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Characterizations of Twenty (2020-2021) Proposed Discrete Distributions

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Abstract

In this paper, certain characterizations of twenty newly proposed discrete distributions: the discrete generalized Lindley distribution of El-Morshedy et al.(2021), the discrete Gumbel distribution of Chakraborty et al.(2020), the skewed geometric distribution of Ong et al.(2020), the discrete Poisson X gamma distribution of Para et al.(2020), the discrete Cos-Poisson distribution of Bakouch et al.(2021), the size biased Poisson Ailamujia distribution of Dar and Para(2021), the generalized Hermite-Genocchi distribution of El-Desouky et al.(2021), the Poisson quasi-xgamma distribution of Altun et al.(2021a), the exponentiated discrete inverse Rayleigh distribution of Mashhadzadeh and MirMostafaee(2020), the Mlynar distribution of Frühwirth et al.(2021), the flexible one-parameter discrete distribution of Eliwa and El-Morshedy(2021), the two-parameter discrete Perks distribution of Tyagi et al.(2020), the discrete Weibull G family distribution of Ibrahim et al.(2021), the discrete Marshall–Olkin Lomax distribution of Ibrahim and Almetwally(2021), the two-parameter exponentiated discrete Lindley distribution of El-Morshedy et al.(2019), the natural discrete one-parameter polynomial exponential distribution of Mukherjee et al.(2020), the zero-truncated discrete Akash distribution of Sium and Shanker(2020), the two-parameter quasi Poisson-Aradhana distribution of Shanker and Shukla(2020), the zero-truncated Poisson-Ishita distribution of Shukla et al.(2020) and the Poisson-Shukla distribution of Shukla and Shanker(2020) are presented to complete, in some way, the authors' works.

Key Words: Discrete distributions; Characterizations; Conditional expectation; Hazard function; Reverse hazard function.

1. Introduction

To understand the behavior of the data obtained through a given process, we need to be able to describe this behavior via its approximate probability law. This, however, requires to establish conditions which govern the required probability law. In other words, we need to have certain conditions under which we may be able to recover the probability law of the data. Therefore, the problem of characterizing a distribution is an important problem in applied sciences, where an investigator is vitally interested to know if their model follows the right distribution. To this end, the investigator relies on conditions under which their model would follow specifically the chosen distribution. El-Morshedy et al.(2021) introduced a new discrete probability model called Discrete Generalized Lindley (DsGLi) distribution; Chakraborty et al.(2020) introduced a new discrete probability model called Discrete Gumbel (DGu) distribution; Ong et al.(2020) proposed a new discrete distribution called Skewed Geometric (SG) distribution; Para et al.(2020) proposed a new discrete probability model called discrete Poisson X Gamma (P-Xgamma) distribution; Bakouch et al.(2021) introduced a new discrete distribution called Cos-Poisson (CosPois) distribution; Dar and Para(2021) proposed a new discrete probability model called Size Biased Poisson Ailamujia (SBPA) distribution; El-Desouky et al.(2021) introduced a new discrete lifetime called Generalized Hermite-Genocchi (GHGD) distribution; Altun et al.(2021a) proposed a new discrete distribution called Poisson Quasi-Xgamma (PQX) distribution; Mashhadzadeh and MirMostafaee(2020) proposed a new discrete distribution called Exponentiated Discrete Inverse Rayleigh (EDIR) distribution; Frühwirth et al.(2021) introduced a new distribution arising from a generalized random game and its asymptotic properties called Mlynar distribution; Eliwa and El-Morshedy(2021) proposed a Discrete Flexible distribution with one parameter called DsFx-I; Tyagi et al.(2020) introduced a two-parameter discrete distribution called Discrete Perks (DP) distribution; Ibrahim et al.(2021) introduced a new discrete analogue of the Weibull class called Discrete Weibull G (DWG) family distribution; Ibrahim and Almetwally(2021) introduced a new discrete distribution called Discrete Marshall–Olkin Lomax distribution (DMOL); El-Morshedy et al.(2019) introduced a new two-parameter discrete distribution called Exponentiated Discrete Lindley (EDLi) distribution; Mukherjee et al.(2020) proposed a Natural Discrete One-Parameter Polynomial Exponential distribution called (NDOPPE); Sium and Shanker(2020) proposed a new discrete distribution as well as its applications and properties called Zero-Truncated Discrete Akash distribution (ZTDAD); Shanker and Shukla(2020) proposed a new Poisson distribution called Quasi Poisson-Aradhana distribution (QPAD); Shukla et al.(2020) proposed a new Poisson distribution and its applications called Zero-Truncated Poisson-Ishita distribution (ZTPID) and Shukla and Shanker(2020) introduced a new Poisson distribution and its applications called the Poisson-Shukla Distribution (PSD). All mentioned authors derived and discussed important mathematical and statistical properties of their distributions. In addition to these properties, El-Morshedy et al.(2021) and Ibrahim and Almetwally(2021) discussed the applications of their distributions to the COVID-19 as well.

In this paper, we present two characterizations of these distributions based on: (i) the conditional expectation of certain function of the random variable; (ii) the hazard rate functions of DsGLi, P-Xgamma, SBPA, GHGD, PQX, EDIR, Mlynar, DsFx-I, DWG, DMOL and ZTDAD and (iii) the reverse hazard functions of DGu, SG, CosPois, DP, EDLi, NDOPPE, QPAD, ZTPID and PSD distributions.

The cumulative distribution functions (cdf), $F(x)$, corresponding probability mass functions (pmf), $f(x)$ and hazard rate functions, $h_F(x)$, or reverse hazard functions, $r_F(x)$ of DsGLi, DGu, SG, P-Xgamma, CosPois, SBPA, GHGD, PQX, EDIR, Mlynar, DsFx-I, DP, DWG, DMOL, EDLi, NDOPPE, ZTDAD, QPAD, ZTPID and PSD are give, respectively, by

$$F(x;\alpha,\eta) = 1 - \frac{\left(1-\ln\eta^{x+1}\right)(\alpha-\alpha\ln\eta+\ln\eta)-\ln\eta}{\alpha(1-\ln\eta)}\eta^{x+1}, \qquad x\in\mathbb{N}^*, \tag{1}$$

$$f(x;\alpha,\eta) = \frac{\eta^x}{(1-\ln\eta)}\times\left\{\begin{array}{c} 1-\eta-\ln\eta\left[1+x-\eta(x+2)\right]+ \\ \left(1-\frac{1}{\alpha}\right)(\ln\eta)^2\left[x-\eta(x+1)\right]\end{array}\right\}, \qquad x\in\mathbb{N}^*, \tag{2}$$

$$h_F(x) = \frac{\alpha\left\{\begin{array}{c} 1-\eta-\ln\eta\left[1+x-\eta(x+2)\right]+ \\ \left(1-\frac{1}{\alpha}\right)(\ln\eta)^2\left[x-\eta(x+1)\right]\end{array}\right\}}{\left\{\left(1-\ln\eta^{x+1}\right)(\alpha-\alpha\ln\eta+\ln\eta)-\ln\eta\right\}\eta}, \qquad x\in\mathbb{N}^*, \tag{3}$$

where $\alpha > 0, \eta \in (0,1)$ are parameters and $\mathbb{N}^*$ is the set of non-negative integers;

$$F(x;\alpha,p) = e^{-\alpha p^{x+1}}, \qquad x \in \mathbb{Z}, \tag{4}$$

$$f(x;\alpha,p) = e^{-\alpha p^{x+1}} - e^{-\alpha p^{x}}, \qquad x \in \mathbb{Z}, \tag{5}$$

$$r_F(x) = 1 - \left(\frac{e^{-\alpha p^{x}}}{e^{-\alpha p^{x+1}}}\right), \qquad x \in \mathbb{Z}, \tag{6}$$

where $\alpha > 0$ and $p \in (0,1)$ are parameters and $\mathbb{Z}$ is the set of integers;

$$F(x;\alpha,p) = C\left\{\left(1-p^{x+1}\right) - pq\left(\frac{1-p^{(\alpha+1)(x+1)}}{1-p^{\alpha+1}}\right)\right\}, \qquad x \in \mathbb{N}^*, \tag{7}$$

$$f(x;\alpha,p) = Cqp^{x}\left(1-p^{\alpha x+1}\right), \qquad x \in \mathbb{N}^*, \tag{8}$$

$$r_F(x) = \frac{q\left(1-p^{\alpha+1}\right)p^{x}\left(1-p^{\alpha x+1}\right)}{\left(1-p^{\alpha+1}\right)\left(1-p^{x+1}\right) - pq\left(1-p^{(\alpha+1)(x+1)}\right)}, \qquad x \in \mathbb{N}^*, \tag{9}$$

where $\alpha > 0,\ p \in (0,1), q = 1-p$ are parameters;

$$F(x;\theta) = 1 - \frac{\theta(\theta+5)x^2 + 2\theta x + \gamma}{2(1+\theta)^{x+4}}, \qquad x \in \mathbb{N}^*, \tag{10}$$

$$f(x;\theta) = \frac{\theta^2\left[2(1+\theta)^2 + \theta(x+1)(x+2)\right]}{2(1+\theta)^{x+4}}, \qquad x \in \mathbb{N}^*, \tag{11}$$

$$h_F(x) = \frac{\theta^2\left[2(1+\theta)^2 + \theta(x+1)(x+2)\right]}{\theta(\theta+5)x^2 + 2\theta x + \gamma}, \qquad x \in \mathbb{N}^*, \tag{12}$$

where $\theta > 0$ is a parameter, $\gamma = 2\left(\theta^3 + 5\theta^2 + 4\theta + 1\right)$;

Remark 1.
Altun et al.(2021b) introduced the discrete Poisson X-Gamma distribution by compounding Poisson distribution and X-gamma distribution. Poisson X-Gamma distribution introduced by Altun et al.(2021b) is quite similar to the P-Xgamma distribution in Para et al.(2020).

$$F(x;\beta,\lambda) = C^*\left\{F_*(x;\lambda) + \Upsilon(x;\beta,\lambda)\right\}, \qquad x \in \mathbb{N}^*, \tag{13}$$

$$f(x;\beta,\lambda) = C\left[\cos(\beta x)\right]^2 \frac{\lambda^x}{x!}, \qquad x \in \mathbb{N}^*, \tag{14}$$

$$r_F(x) = \frac{2e^{-\lambda}\left[\cos(\beta x)\right]^2 \frac{\lambda^x}{x!}}{F_*(x;\lambda) + \Upsilon(x;\beta,\lambda)}, \qquad x \in \mathbb{N}^*, \tag{15}$$

where $\beta > 0,\ \theta > 0$ are parameters, $C = 2e^{-\lambda}\left\{1 + e^{-\lambda[1-\cos(2\beta)]}\cos\left[\lambda\sin(2\beta)\right]\right\}^{-1},\ C^* = Ce^{\lambda}2^{-1}$, $\Upsilon(x;\beta,\lambda) = \mathrm{Re}\left[F_*\left(x;\lambda e^{2i\beta}\right)\right]$ and $F_*(x;\lambda)$ is the cdf of the Poisson distribution with the parameter λ;

$$\begin{aligned} F(x;\eta) &= 1 - \frac{2x^2\eta^2 + 6x\eta^2 + 4\eta^2 + 2x\eta + 4\eta + 1}{(1+2\eta)^{x+2}} \\ &= 1 - \frac{2\eta(x+2)\left[1+\eta(x+1)\right] + 1}{(1+2\eta)^{x+2}}, \qquad x \in \mathbb{N}, \end{aligned} \tag{16}$$

$$f(x;\eta) = \frac{4\eta^3 x(x+1)}{(1+2\eta)^{x+2}}, \qquad x \in \mathbb{N}, \tag{17}$$

$$h_F(x) = \frac{4\eta^3 x(x+1)}{2\eta(x+2)[1+\eta(x+1)]+1}, \qquad x \in \mathbb{N}, \tag{18}$$

where $\eta > 0$ is a parameter and $\mathbb{N}$ is the set of positive integers;

$$F(x;\alpha,\beta,\gamma) = 1 - \frac{\alpha^{x+1} G(\alpha;\beta,x+\gamma+1)}{G(\alpha;\beta,\gamma)}, \qquad x \in \mathbb{N}^*, \tag{19}$$

$$f(x;\alpha,\beta,\gamma) = \frac{\alpha^x H_{n,m}(x+\gamma,\beta)}{G(\alpha;\beta,\gamma)}, \qquad x \in \mathbb{N}^*, \tag{20}$$

$$h_F(x) = \frac{1}{\alpha}\left(\frac{H_{n,m}(x+\gamma,\beta)}{G(\alpha;\beta,x+\gamma+1)}\right), \qquad x \in \mathbb{N}^*, \tag{21}$$

where $\alpha \in (0,1), \beta \geq 0, \gamma \geq 0$ are parameters and $G(\alpha;\beta,\gamma) = \sum_{\ell=0}^{\infty} \alpha^\ell H_{n,m}(\ell+\gamma,\beta)$ and $H_{n,m}(x+\gamma,\beta) = \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{\beta^k}{k!} \frac{n!}{(n-mk)!}(x+\gamma)^{n-mk}$. Note that $G(\alpha;\beta,\gamma)$ is convergent and positive for $\alpha \in (0,1)$;

$$F(x;\alpha,\theta) = 1 - \frac{2\alpha(\theta+1)^2 + \theta(x+3)(\theta(x+2)+2)+2}{2(\alpha+1)(\theta+1)^{x+3}}, \qquad x \in \mathbb{N}^*, \tag{22}$$

$$f(x;\alpha,\theta) = \frac{2\alpha\theta(\theta+1)^2 + \theta^3(x+1)(x+2)}{2(\alpha+1)(\theta+1)^{x+3}}, \qquad x \in \mathbb{N}^*, \tag{23}$$

$$h_F(x) = \frac{2\alpha\theta(\theta+1)^2 + \theta^3(x+1)(x+2)}{2\alpha(\theta+1)^2 + \theta(x+3)(\theta(x+2)+2)+2}, \qquad x \in \mathbb{N}^*, \tag{24}$$

where $\alpha > 0, \theta > 0$ are parameters;

$$F(x;\gamma) = 1 - \left(1 - q^{(x+1)^{-2}}\right)^\gamma, \qquad x \in \mathbb{N}^*, \tag{25}$$

$$f(x;\gamma) = \left(1 - q^{x^{-2}}\right)^\gamma - \left(1 - q^{(x+1)^{-2}}\right)^\gamma, \qquad x \in \mathbb{N}^*, \tag{26}$$

$$h_F(x) = \left(\frac{1 - q^{x^{-2}}}{1 - q^{(x+1)^{-2}}}\right)^\gamma - 1, \qquad x \in \mathbb{N}^*, \tag{27}$$

where $\gamma > 0, q \in (0,1)$ are parameters;

$$F(x;n) = 1 - \frac{(n-1)!}{n^x (n-1-x)!}, \qquad x \in I, \tag{28}$$

$$f(x;n) = \frac{x}{n^x} \cdot \frac{(n-1)!}{(n-x)!}, \qquad x \in I, \tag{29}$$

$$h_F(x) = \frac{x}{n-x}, \qquad x \in I, \tag{30}$$

where $I = \{1,2,...n\}$, $n > 1$ is a positive integer;

$$F(x;\alpha) = 1 - \frac{\alpha^{x+1}}{1-\ln\alpha}\left\{-\ln\alpha + (1-(x+1)\ln\alpha)\alpha^{x+1}\right\}, \qquad x \in \mathbb{N}^*, \tag{31}$$

$$f(x;\alpha) = \frac{(\alpha-1)\alpha^x \ln\alpha}{1-\ln\alpha} + \frac{\alpha^{2x}}{1-\ln\alpha}\left\{1-\alpha^2+\alpha^2\ln\alpha+\left(\alpha^2-1\right)x\ln\alpha\right\}, \quad x \in \mathbb{N}^*, \tag{32}$$

$$h_F(x) = \frac{(\alpha-1)\ln\alpha + \alpha^x\left\{1-\alpha^2+\alpha^2\ln\alpha+\left(\alpha^2-1\right)x\ln\alpha\right\}}{\alpha\left\{-\ln\alpha+(1-(x+1)\ln\alpha)\,\alpha^{x+1}\right\}}, \qquad x\in\mathbb{N}^*, \tag{33}$$

where $\alpha\in(0,1)$ is a parameter;

$$F(x;\alpha,\beta) = \frac{\alpha\left(e^{\beta(x+1)}-1\right)}{1+\alpha e^{\beta(x+1)}}, \qquad x\in\mathbb{N}^*, \tag{34}$$

$$f(x;\alpha,\beta) = \frac{\alpha(1+\alpha)\left(e^{\beta}-1\right)e^{\beta x}}{\left(1+\alpha e^{\beta x}\right)\left(1+\alpha e^{\beta(x+1)}\right)}, \qquad x\in\mathbb{N}^*, \tag{35}$$

$$r_F(x) = \frac{(1+\alpha)\left(e^{\beta}-1\right)e^{\beta x}}{\left(1+\alpha e^{\beta x}\right)\left(e^{\beta(x+1)}-1\right)}, \qquad x\in\mathbb{N}^*, \tag{36}$$

where $\alpha>0,\beta>0$ are parameters;

$$F(x;\beta,p,\gamma) = 1-\exp\left[\left(\frac{G(x+1;\gamma)}{\overline{G}(x+1;\gamma)}\right)^{\beta}\ln(p)\right], \qquad x\in\mathbb{N}^*, \tag{37}$$

