Pak.j.stat.oper.res. Vol.18 No.1 2022 pp 85-97 DOI: http://dx.doi.org/10.18187/pjsor.v18i1.3904

Pakistan Journal of Statistics and Operation Research

Bayesian analysis of an $M|M|1|\infty$ queueing model

Naveen K. Bansal[1*], V.S. Vaidyanathan[2], P. Chandrasekhar[3]

*Corresponding author

1. Department of Mathematical and Statistical Sciences, Marquette University, Wisconsin, USA, naveen.bansal@marquette.edu
2. Department of Statistics, Pondicherry University, Puducherry, India, vaidya.stats@gmail.com
3. Department of Statistics, Loyola College, Chennai, India, drchandrasekharin@yahoo.co.in

Abstract

In this paper, by considering an $M|M|1|\infty$ queueing model, Bayesian estimation of traffic intensity and measures of system performance are worked out under the squared error loss function (SELF) based on the observed data on independent interarrival and service times. Further, minimum posterior risk associated with Bayes estimators of traffic intensity and system performance measures are obtained under SELF. Numerical illustration of the performance of the estimates is given through simulation study. It is shown that Bayes estimators perform better than the maximum likelihood estimators under the influence of prior informations.

Key Words: Bayes estimator, exceedance probability, $M|M|1|\infty$ queue, queue length, queue system size, squared error loss function, traffic intensity.

Mathematical Subject Classification: Primary 60K25; Secondary 90B22.

1. Introduction

Queueing models are often used in the design and analysis of telecommunication systems, traffic systems, service systems and so on. Since the arrival time and service time of entities in the queue are stochastic, it will be of interest to carry out inferential procedures to study and analyze the behaviour of the parameters of the queueing models by assuming suitable probability distributions. This can be done either through the frequentist or Bayesian approach. In case where the probability distribution of either or both of the arrival and service times are not known, non-parametric inferential procedures can be employed to analyze the queueing model; see for example Schweer and Wichelhaus (2015). In this paper, we take a parametric approach by assuming exponential distributions for arrival and service times. A detailed survey on different inferential procedures and its applications to various queueing models can be found in Asanjarani et al. (2021). Among many queueing models available in the literature, $M|M|1|\infty$ model has received more attention primarily due to less model complexities. This model assumes only one service station and does not put a cap on the queue size. The main purpose of this article is to apply certain statistical inference procedures for an $M|M|1|\infty$ queueing model with Poisson input and exponential service times from a Bayesian perspective. It is often the case that information is available on the parameters of the interarrival or service time distribution from prior experiments or from prior analysis of the interarrival or service time data. Bayesian approach provides the methodology by incorporating prior information to the current data.

Maximum likelihood estimator (MLE) for measures of system performance in the case of $M|M|1$ queueing model is discussed in detail in Sharma and Kumar (1999), Yadavalli et al. (2004), Srinivas et al. (2011), Mukherjee and Chowdhury (2005, 2010), Choudhury and Borthakur (2008), and Chowdhury and Mukherjee (2011, 2013). All these authors have studied $M|M|1$ queueing models based on the number of customers present on n iid $M|M|1$ queueing systems. Choudhury and Borthakur (2008) and Mukherjee and Chowdhury (2005, 2010) have also studied Bayes estimation based on the number of customers present in the system. However, in this paper, we consider an $M|M|1$ queueing model based on observed data on interarrival and service times.

Throughout the paper, we consider the queues in which there are m interarrival times and n service times. For the queueing system under consideration, we use appropriate prior distributions for the parameters and evaluate the posterior distributions along with Bayes estimators and minimum posterior risks of the estimators. The rest of the paper is organized as follows: In Section 2, we introduce the model and describe the inferential aspects including Bayes estimators of the parameters and measures of system performance under SELF. Numerical illustration based on simulation study is presented in Section 3. Conclusion of the paper is given in Section 4.

2. $M|M|1|\infty$ queue

2.1. System description of $M|M|1|\infty$ queue and its performance measures

Assume that arrivals follow Poisson process with mean λt. In other words, interarrival times are independent and follow an exponential distribution with mean $\frac{1}{\lambda}$. i.e.,

$$f(t) = \lambda e^{-\lambda t}, 0 < t < \infty; \lambda > 0 \tag{1}$$

For service times, we assume exponential distribution with mean $\frac{1}{\mu}$. The steady state distribution of the number of entities present in the $M|M|1|\infty$ queueing system is given by

$$p_r = (1 - \frac{\lambda}{\mu})(\frac{\lambda}{\mu})^r, r = 0, 1, 2, \ldots; \lambda < \mu \tag{2}$$

see Shortle et al. (2018). Let $\rho = \frac{\lambda}{\mu}$, then (2) can be written as

$$p_r = (1 - \rho)\rho^r, r = 0, 1, 2, \ldots; 0 < \rho < 1 \tag{3}$$

The number of entities in the system (L_S), number of entities in the queue (L_Q) and the probability that there are at least k entities in the queue denoted by ($Q(k)$) are given by

$$L_S = \frac{\lambda}{\mu - \lambda} = \frac{\rho}{1 - \rho} \tag{4}$$

$$L_Q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\rho^2}{1 - \rho} \tag{5}$$

$$Q(k) = \sum_{r=k}^{\infty}(1 - \frac{\lambda}{\mu})(\frac{\lambda}{\mu})^r = (\frac{\lambda}{\mu})^k = \rho^k \tag{6}$$

Let $x_1, x_2, \ldots, x_m$ be a random sample of size m, where x_i is i-th observed interarrival time, $i = 1, 2, \ldots, m$. Similarly, let $y_1, y_2, \ldots, y_n$ be a random sample of size n, where y_j is j-th observed service time, $j = 1, 2, \ldots, n$. Also let $u = \sum_{i=1}^{m} x_i$ and $v = \sum_{j=1}^{n} y_j$. Since maximum likelihood estimators of λ and μ, subject to $\lambda < \mu$ are given by $\hat{\lambda} = \frac{1}{\bar{x}} = \frac{m}{u}$ and $\hat{\mu} = \frac{1}{\bar{y}} = \frac{n}{v}$, where $\bar{x}$ and $\bar{y}$ are the sample means of observed interarrival times and service times respectively, the MLE of ρ is given by $\hat{\rho} = min(1, \frac{\hat{\lambda}}{\hat{\mu}}) = min(1, \frac{mv}{nu})$. In the subsequent sections, Bayes estimators of traffic intensity and measures of system performance are derived under SELF.

