible to choose η as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics. In some cases, one can take $q_1(x) \equiv 1$, which reduces the condition of Theorem 1 to $\mathbf{E}[q_2(X)|X \geq x] = \eta(x)$, $x \in H$. We, however, believe that employing three functions $q_1(x)$, $q_1(x)$ and η will enhance the domain of applicability of Theorem 1.