$$f(x;\beta,p,\gamma) = \exp\left[\left(\frac{G(x;\gamma)}{\overline{G}(x;\gamma)}\right)^{\beta}\ln(p)\right]-\exp\left[\left(\frac{G(x+1;\gamma)}{\overline{G}(x+1;\gamma)}\right)^{\beta}\ln(p)\right], \ x\in\mathbb{N}^*, \tag{38}$$

$$h_F(x) = \exp\left[\left\{\left(\frac{G(x;\gamma)}{\overline{G}(x;\gamma)}\right)^{\beta}-\left(\frac{G(x+1;\gamma)}{\overline{G}(x+1;\gamma)}\right)^{\beta}\right\}\ln(p)\right]-1, \quad x\in\mathbb{N}^*, \tag{39}$$

where $\beta>0, p\in(0,1)$ are parameters and $G(x;\gamma)$ is a baseline cdf, which may depend on the parameter vector γ;

$$F(x;\gamma,\delta,\upsilon) = 1-\frac{\gamma}{\left(1+\frac{x+1}{\upsilon}\right)^{\delta}-1+\gamma}, \qquad x\in\mathbb{N}^*, \tag{40}$$

$$f(x;\gamma,\delta,\upsilon) = \frac{\gamma}{\left(1+\frac{x}{\upsilon}\right)^{\delta}-1+\gamma}-\frac{\gamma}{\left(1+\frac{x+1}{\upsilon}\right)^{\delta}-1+\gamma}, \qquad x\in\mathbb{N}^*, \tag{41}$$

$$h_F(x) = \frac{\left(1+\frac{x+1}{\upsilon}\right)^{\delta}-1+\gamma}{\left(1+\frac{x}{\upsilon}\right)^{\delta}-1+\gamma}-1, \qquad x\in\mathbb{N}^*, \tag{42}$$

where γ,δ,υ are all positive parameters;

$$F(x;a,b) = \frac{\Lambda(x+1;a,b)}{(1-\log a)^{b}}, \qquad x\in\mathbb{N}^*, \tag{43}$$

$$f(x;a,b) = \frac{1}{(1-\log a)^{b}}\left[\Lambda(x+1;a,b)-\Lambda(x;a,b)\right], \qquad x\in\mathbb{N}^*, \tag{44}$$

$$r_F(x) = 1-\frac{\Lambda(x;a,b)}{\Lambda(x+1;a,b)}, \qquad x\in\mathbb{N}^*, \tag{45}$$

where $a\in(0,1), b>0$ are parameters and $\Lambda(x;a,b) = \left(1-a^x+\left[(1+x)a^x-1\right]\log a\right)^{b}$;

$$F(x;\theta,m) = h(\theta)\sum_{j=1}^{m}a_{j-1}\frac{\Gamma(j)}{\theta^j}I_\theta(x,j+1), \qquad x\in\mathbb{N}, \tag{46}$$

$$f(x;\theta,m) = h(\theta)\,p(x)\,(1-\theta)^{x}, \qquad x\in\mathbb{N}, \tag{47}$$

$$r_F(x) = \frac{p(x)(1-\theta)^x}{\sum_{j=1}^{m} a_{j-1}\frac{\Gamma(j)}{\theta^j} I_\theta(x, j+1)} = \frac{p(x)(1-\theta)^x}{Q(x)}, \qquad x \in \mathbb{N}, \tag{48}$$

where $\theta \in (0,1)$, $m \in \mathbb{N}$ are parameters, a_{j-1} $'s$ are known non-negative constants and $I_\theta(a,b) = \frac{1}{B(a,b)}\int_0^\theta t^{a-1}(1-t)^{b-1}dt$ in which a and b must be positive. We should mention that the authors of this paper have allowed $x = 0$, which in view of the definition of $I_\theta(x, j+1)$ is not correct;

$$\begin{aligned} F(x;\theta) &= 1 - \frac{e^{-\theta(x-2)}\left[\left(x^2+2x+2\right) - \left(2x^2+2x+1\right)e^{-\theta} + \left(x^2+1\right)e^{-2\theta}\right]}{2e^{2\theta} - e^{\theta} + 1} \\ &= 1 - \frac{e^{-\theta(x-2)}Q(x)}{2e^{2\theta} - e^{\theta} + 1}, \qquad x \in \mathbb{N}, \end{aligned} \tag{49}$$

$$f(x;\theta) = \frac{\left(e^{\theta}-1\right)^3\left(1+x^2\right)e^{-\theta x}}{2e^{2\theta} - e^{\theta} + 1}, \qquad x \in \mathbb{N}, \tag{50}$$

$$h_F(x) = \frac{\left(e^{\theta}-1\right)^3\left(1+x^2\right)}{e^{2\theta}Q(x)}, \qquad x \in \mathbb{N}, \tag{51}$$

where $\theta > 0$ is a parameter;

$$\begin{aligned} F(x;\alpha,\theta) &= \frac{\theta}{(\alpha^2+2\alpha+2)(\theta+1)^3} \times \\ &\sum_{j=0}^{x} \frac{\left[\theta^2 j^2 + \left(2\alpha\theta^2 + 2\alpha\theta + 3\theta^2\right)j + \left(\alpha^2\theta^2 + 2\alpha^2\theta + 2\alpha\theta^2 + 2\alpha\theta + 2\theta^2 + \alpha^2\right)\right]}{(\theta+1)^j} \\ &= \frac{\theta}{(\alpha^2+2\alpha+2)(\theta+1)^3}\sum_{j=0}^{x}\frac{Q(j)}{(\theta+1)^j}, \qquad x \in \mathbb{N}^*, \end{aligned} \tag{52}$$

$$f(x;\alpha,\theta) = \frac{\theta}{(\alpha^2+2\alpha+2)(\theta+1)^3}\left(\frac{Q(x)}{(\theta+1)^x}\right), \qquad x \in \mathbb{N}^*, \tag{53}$$

$$r_F(x) = \frac{Q(x)(\theta+1)^{-x}}{\sum_{j=0}^{x}\frac{Q(j)}{(\theta+1)^j}}, \qquad x \in \mathbb{N}^*, \tag{54}$$

where $\theta > 0, \alpha$ with $\left(\alpha^2 + 2\alpha + 2 > 0\right)$ are parameters;

$$\begin{aligned} F(x;\theta) &= C(\theta)\sum_{j=1}^{x}\frac{\left[j^2+3j+\left(\theta^3+2\theta^2+\theta+2\right)\right]}{(\theta+1)^j} \\ &= C(\theta)\sum_{j=1}^{x}\frac{Q(j)}{(\theta+1)^j}, \qquad x \in \mathbb{N}, \end{aligned} \tag{55}$$

$$f(x;\theta) = C(\theta)\left(\frac{Q(x)}{(\theta+1)^x}\right), \qquad x \in \mathbb{N}, \tag{56}$$

$$r_F(x) = \frac{Q(x)(\theta+1)^{-x}}{\sum_{j=1}^{x}\frac{Q(j)}{(\theta+1)^j}}, \qquad x \in \mathbb{N}, \tag{57}$$

where $\theta > 0$ is a parameter and $C(\theta) = \frac{\theta^3}{\theta^5+2\theta^4+\theta^3+6\theta^2+6\theta+2}$;

$$F(x;\theta,\alpha) = C(\theta,\alpha)\sum_{j=0}^{x}\frac{\left[\theta(\theta+1)^{\alpha}\Gamma(j+1)+\Gamma(j+\alpha+1)\right]}{(\theta+1)^{j}\Gamma(j+1)}$$
$$= C(\theta,\alpha)\sum_{j=0}^{x}\frac{Q(j)}{(\theta+1)^{j}}, \qquad x \in \mathbb{N}^*, \tag{58}$$

$$f(x;\theta,\alpha) = C(\theta,\alpha)\left(\frac{Q(x)}{(\theta+1)^{x}}\right), \qquad x \in \mathbb{N}^*, \tag{59}$$

$$r_F(x) = \frac{Q(x)(\theta+1)^{-x}}{\sum_{j=0}^{x}\frac{Q(j)}{(\theta+1)^{j}}}, \qquad x \in \mathbb{N}^*, \tag{60}$$

where $\theta > 0, \alpha > 0$ are parameters and $C(\theta,\alpha) = \frac{\theta^{\alpha+1}}{(\theta+1)^{\alpha+1}\left[\theta^{\alpha+1}+\Gamma(\alpha+1)\right]}$.

2. Characterizations results

We present our characterizations (i), (ii) and (iii) via three subsections 2.1., 2.2. and 2.3..

2.1. Characterizations of DsGLi, DGu, SG, P-Xgamma, CosPois, SBPA, GHGD, PQX, EDIR, Mlynar, DsFx-I, DP, DWG, DMOL, EDLi, NDOPPE, ZTDAD, QPAD, ZTPID and PSD respectively, in terms of the conditional expectation of certain function of the random variable

Proposition 2.1.1. Let $X : \Omega \to \mathbb{N}^*$ be a random variable. The pmf of X is (2) if and only if

$$E\left\{\left[\begin{array}{c} 1-\eta-\ln\eta\left[1+X-\eta(X+2)\right]+ \\ \left(1-\frac{1}{\alpha}\right)(\ln\eta)^2\left[X-\eta(X+1)\right]\end{array}\right]^{-1} \mid X > k\right\}$$
$$= \frac{\alpha}{(1-\eta)\left[\left(1-\ln\eta^{k+1}\right)(\alpha-\alpha\ln\eta+\ln\eta)-\ln\eta\right]}, \qquad x \in \mathbb{N}^*, \tag{61}$$

Proof. If X has pmf (2), then the left-hand side of (61) will be

$$(1-F(k))^{-1}\sum_{x=k+1}^{\infty}\left\{\left[\frac{\eta^{x}}{(1-\ln\eta)}\right]\right\}$$
$$= \frac{\alpha(1-\ln\eta)}{\left[\left(1-\ln\eta^{k+1}\right)(\alpha-\alpha\ln\eta+\ln\eta)-\ln\eta\right]\eta^{k+1}}\left[\frac{\eta^{k+1}}{(1-\eta)(1-\ln\eta)}\right]$$
$$= \frac{\alpha}{(1-\eta)\left[\left(1-\ln\eta^{k+1}\right)(\alpha-\alpha\ln\eta+\ln\eta)-\ln\eta\right]}.$$

Conversely, if (61) holds, then

$$\sum_{x=k+1}^{\infty}\left\{\left[\begin{array}{c} 1-\eta-\ln\eta\left[1+X-\eta(X+2)\right]+ \\ \left(1-\frac{1}{\alpha}\right)(\ln\eta)^2\left[X-\eta(X+1)\right]\end{array}\right]^{-1} f(x)\right\}$$
$$= (1-F(k))\times\left[\frac{\alpha}{(1-\eta)\left[\left(1-\ln\eta^{k+1}\right)(\alpha-\alpha\ln\eta+\ln\eta)-\ln\eta\right]}\right]$$
$$= \left[1-F(k+1)+f(k+1)\right]\times\left[\frac{\alpha}{(1-\eta)\left[\left(1-\ln\eta^{k+1}\right)(\alpha-\alpha\ln\eta+\ln\eta)-\ln\eta\right]}\right], \tag{62}$$

where we have used $F(k) = F(k+1) - f(k+1)$. From (61), we also have

$$\sum_{x=k+2}^{\infty}\left\{\left[\begin{array}{c} 1-\eta-\ln\eta\left[1+X-\eta(X+2)\right]+ \\ \left(1-\frac{1}{\alpha}\right)(\ln\eta)^2\left[X-\eta(X+1)\right]\end{array}\right]^{-1} f(x)\right\}$$
$$= \left[1-F(k+1)\right]\times\left[\frac{\alpha}{(1-\eta)\left[\left(1-\ln\eta^{k+2}\right)(\alpha-\alpha\ln\eta+\ln\eta)-\ln\eta\right]}\right]. \tag{63}$$

Now, subtracting (63) from (62), we arrive at

$$\left[1-F(k+1)\right]\left\{\begin{array}{c}\frac{\alpha}{(1-\eta)\left[\left(1-\ln\eta^{k+1}\right)(\alpha-\alpha\ln\eta+\ln\eta)-\ln\eta\right]} \\ \\ -\frac{\alpha}{(1-\eta)\left[\left(1-\ln\eta^{k+2}\right)(\alpha-\alpha\ln\eta+\ln\eta)-\ln\eta\right]}\end{array}\right\}$$

$$= \left\{\begin{array}{c}\frac{1}{\begin{array}{c}1-\eta-\ln\eta\left[1+(k+1)-\eta(k+3)\right]+ \\ \left(1-\frac{1}{\alpha}\right)(\ln\eta)^2\left[(k+1)-\eta(k+2)\right]\end{array}} \\ \\ -\frac{\alpha}{(1-\eta)\left[\left(1-\ln\eta^{k+1}\right)(\alpha-\alpha\ln\eta+\ln\eta)-\ln\eta\right]}\end{array}\right\} f(k+1).$$

From the last equality, after some computations, we arrive at

$$h_F(k+1) = \frac{f(k+1)}{1-F(k+1)} = \frac{\alpha\left\{\begin{array}{c}1-\eta-\ln\eta\left[1+(k+1)-\eta(k+3)\right]+ \\ \left(1-\frac{1}{\alpha}\right)(\ln\eta)^2\left[(k+1)-\eta(k+2)\right]\end{array}\right\}}{\left\{\left(1-\ln\eta^{k+2}\right)(\alpha-\alpha\ln\eta+\ln\eta)-\ln\eta\right\}\eta},$$

which, in view of (3), implies that X has pmf (2).

Proposition 2.1.2. Let $X : \Omega \to \mathbb{Z}$ be a random variable. The pmf of X is (5) if and only if

$$E\left\{\left[e^{-\alpha p^{X+1}} + e^{-\alpha p^{X}}\right] \mid X \le k\right\} = e^{-\alpha p^{k+1}}, \qquad k \in \mathbb{Z}. \tag{64}$$

Proof. If X has pmf (5), then the left-hand side of (64), using the telescoping series property, will be

$$(F(k))^{-1}\sum_{x=-\infty}^{k}\left\{\left[e^{-2\alpha p^{x+1}} - e^{-2\alpha p^{x}}\right]\right\} = e^{\alpha p^{k+1}}\left\{e^{-2\alpha p^{k+1}}\right\} = e^{-\alpha p^{k+1}}.$$

Conversely, if (64) holds, then

$$\sum_{x=-\infty}^{k}\left\{\left[e^{-\alpha p^{x+1}} + e^{-\alpha p^{x}}\right] f(x)\right\} = F(k)\, e^{-\alpha p^{k+1}}$$
$$= \left[F(k+1) - f(k+1)\right] e^{-\alpha p^{k+1}}. \tag{65}$$

From (64), we also have

$$\sum_{x=-\infty}^{k+1}\left\{\left[e^{-\alpha p^{x+1}} + e^{-\alpha p^{x}}\right] f(x)\right\} = F(k+1)\, e^{-\alpha p^{k+2}}. \tag{66}$$

Now, subtracting (65) from (66), we arrive at

$$F(k+1)\left[e^{-\alpha p^{k+2}} - e^{-\alpha p^{k+1}}\right] = \left[e^{-\alpha p^{k+2}}\right] f(k+1).$$

From the last equality, we have

$$r_F(k+1) = \frac{f(k+1)}{F(k+1)} = 1 - \left(\frac{e^{-\alpha p^{k+1}}}{e^{-\alpha p^{k+2}}}\right),$$

which, in view of (6), implies that X has pmf (5).

Proposition 2.1.3. $X : \Omega \to \mathbb{N}^*$ be a random variable. The pmf of X is (8) if and only if

$$E\left\{\left[\left(1-p^{\alpha X+1}\right)^{-1}\right] \mid X \ \leq k\right\} = \frac{\left(1-p^{\alpha+1}\right)\left(1-p^{k+1}\right)}{\left(1-p^{\alpha+1}\right)\left(1-p^{k+1}\right) - pq\left(1-p^{(\alpha+1)(k+1)}\right)}, \qquad k \in \mathbb{N}^*, \tag{67}$$

Proof. If X has pmf (8), then the left-hand side of (67) , using the telescoping series property, will be

$$(F(k))^{-1}\sum_{x=0}^{k}\{Cq[p^x]\} = \frac{1-p^{k+1}}{\left(1-p^{k+1}\right) - pq\left(\frac{1-p^{(\alpha+1)(k+1)}}{1-p^{\alpha+1}}\right)} = \frac{\left(1-p^{\alpha+1}\right)\left(1-p^{k+1}\right)}{\left(1-p^{\alpha+1}\right)\left(1-p^{k+1}\right) - pq\left(1-p^{(\alpha+1)(k+1)}\right)}.$$

Conversely, if (67) holds, then

$$\sum_{x=-\infty}^{k}\left\{\left[\frac{1}{1-p^{\alpha x+1}}\right]\ f(x)\right\} = F(k)\frac{\left(1-p^{\alpha+1}\right)\left(1-p^{k+1}\right)}{\left(1-p^{\alpha+1}\right)\left(1-p^{k+1}\right) - pq\left(1-p^{(\alpha+1)(k+1)}\right)}$$
$$= [F(k+1) - f(k+1)] \times \frac{\left(1-p^{\alpha+1}\right)\left(1-p^{k+1}\right)}{\left(1-p^{\alpha+1}\right)\left(1-p^{k+1}\right) - pq\left(1-p^{(\alpha+1)(k+1)}\right)}, \tag{68}$$

From (67), we also have

$$\sum_{x=-\infty}^{k+1}\left\{\left[\frac{1}{1-p^{\alpha x+1}}\right]\ f(x)\right\} = F(k+1) \times \frac{\left(1-p^{\alpha+1}\right)\left(1-p^{k+2}\right)}{\left(1-p^{\alpha+1}\right)\left(1-p^{k+2}\right) - pq\left(1-p^{(\alpha+1)(k+2)}\right)}. \tag{69}$$

Now, subtracting (68) from (69), we arrive at

$$F(k+1)\left[\frac{pq\left[\left(1-p^{k+1}\right)\left(1-p^{(\alpha+1)(k+2)}\right) - \left(1-p^{k+2}\right)\left(1-p^{(\alpha+1)(k+1)}\right)\right]}{\left(1-p^{\alpha+1}\right)\left(1-p^{k+2}\right) - pq\left(1-p^{(\alpha+1)(k+2)}\right)}\right]$$
$$= \left[\frac{\left(1-p^{k+1}\right) - pq\left(1-p^{(\alpha+1)(k+1)}\right) - \left(1-p^{k+2}\right)\left(1-p^{\alpha(k+1)+1}\right)}{\left(1-p^{\alpha(k+1)+1}\right)}\right]\ f(k+1).$$

From the last equality, after a good deal of simplifications, we have

$$r_F(k+1) = \frac{f(k+1)}{F(k+1)} = \frac{q\left(1-p^{\alpha+1}\right)p^{k+1}\left(1-p^{\alpha(k+1)+1}\right)}{\left(1-p^{\alpha+1}\right)\left(1-p^{k+2}\right) - pq\left(1-p^{(\alpha+1)(k+2)}\right)},$$

which, in view of (9), implies that X has pmf (8).