 DOI: http://dx.doi.org/10.18187/pjsor.v18i1.3904

2.2. Bayes estimation

In this section, we derive Bayes estimators of λ, μ, ρ,L_S, L_Q and $Q(k)$ under SELF. Reparametrizing the parameters (λ, μ) into (ρ, μ), the joint density of $\vec{x} = (x_1, x_2, ..., x_m)^{'}$ and $\vec{y} = (y_1, y_2, ..., y_n)^{'}$ is given by

$$f(\vec{x}, \vec{y}|\rho, \mu) \propto e^{-\mu(\rho u+v)}\rho^m \mu^{m+n}, 0 < \mu < \infty; 0 < \rho < 1 \tag{7}$$

For the prior on (ρ, μ), we assume that ρ and μ are independent, ρ is distributed as Beta(a,b) of first kind and μ is distributed as Gamma(δ,ω). i.e.,

$$\pi(\rho, \mu|a, b, \delta, \omega) \propto \rho^{a-1}(1-\rho)^{b-1}e^{-\delta\mu}\mu^{\omega-1}, 0 < \rho < 1, a, b, \delta, \omega > 0, 0 < \mu < \infty \tag{8}$$

From (7) and (8), the joint pdf of $(\vec{x}, \vec{y}, \rho, \mu)$ is obtained as

$$f(\vec{x}, \vec{y}, \rho, \mu) \propto e^{-(\delta+\rho u+v)\mu}\mu^{(m+n+\omega)-1}\rho^{(m+a)-1}(1-\rho)^{b-1}, 0 < \mu < \infty; 0 < \rho < 1 \tag{9}$$

Hence the posterior density of (ρ, μ) is given by

$$\begin{aligned}
\pi(\rho, \mu|\vec{x}, \vec{y}) &= \frac{f(\vec{x}, \vec{y}, \rho, \mu)}{f(\vec{x}, \vec{y})} \\
&= \frac{e^{-(\delta+\rho u+v)\mu}\mu^{(m+n+\omega)-1}\rho^{(m+a)-1}(1-\rho)^{b-1}}{\int_0^1 \rho^{(m+a)-1}(1-\rho)^{b-1}d\rho \int_0^\infty e^{-(\delta+\rho u+v)\mu}\mu^{(m+n+\omega)-1}d\mu} \\
&= \frac{e^{-(\delta+\rho u+v)\mu}\mu^{(m+n+\omega)-1}\rho^{(m+a)-1}(1-\rho)^{b-1}}{\Gamma(m+n+\omega)\int_0^1 \frac{\rho^{(m+a)-1}(1-\rho)^{b-1}}{(\delta+\rho u+v)^{m+n+\omega}}d\rho}
\end{aligned} \tag{10}$$

Integrating (10) with respect to μ over $(0, \infty)$, the posterior density of ρ is obtained as follows:

$$\pi(\rho|\vec{x}, \vec{y}) = \frac{\frac{\rho^{(m+a)-1}(1-\rho)^{b-1}}{(\delta+\rho u+v)^{m+n+\omega}}}{\int_0^1 \frac{\rho^{(m+a)-1}(1-\rho)^{b-1}}{(\delta+\rho u+v)^{m+n+\omega}}d\rho} \tag{11}$$

Using the notation

$$q(k_1, k_2, k_3|u, v, \delta) = \int_0^1 \frac{\rho^{k_1-1}(1-\rho)^{k_2-1}}{(\delta+\rho u+v)^{k_3}}d\rho,$$

we have

$$\pi(\rho|\vec{x}, \vec{y}) = \frac{1}{q(m+a, b, m+n+\omega)}\frac{\rho^{(m+a)-1}(1-\rho)^{b-1}}{(\rho u+v+\delta)^{m+n+\omega}}, 0 < \rho < 1. \tag{12}$$

The function $q()$ defined above can be evaluated using numerical integration as described in Section 3. Now, the conditional posterior density of μ given ρ is given by

$$\begin{aligned}
\pi(\mu|\rho, \vec{x}, \vec{y}) &= \frac{\pi(\rho, \mu|\vec{x}, \vec{y})}{\pi(\rho|\vec{x}, \vec{y})} \\
&= \frac{(\rho u+v+\delta)^{m+n+\omega}}{\Gamma(m+n+\omega)}e^{-(\delta+\rho u+v)\mu}\mu^{(m+n+\omega)-1}, 0 < \mu < \infty
\end{aligned} \tag{13}$$

It is clear that the conditional distribution of μ given ρ is Gamma with parameters $(\rho u+v+\delta, m+n+\omega)$.

2.2.1. Bayes estimator of λ under SELF

Given the data $(\vec{x},\vec{y})$, the Bayes estimator of λ under SELF denoted as $\hat{\lambda}_B$ is obtained as

$$\begin{aligned}\hat{\lambda}_B &= E[\lambda|\vec{x},\vec{y}]\\ &= E[\rho\mu|\vec{x},\vec{y}]\\ &= E[E(\rho\mu|\rho,\vec{x},\vec{y})|\vec{x},\vec{y}]\\ &= E[\rho E(\mu|\rho,\vec{x},\vec{y})|\vec{x},\vec{y}]\end{aligned}$$

From (13),

$$\begin{aligned}E(\mu|\rho,\vec{x},\vec{y}) &= \int_0^\infty \mu\pi(\mu|\rho,\vec{x},\vec{y})d\mu\\ &= \frac{(\rho u+v+\delta)^{m+n+\omega}}{\Gamma(m+n+\omega)}\int_0^\infty \mu e^{-(\delta+\rho u+v)\mu}\mu^{(m+n+\omega)-1}d\mu\\ &= \frac{(m+n+\omega)}{(\rho u+v+\delta)}\end{aligned} \tag{14}$$

