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Modeling Tri-Model Data With A New Skew Logistic Distribution

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Abstract

This paper considers a new family of the trimodal skew logistic distributions. Some properties of this distribution, including moments, moments generating function, entropy, maximum likelihood estimates of parameters and some other properties, are presented. A simulation study is conducted to examine the performance of the parameters. Numerical optimization is carried out via two real-life datasets. Results show that the new distribution is better fitted in terms of these datasets among logistic, skew logistic and alpha skew logistic distributions based on the value of AIC and BIC.

Key Words: Tri-modal distribution, logistic distribution, skews logistic distribution, maximum likelihood estimation

Mathematical Subject Classification: 60E05, 62E15.

1 Introduction

It is well-known that the bimodal distribution is widely applied for modeling many phenomena. Recently, many researchers introduced families of distributions that are appropriate to model the bimodality of data. Some of the families can fit symmetric data, and others can fit the asymmetric data. Bimodal expansion of skew-normal distribution can be found in Ma and Genton (2004), Huang and Chen (2007), Elal-Olivero et al. (2009), Elal-Olivero (2010), Arellano-Valle et al. (2010), Gómez et al. (2011), Handam (2012), Harandi and Alamatsaz (2013), Hazarika and Chakraborty (2014), among others.

Alpha Skew Normal distribution was another bimodal distribution introduced by Elal-Olivero (2010) and its density is given by

$$f(x) = \frac{\left[(1-\alpha x)^2 + 1\right]}{(2+\alpha^2)}\phi(x), \;\; x \in R, \alpha \in R,$$

where α is the asymmetry parameter which regulates the skewness and $\phi(.)$ is the standard normal pdf. Adopting this concept, Hazarika and Chakraborty (2014) introduced Alpha Skew Logistic distribution and described its distributional properties. The density of the latter is given by

$$f(x) = \frac{3\left[(1-\alpha x)^2+1\right]}{(6+\pi^2\alpha^2)}\frac{e^{-x}}{(e^{-x}+1)^2},\ \ x\in R, \alpha\in R,$$

where, α is the asymmetry parameter. On the other hand Venegas et al. (2016) introduced the logarithmic form of the alpha skew normal distribution of Elal-Olivero (2010) as Log-Alpha Skew Normal distribution with the pdf

$$f(x) = \frac{\left[(1-\alpha y)^2+1\right]}{(\alpha^2+2)\,\sigma x}\phi(y),\ \ x>0, \alpha\in R,$$

where $y = \dfrac{log(x)-\mu}{\sigma}$ is the skewness parameter. Generalized Alpha Skew Normal distribution is another model introduced by Sharafi et al. (2017) and its density function is given by

$$f(x) = \frac{\left[(1-\alpha x)^2+1\right]}{C(\alpha,\gamma)}\phi(x)\Phi(\lambda x),\ \ x\in R, \alpha\in R, \lambda\in R,$$

where $C(\alpha,\lambda) = 1-\alpha b\delta+\dfrac{\alpha^2}{2}, b=\sqrt{\dfrac{2}{\pi}}, \delta=\dfrac{\lambda}{\sqrt{1+\lambda^2}}$ and $\Phi(.)$ is the cdf of the standard normal distribution. There exists some other version of probability distribution model which allows to fit the unimodal as well as bimodal data under the Balakrishnan methodology. Hazarika et al. (2019) introduced Balakrishnan Alpha Skew Normal distribution using this methodology.

Borrowing the same idea, later on Shah et al. (2019) introduced Balakrishnan Alpha Skew Laplace distribution. Shah et al. (2020) also introduced Balakrishnan Alpha Skew Logistic distribution using the same methodology. However, sometimes the datasets have a high value of skewness and kurtosis and trimodal behavior in their frequency distribution which cannot be described by traditional or other bimodal distributions mentioned above.

Mixtures of distributions are always used when it is necessary to model data with more than one mode. The appearance of tri-modal distribution can be obtained by mixing any particular distribution with three parameters. However, the mixing of distributions has some mathematical limitations. Suppose the mixture distribution has a mixing parameter of proportion w_1, w_2 and $w_3 = 1-w_1-w_2$; those are from the three distribution groups. For mixed distribution, we have three groups of parameters from those particular distributions and three more mixing parameters of proportions necessary to estimate. So, an increasing number of parameters are a matter of concern. Therefore, using these distributions has serious estimation problems while conducting the optimization process.