Proposition 2.1.4. Let $X : \Omega \to \mathbb{N}^*$ be a random variable. The pmf of X is (11) if and only if

$$E\left\{\left[2(1+\theta)^2+\theta(X+1)(X+2)\right]^{-1} \mid X > k\right\} = \frac{\theta}{\theta(\theta+5)k^2+2\theta k+\gamma}, \quad k \in \mathbb{N}^*. \tag{70}$$

Proof. If X has pmf (11), then the left-hand side of (70) , using the infinite geometric series property, will be

$$(1-F(k))^{-1}\frac{\theta^2}{2(1+\theta)^4}\sum_{x=k+1}^{\infty}\left(\frac{1}{1+\theta}\right)^x = (1-F(k))^{-1}\frac{\theta}{2(1+\theta)^{k+4}} = \frac{\theta}{\theta(\theta+5)k^2+2\theta k+\gamma}.$$

Conversely, if (70) holds, then

$$\sum_{x=k+1}^{\infty}\left\{\left[2(1+\theta)^2+\theta(x+1)(x+2)\right]^{-1} f(x)\right\} = (1-F(k))\left[\frac{\theta}{\theta(\theta+5)k^2+2\theta k+\gamma}\right] = \left[(1-F(k+1))+f(k+1)\right]\left[\frac{\theta}{\theta(\theta+5)k^2+2\theta k+\gamma}\right]. \tag{71}$$

From (70), we also have

$$\sum_{x=k+2}^{\infty}\left\{\left[2(1+\theta)^2+\theta(x+1)(x+2)\right]^{-1} f(x)\right\} = (1-F(k+1))\left[\frac{\theta}{\theta(\theta+5)(k+1)^2+2\theta(k+1)+\gamma}\right]. \tag{72}$$

Now, subtracting (72) from (71), we arrive at

$$(1-F(k+1))\left\{\frac{\theta}{\theta(\theta+5)k^2+2\theta k+\gamma} - \frac{(\theta)}{(\theta)(\theta+5)(k+1)^2+2\theta(k+1)+\gamma}\right\} = \left\{\frac{1}{2(1+\theta)^2+\theta(k+2)(k+3)} - \frac{\theta}{\theta(\theta+5)(k+1)^2+2\theta k+\gamma}\right\} f(k+1).$$

From the last equality, after a good deal of algebra, we have

$$h_F(k+1) = \frac{f(k+1)}{1-F(k+1)} = \frac{\theta^2\left[2(1+\theta)^2+\theta(k+2)(k+3)\right]}{\theta(\theta+5)(k+1)^2+2\theta(k+1)+\gamma},$$

which, in view of (12), implies that X has pmf (11).

Proposition 2.1.5. Let $X : \Omega \to \mathbb{N}^*$ be a random variable. The pmf of X is (14) if and only if

$$E\left\{\left[\frac{X!}{(\cos(\beta X))^2}\right] \mid X \le k\right\} = \frac{2e^{-\lambda}\left(\lambda^{k+1}-1\right)}{(\lambda-1)\left[F_*(k;\lambda)+\Upsilon(k;\beta,\lambda)\right]}, \quad k \in \mathbb{N}^*. \tag{73}$$

Proof. If X has pmf (14), then the left-hand side of (73) , using the finite geometric series property, will be

$$(F(k))^{-1}\sum_{x=0}^{k}\{C\lambda^x\}=\frac{C}{C^*\{F_*(k;\lambda)+\Upsilon(k;\beta,\lambda)\}}\sum_{x=0}^{k}\lambda^x$$
$$=\frac{2e^{-\lambda x}}{F_*(k;\lambda)+\Upsilon(k;\beta,\lambda)}\sum_{x=0}^{k}\lambda^x=\left[\frac{2e^{-\lambda}}{F_*(k;\lambda)+\Upsilon(k;\beta,\lambda)}\right]\left(\frac{\lambda^{k+1}-1}{\lambda-1}\right)$$
$$=\frac{2e^{-\lambda}\left(\lambda^{k+1}-1\right)}{(\lambda-1)\left[F_*(k;\lambda)+\Upsilon(k;\beta,\lambda)\right]}.$$

Conversely, if (73) holds, then

$$\sum_{x=0}^{k}\left\{\left[\frac{x!}{(\cos(\beta x))^2}\right]f(x)\right\}=F(k)\frac{2e^{-\lambda}\left(\lambda^{k+1}-1\right)}{(\lambda-1)\left[F_*(k;\lambda)+\Upsilon(k;\beta,\lambda)\right]}$$
$$=\left[F(k+1)-f(k+1)\right]\left[\frac{2e^{-\lambda}\left(\lambda^{k+1}-1\right)}{(\lambda-1)\left[F_*(k;\lambda)+\Upsilon(k;\beta,\lambda)\right]}\right], \tag{74}$$

From (73), we also have

$$\sum_{x=0}^{k+1}\left\{\left[\frac{x!}{(\cos(\beta x))^2}\right]f(x)\right\}$$
$$=F(k+1)\left[\frac{2e^{-\lambda}\left(\lambda^{k+2}-1\right)}{(\lambda-1)\left[F_*(k+1;\lambda)+\Upsilon(k+1;\beta,\lambda)\right]}\right], \tag{75}$$

Now, subtracting (74) from (75), we arrive at

$$F(k+1)\left[\frac{2e^{-\lambda}\left(\lambda^{k+2}-1\right)}{(\lambda-1)\left[F_*(k+1;\lambda)+\Upsilon(k+1;\beta,\lambda)\right]}-\frac{2e^{-\lambda}\left(\lambda^{k+1}-1\right)}{(\lambda-1)\left[F_*(k;\lambda)+\Upsilon(k;\beta,\lambda)\right]}\right]$$
$$=\left[\frac{(k+1)!}{(\cos(\beta(k+1)))^2}-\frac{2e^{-\lambda}\left(\lambda^{k+1}-1\right)}{(\lambda-1)\left[F_*(k;\lambda)+\Upsilon(k;\beta,\lambda)\right]}\right]f(k+1).$$

From the last equality, after a good deal of simplifications, we have

$$r_F(k+1)=\frac{f(k+1)}{F(k+1)}=\frac{2e^{-\lambda}\left[\cos(\beta(k+1))\right]^2\frac{\lambda^{k+1}}{(k+1)!}}{F_*(k+1;\lambda)+\Upsilon(k+1;\beta,\lambda)},$$

which, in view of (15), implies that X has pmf (14).

Proposition 2.1.6. Let $X:\Omega\to\mathbb{N}$ be a random variable. The pmf of X is (17) if and only if

$$E\left\{\left[X(X+1)\right]^{-1}\mid X>k\right\}=\frac{2\eta^2}{2(k+2)\eta\left[1+(k+1)\eta\right]+1},\quad k\in\mathbb{N}. \tag{76}$$

Proof. If X has pmf (17), then the left-hand side of (76) will be

$$\begin{aligned}&(1-F(k))^{-1}\sum_{x=k+1}^{\infty}\left\{\left[\frac{4\eta^3}{(1+2\eta)^{x+2}}\right]\right\}\\&=\frac{(1+2\eta)^{k+2}}{2\eta(k+2)[1+\eta(k+1)]+1}\left[\frac{2\eta^2}{(1+2\eta)^{k+2}}\right]\\&=\frac{2\eta^2}{2(k+2)\eta[1+(k+1)\eta]+1}.\end{aligned}$$

Conversely, if (76) holds, then

$$\begin{aligned}&\sum_{x=k+1}^{\infty}\left\{\left[\frac{1}{x(x+1)}\right]\ f(x)\right\}=(1-F(k))\left[\frac{2\eta^2}{2(k+2)\eta[1+(k+1)\eta]+1}\right]\\&=[1-F(k+1)+f(k+1)]\left[\frac{2\eta^2}{2(k+2)\eta[1+(k+1)\eta]+1}\right].\end{aligned}\tag{77}$$

From (76), we also have

$$\sum_{x=k+2}^{\infty}\left\{\left[\frac{1}{x(x+1)}\right]\ f(x)\right\}=[1-F(k+1)]\left[\frac{2\eta^2}{2(k+3)\eta[1+(k+2)\eta]+1}\right].\tag{78}$$

Now, subtracting (78) from (77), we have

$$\begin{aligned}&[1-F(k+1)]\,2\eta^2\left\{\frac{1}{2(k+2)\eta[1+(k+1)\eta]+1}-\frac{1}{2(k+3)\eta[1+(k+2)\eta]+1}\right\}\\&=\left\{\frac{1}{(k+1)(k+2)}-\frac{2\eta^2}{2(k+2)\eta[1+(k+1)\eta]+1}\right\}\ f(k+1).\end{aligned}$$

From the last equality, after some computations, we arrive at

$$h_F(k+1)=\frac{f(k+1)}{1-F(k+1)}=\frac{\frac{4\eta^3}{2(k+3)\eta[1+(k+2)\eta]+1}}{\frac{1}{(k+1)(k+2)}}=\frac{4\eta^3(k+1)(k+2)}{2(k+3)\eta[1+(k+2)\eta]+1},$$

which, in view of (18), implies that X has pmf (17).

Proposition 2.1.7. Let $X:\Omega\to\mathbb{N}^*$ be a random variable. The pmf of X is (20) if and only if

$$E\left\{[H_{n,m}(X+\gamma,\beta)]^{-1}\ \mid X\ >k\right\}=\frac{1}{(1-\alpha)G(\alpha;\beta,k+\gamma+1)},\quad k\in\mathbb{N}^*.\tag{79}$$

Proof. If X has pmf (20), then the left-hand side of (79) will be

$$\begin{aligned}&(1-F(k))^{-1}\sum_{x=k+1}^{\infty}\left\{\left[\frac{\alpha^k}{G(\alpha;\beta,\gamma)}\right]\right\}\\&=\frac{G(\alpha;\beta,\gamma)}{\alpha^{k+1}G(\alpha;\beta,k+\gamma+1)}\left[\frac{\alpha^{k+1}}{(1-\alpha)G(\alpha;\beta,\gamma)}\right]\\&=\frac{1}{(1-\alpha)G(\alpha;\beta,k+\gamma+1)}.\end{aligned}$$

Conversely, if (79) holds, then

$$\sum_{x=k+1}^{\infty}\left\{\left[\frac{1}{H_{n,m}(x+\gamma,\beta)}\right]\ f(x)\right\}=(1-F(k))\left[\frac{1}{(1-\alpha)G(\alpha;\beta,k+\gamma+1)}\right]$$
$$=[1-F(k+1)+f(k+1)]\left[\frac{1}{(1-\alpha)G(\alpha;\beta,k+\gamma+1)}\right]. \tag{80}$$

From (79), we also have

$$\sum_{x=k+2}^{\infty}\left\{\left[\frac{1}{H_{n,m}(x+\gamma,\beta)}\right]\ f(x)\right\}$$
$$=[1-F(k+1)]\left[\frac{1}{(1-\alpha)G(\alpha;\beta,k+1+\gamma+1)}\right]. \tag{81}$$

Now, subtracting (81) from (80), we have

$$[1-F(k+1)]\left\{\frac{1}{(1-\alpha)G(\alpha;\beta,k+\gamma+1)}-\frac{1}{(1-\alpha)G(\alpha;\beta,k+1+\gamma+1)}\right\}$$
$$=\left\{\frac{1}{H_{n,m}(x+\gamma,\beta)}-\frac{1}{(1-\alpha)G(\alpha;\beta,k+\gamma+1)}\right\}\ f(k+1).$$

From the last equality, after some computations, we arrive at

$$h_F(k+1)=\frac{f(k+1)}{1-F(k+1)}=\frac{\frac{1}{G(\alpha;\beta,k+\gamma+1)(1-\alpha)}-\frac{1}{G(\alpha;\beta,k+1+\gamma+1)(1-\alpha)}}{\frac{1}{H_{n,m}(x+\gamma,\beta)}-\frac{1}{(1-\alpha)G(\alpha;\beta,k+\gamma+1)}}$$
$$=\frac{1}{\alpha}\left(\frac{H_{n,m}(k+1+\gamma,\beta)}{G(\alpha;\beta,k+1+\gamma+1)}\right),$$

which, in view of (21), implies that X has pmf (20).

Proposition 2.1.8. Let $X:\Omega\to\mathbb{N}^*$ be a random variable. The pmf of X is (23) if and only if

$$E\left\{\left[\frac{1}{2\alpha\theta(\theta+1)^2+\theta^3(X+1)(X+2)}\right]\mid X>k\right\}$$
$$=\frac{1}{\theta\left\{2\alpha(\theta+1)^2+\theta(k+3)(\theta(k+2)+2)+2\right\}}. \tag{82}$$

Proof. If X has pmf (23), then for $k\in\mathbb{N}^*$, the left-hand side of (82) , using infinite geometric series formula, will be

$$(1-F(k))^{-1}\sum_{x=k+1}^{\infty}\frac{1}{2(\alpha+1)}(\theta+1)^{-(x+3)}$$
$$=\frac{2(\alpha+1)(\theta+1)^{k+3}}{2\alpha(\theta+1)^2+\theta(k+3)(\theta(k+2)+2)+2}\times\frac{(\theta+1)^{-(k+3)}}{2\theta(\alpha+1)}$$
$$=\frac{1}{\theta\left\{2\alpha(\theta+1)^2+\theta(k+3)(\theta(k+2)+2)+2\right\}}.$$

Conversely, if (82) holds, then

$$\sum_{x=k+1}^{\infty}\left\{\left[\frac{1}{2\alpha\theta\left(\theta+1\right)^{2}+\theta^{3}\left(x+1\right)\left(x+2\right)}\right]\ f\left(x\right)\right\}$$
$$=\frac{\left(1-F\left(k\right)\right)}{\theta\left\{2\alpha\left(\theta+1\right)^{2}+\theta\left(k+3\right)\left(\theta\left(k+2\right)+2\right)+2\right\}}$$
$$=\frac{\left[\left(1-F\left(k+1\right)\right)+f\left(k+1\right)\right]}{\theta\left\{2\alpha\left(\theta+1\right)^{2}+\theta\left(k+3\right)\left(\theta\left(k+2\right)+2\right)+2\right\}}. \tag{83}$$

From (82), we also have

$$\sum_{x=k+2}^{\infty}\left\{\left[\frac{1}{2\alpha\theta\left(\theta+1\right)^{2}+\theta^{3}\left(x+1\right)\left(x+2\right)}\right]\ f\left(x\right)\right\}$$
$$=\frac{\left(1-F\left(k+1\right)\right)}{\theta\left\{2\alpha\left(\theta+1\right)^{2}+\theta\left(k+4\right)\left(\theta\left(k+3\right)+2\right)+2\right\}}. \tag{84}$$

Now, subtracting (84) from (83), yields

$$\left[\frac{1}{2\alpha\theta\left(\theta+1\right)^{2}+\theta^{3}\left(k+2\right)\left(k+3\right)}\right]f\left(k+1\right)$$
$$=\left(1-F\left(k+1\right)\right)\left\{\begin{array}{c}\frac{1}{\theta\left\{2\alpha(\theta+1)^{2}+\theta(k+3)(\theta(k+2)+2)+2\right\}}-\\ \frac{1}{\theta\left\{2\alpha(\theta+1)^{2}+\theta(k+4)(\theta(k+3)+2)+2\right\}}\end{array}\right\}$$
$$+f\left(k+1\right)\left[\frac{1}{\theta\left\{2\alpha\left(\theta+1\right)^{2}+\theta\left(k+3\right)\left(\theta\left(k+2\right)+2\right)+2\right\}}\right].$$

From the above equality, after some regrouping the terms and simplifications, we have

$$\frac{f\left(k+1\right)}{1-F\left(k+1\right)}=\frac{2\alpha\theta\left(\theta+1\right)^{2}+\theta^{3}\left(k+2\right)\left(k+3\right)}{2\alpha\left(\theta+1\right)^{2}+\theta\left(k+4\right)\left(\theta\left(k+3\right)+2\right)+2},$$

which, in view of (24), implies that X has pmf (23).