Hence, from (12), we get,

$$\begin{aligned}\hat{\lambda}_B &= E\big[\rho\frac{(m+n+\omega)}{(\rho u+v+\delta)}|\vec{x},\vec{y}\big]\\ &= (m+n+\omega)E\big[\frac{\rho}{(\rho u+v+\delta)}|\vec{x},\vec{y}\big]\\ &= (m+n+\omega)\int_0^1 \frac{\rho}{(\rho u+v+\delta)}\pi(\rho|\vec{x},\vec{y})d\rho\\ &= (m+n+\omega)\frac{q(m+a+1,b,m+n+\omega+1)}{q(m+a,b,m+n+\omega)}\end{aligned} \tag{15}$$

2.2.2. Bayes estimator of μ under SELF

The Bayes estimator of μ under SELF denoted by $\hat{\mu}_B$ is obtained as

$$\begin{aligned}\hat{\mu}_B &= E[\mu|\vec{x},\vec{y}]\\ &= E[E(\mu|\rho,\vec{x},\vec{y})]\\ &= E\big[\frac{(m+n+\omega)}{(\rho u+v+\delta)}\big]\\ &= \int_0^1 \frac{(m+n+\omega)}{(\rho u+v+\delta)}\pi(\rho|\vec{x},\vec{y})d\rho\\ &= (m+n+\omega)\frac{q(m+a,b,m+n+\omega+1)}{q(m+a,b,m+n+\omega)}\end{aligned} \tag{16}$$

2.2.3. Bayes estimator of ρ under SELF

Bayes estimator of ρ under SELF denoted by $\hat{\rho}_B$ is given by

$$\begin{aligned}\hat{\rho}_B &= E[\rho|\vec{x},\vec{y}]\\ &= \int_0^1 \rho\pi(\rho|\vec{x},\vec{y})d\rho\\ &= \frac{q(m+a+1,b,m+n+\omega)}{q(m+a,b,m+n+\omega)}\end{aligned} \tag{17}$$

2.3. Bayes estimators of measures of system performance

In this section, Bayes estimators of different queueing performance measures are derived in the steady state. More specifically, we estimate measures of system performance namely, the expected number of entities in the system, average queue length, mean sojourn time and the probability of minimum queue size by observing the arrival and service times of entities.

2.3.1. Bayes estimator of L_S under SELF

Bayes estimator of L_S under SELF denoted by $\hat{L_{SB}}$ is obtained as

$$\begin{aligned}
\hat{L_{SB}} &= E[L_S|\vec{x},\vec{y}] \\
&= E\Big[\frac{\rho}{(1-\rho)}|\vec{x},\vec{y}\Big] \\
&= \int_0^1 \frac{\rho}{(1-\rho)}\pi(\rho|\vec{x},\vec{y})d\rho \\
&= \frac{q(m+a+1,b-1,m+n+\omega)}{q(m+a,b,m+n+\omega)}
\end{aligned} \tag{18}$$

2.3.2. Bayes estimator of L_Q under SELF

Bayes estimator of average queue length L_Q under SELF denoted by $\hat{L_{QB}}$ is given by

$$\begin{aligned}
\hat{L_{QB}} &= E[L_Q|\vec{x},\vec{y}] \\
&= E\Big[\frac{\rho^2}{(1-\rho)}|\vec{x},\vec{y}\Big] \\
&= \int_0^1 \frac{\rho^2}{(1-\rho)}\pi(\rho|\vec{x},\vec{y})d\rho \\
&= \frac{q(m+a+2,b-1,m+n+\omega)}{q(m+a,b,m+n+\omega)}
\end{aligned} \tag{19}$$

2.3.3. Bayes estimator of $Q(k)$ under SELF

Bayes estimator of probability of minimum queue size $Q(k)$ under SELF denoted by $\hat{Q(k)}_B$ is obtained as

$$\begin{aligned}
\hat{Q(k)}_B &= E[Q(k)|\vec{x},\vec{y}] \\
&= E[\rho^k|\vec{x},\vec{y}] \\
&= \frac{q(m+a+k,b,m+n+\omega)}{q(m+a,b,m+n+\omega)}
\end{aligned} \tag{20}$$

2.4. Minimum posterior risks of the parameters and system performance measures

In this section, we derive the minimum posterior risks of the parameters λ, μ and ρ and system performance measures L_S, L_Q and $Q(k)$.

2.4.1. Minimum posterior risk of λ

By definition,

$$\begin{aligned}
Var(\lambda|\vec{x},\vec{y}) &= E[(\lambda-E(\lambda))^2|\vec{x},\vec{y}] \\
&= E[\lambda^2|\vec{x},\vec{y}] - \hat{\lambda}_B^2
\end{aligned} \tag{21}$$

Now, consider

$$\begin{aligned} E[\lambda^2|\vec{x},\vec{y}] &= E[\rho^2\mu^2|\vec{x},\vec{y}] \\ &= E[\rho^2 E(\mu^2|\rho,\vec{x},\vec{y})|\vec{x},\vec{y}] \end{aligned}$$

Also,

$$\begin{aligned} E(\mu^2|\rho,\vec{x},\vec{y}) &= \int_0^\infty \mu^2\pi(\mu|\rho,\vec{x},\vec{y})d\mu \\ &= \frac{(\rho u+v+\delta)^{m+n+\omega}}{\Gamma(m+n+\omega)}\int_0^\infty e^{-(\delta+\rho u+v)\mu}\mu^{(m+n+\omega+2)-1}d\mu \\ &= \frac{(m+n+\omega+1)(m+n+\omega)}{(\rho u+v+\delta)^2} \end{aligned}$$

Therefore,

$$E[\rho^2 E(\mu^2|\rho,\vec{x},\vec{y})|\vec{x},\vec{y}] = (m+n+\omega+1)(m+n+\omega)E\big[\frac{\rho^2}{(\rho u+v+\delta)^2}|\vec{x},\vec{y}\big]$$