Accordingly, it is essential to introduce a new family of trimodal distributions as an alternative to the mixture of distributions that present multiple modes and manage to fit the asymmetry of any dataset. The primary motivation of this paper is to propose a new family of logistic distributions that allows for fitting both symmetric and asymmetric behavior of any trimodel datasets.

The rest of the article is organized as follows. In section 2, we introduced a new trimodal skew logistic distribution along with the visualization of the density and its properties. Section 3 presents some important distributional properties of the new distribution. Certain characterizations of the Tri-modal Skew Logistic (TSLG) are presented in section 4. Section 5 deals with parameter estimation of TSLG distribution. Section 6 is devoted to the simulation and section 7 provides the applications. Finally, section 8 concludes the papers.

2 A Tri-modal Skew Logistic Distribution

This section introduces a new form of tri-modal skew logistic distribution and investigates its basic properties.

Definition

A random variable X follows tri-modal skew logistic distribution, if its pdf is given by

$$f(x)=\frac{2\left(\left(\left(\frac{x}{\beta}\right)^{2}-1\right)^{2}+2\right)\exp(-\frac{x}{\beta})}{\beta \mathrm{C}_1\left\{\exp\left(-\frac{x}{\beta}\right)+1\right\}^{2}\left\{\exp\left(-\frac{\lambda x}{\beta}\right)+1\right\}},\quad x\in R,\lambda\in R,\beta>0,\alpha\geq 0, \tag{1}$$

where, $C_1=\frac{1}{15}\left(45-10\pi^2+7\pi^4\right)$ and the real number λ is the shape parameter. Using Taylor Series Expansion for $(1+z)^{-1}$ one can write single series representation of the tri-modal skew logistic density as

$$f(x)=\begin{cases}\frac{2}{\beta C_1\left(1+\exp-\frac{x}{\beta}\right)^2}\left\{\left(\left(\frac{x}{\beta}\right)^2-1\right)^2+2\right\} \\ \sum_{j=0}^{\infty}\binom{-1}{j}\exp\left(-\frac{1+\lambda j}{\beta}x\right), & x\geq 0,\\ \\ \frac{2}{\beta C_1\left(1+\exp-\frac{x}{\beta}\right)^2}\left\{\left(\left(\frac{x}{\beta}\right)^2-1\right)^2+2\right\} \\ \sum_{j=0}^{\infty}\binom{-1}{j}\exp\left(-\frac{1-\lambda-\lambda j}{\beta}x\right), & x<0,\end{cases} \tag{2}$$

By expanding the terms of equation (2), the double series representation can be written as

$$f(x)=\begin{cases}\frac{2}{\beta \mathrm{C}_1}\left\{\left(\left(\frac{x}{\beta}\right)^2-1\right)^2+2\right\} \\ \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}\exp\left(-C_2x\right), & x\geq 0,\\ \\ \frac{2}{\beta \mathrm{C}_1}\left\{\left(\left(\frac{x}{\beta}\right)^2-1\right)^2+2\right\} \\ \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}\exp\left(Dx\right), & x<0,\end{cases} \tag{3}$$

where, $C_2=\frac{1+\lambda j+k}{\beta}$, $\frac{1+\lambda+\lambda j+k}{\beta}=\left(C_2+\frac{\lambda}{\beta}\right)$ and $D=\left(C_2+\frac{\lambda}{\beta}\right)$.