Proposition 2.1.9. Let $X:\Omega\rightarrow\mathbb{N}^{*}$ be a random variable. The pmf of X is (26) if and only if

$$E\left\{\left[\left(1-q^{X^{-2}}\right)^{\gamma}+\left(1-q^{(X+1)^{-2}}\right)^{\gamma}\right]\ \mid X\ >k\right\}$$
$$=\left(1-q^{(k+1)^{-2}}\right)^{\gamma},\qquad k\in\mathbb{N}^{*}, \tag{85}$$

Proof. If X has pmf (26), then the left-hand side of (85) will be

$$\left(1-F\left(k\right)\right)^{-1}\sum_{x=k+1}^{\infty}\left\{\left[\left(1-q^{x^{-2}}\right)^{2\gamma}-\left(1-q^{(k+1)^{-2}}\right)^{2\gamma}\right]\right\}$$
$$=\left(1-q^{(k+1)^{-2}}\right)^{-\gamma}\left[\left(1-q^{(k+1)^{-2}}\right)^{2\gamma}\right]=\left(1-q^{(k+1)^{-2}}\right)^{\gamma}.$$

Conversely, if (85) holds, then

$$\sum_{x=k+1}^{\infty}\left\{\left[\left(1-q^{x^{-2}}\right)^{\gamma}+\left(1-q^{(x+1)^{-2}}\right)^{\gamma}\right]\ f\left(x\right)\right\}=\left(1-F\left(k\right)\right)\left(1-q^{(k+1)^{-2}}\right)^{\gamma}$$
$$=\left[1-F\left(k+1\right)+f\left(k+1\right)\right]\left(1-q^{(k+1)^{-2}}\right)^{\gamma}. \tag{86}$$

From (85), we also have

$$\sum_{x=k+2}^{\infty}\left\{\left[\left(1-q^{x^{-2}}\right)^{\gamma}+\left(1-q^{(x+1)^{-2}}\right)^{\gamma}\right]\ f\left(x\right)\right\}$$
$$=\left[1-F\left(k+1\right)\right]\left(1-q^{(k+2)^{-2}}\right)^{\gamma}. \tag{87}$$

Now, subtracting (87) from (86), we arrive at

$$\left[1-F\left(k+1\right)\right]\left\{\left(1-q^{(k+1)^{-2}}\right)^{\gamma}-\left(1-q^{(k+2)^{-2}}\right)^{\gamma}\right\}$$
$$=\left\{\left[\left(1-q^{(k+1)^{-2}}\right)^{\gamma}+\left(1-q^{(k+2)^{-2}}\right)^{\gamma}\right]-\left(1-q^{(k+1)^{-2}}\right)^{\gamma}\right\}\ f\left(k+1\right)$$
$$=\left(1-q^{(k+2)^{-2}}\right)^{\gamma}\ f\left(k+1\right).$$

From the last equality, after some computations, we arrive at

$$h_F\left(k+1\right)=\frac{f\left(k+1\right)}{1-F\left(k+1\right)}=\left(\frac{1-q^{(k+1)^{-2}}}{1-q^{(k+2)^{-2}}}\right)^{\gamma}-1,$$

which, in view of (27), implies that X has pmf (26).

Proposition 2.1.10. Let $X:\Omega\rightarrow I$ be a random variable. The pmf of X is (29) if and only if

$$E\left\{\left[\frac{(n-X)!}{X}\right]\ \mid X\ >k\right\}=\frac{(n-1-k)!\left(n^{n}-n^{k}\right)}{n^{n}\left(n-1\right)},\qquad k\in\mathbb{N}^{*}, \tag{88}$$

Proof. If X has pmf (29), then the left-hand side of (88), using finite geometric sum formula, will be

$$\left(1-F\left(k\right)\right)^{-1}\sum_{x=k+1}^{n}\left\{\left[(n-1)!\left(\frac{1}{n}\right)^{x}\right]\right\}$$
$$=\left(\frac{n^{k}\left(n-1-k\right)!}{(n-1)!}\right)\left[(n-1)!\left(\frac{n^{n-k}-1}{n^{n}\left(n-1\right)}\right)\right]=\frac{(n-1-k)!\left(n^{n}-n^{k}\right)}{n^{n}\left(n-1\right)}.$$

Conversely, if (88) holds, then

$$\sum_{x=k+1}^{n}\left\{\left[\frac{(n-x)!}{x}\right]\ f\left(x\right)\right\}=\left(1-F\left(k\right)\right)\left[\frac{(n-1-k)!\left(n^{n}-n^{k}\right)}{n^{n}\left(n-1\right)}\right]$$
$$=\left[1-F\left(k+1\right)+f\left(k+1\right)\right]\left[\frac{(n-1-k)!\left(n^{n}-n^{k}\right)}{n^{n}\left(n-1\right)}\right]. \tag{89}$$

From (88), we also have

$$\sum_{x=k+2}^{n}\left\{\left[\frac{(n-x)!}{x}\right]\ f\left(x\right)\right\}=\left[1-F\left(k+1\right)\right]\left[\frac{(n-2-k)!\left(n^{n}-n^{k+1}\right)}{n^{n}\left(n-1\right)}\right]. \tag{90}$$

Now, subtracting (90) from (89), we arrive at

$$\begin{aligned}
&[1-F(k+1)]\left\{\left[\frac{(n-1-k)!\left(n^n-n^k\right)}{n^n(n-1)}\right]-\left[\frac{(n-2-k)!\left(n^n-n^{k+1}\right)}{n^n(n-1)}\right]\right\} \\
&=\left\{\left[\frac{(n-1-k)!}{k+1}\right]-\left[\frac{(n-1-k)!\left(n^n-n^k\right)}{n^n(n-1)}\right]\right\} f(k+1) \\
&=(n-1-k)!\left\{\frac{n^n(n-1)-(k+1)\left(n^n-n^k\right)}{(k+1)(n-1)n^n}\right\} f(k+1).
\end{aligned}$$

From the last equality, after some computations, we arrive at

$$h_F(k+1)=\frac{f(k+1)}{1-F(k+1)}=\frac{k+1}{n-1-k},$$

which, in view of (30), implies that X has pmf (29).

Proposition 2.1.11. Let $X:\Omega\rightarrow\mathbb{N}^*$ be a random variable. The pmf of X is (32) if and only if

$$\begin{aligned}
&E\left\{\left[(\alpha-1)\ln\alpha+\alpha^X\left\{1-\alpha^2+\alpha^2\ln\alpha+\left(\alpha^2-1\right)X\ln\alpha\right\}\right]^{-1}\mid X>k\right\} \\
&=\frac{1}{(1-\alpha)\left\{-\ln\alpha+(1-(k+1)\ln\alpha)\alpha^{k+1}\right\}}, \qquad k\in\mathbb{N}^*, \qquad (91)
\end{aligned}$$

Proof. If X has pmf (32), then the left-hand side of (91), using finite geometric sum formula, will be

$$\begin{aligned}
&(1-F(k))^{-1}\sum_{x=k+1}^{\infty}\left\{\left[\frac{\alpha^x}{1-\ln\alpha}\right]\right\} \\
&=\frac{1-\ln\alpha}{\alpha^{k+1}\left\{-\ln\alpha+(1-(k+1)\ln\alpha)\alpha^{k+1}\right\}}\left[\frac{\frac{\alpha^{k+1}}{1-\alpha}}{1-\ln\alpha}\right] \\
&=\frac{1}{(1-\alpha)\left\{-\ln\alpha+(1-(k+1)\ln\alpha)\alpha^{k+1}\right\}}.
\end{aligned}$$

Conversely, if (91) holds, then

$$\begin{aligned}
&\sum_{x=k+1}^{\infty}\left\{\left[\left[(\alpha-1)\ln\alpha+\alpha^x\left\{1-\alpha^2+\alpha^2\ln\alpha+\left(\alpha^2-1\right)x\ln\alpha\right\}\right]^{-1}\right] f(x)\right\} \\
&=(1-F(k))\left[\frac{1}{(1-\alpha)\left\{-\ln\alpha+(1-(k+1)\ln\alpha)\alpha^{k+1}\right\}}\right] \\
&=[1-F(k+1)+f(k+1)]\left[\frac{1}{(1-\alpha)\left\{-\ln\alpha+(1-(k+1)\ln\alpha)\alpha^{k+1}\right\}}\right]. \qquad (92)
\end{aligned}$$

From (91), we also have

$$\begin{aligned}
&\sum_{x=k+2}^{\infty}\left\{\left[(\alpha-1)\ln\alpha+\alpha^x\left\{1-\alpha^2+\alpha^2\ln\alpha+\left(\alpha^2-1\right)x\ln\alpha\right\}\right]^{-1} f(x)\right\} \\
&=[1-F(k+1)]\left[\frac{1}{(1-\alpha)\left\{-\ln\alpha+(1-(k+2)\ln\alpha)\alpha^{k+2}\right\}}\right]. \qquad (93)
\end{aligned}$$

Now, subtracting (93) from (92), we arrive at

$$[1-F(k+1)]\left\{\begin{array}{c}\left[\frac{1}{(1-\alpha)\{-\ln\alpha+(1-(k+1)\ln\alpha)\alpha^{k+1}\}}\right]\\-\left[\frac{1}{(1-\alpha)\{-\ln\alpha+(1-(k+2)\ln\alpha)\alpha^{k+2}\}}\right]\end{array}\right\}$$
$$=\left\{\begin{array}{c}\left[\frac{1}{(\alpha-1)\ln\alpha+\alpha^{k+1}\{1-\alpha^2+\alpha^2\ln\alpha+(\alpha^2-1)(k+1)\ln\alpha\}}\right]\\-\left[\frac{1}{(1-\alpha)\{-\ln\alpha+(1-(k+1)\ln\alpha)\alpha^{k+1}\}}\right]\end{array}\right\}f(k+1).$$

From the last equality, after some computations, we arrive at

$$\begin{aligned}h_F(k+1)&=\frac{f(k+1)}{1-F(k+1)}\\&=\frac{(\alpha-1)\ln\alpha+\alpha^{k+1}\left\{1-\alpha^2+\alpha^2\ln\alpha+\left(\alpha^2-1\right)(k+1)\ln\alpha\right\}}{\alpha\left\{-\ln\alpha+(1-(k+2)\ln\alpha)\alpha^{k+2}\right\}},\end{aligned}$$

which, in view of (33), implies that X has pmf (32).

Proposition 2.1.12. Let $X:\Omega\rightarrow\mathbb{N}^*$ be a random variable. The pmf of X is (35) if and only if

$$\begin{aligned}&E\left\{\left[\left(1+\alpha e^{\beta X}\right)\left(1+\alpha e^{\beta(X+1)}\right)\right]\mid X\ \leq k\right\}\\&=(1+\alpha)\left(1+\alpha e^{\beta(k+1)}\right).\end{aligned}\tag{94}$$

Proof. If X has pmf (35), then the left-hand side of (94), using finite geometric sum, will be

$$\begin{aligned}&(F(k))^{-1}\sum_{x=0}^{k}\left\{\left[\alpha(1+\alpha)\left(e^{\beta}-1\right)e^{\beta x}\right]\right\}\\&=\frac{1+\alpha e^{\beta(k+1)}}{\alpha\left(e^{\beta(k+1)}-1\right)}\left\{\alpha(1+\alpha)\left(e^{\beta(k+1)}-1\right)\right\}\\&=(1+\alpha)\left(1+\alpha e^{\beta(k+1)}\right).\end{aligned}$$

Conversely, if (94) holds, then

$$\begin{aligned}&\sum_{x=0}^{k}\left\{\left[\left(1+\alpha e^{\beta x}\right)\left(1+\alpha e^{\beta(x+1)}\right)\right]\ f(x)\right\}\\&=F(k)\left\{(1+\alpha)\left(1+\alpha e^{\beta(k+1)}\right)\right\}\\&=[F(k+1)-f(k+1)]\left\{(1+\alpha)\left(1+\alpha e^{\beta(k+1)}\right)\right\}.\end{aligned}\tag{95}$$

From (94), we also have

$$\begin{aligned}&\sum_{x=0}^{k+1}\left\{\left[\left(1+\alpha e^{\beta x}\right)\left(1+\alpha e^{\beta(x+1)}\right)\right]\ f(x)\right\}\\&=F(k+1)\left\{(1+\alpha)\left(1+\alpha e^{\beta(k+2)}\right)\right\}.\end{aligned}\tag{96}$$

Now, subtracting (95) from (96), we arrive at

$$
\begin{aligned}
&F(k+1)(1+\alpha)\left[(1+\alpha e^{\beta(k+2)})-(1+\alpha e^{\beta(k+1)})\right]\\
&=F(k+1)\,\alpha(1+\alpha)\left[e^{\beta(k+2)}-e^{\beta(k+1)}\right]\\
&=\left[(1+\alpha e^{\beta(k+1)})(1+\alpha e^{\beta(k+2)})-(1+\alpha)(1+\alpha e^{\beta(k+1)})\right]\ f(k+1)\\
&=(1+\alpha e^{\beta(k+1)})\left[(1+\alpha e^{\beta(k+2)})-(1+\alpha)\right]f(k+1)\\
&=\alpha(1+\alpha e^{\beta(k+1)})\left[(e^{\beta(k+2)}-1)\right]f(k+1).
\end{aligned}
$$

From the last equality, we have

$$
\begin{aligned}
r_F(k+1)=\frac{f(k+1)}{F(k+1)}&=\frac{(1+\alpha)\left[e^{\beta(k+2)}-e^{\beta(k+1)}\right]}{(1+\alpha e^{\beta(k+1)})\left[(e^{\beta(k+2)}-1)\right]}\\
&=\frac{(1+\alpha)(e^{\beta}-1)e^{\beta(k+1)}}{(1+\alpha e^{\beta(k+1)})\left[(e^{\beta(k+2)}-1)\right]},
\end{aligned}
$$

which, in view of (36), implies that X has pmf (35).

Proposition 2.1.13. Let $X:\Omega\to\mathbb{N}^*$ be a random variable. The pmf of X is (38) if and only if

$$
\begin{aligned}
&E\left\{\left[\exp\left[\left(\frac{G(X;\gamma)}{\overline{G}(X;\gamma)}\right)^{\beta}\ln(p)\right]+\exp\left[\left(\frac{G(X+1;\gamma)}{\overline{G}(X+1;\gamma)}\right)^{\beta}\ln(p)\right]\right]\ \mid X\ >k\right\}\\
&=\exp\left[\left(\frac{G(k+1;\gamma)}{\overline{G}(k+1;\gamma)}\right)^{\beta}\ln(p)\right]. \qquad (97)
\end{aligned}
$$

Proof. If X has pmf (38), then the left-hand side of (97) , using telescoping sum, will be

$$
\begin{aligned}
&[1-F(k)]^{-1}\sum_{x=k+1}^{\infty}\left\{\left[\exp\left[2\left(\frac{G(x;\gamma)}{\overline{G}(x;\gamma)}\right)^{\beta}\ln(p)\right]-\exp\left[2\left(\frac{G(x+1;\gamma)}{\overline{G}(x+1;\gamma)}\right)^{\beta}\ln(p)\right]\right]\right\}\\
&=\exp\left[-\left(\frac{G(k+1;\gamma)}{\overline{G}(k+1;\gamma)}\right)^{\beta}\ln(p)\right]\left\{\exp\left[2\left(\frac{G(k+1;\gamma)}{\overline{G}(k+1;\gamma)}\right)^{\beta}\ln(p)\right]-0\right\}\\
&=\exp\left[\left(\frac{G(k+1;\gamma)}{\overline{G}(k+1;\gamma)}\right)^{\beta}\ln(p)\right].
\end{aligned}
$$

Conversely, if (97) holds, then

$$
\begin{aligned}
&\sum_{x=k+1}^{\infty}\left\{\left[\exp\left[\left(\frac{G(x;\gamma)}{\overline{G}(x;\gamma)}\right)^{\beta}\ln(p)\right]+\exp\left[\left(\frac{G(x+1;\gamma)}{\overline{G}(x+1;\gamma)}\right)^{\beta}\ln(p)\right]\right]f(x)\right\}\\
&=[1-F(k)]\exp\left[\left(\frac{G(k+1;\gamma)}{\overline{G}(k+1;\gamma)}\right)^{\beta}\ln(p)\right]\\
&=[1-F(k+1)+f(k+1)]\left\{\exp\left[\left(\frac{G(k+1;\gamma)}{\overline{G}(k+1;\gamma)}\right)^{\beta}\ln(p)\right]\right\}. \qquad (98)
\end{aligned}
$$

From (97), we also have

$$\sum_{x=k+2}^{\infty}\left\{\left[\exp\left[\left(\frac{G(x;\gamma)}{\overline{G}(x;\gamma)}\right)^{\beta}\ln(p)\right]+\exp\left[\left(\frac{G(x+1;\gamma)}{\overline{G}(x+1;\gamma)}\right)^{\beta}\ln(p)\right]\right]\ f(x)\right\}$$
$$=[1-F(k+1)]\left\{\exp\left[\left(\frac{G(k+2;\gamma)}{\overline{G}(k+2;\gamma)}\right)^{\beta}\ln(p)\right]\right\}. \tag{99}$$

Now, subtracting (99) from (98), we arrive at

$$[1-F(k+1)]\left\{\left[\exp\left(\frac{G(k+1;\gamma)}{\overline{G}(k+1;\gamma)}\right)^{\beta}\ln(p)\right]-\left[\exp\left(\frac{G(k+2;\gamma)}{\overline{G}(k+2;\gamma)}\right)^{\beta}\ln(\ p\ )\right]\right\}$$
$$=\left[\exp\left(\frac{G(k+2;\gamma)}{\overline{G}(k+2;\gamma)}\right)^{\beta}\ln(p)\right]\ f(k+1).$$

From the last equality, we have

$$h_F(k+1)=\frac{f(k+1)}{1-F(k+1)}=\exp\left[\left\{\left(\frac{G(k+1;\gamma)}{\overline{G}(k+1;\gamma)}\right)^{\beta}-\left(\frac{G(k+2;\gamma)}{\overline{G}(k+2;\gamma)}\right)^{\beta}\right\}\ln(p)\right]-1,$$

which, in view of (39), implies that X has pmf (38).