It can be shown that

$$E\big[\frac{\rho^2}{(\rho u+v+\delta)^2}|\vec{x},\vec{y}\big] = \frac{q(m+a+2,b,m+n+\omega+2)}{q(m+a,b,m+n+\omega)}$$

Hence,

$$E[\rho^2 E(\mu^2|\rho,\vec{x},\vec{y})|\vec{x},\vec{y}] = (m+n+\omega+1)(m+n+\omega)\frac{q(m+a+2,b,m+n+\omega+2)}{q(m+a,b,m+n+\omega)}$$

Thus,

$$E[\lambda^2|\vec{x},\vec{y}] = (m+n+\omega+1)(m+n+\omega)\frac{q(m+a+2,b,m+n+\omega+2)}{q(m+a,b,m+n+\omega)} \tag{22}$$

Using (22) and (15) in (21) and simplifying it, we get

$$\begin{aligned} Var(\lambda|\vec{x},\vec{y}) = \frac{(m+n+\omega)}{[q(m+a,b,m+n+\omega)]^2}&[(m+n+\omega+1)q(m+a+2,b,m+n+\omega+2) \\ & q(m+a,b,m+n+\omega) - (m+n+\omega)[q(m+a+1,b,m+n+\omega+1)]^2] \end{aligned} \tag{23}$$

2.4.2. Minimum posterior risk of μ

The minimum posterior risk of the parameter μ is given by

$$Var(\mu|\vec{x},\vec{y}) = E[\mu^2|\vec{x},\vec{y}] - \hat{\mu}_B^2 \tag{24}$$

Now, $E[\mu^2|\vec{x},\vec{y}]$ can be expressed as $E[\mu^2|\vec{x},\vec{y}] = E_\rho[E(\mu^2|\rho,\vec{x},\vec{y})]$. Consider

$$\begin{aligned} E_\rho[E(\mu^2|\rho,\vec{x},\vec{y})] &= E_\rho\big[\frac{(m+n+\omega+1)(m+n+\omega)}{(\rho u+v+\delta)^2}\big] \\ &= (m+n+\omega+1)(m+n+\omega)E_\rho\big[\frac{1}{(\rho u+v+\delta)^2}\big] \\ &= (m+n+\omega+1)(m+n+\omega)\int_0^1 \frac{1}{(\rho u+v+\delta)^2}\pi(\rho|\vec{x},\vec{y})d\rho \\ &= (m+n+\omega+1)(m+n+\omega)\frac{q(m+a,b,m+n+\omega+2)}{q(m+a,b,m+n+\omega)} \end{aligned} \tag{25}$$

Using (25) and (16) in (24) and simplifying it, we get

$$\begin{aligned} Var(\mu|\vec{x},\vec{y}) = {} & \frac{(m+n+\omega)}{[q(m+a,b,m+n+\omega)]^2}[(m+n+\omega+1)q(m+a,b,m+n+\omega) \\ & q(m+a,b,m+n+\omega+2) - (m+n+\omega)[q(m+a,b,m+n+\omega+1)]^2] \end{aligned} \tag{26}$$

2.4.3. Minimum posterior risk of ρ

The minimum posterior risk of the traffic intensity parameter ρ is given by

$$\begin{aligned} Var(\rho|\vec{x},\vec{y}) &= E[\rho^2|\vec{x},\vec{y}] - \hat{\rho}_B^2 \\ &= \frac{q(m+a+2,b,m+n+\omega)}{q(m+a,b,m+n+\omega)} - \left[\frac{q(m+a+1,b,m+n+\omega)}{q(m+a,b,m+n+\omega)}\right]^2 \\ &= \frac{q(m+a,b,m+n+\omega)q(m+a+2,b,m+n+\omega) - [q(m+a+1,b,m+n+\omega)]^2}{[q(m+a,b,m+n+\omega)]^2} \end{aligned} \tag{27}$$

2.4.4. Minimum posterior risks of system performance measures

By proceeding as explained in the previous sub sections, the expressions for minimum posterior risks of L_S, L_Q and $Q(k)$ are derived as follows:

$$Var(L_S|\vec{x},\vec{y}) = \frac{q(m+a,b,m+n+\omega)q(m+a+2,b-2,m+n+\omega) - [q(m+a+1,b-1,m+n+\omega)]^2}{[q(m+a,b,m+n+\omega)]^2} \tag{28}$$

$$Var(L_Q|\vec{x},\vec{y}) = \frac{q(m+a,b,m+n+\omega)q(m+a+4,b-2,m+n+\omega) - [q(m+a+2,b-1,m+n+\omega)]^2}{[q(m+a,b,m+n+\omega)]^2} \tag{29}$$

$$Var(Q(k)|\vec{x},\vec{y}) = \frac{q(m+a,b,m+n+\omega)q(m+a+2k,b,m+n+\omega) - [q(m+a+k,b,m+n+\omega)]^2}{[q(m+a,b,m+n+\omega)]^2} \tag{30}$$

3. Numerical illustration and the choice of the prior

For the numerical illustration, we compare the proposed Bayes estimators with the maximum likelihood (ML) estimators through simulation. All computations are carried out in R version 3.5.3. The function Integrate() in R is used to evaluate the integral function $q()$ given in Section 2.2. Bias and mean square errors are computed and compared for the both estimators under different sets of parameters and different sample sizes. We fix the service time parameter $\mu = 4$ but vary the interarrival time parameter λ such that $0 < \rho(= \frac{\lambda}{\mu}) < 1$. The hyperparameters of the distribution of $\mu, \omega = 4, \delta = 1$, are chosen in such a way that $E[\mu] = \frac{\omega}{\delta} = 4$. The samples $(x_1, x_2, \ldots, x_m)$ and $(y_1, y_2, \ldots, y_n)$ were simulated 10000 times from the $exp(\lambda)$ and $exp(\mu)$ distributions, respectively.