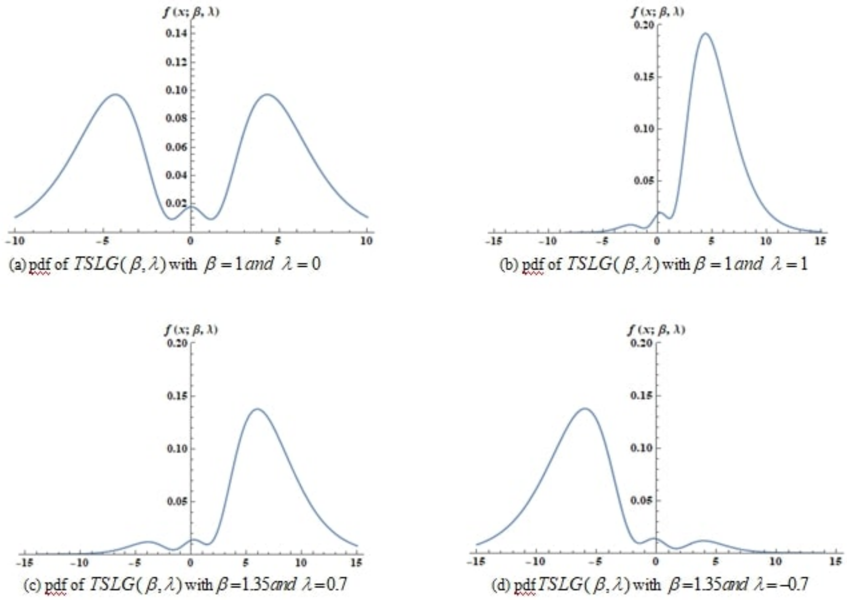

Figure 1: Plots of the probability density function of $TSLG(\beta, \lambda)$ for different choices of β and λ

2.1 Special case of $TSLG(\beta, \lambda)$distribution

1. If $\beta = 1$, then we get

$$f(x) = \frac{2\left(\left(x^2-1\right)^2+2\right)\exp(-x)}{C_1\{\exp(-x)+1\}^2\{\exp(-\lambda x)+1\}};\ \ x \in R, \alpha \in R, \lambda \in R.$$

2. If $\lambda = 0$, then we get the tri-modal logistic distribution and is given by

$$f(x) = \frac{\left(\left(\left(\frac{x}{\beta}\right)^2-1\right)^2+2\right)\exp\left(-\frac{x}{\beta}\right)}{\beta \mathrm{C}_1\left\{\exp\left(-\frac{x}{\beta}\right)+1\right\}^2}.$$

2.2 Plots of density function

The density functions of $TSLG(\beta, \lambda)$ for different choice of β and λ are plotted in Figure 1. Figure 1 shows that the density may have three modes for some values of β and λ . It is noted that the model is positively skewed when $\lambda > 0$ and negatively skewed when $\lambda < 0$.

3 Distributional Properties

In this section we investigate some of the distributional properties related to the TSLG distribution.

3.1 Cumulative distribution function

Theorem 1.

The cdf of $TSLG(\beta,\lambda)$ distribution is given by

$$F(x)=\begin{cases}\frac{2}{\beta C_1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\begin{pmatrix}-1\\ j\end{pmatrix}\begin{pmatrix}-2\\ k\end{pmatrix} \\ \Big[\frac{1}{\beta^4 C_2^5}\left\{24-\exp\left(-C_2x\right)(24+C_2x\right. \\ \left.\left(24+12C_2x+\left(C_2x\right)^3+4\left(C_2x\right)^2\right)\right\} \\ -\frac{2}{\beta^2C_2^3}\left\{2-\left(\left(C_2x\right)^2+C_2x+2\right)\exp\left(-C_2x\right)\right\} \\ +\frac{3}{C_2}\left\{1-\exp\left(-C_2x\right)\right\}\Big], & x\geq 0, \\ \\ \frac{2}{\beta C_1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\begin{pmatrix}-1\\ j\end{pmatrix}\begin{pmatrix}-2\\ k\end{pmatrix}\Big[\frac{1}{\beta^4 D^5}\left\{\exp\left(Dx\right)(24+Dx\right. \\ \left(-24+12Dx-4\left(Dx\right)^2+\left(Dx\right)^3\right)-\frac{2}{\beta^2\left(D\right)^3}\left\{\exp\left(Dx\right)\right. \\ \left.\left(\left(Dx\right)^2-\left(Dx\right)+2\right)\right\}+\frac{3}{D}\exp\left(D\right)x\Big], & x<0,\end{cases} \tag{4}$$

where, $D=\left(C_2+\dfrac{\lambda}{\beta}\right)$.