Proposition 2.1.14. Let $X:\Omega\to\mathbb{N}^{*}$ be a random variable. The pmf of X is (41) if and only if

$$E\left\{\left[\frac{1}{\left(1+\frac{X}{\upsilon}\right)^{\delta}-1+\gamma}+\frac{1}{\left(1+\frac{X+1}{\upsilon}\right)^{\delta}-1+\gamma}\right]\ \mid X\ >k\right\}$$
$$=\frac{1}{\left(1+\frac{k+1}{\upsilon}\right)^{\delta}-1+\gamma}. \tag{100}$$

Proof. If X has pmf (41), then the left-hand side of (100) , using telescoping sum, will be

$$[1-F(k)]^{-1}\sum_{x=k+1}^{\infty}\gamma\left\{\left[\left(\frac{1}{\left(1+\frac{x}{\upsilon}\right)^{\delta}-1+\gamma}\right)^{2}-\left(\frac{1}{\left(1+\frac{x+1}{\upsilon}\right)^{\delta}-1+\gamma}\right)^{2}\right]\right\}$$
$$=\left[\left(1+\frac{k+1}{\upsilon}\right)^{\delta}-1+\gamma\right]\left\{\left(\frac{1}{\left(1+\frac{k+1}{\upsilon}\right)^{\delta}-1+\gamma}\right)^{2}-0\right\}$$
$$=\frac{1}{\left(1+\frac{k+1}{\upsilon}\right)^{\delta}-1+\gamma}.$$

Conversely, if (100) holds, then

$$\sum_{x=k+1}^{\infty}\left\{\left[\frac{1}{\left(1+\frac{X}{\upsilon}\right)^{\delta}-1+\gamma}+\frac{1}{\left(1+\frac{x+1}{\upsilon}\right)^{\delta}-1+\gamma}\right]f(x)\right\}$$
$$=[1-F(k)]\left(\frac{1}{\left(1+\frac{k+1}{\upsilon}\right)^{\delta}-1+\gamma}\right)$$
$$=[1-F(k+1)+f(k+1)]\left(\frac{1}{\left(1+\frac{k+1}{\upsilon}\right)^{\delta}-1+\gamma}\right). \tag{101}$$

From (100), we also have

$$\sum_{x=k+2}^{\infty}\left\{\left[\frac{1}{\left(1+\frac{X}{\upsilon}\right)^{\delta}-1+\gamma}+\frac{1}{\left(1+\frac{x+1}{\upsilon}\right)^{\delta}-1+\gamma}\right]f\left(x\right)\right\}$$
$$=\left[1-F\left(k+1\right)\right]\left(\frac{1}{\left(1+\frac{k+2}{\upsilon}\right)^{\delta}-1+\gamma}\right). \qquad (102)$$

Now, subtracting (102) from (101), we arrive at

$$\left[1-F\left(k+1\right)\right]\left\{\left(\frac{1}{\left(1+\frac{k+1}{\upsilon}\right)^{\delta}-1+\gamma}\right)-\left(\frac{1}{\left(1+\frac{k+2}{\upsilon}\right)^{\delta}-1+\gamma}\right)\right\}$$
$$=\left(\frac{1}{\left(1+\frac{k+2}{\upsilon}\right)^{\delta}-1+\gamma}\right)\ f\left(k+1\right).$$

From the last equality, we have

$$h_F\left(k+1\right)=\frac{f\left(k+1\right)}{1-F\left(k+1\right)}=\frac{\left(1+\frac{k+2}{\upsilon}\right)^{\delta}-1+\gamma}{\left(1+\frac{k+1}{\upsilon}\right)^{\delta}-1+\gamma}-1,$$

which, in view of (42), implies that X has pmf (41).

Proposition 2.1.15. Let $X:\Omega\rightarrow\mathbb{N}^*$ be a random variable. The pmf of X is (44) if and only if

$$E\left\{\left[\Lambda\left(X+1;a,b\right)+\Lambda\left(X;a,b\right)\right]\ \mid X\ \leq k\right\}=\Lambda\left(k+1;a,b\right). \qquad (103)$$

Proof. If X has pmf (44), then the left-hand side of (103) , using telescoping sum, will be

$$\left[F\left(k\right)\right]^{-1}\sum_{x=0}^{k}\frac{1}{\left(1-\log a\right)^{b}}\left\{\left[\Lambda^{2}\left(x+1;a,b\right)-\Lambda^{2}\left(x;a,b\right)\right]\right\}$$
$$=\frac{1}{\Lambda\left(k+1;a,b\right)}\left\{\Lambda^{2}\left(k+1;a,b\right)-\Lambda\left(0;a,b\right)\right\}$$
$$=\Lambda\left(k+1;a,b\right).$$

Conversely, if (103) holds, then

$$\sum_{x=0}^{k}\left\{\left[\Lambda\left(x+1;a,b\right)+\Lambda\left(x;a,b\right)\right]f\left(x\right)\right\}$$
$$=F\left(k\right)\Lambda\left(k+1;a,b\right)=\left[F\left(k+1\right)-f\left(k+1\right)\right]\Lambda\left(k+1;a,b\right). \qquad (104)$$

From (103), we also have

$$\sum_{x=0}^{k+1}\left\{\left[\Lambda\left(x+1;a,b\right)+\Lambda\left(x;a,b\right)\right]f\left(x\right)\right\}$$
$$=F\left(k+1\right)\Lambda\left(k+2;a,b\right). \qquad (105)$$

Now, subtracting (104) from (105), we arrive at

$$F\left(k+1\right)\left\{\Lambda\left(k+2;a,b\right)-\Lambda\left(k+1;a,b\right)\right\}=\Lambda\left(k+2;a,b\right)\ f\left(k+1\right).$$

From the last equality, we have

$$r_F(k+1) = \frac{f(k+1)}{F(k+1)} = 1 - \frac{\Lambda(k+1;a,b)}{\Lambda(k+2;a,b)},$$

which, in view of (45), implies that X has pmf (44).

Proposition 2.1.16. Let $X : \Omega \to \mathbb{N}$ be a random variable. The pmf of X is (47) if and only if

$$E\left\{\left[(p(X))^{-1}\right] \mid X \le k\right\} = \frac{1-(1-\theta)^{k+1}}{\theta Q(k)}. \tag{106}$$

Proof. If X has pmf (47), then the left-hand side of (106) , using finite geometric sum, will be

$$[F(k)]^{-1}\sum_{x=0}^{k} h(\theta)(1-\theta)^x = \frac{1}{Q(k)}\sum_{x=0}^{k}(1-\theta)^x = \frac{1-(1-\theta)^{k+1}}{\theta Q(k)}.$$

Conversely, if (106) holds, then

$$\sum_{x=0}^{k}\left\{\left[(p(x))^{-1}\right] f(x)\right\} = F(k)\left(\frac{1-(1-\theta)^{k+1}}{\theta Q(k)}\right)$$
$$= [F(k+1) - f(k+1)]\left(\frac{1-(1-\theta)^{k+1}}{\theta Q(k)}\right). \tag{107}$$

From (106), we also have

$$\sum_{x=0}^{k+1}\left\{\left[(p(x))^{-1}\right] f(x)\right\} = F(k+1)\left(\frac{1-(1-\theta)^{k+2}}{\theta Q(k+1)}\right). \tag{108}$$

Now, subtracting (107) from (108), we arrive at

$$F(k+1)\left\{\left(\frac{1-(1-\theta)^{k+2}}{\theta Q(k+1)}\right) - \left(\frac{1-(1-\theta)^{k+1}}{\theta Q(k)}\right)\right\}$$
$$= f(k+1)\left\{\frac{1}{p(k+1)} - \left(\frac{1-(1-\theta)^{k+1}}{\theta Q(k)}\right)\right\}.$$

From the last equality, after some computations, we have

$$r_F(k+1) = \frac{f(k+1)}{F(k+1)} = \frac{p(k+1)(1-\theta)^{k+1}}{Q(k+1)},$$

which, in view of (48), implies that X has pmf (47).

Proposition 2.1.17. Let $X : \Omega \to \mathbb{N}$ be a random variable. The pmf of X is (50) if and only if

$$E\left\{\left[\frac{1}{1+X^2}\right] \mid X > k\right\} = \frac{e^{-2\theta}\left(e^{\theta}-1\right)^2}{Q(k)}. \tag{109}$$

Proof. If X has pmf (50), then the left-hand side of (109) , using infinite geometric sum, will be

$$[1-F(k)]^{-1}\sum_{x=k+1}^{\infty}\left\{\left[\frac{\left(e^{\theta}-1\right)^{3}e^{-\theta x}}{2e^{2\theta}-e^{\theta}+1}\right]\right\}$$
$$=\left[\frac{2e^{2\theta}-e^{\theta}+1}{e^{-\theta(k-2)}Q(k)}\right]\left\{\left(\frac{\left(e^{\theta}-1\right)^{3}}{2e^{2\theta}-e^{\theta}+1}\right)\left(\frac{e^{-\theta(k+1)}}{1-e^{-\theta}}\right)\right\}=\frac{e^{-2\theta}\left(e^{\theta}-1\right)^{2}}{Q(k)}.$$

Conversely, if (109) holds, then

$$\sum_{x=k+1}^{\infty}\left\{\left[\frac{1}{1+x^{2}}\right]f(x)\right\}=[1-F(k)]\left(\frac{e^{-2\theta}\left(e^{\theta}-1\right)^{2}}{Q(k)}\right)$$
$$=[1-F(k+1)+f(k+1)]\left(\frac{e^{-2\theta}\left(e^{\theta}-1\right)^{2}}{Q(k)}\right). \tag{110}$$

From (109), we also have

$$\sum_{x=k+2}^{\infty}\left\{\left[\frac{1}{1+x^{2}}\right]f(x)\right\}=[1-F(k+1)]\left(\frac{e^{-2\theta}\left(e^{\theta}-1\right)^{2}}{Q(k+1)}\right). \tag{111}$$

Now, subtracting (111) from (110), we arrive at

$$[1-F(k+1)]\left\{\left(\frac{e^{-2\theta}\left(e^{\theta}-1\right)^{2}}{Q(k)}\right)-\left(\frac{e^{-2\theta}\left(e^{\theta}-1\right)^{2}}{Q(k+1)}\right)\right\}$$
$$=\left(\frac{1}{1+(k+1)^{2}}-\frac{e^{-2\theta}\left(e^{\theta}-1\right)^{2}}{Q(k)}\right)f(k+1).$$

From the last equality, after some computations, we have

$$h_F(k+1)=\frac{f(k+1)}{1-F(k+1)}=\frac{\left(\frac{e^{-2\theta}\left(e^{\theta}-1\right)^{2}}{Q(k)}\right)-\left(\frac{e^{-2\theta}\left(e^{\theta}-1\right)^{2}}{Q(k+1)}\right)}{\frac{1}{1+(k+1)^{2}}-\frac{e^{-2\theta}\left(e^{\theta}-1\right)^{2}}{Q(k)}}$$
$$=\frac{\left(e^{\theta}-1\right)^{3}\left(1+(k+1)^{2}\right)}{e^{2\theta}Q(k+1)},$$

which, in view of (51), implies that X has pmf (50).

Proposition 2.1.18. Let $X:\Omega\to\mathbb{N}^{*}$ be a random variable. The pmf of X is (53) if and only if

$$E\left\{\left[(Q(X))^{-1}\right]\mid X\ \leq k\right\}=\frac{(\theta+1)\left(1-(\theta+1)^{-(k+1)}\right)}{\theta\sum_{j=0}^{k}\frac{Q(j)}{(\theta+1)^{j}}}. \tag{112}$$

Proof. If X has pmf (53), then the left-hand side of (112) , using finite geometric sum, will be

$$[F(k)]^{-1}\sum_{x=0}^{k}\frac{\theta}{(\alpha^2+2\alpha+2)(\theta+1)^3}\left(\frac{1}{(\theta+1)^x}\right)$$
$$=\frac{1}{\sum_{j=0}^{k}\frac{Q(j)}{(\theta+1)^j}}\sum_{x=0}^{k}\left(\frac{1}{\theta+1}\right)^x=\frac{1}{\sum_{j=0}^{k}\frac{Q(j)}{(\theta+1)^j}}\left(\frac{(\theta+1)^{-(k+1)}-1}{(\theta+1)^{-1}-1}\right)$$
$$=\frac{(\theta+1)\left(1-(\theta+1)^{-(k+1)}\right)}{\theta\sum_{j=0}^{k}\frac{Q(j)}{(\theta+1)^j}}.$$

Conversely, if (112) holds, then

$$\sum_{x=0}^{k}\left\{\left[(Q(x))^{-1}\right]f(x)\right\}=F(k)\left(\frac{(\theta+1)\left(1-(\theta+1)^{-(k+1)}\right)}{\theta\sum_{j=0}^{k}\frac{Q(j)}{(\theta+1)^j}}\right)$$
$$=[F(k+1)-f(k+1)]\left(\frac{(\theta+1)\left(1-(\theta+1)^{-(k+1)}\right)}{\theta\sum_{j=0}^{k}\frac{Q(j)}{(\theta+1)^j}}\right). \tag{113}$$

From (112), we also have

$$\sum_{x=0}^{k+1}\left\{\left[(Q(x))^{-1}\right]f(x)\right\}=F(k+1)\left(\frac{(\theta+1)\left(1-(\theta+1)^{-(k+2)}\right)}{\theta\sum_{j=0}^{k+1}\frac{Q(j)}{(\theta+1)^j}}\right). \tag{114}$$

Now, subtracting (113) from (114), we arrive at

$$F(k+1)\left\{\left(\frac{(\theta+1)\left(1-(\theta+1)^{-(k+2)}\right)}{\theta\sum_{j=0}^{k+1}\frac{Q(j)}{(\theta+1)^j}}\right)-\left(\frac{(\theta+1)\left(1-(\theta+1)^{-(k+1)}\right)}{\theta\sum_{j=0}^{k}\frac{Q(j)}{(\theta+1)^j}}\right)\right\}$$
$$=\ f(k+1)\left\{\frac{1}{Q(k+1)}-\left(\frac{(\theta+1)\left(1-(\theta+1)^{-(k+1)}\right)}{\theta\sum_{j=0}^{k}\frac{Q(j)}{(\theta+1)^j}}\right)\right\}.$$

From the last equality, after some computations, we have

$$r_F(k+1)=\frac{f(k+1)}{F(k+1)}=\frac{\left(\frac{(\theta+1)\left(1-(\theta+1)^{-(k+2)}\right)}{\theta\sum_{j=0}^{k+1}\frac{Q(j)}{(\theta+1)^j}}\right)-\left(\frac{(\theta+1)\left(1-(\theta+1)^{-(k+1)}\right)}{\theta\sum_{j=0}^{k}\frac{Q(j)}{(\theta+1)^j}}\right)}{\frac{1}{Q(k+1)}-\left(\frac{(\theta+1)\left(1-(\theta+1)^{-(k+1)}\right)}{\theta\sum_{j=0}^{k}\frac{Q(j)}{(\theta+1)^j}}\right)}$$
$$=\frac{Q(k+1)(\theta+1)^{-(k+1)}}{\sum_{j=0}^{k+1}\frac{Q(j)}{(\theta+1)^j}},$$

which, in view of (54), implies that X has pmf (53).

Proposition 2.1.19. Let $X:\Omega\to\mathbb{N}$ be a random variable. The pmf of X is (56) if and only if

$$E\left\{\left[(Q(X))^{-1}\right]\ \mid X\ \leq k\right\}=\frac{\left(1-(\theta+1)^{-k}\right)}{\theta\sum_{j=1}^{k}\frac{Q(j)}{(\theta+1)^j}}. \tag{115}$$

Proof. If X has pmf (56), then the left-hand side of (115) , using finite geometric sum, will be

$$[F(k)]^{-1}\sum_{x=1}^{k} C(\theta)\left(\frac{1}{(\theta+1)^{x}}\right)$$
$$=\frac{1}{\sum_{j=1}^{k}\frac{Q(j)}{(\theta+1)^{j}}}\sum_{x=1}^{k}\left(\frac{1}{\theta+1}\right)^{x}=\frac{1}{\sum_{j=1}^{k}\frac{Q(j)}{(\theta+1)^{j}}}\left((\theta+1)^{-1}\frac{(\theta+1)^{-k}-1}{(\theta+1)^{-1}-1}\right)$$
$$=\frac{\left(1-(\theta+1)^{-k}\right)}{\theta\sum_{j=1}^{k}\frac{Q(j)}{(\theta+1)^{j}}}.$$

Conversely, if (115) holds, then

$$\sum_{x=1}^{k}\left\{\left[(Q(x))^{-1}\right]f(x)\right\}=F(k)\left(\frac{\left(1-(\theta+1)^{-k}\right)}{\theta\sum_{j=1}^{k}\frac{Q(j)}{(\theta+1)^{j}}}\right)$$
$$=[F(k+1)-f(k+1)]\left(\frac{\left(1-(\theta+1)^{-k}\right)}{\theta\sum_{j=1}^{k}\frac{Q(j)}{(\theta+1)^{j}}}\right). \tag{116}$$

From (115), we also have

$$\sum_{x=1}^{k+1}\left\{\left[(Q(x))^{-1}\right]f(x)\right\}=F(k+1)\left(\frac{\left(1-(\theta+1)^{-(k+1)}\right)}{\theta\sum_{j=1}^{k+1}\frac{Q(j)}{(\theta+1)^{j}}}\right). \tag{117}$$

Now, subtracting (116) from (117), we arrive at

$$F(k+1)\left\{\left(\frac{\left(1-(\theta+1)^{-(k+1)}\right)}{\theta\sum_{j=1}^{k+1}\frac{Q(j)}{(\theta+1)^{j}}}\right)-\left(\frac{\left(1-(\theta+1)^{-k}\right)}{\theta\sum_{j=1}^{k}\frac{Q(j)}{(\theta+1)^{j}}}\right)\right\}$$
$$=\ f(k+1)\left\{\frac{1}{Q(k+1)}-\left(\frac{\left(1-(\theta+1)^{-k}\right)}{\theta\sum_{j=1}^{k}\frac{Q(j)}{(\theta+1)^{j}}}\right)\right\}.$$

From the last equality, after some computations, we have

$$r_F(k+1)=\frac{f(k+1)}{F(k+1)}=\frac{\left(\frac{\left(1-(\theta+1)^{-(k+1)}\right)}{\theta\sum_{j=1}^{k+1}\frac{Q(j)}{(\theta+1)^{j}}}\right)-\left(\frac{\left(1-(\theta+1)^{-k}\right)}{\theta\sum_{j=1}^{k}\frac{Q(j)}{(\theta+1)^{j}}}\right)}{\frac{1}{Q(k+1)}-\left(\frac{\left(1-(\theta+1)^{-k}\right)}{\theta\sum_{j=1}^{k}\frac{Q(j)}{(\theta+1)^{j}}}\right)}$$
$$=\frac{Q(k+1)(\theta+1)^{-(k+1)}}{\sum_{j=1}^{k+1}\frac{Q(j)}{(\theta+1)^{j}}},$$

which, in view of (57), implies that X has pmf (56).