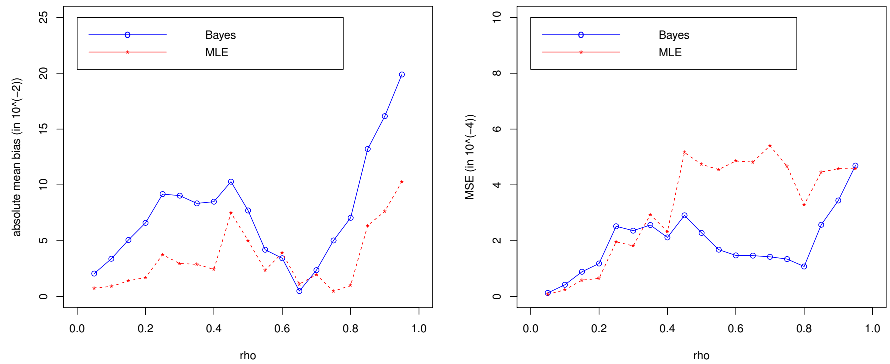

Figure 1: Absolute mean bias and MSE of Bayes, ML estimator of ρ

Figure 1 shows the comparison of the Bayes estimator of ρ against the ML estimator of ρ for the sample sizes $m = 10, n = 15$ in terms of the absolute mean bias and the mean square error (MSE). For this figure, the hyperparameters of the distribution of $\lambda, a = 1, b = 1$, are chosen so that the prior of ρ is non-informative, i.e. $U[0, 1]$. The figure shows that the bias of the Bayes estimator is slightly worse than the bias of the ML estimator as it is usually the case in Bayes estimation. However, the MSE of Bayes estmator is far superior for higher values of $\rho(> 0.35)$, and it is only slightly worse for the smaller values of ρ.

In Tables 1-6, besides ρ, we also compare the Bayes and ML estimators of the interarrival time parameter λ, the service time parameter μ, the mean number of entities in the system L_S, and the mean number of entities in the queue L_Q for different sets of parameters. Note that the bias and the MSEs of ML estimators of L_S and L_Q are not presented in the tables because they are not computable due to the fact that the ML estimator of ρ have value 1 for some simulated samples; see equations (4) and (5). Note that ML estimator $\hat{\rho}_{ML} = 1$ when $\bar{x} \leq \bar{y}$.

We consider different sets of sample sizes $(m, n) \in \{(10, 15), (20, 30), (15, 10), (30, 20)\}$. We vary the hyperparameters (a, b) of the distribution of ρ so that one set of (a, b) yields $\rho(= \frac{\lambda}{\mu})$ close to $E[\rho](= \frac{a}{(a+b)})$, one set yields ρ away from $E[\rho]$, and one set matches with the non-informative prior $U[0, 1]$. The reason of doing so was that, in practice, an expert opinion or empirical estimates of (a, b) may be based on the average of the past history or past data but the actual ρ may or may not be close to $E[\rho]$.

Tables 1-6 show that the absolute mean bias of the Bayes estimators $\hat{\lambda}_B$ and $\hat{\mu}_B$ are smaller compared to those of the ML estimators when $E[\rho]$ is closer to ρ, but larger when it is further away from ρ. The bias of the ML estimator of ρ is slightly better than the bias of the Bayes estimator for all prior choices. In terms of the mean square errors, when actual ρ is closer to zero, Bayes estimators of $\hat{\lambda}_B$, $\hat{\mu}_B$ and $\hat{\rho}_B$ perform better under the prior when $E[\rho] \approx \rho$ but not as good as when $E[\rho]$ is further away from ρ. However, when the actual ρ is near 0.5, then $\hat{\lambda}_B$, $\hat{\mu}_B$ and $\hat{\rho}_B$ are better than the ML estimators under the prior when $E[\rho] = 0.5$ (non-informative prior) or when $E[\rho] < 0.5$. When the actual $\rho > 0.5$, $\hat{\lambda}_B$, $\hat{\mu}_B$ and $\hat{\rho}_B$ perform better under both noninformative prior or when $E[\rho] \approx \rho$.

In summary, the simulation results show that when ρ is closer to zero, ML estimators are overall superior. However whenever ρ is near 0.5 or higher, the Bayes estimators are superior under non-informative priors or when $E[\rho] \approx \rho$. For the estimators of L_S and L_Q, the ML estimators are not comparable, but the Bayes estimators produce reasonably good results in terms of both bias and MSE under non-informative prior or when the prior is such that $E[\rho]$ is near 0.

Table 1: Absolute mean bias of estimates of queue parameters under Bayes and ML for $\lambda = 0.5$, $\mu = 4$

(m,n)	$\hat{\lambda}_B$	$\hat{\lambda}_{ML}$	$\hat{\mu}_B$	$\hat{\mu}_{ML}$	$\hat{\rho}_B$	$\hat{\rho}_{ML}$	$\hat{L_{SB}}$	$\hat{L_{QB}}$
				$a = 0.4$; $b = 4.0$; $\frac{a}{(a+b)} = 0.0909$				
(10,15)	0.0470	0.0582	0.1962	0.2907	0.0162	0.0143	0.0325	0.0163
(20,30)	0.0250	0.0280	0.1112	0.1350	0.0095	0.0070	0.0173	0.0078
(15,10)	0.0295	0.0349	0.2320	0.4472	0.0141	0.0084	0.0275	0.0134
(30,20)	0.0150	0.0167	0.1602	0.2214	0.0081	0.0038	0.0149	0.0068
				$a = 1$; $b = 1$; $\frac{a}{(a+b)} = 0.5$				
(10,15)	0.1128	0.0573	0.0528	0.2818	0.0460	0.0142	0.01025	0.0565
(20,30)	0.0529	0.0266	0.0163	0.1361	0.02198	0.0067	0.0364	0.0144
(15,10)	0.0695	0.0340	0.0948	0.4410	0.0410	0.0084	0.0833	0.0423
(30,20)	0.0333	0.0161	0.0265	0.2165	0.0203	0.0038	0.0332	0.0128
				$a = 4$; $b = 0.4$; $\frac{a}{(a+b)} = 0.9090$				
(10,15)	0.2972	0.0566	0.7722	0.2735	0.1544	0.0146	2.8381	2.6838
(20,30)	0.1356	0.0262	0.3818	0.1545	0.0621	0.0062	0.1110	0.0489
(15,10)	0.1877	0.0342	1.0724	0.4490	0.1552	0.0086	2.8702	2.7150
(30,20)	0.0885	0.0169	0.5830	0.1953	0.0632	0.0046	0.1131	0.0498