Proof

Case 1: If $X\geq 0$ then cdf F(x) can be written as

$$\begin{aligned}F(x)&=P(X\leq x)\\ &=\int_0^x\frac{2}{\beta C_1}\left\{\left(\left(\frac{z}{\beta}\right)^2-1\right)^2+2\right\}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\begin{pmatrix}-1\\ j\end{pmatrix}\begin{pmatrix}-2\\ k\end{pmatrix}\exp\left(-C_2z\right)dz\\ &=\frac{2}{\beta C_1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\begin{pmatrix}-1\\ j\end{pmatrix}\begin{pmatrix}-2\\ k\end{pmatrix}\left[\frac{1}{\beta^4}\int_0^x z^4\exp\left(-C_2z\right)\,dz\right.\\ &\left.-\frac{2}{\beta^2}\int_0^x z^2\exp\left(-C_2z\right)\,dz+3\int_0^x\exp\left(-C_2z\right)\,dz\right].\end{aligned} \tag{5}$$

we can also write it as

$$F(x)=\frac{2}{\beta C_1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\begin{pmatrix}-1\\ j\end{pmatrix}\begin{pmatrix}-2\\ k\end{pmatrix}I_1. \tag{6}$$

applying (3.351.1) and (3.351.8) in Gradshteyn and Ryzhik (2000), the integral I_1 can be calculated as

$$\begin{aligned}I_1&=\frac{1}{\beta^4C_2^5}\left\{24-\exp\left[\left(-C_2x\right)\left(C_2x\left(\left(C_2x\right)^3+4\left(C_2x\right)^2+12C_2x+24\right)+24\right)\right\}\right.\\ &-\frac{2}{\beta^2C_2^3}\left\{2-\exp\left[\left(-C_2x\right)\left(\left(C_2x\right)^2+C_2x+2\right)\right\}\right.+\frac{3}{C_2}\left\{1-\exp\left[-C_2x\right\}\right..\end{aligned} \tag{7}$$

From (6) and (7) one can obtain the cdf $F(x)$ for $X \geq 0$ as

$$F(x) = \frac{2}{\beta C_1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \begin{pmatrix} -1 \\ j \end{pmatrix} \begin{pmatrix} -2 \\ k \end{pmatrix} \Bigg[\frac{1}{\beta^4 C_2^5} \{24 - \exp(-C_2 x)(24 + C_2 x \left(24 + 12C_2 x + (C_2 x)^3 + 4(C_2 x)^2\right)\} - \frac{2}{\beta^2 C_2^3} \left\{2 - \left((C_2 x)^2 + C_2 x + 2\right) \exp(-C_2 x)\right\} + \frac{3}{C_2} \{1 - \exp(-C_2 x)\} \Bigg].$$

Case 2: If $X < 0$ the cdf $F(x)$ can be written as

$$F(x) = \int_{-\infty}^{x} \frac{1}{2\beta C_1} \left\{ \left(\left(\frac{z}{\beta} \right)^2 - 1 \right)^2 + 2 \right\} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \begin{pmatrix} -1 \\ j \end{pmatrix} \begin{pmatrix} -2 \\ k \end{pmatrix} \exp\left(z \left(\frac{\lambda}{\beta} + C_2 \right) \right) dz.$$

Similarly, the cdf of $TSLG(\lambda)$ for $x < 0$ can be written as

$$F(x) = \frac{2}{\beta C_1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \begin{pmatrix} -1 \\ j \end{pmatrix} \begin{pmatrix} -2 \\ k \end{pmatrix} \Bigg[\frac{1}{\beta^4 D^5} \{\exp(Dx)(24 + Dx \left(-24 + 12Dx - 4(Dx)^2 + (Dx)^3\right) - \frac{2}{\beta^2 (D)^3} \{\exp(Dx) \left((Dx)^2 - (Dx) + 2\right)\} + \frac{3}{D} \exp(D)x \Bigg].$$

3.2 Moment Generating Function (mgf)

In this section, we derive the moment generating function of the tri-modal skew logistic distribution.

Theorem 2.