Proposition 2.1.20. Let $X:\Omega\to\mathbb{N}$ be a random variable. The pmf of X is (57) if and only if

$$E\left\{\left[(Q(X))^{-1}\right]\ \mid X\ \leq k\right\}=\frac{(\theta+1)\left(1-(\theta+1)^{-(k+1)}\right)}{\theta\sum_{j=0}^{k}\frac{Q(j)}{(\theta+1)^{j}}}. \tag{118}$$

Proof. If X has pmf (59), then the left-hand side of (118) , using finite geometric sum, will be

$$\begin{aligned}
&[F(k)]^{-1}\sum_{x=0}^{k} C(\theta,\alpha)\left(\frac{1}{(\theta+1)^{x}}\right)\\
&=\frac{1}{\sum_{j=0}^{k}\frac{Q(j)}{(\theta+1)^{j}}}\sum_{x=0}^{k}\left(\frac{1}{\theta+1}\right)^{x}=\frac{1}{\sum_{j=0}^{k}\frac{Q(j)}{(\theta+1)^{j}}}\left(\frac{(\theta+1)^{-(k+1)}-1}{(\theta+1)^{-1}-1}\right)\\
&=\frac{(\theta+1)\left(1-(\theta+1)^{-(k+1)}\right)}{\theta\sum_{j=0}^{k}\frac{Q(j)}{(\theta+1)^{j}}}.
\end{aligned}$$

Conversely, if (118) holds, then

$$\begin{aligned}
&\sum_{x=0}^{k}\left\{\left[(Q(x))^{-1}\right]f(x)\right\}=F(k)\left(\frac{(\theta+1)\left(1-(\theta+1)^{-(k+1)}\right)}{\theta\sum_{j=0}^{k}\frac{Q(j)}{(\theta+1)^{j}}}\right)\\
&=[F(k+1)-f(k+1)]\left(\frac{(\theta+1)\left(1-(\theta+1)^{-(k+1)}\right)}{\theta\sum_{j=0}^{k}\frac{Q(j)}{(\theta+1)^{j}}}\right). \qquad (119)
\end{aligned}$$

From (118), we also have

$$\sum_{x=0}^{k+1}\left\{\left[(Q(x))^{-1}\right]f(x)\right\}=F(k+1)\left(\frac{(\theta+1)\left(1-(\theta+1)^{-(k+2)}\right)}{\theta\sum_{j=0}^{k+1}\frac{Q(j)}{(\theta+1)^{j}}}\right). \qquad (120)$$

Now, subtracting (119) from (120), we arrive at

$$\begin{aligned}
&F(k+1)\left(\frac{\theta+1}{\theta}\right)\left\{\left(\frac{\left(1-(\theta+1)^{-(k+2)}\right)}{\sum_{j=0}^{k+1}\frac{Q(j)}{(\theta+1)^{j}}}\right)-\left(\frac{\left(1-(\theta+1)^{-(k+1)}\right)}{\sum_{j=0}^{k}\frac{Q(j)}{(\theta+1)^{j}}}\right)\right\}\\
&=\ f(k+1)\left\{\frac{1}{Q(k+1)}-\left(\frac{\theta+1}{\theta}\right)\left(\frac{\left(1-(\theta+1)^{-(k+1)}\right)}{\sum_{j=0}^{k}\frac{Q(j)}{(\theta+1)^{j}}}\right)\right\}.
\end{aligned}$$

From the last equality, after some computations, we have

$$\begin{aligned}
r_F(k+1)=\frac{f(k+1)}{F(k+1)}&=\frac{\left(\frac{\theta+1}{\theta}\right)\left\{\left(\frac{\left(1-(\theta+1)^{-(k+2)}\right)}{\sum_{j=0}^{k+1}\frac{Q(j)}{(\theta+1)^{j}}}\right)-\left(\frac{\left(1-(\theta+1)^{-(k+1)}\right)}{\sum_{j=0}^{k}\frac{Q(j)}{(\theta+1)^{j}}}\right)\right\}}{\frac{1}{Q(k+1)}-\left(\frac{\theta+1}{\theta}\right)\left(\frac{\left(1-(\theta+1)^{-(k+1)}\right)}{\sum_{j=0}^{k}\frac{Q(j)}{(\theta+1)^{j}}}\right)}\\
&=\frac{Q(k+1)(\theta+1)^{-(k+1)}}{\sum_{j=0}^{k+1}\frac{Q(j)}{(\theta+1)^{j}}},
\end{aligned}$$

which, in view of (60), implies that X has pmf (59).

2.2. Characterizations of DsGLi, P-Xgamma, SBPA, GHGD, PQX, EDIR, Mlynar, DsFx-I, DWG, DMOL and ZTDAD respectively, based on hazard function

Proposition 2.2.1. Let $X:\Omega\to\mathbb{N}^{*}$ be a random variable. The pmf of X is (2) if and only if its hazard

rate function satisfies the difference equation

$$h_F(k+1) - h_F(k) = \frac{\alpha \left\{ \begin{array}{c} 1-\eta-\ln\eta\left[1+(k+1)-\eta(k+3)\right]+ \\ \left(1-\frac{1}{\alpha}\right)(\ln\eta)^2\left[(k+1)-\eta(k+2)\right] \end{array} \right\}}{\left\{\left(1-\ln\eta^{k+2}\right)(\alpha-\alpha\ln\eta+\ln\eta)-\ln\eta\right\}\eta} - \frac{\alpha \left\{ \begin{array}{c} 1-\eta-\ln\eta\left[1+k-\eta(k+2)\right]+ \\ \left(1-\frac{1}{\alpha}\right)(\ln\eta)^2\left[k-\eta(k+1)\right] \end{array} \right\}}{\left\{\left(1-\ln\eta^{k+1}\right)(\alpha-\alpha\ln\eta+\ln\eta)-\ln\eta\right\}\eta}, \qquad k\in\mathbb{N}^*, \tag{121}$$

with the initial condition $h_F(0) = \frac{\alpha\left\{1-\eta-\ln\eta[1-2\eta]-\eta\left(1-\frac{1}{\alpha}\right)(\ln\eta)^2\right\}}{\{(1-\ln\eta)(\alpha-\alpha\ln\eta+\ln\eta)-\ln\eta\}\eta}$.

Proof. If X has pmf (2), then clearly (121) holds. Now, if (121) holds, then for every $x\in\mathbb{N}^*$, we have

$$\sum_{k=0}^{x-1}\left\{h_F(k+1)-h_F(k)\right\} = \sum_{k=0}^{x-1}\left\{ \begin{array}{c} \frac{\alpha \left\{ \begin{array}{c} 1-\eta-\ln\eta\left[1+(k+1)-\eta(k+3)\right]+ \\ \left(1-\frac{1}{\alpha}\right)(\ln\eta)^2\left[(k+1)-\eta(k+2)\right] \end{array} \right\}}{\left\{\left(1-\ln\eta^{k+2}\right)(\alpha-\alpha\ln\eta+\ln\eta)-\ln\eta\right\}\eta} \\ -\frac{\alpha \left\{ \begin{array}{c} 1-\eta-\ln\eta\left[1+k-\eta(k+2)\right]+ \\ \left(1-\frac{1}{\alpha}\right)(\ln\eta)^2\left[k-\eta(k+1)\right] \end{array} \right\}}{\left\{\left(1-\ln\eta^{k+1}\right)(\alpha-\alpha\ln\eta+\ln\eta)-\ln\eta\right\}\eta} \end{array} \right\},$$

or

$$h_F(x)-h_F(0) = \frac{\alpha \left\{ \begin{array}{c} 1-\eta-\ln\eta\left[1+x-\eta(x+2)\right]+ \\ \left(1-\frac{1}{\alpha}\right)(\ln\eta)^2\left[x-\eta(x+1)\right] \end{array} \right\}}{\left\{\left(1-\ln\eta^{x}\right)(\alpha-\alpha\ln\eta+\ln\eta)-\ln\eta\right\}\eta} - \frac{\alpha\left\{1-\eta-\ln\eta\left[1-2\eta\right]-\eta\left(1-\frac{1}{\alpha}\right)(\ln\eta)^2\right\}}{\left\{(1-\ln\eta)(\alpha-\alpha\ln\eta+\ln\eta)-\ln\eta\right\}\eta}.$$

In view of the fact that $h_F(0) = \frac{\alpha\left\{1-\eta-\ln\eta[1-2\eta]-\eta\left(1-\frac{1}{\alpha}\right)(\ln\eta)^2\right\}}{\{(1-\ln\eta)(\alpha-\alpha\ln\eta+\ln\eta)-\ln\eta\}\eta}$, from the last equation we have

$$h_F(x) = \frac{\alpha \left\{ \begin{array}{c} 1-\eta-\ln\eta\left[1+x-\eta(x+2)\right]+ \\ \left(1-\frac{1}{\alpha}\right)(\ln\eta)^2\left[x-\eta(x+1)\right] \end{array} \right\}}{\left\{\left(1-\ln\eta^{x}\right)(\alpha-\alpha\ln\eta+\ln\eta)-\ln\eta\right\}\eta}, \qquad x\in\mathbb{N}^*,$$

which, in view of (3), implies that X has pmf (2).

Proposition 2.2.2. Let $X:\Omega\to\mathbb{N}^*$ be a random variable. The pmf of X is (11) if and only if its hazard rate function satisfies the difference equation

$$h_F(k+1)-h_F(k) = \frac{\theta^2\left[2(1+\theta)^2+\theta(k+2)(k+3)\right]}{\theta(\theta+5)(k+1)^2+2\theta(k+1)+\gamma} - \frac{\theta^2\left[2(1+\theta)^2+\theta(k+1)(k+2)\right]}{\theta(\theta+5)(k)^2+2\theta(k)+\gamma}, \tag{122}$$

$k\in\mathbb{N}^*$, with the boundary condition $h_F(0) = 2\gamma^{-1}\theta^2\left[1+4\theta+\theta^2\right]$.

Proof. If X has pmf (11), then clearly (122) holds. Now, if (122) holds, then for every $x\in\mathbb{N}^*$, we have

$$\sum_{k=0}^{x-1}\left\{h_F(k+1)-h_F(k)\right\} = \sum_{k=0}^{x-1}\left\{\frac{\theta^2\left[2(1+\theta)^2+\theta(k+2)(k+3)\right]}{\theta(\theta+5)(k+1)^2+2\theta(k+1)+\gamma} - \frac{\theta^2\left[2(1+\theta)^2+\theta(k+1)(k+2)\right]}{\theta(\theta+5)(k)^2+2\theta(k)+\gamma}\right\},$$

or

$$h_F(x) - h_F(0) = \frac{\theta^2\left[2(1+\theta)^2 + \theta(x+1)(x+2)\right]}{\theta(\theta+5)(x)^2 + 2\theta(x) + \gamma} - \frac{\theta^2\left[2(1+\theta)^2 + 2\theta\right]}{\gamma}.$$

In view of the fact that $h_F(0) = \frac{\theta^2\left[2(1+\theta)^2+2\theta\right]}{\gamma}$, from the last equation we have

$$h_F(x) = \frac{\theta^2\left[2(1+\theta)^2 + \theta(x+1)(x+2)\right]}{\theta(\theta+5)(x)^2 + 2\theta(x) + \gamma},$$

which, in view of (12), implies that X has pmf (11).

Proposition 2.2.3. Let $X : \Omega \to \mathbb{N}$ be a random variable. The pmf of X is (17) if and only if its hazard rate function satisfies the difference equation

$$\begin{aligned} &h_F(k+1) - h_F(k) \\ &= \frac{4\eta^3(k+1)(k+2)}{2(k+3)\eta[1+(k+2)\eta]+1} - \frac{4\eta^3 k(k+1)}{2(k+2)\eta[1+(k+1)\eta]+1},\ k \in \mathbb{N}, \end{aligned} \tag{123}$$

with the initial condition $h_F(1) = \frac{8\eta^3}{6\eta[1+2\eta]+1}$.

Proof. If X has pmf (17), then clearly (123) holds. Now, if (123) holds, then for every $x \in \mathbb{N}$, we have

$$\begin{aligned} &\sum_{k=1}^{x-1}\{h_F(k+1) - h_F(k)\} \\ &= 4\eta^3 \sum_{k=1}^{x-1}\left\{\frac{(k+1)(k+2)}{2(k+3)\eta[1+(k+2)\eta]+1} - \frac{k(k+1)}{2(k+2)\eta[1+(k+1)\eta]+1}\right\}, \end{aligned}$$

or

$$h_F(x) - h_F(1) = \frac{4\eta^3 x(x+1)}{2(x+2)\eta[1+(x+1)\eta]+1} - \frac{8\eta^3}{6\eta[1+2\eta]+1}.$$

In view of the fact that $h_F(1) = \frac{8\eta^3}{6\eta[1+2\eta]+1}$, from the last equation we have

$$h_F(x) = \frac{4\eta^3 x(x+1)}{2(x+2)\eta[1+(x+1)\eta]+1}, \qquad x \in \mathbb{N},$$

which, in view of (18), implies that X has pmf (17).

Proposition 2.2.4. Let $X : \Omega \to \mathbb{N}^*$ be a random variable. The pmf of X is (20) if and only if its hazard rate function satisfies the difference equation

$$\begin{aligned} &h_F(k+1) - h_F(k) \\ &= \frac{1}{\alpha}\left\{\left(\frac{H_{n,m}(k+1+\gamma,\beta)}{G(\alpha;\beta,k+1+\gamma+1)}\right) - \left(\frac{H_{n,m}(k+\gamma,\beta)}{G(\alpha;\beta,k+\gamma+1)}\right)\right\}, \qquad k \in \mathbb{N}^*, \end{aligned} \tag{124}$$

with the initial condition $h_F(0) = \frac{1}{\alpha}\left(\frac{H_{n,m}(\gamma,\beta)}{G(\alpha;\beta,\gamma+1)}\right)$.

Proof. If X has pmf (20), then clearly (124) holds. Now, if (124) holds, then for every $x \in \mathbb{N}^*$, we have

$$\sum_{k=0}^{x-1} \{h_F(k+1) - h_F(k)\}$$
$$= \frac{1}{\alpha} \sum_{k=0}^{x-1} \left\{ \left(\frac{H_{n,m}(k+1+\gamma,\beta)}{G(\alpha;\beta,k+1+\gamma+1)} \right) - \left(\frac{H_{n,m}(k+\gamma,\beta)}{G(\alpha;\beta,k+\gamma+1)} \right) \right\}$$
$$= \frac{1}{\alpha} \left\{ \left(\frac{H_{n,m}(x+\gamma,\beta)}{G(\alpha;\beta,x+\gamma+1)} \right) - \left(\frac{H_{n,m}(\gamma,\beta)}{G(\alpha;\beta,\gamma+1)} \right) \right\},$$

or

$$h_F(x) - h_F(0) = \frac{1}{\alpha} \left\{ \left(\frac{H_{n,m}(x+\gamma,\beta)}{G(\alpha;\beta,x+\gamma+1)} \right) - \left(\frac{H_{n,m}(\gamma,\beta)}{G(\alpha;\beta,\gamma+1)} \right) \right\}.$$

In view of the fact that $h_F(0) = \frac{1}{\alpha}\left(\frac{H_{n,m}(\gamma,\beta)}{G(\alpha;\beta,\gamma+1)}\right)$, from the last equation we have

$$h_F(x) = \frac{1}{\alpha} \left(\frac{H_{n,m}(x+\gamma,\beta)}{G(\alpha;\beta,x+\gamma+1)} \right), \qquad x \in \mathbb{N}^*,$$

which, in view of (21), implies that X has pmf (20).