Table 2: Mean Square Error of estimates of queue parameters under Bayes and ML for $\lambda = 0.5$, $\mu = 4$

(m,n)	$\hat{\lambda}_B$	$\hat{\lambda}_{ML}$	$\hat{\mu}_B$	$\hat{\mu}_{ML}$	$\hat{\rho}_B$	$\hat{\rho}_{ML}$	$\hat{L_{SB}}$	$\hat{L_{QB}}$
				$a = 0.4$; $b = 4.0$; $\frac{a}{(a+b)} = 0.0909$				
(10,15)	0.0341	0.0428	0.7483	1.4738	0.0028	0.0040	0.0076	0.0012
(20,30)	0.0147	0.0159	0.4573	0.6337	0.0014	0.0016	0.0030	0.0003
(15,10)	0.0200	0.0227	0.9146	2.6866	0.0021	0.0032	0.0052	0.0007
(30,20)	0.0093	0.0098	0.6220	1.0344	0.0012	0.0014	0.0025	0.0002
				$a = 1$; $b = 1$; $\frac{a}{(a+b)} = 0.5$				
(10,15)	0.0585	0.0415	0.6758	1.4959	0.0068	0.0039	0.0700	0.0405
(20,30)	0.0199	0.0162	0.4249	0.6238	0.0023	0.0016	0.0059	0.0008
(15,10)	0.0293	0.0227	0.7850	2.5744	0.0052	0.0033	0.0306	0.0127
(30,20)	0.0112	0.0097	0.5773	1.0449	0.0019	0.0014	0.0045	0.0005
				$a = 4$; $b = 0.4$; $\frac{a}{(a+b)} = 0.9090$				
(10,15)	0.1741	0.0427	1.0591	1.4421	0.0389	0.0042	80.6367	78.2085
(20,30)	0.0415	0.0162	0.5091	0.6515	0.0069	0.0016	0.0969	0.0730
(15,10)	0.0749	0.0241	1.6326	2.6991	0.0361	0.0034	51.8392	49.8167
(30,20)	0.0205	0.0099	0.7691	1.0141	0.0066	0.0015	0.0422	0.0210

Table 3: Absolute mean bias of estimates of queue parameters under Bayes and ML for $\lambda = 2$, $\mu = 4$

			$a = 0.4$; $b = 4.0$; $\frac{a}{(a+b)} = 0.0909$					
(m,n)	$\hat{\lambda}_B$	$\hat{\lambda}_{ML}$	$\hat{\mu}_B$	$\hat{\mu}_{ML}$	$\hat{\rho}_B$	$\hat{\rho}_{ML}$	$\hat{L_{SB}}$	$\hat{L_{QB}}$
(10,15)	0.2723	0.2250	0.5734	0.2776	0.1068	0.0438	0.2127	0.1059
(20,30)	0.1631	0.0993	0.3608	0.1246	0.0654	0.0251	0.0983	0.0328
(15,10)	0.1739	0.1470	0.7805	0.4443	0.0960	0.0285	0.1854	0.0893
(30,20)	0.1039	0.0652	0.5218	0.2115	0.0615	0.0156	0.0914	0.0299
			$a = 1$; $b = 1$; $\frac{a}{(a+b)} = 0.5$					
(10,15)	0.2278	0.2292	0.1006	0.2896	0.0714	0.0435	5.2666	5.1952
(20,30)	0.1557	0.1066	0.0436	0.1553	0.0561	0.0228	3.1586	3.1025
(15,10)	0.1673	0.1459	0.0770	0.4346	0.0715	0.0301	4.9098	4.8383
(30,20)	0.1074	0.0684	0.0295	0.2204	0.0560	0.0152	2.8941	2.8381
			$a = 4$; $b = 0.4$; $\frac{a}{(a+b)} = 0.9090$					
(10,15)	0.7530	0.2249	0.5003	0.2921	0.3079	0.0429	100.3465	100.0385
(20,30)	0.5275	0.1022	0.4235	0.1313	0.2324	0.0246	60.1449	59.9125
(15,10)	0.5283	0.1405	0.7611	0.4413	0.3104	0.0273	98.9656	98.6552
(30,20)	0.3594	0.0698	0.6132	0.2208	0.2343	0.0149	59.5554	59.3210

Table 4: Mean Square Error of estimates of queue parameters under Bayes and ML for $\lambda = 2$, $\mu = 4$

			$a = 0.4$; $b = 4.0$; $\frac{a}{(a+b)} = 0.0909$					
(m,n)	$\hat{\lambda}_B$	$\hat{\lambda}_{ML}$	$\hat{\mu}_B$	$\hat{\mu}_{ML}$	$\hat{\rho}_B$	$\hat{\rho}_{ML}$	$\hat{L_{SB}}$	$\hat{L_{QB}}$
(10,15)	0.2422	0.6637	1.0663	1.4643	0.0179	0.0456	0.1391	0.0631
(20,30)	0.1395	0.2519	0.5631	0.6270	0.0106	0.0238	0.1265	0.0714
(15,10)	0.1864	0.3759	1.4864	2.6485	0.0157	0.0433	0.1333	0.0638
(30,20)	0.1055	0.1545	0.8433	1.0346	0.0100	0.0226	0.1211	0.0683
			$a = 1$; $b = 1$; $\frac{a}{(a+b)} = 0.5$					
(10,15)	0.3924	0.6788	0.6333	1.4990	0.0224	0.0452	59.6182	57.5627
(20,30)	0.2260	0.2659	0.4039	0.6398	0.0185	0.0242	32.7678	31.4387
(15,10)	0.2791	0.3749	0.6830	2.6538	0.0210	0.0440	51.3451	49.4871
(30,20)	0.1478	0.1575	0.5300	1.0566	0.0171	0.0226	25.8031	24.6629
			$a = 4$; $b = 0.4$; $\frac{a}{(a+b)} = 0.9090$					
(10,15)	0.9469	0.6904	0.6525	1.5423	0.1065	0.0466	12294.75	12223.55
(20,30)	0.5161	0.2567	0.4707	0.6105	0.0719	0.0240	5833.374	5793.950
(15,10)	0.5623	0.3706	0.9162	2.6261	0.1052	0.0426	11517.82	11449.13
(30,20)	0.2896	0.1600	0.7116	1.0236	0.0705	0.0222	5370.813	5333.135