Proposition 2.2.5. Let $X : \Omega \to \mathbb{N}^*$ be a random variable. The pmf of X is (23) if and only if its hazard function satisfies the difference equation

$$h_F(k+1) - h_F(k)$$
$$= \frac{2\alpha\theta(\theta+1)^2 + \theta^3(k+2)(k+3)}{2\alpha(\theta+1)^2 + \theta(k+4)(\theta(k+3)+2)+2}$$
$$- \frac{2\alpha\theta(\theta+1)^2 + \theta^3(k+1)(k+2)}{2\alpha(\theta+1)^2 + \theta(k+3)(\theta(k+2)+2)+2}, \tag{125}$$

$k \in \mathbb{N}^*$, with the initial condition $h_F(0) = \frac{\alpha\theta(\theta+1)^2+\theta^3}{\alpha(\theta+1)^2+3\theta(\theta+1)+1}$.

Proof. If X has pmf (23), then clearly (125) holds. Now, if (125) holds, then for every $x \in \mathbb{N}^*$, we have

$$\sum_{k=0}^{x-1} \{h_F(k+1) - h_F(k)\}$$
$$= \sum_{k=0}^{x-1} \left\{ \begin{array}{c} \frac{2\alpha\theta(\theta+1)^2+\theta^3(k+2)(k+3)}{2\alpha(\theta+1)^2+\theta(k+4)(\theta(k+3)+2)+2} \\ -\frac{2\alpha\theta(\theta+1)^2+\theta^3(k+1)(k+2)}{2\alpha(\theta+1)^2+\theta(k+3)(\theta(k+2)+2)+2} \end{array} \right\},$$

or

$$h_F(x) - h_F(0) = \frac{2\alpha\theta(\theta+1)^2 + \theta^3(x+1)(x+2)}{2\alpha(\theta+1)^2 + \theta(x+3)(\theta(x+2)+2)+2}$$
$$- \frac{\alpha\theta(\theta+1)^2 + \theta^3}{\alpha(\theta+1)^2 + 3\theta(\theta+1)+1},$$

or

$$h_F(x) = \frac{2\alpha\theta(\theta+1)^2 + \theta^3(x+1)(x+2)}{2\alpha(\theta+1)^2 + \theta(x+3)(\theta(x+2)+2)+2}, \qquad x \in \mathbb{N}^*,$$

which in view of (24) is the hazard function corresponding to the pmf (23) .

Proposition 2.2.6. Let $X : \Omega \to \mathbb{N}^*$ be a random variable. The pmf of X is (26) if and only if its hazard

rate function satisfies the difference equation

$$h_F(k+1) - h_F(k) = \left(\frac{1-q^{(k+1)^{-2}}}{1-q^{(k+2)^{-2}}}\right)^{\gamma} - \left(\frac{1-q^{k^{-2}}}{1-q^{(k+1)^{-2}}}\right)^{\gamma}, \qquad k \in \mathbb{N}^*, \tag{126}$$

with the initial condition $h_F(0) = (1-q)^{-\gamma} - 1$.

Proof. If X has pmf (26), then clearly (126) holds. Now, if (126) holds, then for every $x \in \mathbb{N}^*$, we have

$$\sum_{k=0}^{x-1} \{h_F(k+1) - h_F(k)\} = \sum_{k=0}^{x-1} \left\{ \left(\frac{1-q^{(k+1)^{-2}}}{1-q^{(k+2)^{-2}}}\right)^{\gamma} - \left(\frac{1-q^{k^{-2}}}{1-q^{(k+1)^{-2}}}\right)^{\gamma} \right\},$$

or

$$h_F(x) - h_F(0) = \left(\frac{1-q^{x^{-2}}}{1-q^{(x+1)^{-2}}}\right)^{\gamma} - \left(\frac{1}{1-q}\right)^{\gamma}.$$

In view of the fact that $h_F(0) = (1-q)^{-\gamma} - 1$, from the last equation we have

$$h_F(x) = \left(\frac{1-q^{x^{-2}}}{1-q^{(x+1)^{-2}}}\right)^{\gamma} - 1, \qquad x \in \mathbb{N}^*,$$

which, in view of (27), implies that X has pmf (26).

Proposition 2.2.7. Let $X : \Omega \to I$ be a random variable. The pmf of X is (29) if and only if its hazard rate function satisfies the difference equation

$$h_F(k+1) - h_F(k) = \frac{k+1}{n-1-k} - \frac{k}{n-k}, \qquad k \in I, \tag{127}$$

with the initial condition $h_F(1) = \frac{1}{n-1}$.

Proof. If X has pmf (29), then clearly (127) holds. Now, if (127) holds, then for every $x \in I$, we have

$$\sum_{k=1}^{x-1} \{h_F(k+1) - h_F(k)\} = \sum_{k=1}^{x-1} \left\{ \frac{k+1}{n-1-k} - \frac{k}{n-k} \right\},$$

or

$$h_F(x) - h_F(1) = \frac{x}{n-x} - \frac{1}{n-1}.$$

In view of the fact that $h_F(1) = \frac{1}{n-1}$, from the last equation we have

$$h_F(x) = \frac{x}{n-x}, \quad x \in I,$$

which, in view of (30), implies that X has pmf (29).

Proposition 2.2.8. Let $X : \Omega \to \mathbb{N}^*$ be a random variable. The pmf of X is (32) if and only if its hazard rate function satisfies the difference equation

$$h_F(k+1) - h_F(k) = \frac{(\alpha-1)\ln\alpha + \alpha^{k+1}\left\{1-\alpha^2+\alpha^2\ln\alpha+\left(\alpha^2-1\right)(k+1)\ln\alpha\right\}}{\alpha\left\{-\ln\alpha+(1-(k+2)\ln\alpha)\,\alpha^{k+2}\right\}} - \frac{(\alpha-1)\ln\alpha + \alpha^{k}\left\{1-\alpha^2+\alpha^2\ln\alpha+\left(\alpha^2-1\right)k\ln\alpha\right\}}{\alpha\left\{-\ln\alpha+(1-(k+1)\ln\alpha)\,\alpha^{k+1}\right\}}, \tag{128}$$

$k \in \mathbb{N}^*$, with the initial condition $h_F(0) = \frac{(\alpha-1)\ln\alpha+\left\{1-\alpha^2+\alpha^2\ln\alpha\right\}}{\alpha\{-\ln\alpha+(1-\ln\alpha)\alpha\}}$.

Proof. If X has pmf (32), then clearly (128) holds. Now, if (128) holds, then for every $x \in \mathbb{N}^*$, we have

$$\sum_{k=0}^{x-1} \{h_F(k+1) - h_F(k)\}$$
$$= \sum_{k=0}^{x-1} \left\{ \begin{array}{c} \frac{(\alpha-1)\ln\alpha + \alpha^{k+1}\left\{1-\alpha^2+\alpha^2\ln\alpha+\left(\alpha^2-1\right)(k+1)\ln\alpha\right\}}{\alpha\{-\ln\alpha+(1-(k+2)\ln\alpha)\alpha^{k+2}\}} - \\ \frac{(\alpha-1)\ln\alpha+\alpha^k\left\{1-\alpha^2+\alpha^2\ln\alpha+\left(\alpha^2-1\right)k\ln\alpha\right\}}{\alpha\{-\ln\alpha+(1-(k+1)\ln\alpha)\alpha^{k+1}\}} \end{array} \right\},$$

or

$$h_F(x) - h_F(0) = \frac{(\alpha-1)\ln\alpha + \alpha^x\left\{1-\alpha^2+\alpha^2\ln\alpha+\left(\alpha^2-1\right)x\ln\alpha\right\}}{\alpha\left\{-\ln\alpha + (1-(x+1)\ln\alpha)\alpha^{x+1}\right\}} - \frac{(\alpha-1)\ln\alpha + \left\{1-\alpha^2+\alpha^2\ln\alpha\right\}}{\alpha\left\{-\ln\alpha+(1-\ln\alpha)\alpha\right\}}.$$

In view of the fact that $h_F(0) = \frac{(\alpha-1)\ln\alpha+\left\{1-\alpha^2+\alpha^2\ln\alpha\right\}}{\alpha\{-\ln\alpha+(1-\ln\alpha)\alpha\}}$, from the last equation we have

$$h_F(x) = \frac{(\alpha-1)\ln\alpha + \alpha^x\left\{1-\alpha^2+\alpha^2\ln\alpha+\left(\alpha^2-1\right)x\ln\alpha\right\}}{\alpha\left\{-\ln\alpha + (1-(x+1)\ln\alpha)\alpha^{x+1}\right\}}, \quad x \in \mathbb{N}^*,$$

which, in view of (33), implies that X has pmf (32).

Proposition 2.2.9. Let $X : \Omega \to \mathbb{N}^*$ be a random variable. The pmf of X is (38) if and only if its hazard rate function satisfies the difference equation

$$h_F(k+1) - h_F(k) = \exp\left[\left\{\left(\frac{G(k+1;\gamma)}{\overline{G}(k+1;\gamma)}\right)^\beta - \left(\frac{G(k+2;\gamma)}{\overline{G}(k+2;\gamma)}\right)^\beta\right\}\ln(p)\right] - \exp\left[\left\{\left(\frac{G(k;\gamma)}{\overline{G}(k;\gamma)}\right)^\beta - \left(\frac{G(k+1;\gamma)}{\overline{G}(k+1;\gamma)}\right)^\beta\right\}\ln(p)\right], \tag{129}$$

$k \in \mathbb{N}^*$, with the initial condition $h_F(0) = \exp\left[\left\{\left(\frac{G(0;\gamma)}{\overline{G}(0;\gamma)}\right)^\beta - \left(\frac{G(1;\gamma)}{\overline{G}(1;\gamma)}\right)^\beta\right\}\ln(p)\right] - 1$.

Proof. If X has pmf (38), then clearly (129) holds. Now, if (129) holds, using telescoping sum, for every $x \in \mathbb{N}^*$, we have

$$\sum_{k=0}^{x-1} \{h_F(k+1) - h_F(k)\}$$
$$= \exp\left[\left\{\left(\frac{G(x;\gamma)}{\overline{G}(x;\gamma)}\right)^\beta - \left(\frac{G(x+1;\gamma)}{\overline{G}(x+1;\gamma)}\right)^\beta\right\}\ln(p)\right] - \exp\left[\left\{\left(\frac{G(0;\gamma)}{\overline{G}(0;\gamma)}\right)^\beta - \left(\frac{G(1;\gamma)}{\overline{G}(1;\gamma)}\right)^\beta\right\}\ln(p)\right].$$

So, we have

$$h_F(x) - h_F(0) = \exp\left[\left\{\left(\frac{G(x;\gamma)}{\overline{G}(x;\gamma)}\right)^\beta - \left(\frac{G(x+1;\gamma)}{\overline{G}(x+1;\gamma)}\right)^\beta\right\}\ln(p)\right] - \exp\left[\left\{\left(\frac{G(0;\gamma)}{\overline{G}(0;\gamma)}\right)^\beta - \left(\frac{G(1;\gamma)}{\overline{G}(1;\gamma)}\right)^\beta\right\}\ln(p)\right].$$

In view of the fact that $h_F(0) = \exp\left[\left\{\left(\frac{G(0;\gamma)}{\overline{G}(0;\gamma)}\right)^{\beta} - \left(\frac{G(1;\gamma)}{\overline{G}(1;\gamma)}\right)^{\beta}\right\}\ln(p)\right] - 1$, from the last equation we have

$$h_F(x) - \left\{\exp\left[\left\{\left(\frac{G(0;\gamma)}{\overline{G}(0;\gamma)}\right)^{\beta} - \left(\frac{G(1;\gamma)}{\overline{G}(1;\gamma)}\right)^{\beta}\right\}\ln(p)\right] - 1\right\}$$
$$= \exp\left[\left\{\left(\frac{G(x;\gamma)}{\overline{G}(x;\gamma)}\right)^{\beta} - \left(\frac{G(x+1;\gamma)}{\overline{G}(x+1;\gamma)}\right)^{\beta}\right\}\ln(p)\right]$$
$$- \exp\left[\left\{\left(\frac{G(0;\gamma)}{\overline{G}(0;\gamma)}\right)^{\beta} - \left(\frac{G(1;\gamma)}{\overline{G}(1;\gamma)}\right)^{\beta}\right\}\ln(p)\right],$$

or

$$h_F(x) = \exp\left[\left\{\left(\frac{G(x;\gamma)}{\overline{G}(x;\gamma)}\right)^{\beta} - \left(\frac{G(x+1;\gamma)}{\overline{G}(x+1;\gamma)}\right)^{\beta}\right\}\ln(p)\right] - 1, \quad x \in \mathbb{N}^*,$$

which, in view of (39), implies that X has pmf (38).

Proposition 2.2.10. Let $X : \Omega \to \mathbb{N}^*$ be a random variable. The pmf of X is (41) if and only if its hazard rate function satisfies the difference equation

$$h_F(k+1) - h_F(k) = \left(\frac{\left(1+\frac{k+2}{\upsilon}\right)^{\delta} - 1 + \gamma}{\left(1+\frac{k+1}{\upsilon}\right)^{\delta} - 1 + \gamma}\right) - \left(\frac{\left(1+\frac{k+1}{\upsilon}\right)^{\delta} - 1 + \gamma}{\left(1+\frac{k}{\upsilon}\right)^{\delta} - 1 + \gamma}\right), \quad (130)$$

$k \in \mathbb{N}^*$, with the initial condition $h_F(0) = \frac{\left(1+\frac{1}{\upsilon}\right)^{\delta} - 1 + \gamma}{\gamma} - 1$.

Proof. If X has pmf (41), then clearly (130) holds. Now, if (130) holds, using telescoping sum, for every $x \in \mathbb{N}^*$, we have

$$\sum_{k=1}^{x-1}\{h_F(k+1) - h_F(k)\}$$
$$= \left(\frac{\left(1+\frac{x+1}{\upsilon}\right)^{\delta} - 1 + \gamma}{\left(1+\frac{x}{\upsilon}\right)^{\delta} - 1 + \gamma}\right) - \left(\frac{\left(1+\frac{1}{\upsilon}\right)^{\delta} - 1 + \gamma}{\gamma}\right).$$

So, we have

$$h_F(x) - h_F(0) = \left(\frac{\left(1+\frac{x+1}{\upsilon}\right)^{\delta} - 1 + \gamma}{\left(1+\frac{x}{\upsilon}\right)^{\delta} - 1 + \gamma}\right) - \left(\frac{\left(1+\frac{1}{\upsilon}\right)^{\delta} - 1 + \gamma}{\gamma}\right).$$

In view of the fact that $h_F(0) = \left(\frac{\left(1+\frac{1}{\upsilon}\right)^{\delta} - 1 + \gamma}{\gamma}\right) - 1$, from the last equation we have

$$h_F(x) - \left\{\left(\frac{\left(1+\frac{1}{\upsilon}\right)^{\delta} - 1 + \gamma}{\gamma}\right) - 1\right\}$$
$$= \left(\frac{\left(1+\frac{x+1}{\upsilon}\right)^{\delta} - 1 + \gamma}{\left(1+\frac{x}{\upsilon}\right)^{\delta} - 1 + \gamma}\right) - \left(\frac{\left(1+\frac{1}{\upsilon}\right)^{\delta} - 1 + \gamma}{\gamma}\right),$$

or

$$h_F(x) = \left(\frac{\left(1+\frac{x+1}{\upsilon}\right)^{\delta} - 1 + \gamma}{\left(1+\frac{x}{\upsilon}\right)^{\delta} - 1 + \gamma}\right) - 1, \quad x \in \mathbb{N}^*,$$

which, in view of (42), implies that X has pmf (41).

Proposition 2.2.11. Let $X : \Omega \to \mathbb{N}$ be a random variable. The pmf of X is (50) if and only if its hazard

rate function satisfies the difference equation

$$h_F(k+1) - h_F(k) = \frac{\left(e^{\theta}-1\right)^3}{e^{2\theta}} \left(\frac{1+(k+1)^2}{Q(k+1)} - \frac{1+k^2}{Q(k)} \right), \qquad k \in \mathbb{N}, \tag{131}$$

with the initial condition $h_F(1) = \frac{2\left(e^{\theta}-1\right)^3}{5e^{2\theta}-5e^{\theta}+2}$.

Proof. If X has pmf (50), then clearly (131) holds. Now, if (131) holds, using telescoping sum, for every $x \in \mathbb{N}$, we have

$$\sum_{k=1}^{x-1} \{h_F(k+1) - h_F(k)\} = \frac{\left(e^{\theta}-1\right)^3}{e^{2\theta}} \left(\frac{1+x^2}{Q(x)} - \frac{2}{Q(1)} \right).$$

So, we have

$$h_F(x) - h_F(1) = \frac{\left(e^{\theta}-1\right)^3}{e^{2\theta}} \left(\frac{1+x^2}{Q(x)} - \frac{2}{Q(1)} \right).$$

In view of the fact that $h_F(1) = \frac{2\left(e^{\theta}-1\right)^3}{5e^{2\theta}-5e^{\theta}+2}$, from the last equation we have

$$h_F(x) - \frac{2\left(e^{\theta}-1\right)^3}{5e^{2\theta}-5e^{\theta}+2} = \frac{\left(e^{\theta}-1\right)^3}{e^{2\theta}} \left(\frac{1+x^2}{Q(x)} - \frac{2}{Q(1)} \right),$$

or

$$h_F(x) = \frac{\left(e^{\theta}-1\right)^3 \left(1+x^2\right)}{e^{2\theta} Q(x)} \quad x \in \mathbb{N}^*,$$

which, in view of (51), implies that X has pmf (50).