Table 5: Absolute mean bias of estimates of queue parameters under Bayes and ML for $\lambda = 3.5$, $\mu = 4$

(m,n)	$\hat{\lambda}_B$	$\hat{\lambda}_{ML}$	$\hat{\mu}_B$	$\hat{\mu}_{ML}$	$\hat{\rho}_B$	$\hat{\rho}_{ML}$	$\hat{L_{SB}}$	$\hat{L_{QB}}$
				$a = 0.4$; $b = 4.0$; $\frac{a}{(a+b)} = 0.0909$				
(10,15)	1.0791	0.4184	0.9134	0.2956	0.3701	0.0526	5.7088	5.3386
(20,30)	0.7862	0.1888	0.6991	0.1492	0.2875	0.0316	5.2104	4.9229
(15,10)	0.7855	0.2422	1.3092	0.4565	0.3462	0.0666	5.5850	5.2388
(30,20)	0.5476	0.1222	1.0193	0.2125	0.2723	0.0375	5.0980	4.8256
				$a = 1$; $b = 1$; $\frac{a}{(a+b)} = 0.5$				
(10,15)	0.3197	0.3717	0.4113	0.2562	0.1379	0.0522	9.3156	9.4536
(20,30)	0.2073	0.1928	0.3111	0.1491	0.0984	0.0311	12.0354	12.1339
(15,10)	0.1859	0.2547	0.5738	0.4442	0.1282	0.0644	9.8269	9.9551
(30,20)	0.1229	0.1340	0.4231	0.2180	0.0933	0.0380	12.3670	12.4604
				$a = 4$; $b = 0.4$; $\frac{a}{(a+b)} = 0.9090$				
(10,15)	0.1963	0.3917	0.1010	0.2952	0.0322	0.0544	156.5448	156.5125
(20,30)	0.1506	0.2078	0.0400	0.1294	0.0346	0.0265	156.0456	156.0110
(15,10)	0.1673	0.2507	0.0732	0.4340	0.0336	0.0666	156.8547	156.8211
(30,20)	0.1192	0.1207	0.0188	0.2103	0.0345	0.0385	154.7707	154.7362

Table 6: Mean Square Error of estimates of queue parameters under Bayes and ML for $\lambda = 3.5$, $\mu = 4$

(m,n)	$\hat{\lambda}_B$	$\hat{\lambda}_{ML}$	$\hat{\mu}_B$	$\hat{\mu}_{ML}$	$\hat{\rho}_B$	$\hat{\rho}_{ML}$	$\hat{L_{SB}}$	$\hat{L_{QB}}$
				$a = 0.4$; $b = 4.0$; $\frac{a}{(a+b)} = 0.0909$				
(10,15)	1.3962	2.2029	1.6621	1.5030	0.1416	0.0406	32.7403	28.6038
(20,30)	0.7840	0.7880	0.9342	0.6235	0.0872	0.0262	27.4551	24.4726
(15,10)	0.8802	1.1108	2.7190	2.7731	0.1250	0.0459	31.3903	27.5855
(30,20)	0.4752	0.4767	1.6141	1.0198	0.0790	0.0279	26.3611	23.5808
				$a = 1$; $b = 1$; $\frac{a}{(a+b)} = 0.5$				
(10,15)	0.4899	1.9824	0.7934	1.4477	0.0267	0.0398	156.6287	157.8606
(20,30)	0.2926	0.7810	0.4745	0.6545	0.0167	0.0262	262.7639	263.4897
(15,10)	0.4364	1.1426	1.0177	2.6633	0.0239	0.0449	173.1569	174.3057
(30,20)	0.2541	0.4995	0.6126	1.0443	0.0156	0.0282	279.3706	280.0019
				$a = 4$; $b = 0.4$; $\frac{a}{(a+b)} = 0.9090$				
(10,15)	0.5065	2.0689	0.5339	1.5537	0.0027	0.0405	25807.57	25794.67
(20,30)	0.2808	0.8006	0.2932	0.6222	0.0035	0.0256	26507.51	26492.48
(15,10)	0.4761	1.1765	0.5010	2.5498	0.0026	0.0454	25881.82	25868.67
(30,20)	0.2586	0.4717	0.2946	1.0346	0.0033	0.0278	26030.22	26015.54

In general, for Bayes estimation, our recommendation is to choose the prior (8) such that the prior on ρ is non-informative, i.e., $a = 1, b = 1$, and to choose prior of μ to be Gamma distribution with hyperparameters ω and δ which can be based on expert opinion. ω and δ can also be estimated from the marginal likelihood of $(y_1, y_2, \ldots, y_n)$. Another choice is to estimate the hyperparameters of (8) using the marginal likelihood of $(x_1, x_2, \ldots, x_m)$ and $(y_1, y_2, \ldots, y_n)$ as we illustrate below.

3.1. Estimation of hyperparameters

The hyperparameter vector $\vec{\theta} = (a, b, \delta, \omega)'$ can be estimated by maximizing the marginal likelihood. The marginal likelihood is given by

$$\begin{aligned} L(\vec{\theta}|\vec{x},\vec{y}) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\frac{\delta^{\omega}}{\Gamma(\omega)}\int_0^1\int_0^{\infty} e^{-\mu(\rho u+v+\delta)}\rho^{m+a-1}(1-\rho)^{b-1}\mu^{m+n+\omega-1}d\rho d\mu \\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\frac{\delta^{\omega}}{\Gamma(\omega)}\int_0^1 \frac{\Gamma(m+n+\omega)}{(\rho u+v+\delta)^{m+n+\omega}}\rho^{m+a-1}(1-\rho)^{b-1}d\rho. \end{aligned}$$

The parameter vector $\vec{\theta}$ can be estimated by solving the system of equations $\frac{\partial LogL}{\partial \vec{\theta}} = \vec{0}$. However, the system of equations will be highly non-linear due to the presence of gamma functions involved in the integration. The issue of integration can be resolved by using EM algorithm, but the system of equations will still be complex due to the gamma function.Thus, we propose, the method of moments estimator that involve a tractable system of equations.

Note that the method of moments estimators are not as efficient as the marginal ML estimators. The first two moments of the distributions of X and Y are given by

$$\begin{aligned} E(X) &= E\Big(\frac{1}{\lambda}\Big) = E\Big(\frac{1}{\rho\mu}\Big) \\ &= \frac{\delta^{\omega-1}}{\Gamma(\omega)}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\frac{\Gamma(a-1)\Gamma(b)}{\Gamma(a+b-1)}\frac{\Gamma(\omega-1)}{\delta^{\omega-1}} \\ &= \frac{\delta(a+b-1)}{(\omega-1)(a-1)}, \\ E(X^2) &= E\Big(\frac{2}{\rho^2\mu^2}\Big) \\ &= \frac{2\delta^2(a+b-1)(a+b-2)}{(\omega-1)(\omega-2)(a-1)(a-2)}, \\ E(Y) &= E\Big(\frac{1}{\mu}\Big) = \frac{\delta}{\omega-1}, \\ E(Y^2) &= E\Big(\frac{2}{\mu^2}\Big) = \frac{2\delta^2}{(\omega-1)(\omega-2)}. \end{aligned}$$

Now by replacing $E(X)$, $E(X^2)$, $E(Y)$ and $E(Y^2)$ in the above equations by its corresponding sample moments m_{1X}, m_{2X}, m_{1Y} and m_{2Y} and solving for a, b, ω and δ, we get the moment estimators. After solving for a, b, ω and δ, we get the following expressions for the estimators of a, b, ω and δ.

$$\begin{aligned} \hat{a} &= \frac{(m_{2X}m_{1Y} - m_{2Y}m_{1X})m_{1Y}}{m_{2X}{m_{1Y}}^2 - m_{2Y}{m_{1X}}^2} + 1, \\ \hat{b} &= \frac{(m_{1X} - m_{1Y})(m_{2X}m_{1Y} - m_{2Y}m_{1X})}{m_{2X}{m_{1Y}}^2 - m_{2Y}{m_{1X}}^2}, \\ \hat{\omega} &= \frac{2(m_{2Y} - {m_{1Y}}^2)}{m_{2Y} - 2{m_{1Y}}^2}, \\ \hat{\delta} &= \frac{m_{1Y}m_{2Y}}{m_{2Y} - 2{m_{1Y}}^2}. \end{aligned}$$

4. Conclusion

An attempt is made in this paper to study the Bayesian estimation of $M|M|1|\infty$ queueing model when sample observations are available on independent interarrival and service times of entities at some points of times. Bayes estimators of traffic intensity and system performance measures have been derived assuming squared error loss function. Bayes estimators, studied in the past, assume available observations on the queue sizes which are sometimes not practical. We obtain Bayes estimators when observations are available on arrival times and service times which are sometimes easy to document. These estimators show a superior performance for the most part against the ML estimators. A part of deficiency in the ML estimators is due to the constraint $\lambda < \mu$. In this case, the ML estimates of L_S and L_Q does not even exist if $\bar{x} \leq \bar{y}$, whereas, Bayes estimates can be obtained.

Acknowledgement

The authors thank the referees for their useful comments and suggestions, which have helped us in improving the presentation of the paper.

References

1. Asanjarani, A., Nazarathy, Y., and Taylor, P. (2021). A survey of parameter and state estimation in queues. *Queueing Systems*, 97(1):39–80.
2. Choudhury, A. and Borthakur, A. C. (2008). Bayesian Inference and prediction in the single server Markovian queue. *Metrika*, 67(3):371–383.
3. Chowdhury, S. and Mukherjee, S. (2011). Estimation of waiting time distribution in an $M|M|1$ queue. *OPSEARCH*, 48(4):306–317.
4. Chowdhury, S. and Mukherjee, S. (2013). Estimation of traffic intensity based on queue length in a single $M|M|1$ queue. *Communications in Statistics-Theory and Methods*, 42(13):2376–2390.
5. Mukherjee, S. and Chowdhury, S. (2005). Maximum likelihood and Bayes estimation in $M|M|1$ queue. *Stochastic Modeling and Applications*, 8:47–55.
6. Mukherjee, S. and Chowdhury, S. (2010). Bayes estimation of measures of effectiveness in an $M|M|1$ queueing model. *Calcutta Statistical Association Bulletin*, 62(1-2):97–108.
7. Schweer, S. and Wichelhaus, C. (2015). Nonparametric estimation of the service time distribution in the discrete-time $GI|G|\infty$ queue with partial information. *Stochastic Processes and their Applications*, 125(1):233–253.
8. Sharma, K. and Kumar, V. (1999). Inferences on $M|M|1|\infty|FIFO$ queue system. *Opsearch*, 36(1):26–34.
9. Shortle, J. F., Thompson, J. M., Gross, D., and Harris, C. M. (2018). *Fundamentals of Queueing Theory*. John Wiley & Sons.
10. Srinivas, V., Subba Rao, S., and Kale, B. (2011). Estimation of measures in $M|M|1$ queue. *Communications in Statistics-Theory and Methods*, 40(18):3327–3336.
11. Yadavalli, V., Adendorff, K., Erasmus, G., Chandrasekhar, P., and Deepa, S. (2004). Confidence limits for expected waiting time of two queuing models. *ORiON*, 20(1):1–6.