2.3. Characterizations of DGu, SG, CosPois, DP, EDLi, NDOPPE, QPAD, ZTPID and PSD respectively, based on reverse hazard function

Proposition 2.3.1. Let $X : \Omega \to \mathbb{Z}$ be a random variable. The pmf of X is (5) if and only if its reverse hazard rate function satisfies the difference equation

$$r_F(k+1) - r_F(k) = \left(\frac{e^{-\alpha p^k}}{e^{-\alpha p^{k+1}}} \right) - \left(\frac{e^{-\alpha p^{k+1}}}{e^{-\alpha p^{K+2}}} \right), \qquad k \in \mathbb{Z}\ , \tag{132}$$

with the boundary condition $r_F(0) = 1 - e^{\alpha(p-1)}$.

Proof. If X has pmf (5), then clearly (132) holds. Now, if (132) holds, then for every $x \in \mathbb{Z}$, we have

$$\sum_{k=0}^{x-1} \{r_F(k+1) - r_F(k)\} = \sum_{k=0}^{x-1} \left\{ \left(\frac{e^{-\alpha p^k}}{e^{-\alpha p^{k+1}}} \right) - \left(\frac{e^{-\alpha p^{k+1}}}{e^{-\alpha p^{K+2}}} \right) \right\},$$

or

$$r_F(x) - r_F(0) = e^{\alpha(p-1)} - \left(\frac{e^{-\alpha p^x}}{e^{-\alpha p^{x+1}}} \right).$$

In view of the fact that $r_F(0) = 1 - e^{\alpha(p-1)}$, from the last equation we have

$$r_F(x) = 1 - \left(\frac{e^{-\alpha p^x}}{e^{-\alpha p^{x+1}}} \right),$$

which, in view of (6), implies that X has pmf (5).

Proposition 2.3.2. Let $X:\Omega\to\mathbb{Z}^*$, be a random variable. The pmf of X is (8) if and only if its reverse hazard rate function satisfies the difference equation

$$\begin{aligned}&r_F(k+1)-r_F(k)\\&=\left(\frac{q(1-p^{\alpha+1})p^{k+1}(1-p^{\alpha(k+1)+1})}{(1-p^{\alpha+1})(1-p^{k+2})-pq(1-p^{(\alpha+1)(k+2)})}\right)\\&-\left(\frac{q(1-p^{\alpha+1})p^{k}(1-p^{\alpha k+1})}{(1-p^{\alpha+1})(1-p^{k+1})-pq(1-p^{(\alpha+1)(k+1)})}\right),\qquad k\in\mathbb{Z}^*,\end{aligned}\tag{133}$$

with the boundary condition $r_F(0)=\left(\frac{q(1-p^{\alpha+1})(1-p)}{(1-p)(1-p^{\alpha+1})-pq(1-p^{\alpha+1})}\right)$.

Proof. If X has pmf (8), then clearly (133) holds. Now, if (133) holds, then for every $x\in\mathbb{Z}^*$, we have

$$\sum_{k=0}^{x-1}\left\{h_F(k+1)-h_F(k)\right\}=\sum_{k=0}^{x-1}\left\{\begin{array}{c}\frac{q(1-p^{\alpha+1})p^{k+1}(1-p^{\alpha(k+1)+1})}{(1-p^{\alpha+1})(1-p^{k+2})-pq(1-p^{(\alpha+1)(k+2)})}\\-\frac{q(1-p^{\alpha+1})p^{k}(1-p^{\alpha k+1})}{(1-p^{\alpha+1})(1-p^{k+1})-pq(1-p^{(\alpha+1)(k+1)})}\end{array}\right\},$$

or

$$\begin{aligned}&r_F(x)-r_F(0)\\&=\left(\frac{q(1-p^{\alpha+1})p^{x}(1-p^{\alpha x+1})}{(1-p^{\alpha+1})(1-p^{x+1})-pq(1-p^{(\alpha+1)(x+1)})}\right)-\left(\frac{q(1-p^{\alpha+1})(1-p)}{(1-p)(1-p^{\alpha+1})-pq(1-p^{\alpha+1})}\right).\end{aligned}$$

In view of the fact that $r_F(0)=\left(\frac{q(1-p^{\alpha+1})(1-p)}{(1-p)(1-p^{\alpha+1})-pq(1-p^{\alpha+1})}\right)$, from the last equation we have

$$r_F(x)=\left(\frac{q(1-p^{\alpha+1})p^{x}(1-p^{\alpha x+1})}{(1-p^{\alpha+1})(1-p^{x+1})-pq(1-p^{(\alpha+1)(x+1)})}\right),$$

which, in view of (9), implies that X has pmf (8).

Proposition 2.3.3. Let $X:\Omega\to\mathbb{N}^*$ be a random variable. The pmf of X is (14) if and only if its reverse hazard rate function satisfies the difference equation

$$\begin{aligned}&r_F(k+1)-r_F(k)\\&=\frac{2e^{-\lambda}\left[\cos\left(\beta\left(k+1\right)\right)\right]^2\frac{\lambda^{k+1}}{(k+1)!}}{F_*(k+1;\lambda)+\Upsilon(k+1;\beta,\lambda)}-\frac{2e^{-\lambda}\left[\cos\left(\beta k\right)\right]^2\frac{\lambda^{k}}{k!}}{F_*(k;\lambda)+\Upsilon(k;\beta,\lambda)},\quad k\in\mathbb{N},\end{aligned}\tag{134}$$

with the boundary condition $r_F(0)=\frac{2}{1+e^{\lambda}\Upsilon(0;\beta,\lambda)}$.

Proof. If X has pmf (14), then clearly (134) holds. Now, if (134) holds, then for every $x\in\mathbb{N}^*$, we have

$$\begin{aligned}&\sum_{k=0}^{x-1}\left\{r_F(k+1)-r_F(k)\right\}\\&=\sum_{k=0}^{x-1}\left\{\frac{2e^{-\lambda}\left[\cos\left(\beta\left(k+1\right)\right)\right]^2\frac{\lambda^{k+1}}{(k+1)!}}{F_*(k+1;\lambda)+\Upsilon(k+1;\beta,\lambda)}-\frac{2e^{-\lambda}\left[\cos\left(\beta k\right)\right]^2\frac{\lambda^{k}}{k!}}{F_*(k;\lambda)+\Upsilon(k;\beta,\lambda)}\right\},\end{aligned}$$

or

$$r_F(x)-r_F(0)=\frac{2e^{-\lambda}\left[\cos\left(\beta x\right)\right]^2\frac{\lambda^{x}}{x!}}{F_*(x;\lambda)+\Upsilon(x;\beta,\lambda)}-\frac{2}{1+e^{\lambda}\Upsilon(0;\beta,\lambda)}.$$

In view of the fact that $r_F(0) = \frac{2}{1+e^{\lambda}\Upsilon(0;\beta,\lambda)}$, from the last equation we have

$$r_F(x) = \frac{2e^{-\lambda}\left[\cos(\beta x)\right]^2 \frac{\lambda^x}{x!}}{F_*(x;\lambda) + \Upsilon(x;\beta,\lambda)},$$

which, in view of (15), implies that X has pmf (14).

Proposition 2.3.4. Let $X : \Omega \to \mathbb{N}^*$ be a random variable. The pmf of X is (35) if and only if its reverse hazard rate function satisfies the difference equation

$$\begin{aligned} r_F(k+1) - r_F(k) &= \frac{(1+\alpha)(e^{\beta}-1)e^{\beta(k+1)}}{\left(1+\alpha e^{\beta(k+1)}\right)\left[\left(e^{\beta(k+2)}-1\right)\right]} \\ &- \frac{(1+\alpha)(e^{\beta}-1)e^{\beta k}}{\left(1+\alpha e^{\beta k}\right)\left[\left(e^{\beta(k+1)}-1\right)\right]}, \end{aligned} \tag{135}$$

$k \in \mathbb{N}^*$, with the boundary condition $r_F(0) = 1$.

Proof. If X has pmf (35), then clearly (135) holds. Now, if (135) holds, then for every $x \in \mathbb{N}^*$, we have

$$\begin{aligned} &\sum_{k=0}^{x-1}\left\{r_F(k+1) - r_F(k)\right\} \\ &= \sum_{k=0}^{x-1}\left\{\frac{(1+\alpha)(e^{\beta}-1)e^{\beta(k+1)}}{\left(1+\alpha e^{\beta(k+1)}\right)\left[\left(e^{\beta(k+2)}-1\right)\right]} - \frac{(1+\alpha)(e^{\beta}-1)e^{\beta k}}{\left(1+\alpha e^{\beta k}\right)\left[\left(e^{\beta(k+1)}-1\right)\right]}\right\}, \end{aligned}$$

or

$$r_F(x) - r_F(0) = \frac{(1+\alpha)(e^{\beta}-1)e^{\beta x}}{\left(1+\alpha e^{\beta x}\right)\left[\left(e^{\beta(x+1)}-1\right)\right]} - 1.$$

In view of the fact that $r_F(0) = 1$, from the last equation we have

$$r_F(x) = \frac{(1+\alpha)(e^{\beta}-1)e^{\beta x}}{\left(1+\alpha e^{\beta x}\right)\left[\left(e^{\beta(x+1)}-1\right)\right]},$$

which, in view of (36), implies that X has pmf (35).

Proposition 2.3.5. Let $X : \Omega \to \mathbb{N}^*$ be a random variable. The pmf of X is (44) if and only if its reverse hazard rate function satisfies the difference equation

$$r_F(k+1) - r_F(k) = \frac{\Lambda(k+1;a,b)}{\Lambda(k+2;a,b)} - \frac{\Lambda(k;a,b)}{\Lambda(k+1;a,b)}, \qquad k \in \mathbb{N}^*, \tag{136}$$

with the initial condition $r_F(0) = 1$.

Proof. If X has pmf (44), then clearly (136) holds. Now, if (136) holds, using telescoping sum, for every $x \in \mathbb{N}^*$, we have

$$\begin{aligned} &\sum_{k=0}^{x-1}\left\{r_F(k+1) - r_F(k)\right\} \\ &= \frac{\Lambda(0;a,b)}{\Lambda(1;a,b)} - \frac{\Lambda(x;a,b)}{\Lambda(x+1;a,b)} = -\frac{\Lambda(x;a,b)}{\Lambda(x+1;a,b)}. \end{aligned}$$

So, we have

$$r_F(x) - r_F(0) = -\frac{\Lambda(x;a,b)}{\Lambda(x+1;a,b)}.$$

In view of the fact that $r_F(0) = 1$, from the last equation we have

$$r_F(x) - 1 = -\frac{\Lambda(x;a,b)}{\Lambda(x+1;a,b)},$$

or

$$r_F(x) = 1 - \frac{\Lambda(x;a,b)}{\Lambda(x+1;a,b)}, \quad x \in \mathbb{N}^*,$$

which, in view of (45), implies that X has pmf (44).

Proposition 2.3.6. Let $X : \Omega \to \mathbb{N}$ be a random variable. The pmf of X is (47) if and only if its reverse hazard rate function satisfies the difference equation

$$\begin{aligned} &r_F(k+1) - r_F(k) \\ &= \frac{p(k+1)(1-\theta)^{k+1}}{Q(k+1)} - \frac{p(k)(1-\theta)^k}{Q(k)}, \qquad k \in \mathbb{N}, \end{aligned} \tag{137}$$

with the initial condition $r_F(1) = \frac{p(1)(1-\theta)}{Q(1)}$.

Proof. If X has pmf (47), then clearly (137) holds. Now, if (137) holds, using telescoping sum, for every $x \in \mathbb{N}$, we have

$$\sum_{k=1}^{x-1}\{r_F(k+1) - r_F(k)\} = \frac{p(x)(1-\theta)^x}{Q(x)} - \frac{p(1)(1-\theta)}{Q(1)}.$$

So, we have

$$r_F(x) - r_F(1) = \frac{p(x)(1-\theta)^x}{Q(x)} - \frac{p(1)(1-\theta)}{Q(1)}.$$

In view of the fact that $h_F(1) = \frac{p(1)(1-\theta)}{Q(1)}$, from the last equation we have

$$r_F(x) = \frac{p(x)(1-\theta)^x}{Q(x)},$$

which, in view of (48), implies that X has pmf (47).

Proposition 2.3.7. Let $X : \Omega \to \mathbb{N}^*$ be a random variable. The pmf of X is (53) if and only if its reverse hazard rate function satisfies the difference equation

$$\begin{aligned} &r_F(k+1) - r_F(k) \\ &= \frac{Q(k+1)(\theta+1)^{-(k+1)}}{\sum_{j=0}^{k+1}\frac{Q(j)}{(\theta+1)^j}} - \frac{Q(k)(\theta+1)^{-k}}{\sum_{j=0}^{k}\frac{Q(j)}{(\theta+1)^j}}, \qquad k \in \mathbb{N}^*, \end{aligned} \tag{138}$$

with the initial condition $r_F(0) = 1$.

Proof. If X has pmf (53), then clearly (138) holds. Now, if (138) holds, using telescoping sum, for every $x \in \mathbb{N}^*$, we have

$$\begin{aligned} \sum_{k=0}^{x-1}\{r_F(k+1) - r_F(k)\} &= \frac{Q(x)(\theta+1)^{-x}}{\sum_{j=0}^{x}\frac{Q(j)}{(\theta+1)^j}} - \frac{Q(0)(\theta+1)^{-0}}{\sum_{j=0}^{0}\frac{Q(j)}{(\theta+1)^j}} \\ &= \frac{Q(x)(\theta+1)^{-x}}{\sum_{j=0}^{x}\frac{Q(j)}{(\theta+1)^j}} - 1. \end{aligned}$$

So, we have

$$r_F(x) - r_F(0) = \frac{Q(x)(\theta+1)^{-x}}{\sum_{j=0}^{x}\frac{Q(j)}{(\theta+1)^j}} - 1.$$

In view of the fact that $h_F(0)=1$, from the last equation we have

$$r_F(x)=\frac{Q(x)(\theta+1)^{-x}}{\sum_{j=0}^{x}\frac{Q(j)}{(\theta+1)^j}},$$

which, in view of (54), implies that X has pmf (53).

Proposition 2.3.8. Let $X:\Omega\to\mathbb{N}$ be a random variable. The pmf of X is (57) if and only if its reverse hazard rate function satisfies the difference equation

$$r_F(k+1)-r_F(k)$$
$$=\frac{Q(k+1)(\theta+1)^{-(k+1)}}{\sum_{j=1}^{k+1}\frac{Q(j)}{(\theta+1)^j}}-\frac{Q(k)(\theta+1)^{-k}}{\sum_{j=1}^{k}\frac{Q(j)}{(\theta+1)^j}},\qquad k\in\mathbb{N},\tag{139}$$

with the initial condition $r_F(1)=1$.

Proof. If X has pmf (56), then clearly (139) holds. Now, if (139) holds, using telescoping sum, for every $x\in\mathbb{N}$, we have

$$\sum_{k=1}^{x-1}\{r_F(k+1)-r_F(k)\}=\frac{Q(x)(\theta+1)^{-x}}{\sum_{j=1}^{x}\frac{Q(j)}{(\theta+1)^j}}-\frac{Q(1)(\theta+1)^{-1}}{\sum_{j=1}^{1}\frac{Q(j)}{(\theta+1)^j}}$$
$$=\frac{Q(x)(\theta+1)^{-x}}{\sum_{j=1}^{x}\frac{Q(j)}{(\theta+1)^j}}-1.$$

So, we have

$$r_F(x)-r_F(1)=\frac{Q(x)(\theta+1)^{-x}}{\sum_{j=1}^{x}\frac{Q(j)}{(\theta+1)^j}}-1.$$

In view of the fact that $r_F(1)=1$, from the last equation we have

$$r_F(x)=\frac{Q(x)(\theta+1)^{-x}}{\sum_{j=1}^{x}\frac{Q(j)}{(\theta+1)^j}},$$

which, in view of (57), implies that X has pmf (56).

Proposition 2.3.9. Let $X:\Omega\to\mathbb{N}^*$ be a random variable. The pmf of X is (59) if and only if its reverse hazard rate function satisfies the difference equation

$$r_F(k+1)-r_F(k)$$
$$=\frac{Q(k+1)(\theta+1)^{-(k+1)}}{\sum_{j=0}^{k+1}\frac{Q(j)}{(\theta+1)^j}}-\frac{Q(k)(\theta+1)^{-k}}{\sum_{j=0}^{k}\frac{Q(j)}{(\theta+1)^j}},\qquad k\in\mathbb{N},\tag{140}$$

with the initial condition $r_F(0)=1$.

Proof. If X has pmf (59), then clearly (140) holds. Now, if (140) holds, using telescoping sum, for every $x\in\mathbb{N}^*$, we have

$$\sum_{k=0}^{x-1}\{r_F(k+1)-r_F(k)\}=\frac{Q(x)(\theta+1)^{-x}}{\sum_{j=0}^{x}\frac{Q(j)}{(\theta+1)^j}}-\frac{Q(1)(\theta+1)^{-0}}{\sum_{j=0}^{0}\frac{Q(j)}{(\theta+1)^j}}$$
$$=\frac{Q(x)(\theta+1)^{-x}}{\sum_{j=0}^{x}\frac{Q(j)}{(\theta+1)^j}}-1.$$

So, we have

$$r_F(x) - r_F(0) = \frac{Q(x)(\theta+1)^{-x}}{\sum_{j=0}^{x} \frac{Q(j)}{(\theta+1)^j}} - 1.$$

In view of the fact that $r_F(0) = 1$, from the last equation we have

$$r_F(x) = \frac{Q(x)(\theta+1)^{-x}}{\sum_{j=0}^{x} \frac{Q(j)}{(\theta+1)^j}},$$

which, in view of (60), implies that X has pmf (59).

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