The moment generating function (mgf) of $TSLG(\lambda)$ distribution is given by

$$M(t) = \frac{1}{2C_1} \Bigg[24 \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \begin{pmatrix} -1 \\ j \end{pmatrix} \begin{pmatrix} -2 \\ k \end{pmatrix} \left\{ \left(\frac{1}{D+t} \right)^5 + \left(\frac{1}{C_2 - t} \right)^5 \right\} - 4 \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \begin{pmatrix} -1 \\ j \end{pmatrix} \begin{pmatrix} -2 \\ k \end{pmatrix} \left\{ \left(\frac{1}{D+t} \right)^3 + \left(\frac{1}{C_2 - t} \right)^3 \right\} + 3 \sum_{j=0}^{\infty} \begin{pmatrix} -1 \\ j \end{pmatrix} (1 + (\beta t - \lambda j) B(1 - \beta t + \lambda j) - (\beta t + \lambda + \lambda j) B(\beta t + \lambda + \lambda j + 1)) \Bigg]. \tag{8}$$

Proof:

On using single and double series expansion of (2) and (3) one can obtain

$$M(t) = E(exp(tx)) = \frac{2}{\beta C_1}(I_2 + I_3) \quad where, \tag{9}$$

$$\begin{aligned} I_2 = & \frac{1}{\beta^4}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}\int_0^{\infty} x^4 \exp\left(t - C_2 x\right)\, dx \\ & - \frac{2}{\beta^2}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}\int_0^{\infty} x^2 \exp\left(t - C_2 x\right)\, dx \\ & + 3\sum_{j=0}^{\infty}\binom{-1}{j}\int_0^{\infty}\frac{\exp\left(x\left(t - \frac{1+\lambda j}{\beta}\right)\right)}{\left\{\exp\left(-\frac{x}{\beta}\right) + 1\right\}^2} dx \end{aligned}$$

and

$$\begin{aligned} I_3 = & \frac{1}{\beta^4}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}\int_0^{\infty} x^4 \exp\left((t + D)x\right)\, dx \\ & - \frac{2}{\beta^2}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}\int_0^{\infty} x^2 \exp\left((t + D)x\right)\, dx \\ & + 3\sum_{j=0}^{\infty}\binom{-1}{j}\int_0^{\infty}\frac{x^4\exp\left(x\left(t - \frac{-j\lambda - \lambda + 1}{\beta}\right)\right)}{\left\{\exp\left(-\frac{x}{\beta}\right) + 1\right\}^2} dx. \end{aligned}$$

The last part of both of the integrals can be obtained by applying (2.2.4.4) in Prudnikov et al. (1986) after using the method of substitution and the remaining parts of integrals I_2 and I_3 can be calculated by using (3.351.1) and (3.326.2) in Gradshteyn and Ryzhik (2000). Therefore,

$$\begin{aligned} I_2 = & \frac{24}{\beta^4}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}\left(\frac{\beta}{j\lambda + k - \beta t + 1}\right)^5 \\ & - \frac{4}{\beta^2}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}\left(\frac{\beta}{j\lambda + k - \beta t + 1}\right)^3 \\ & + 3\beta\sum_{j=0}^{\infty}\binom{-1}{j}\left\{B(j\lambda + \beta(-t) + 1)(\beta t - j\lambda) + \frac{1}{2}\right\} \end{aligned} \tag{10}$$

and

$$\begin{aligned} I_3 = & \frac{24}{\beta^4}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}\left(\frac{1}{D + t}\right)^5 \\ & - \frac{4}{\beta^2}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}\left(\frac{1}{D + t}\right)^3 \\ & + 3\beta\sum_{j=0}^{\infty}\binom{-1}{j}\left\{\frac{1}{2} - B(j\lambda + \lambda + \beta t)(j\lambda + \lambda + \beta t + 1)\right\}. \end{aligned} \tag{11}$$

Substituting (10) and (11) in (8), we can get the mgf of tri-modal skew logistic distribution as

$$M(t) = \frac{1}{2C_1}\left[24\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}\left\{\left(\frac{1}{D+t}\right)^5 + \left(\frac{1}{C_2-t}\right)^5\right\}\right.$$
$$- 4\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}\left\{\left(\frac{1}{D+t}\right)^3 + \left(\frac{1}{C_2-t}\right)^3\right\}$$
$$+ 3\sum_{j=0}^{\infty}\binom{-1}{j}\ (1 + (\beta t - \lambda j)B(1 - \beta t + \lambda j)$$
$$\left. - (\beta t + \lambda + \lambda j)B(\beta t + \lambda + \lambda j + 1))\right],$$

where, $D = \left(C_2 + \frac{\lambda}{\beta}\right)$.

3.3 Moments

In this section, we derive the n^{th} order moment of the tri-modal skew logistics distribution. Using single-series and double-series expansion of the new distribution one can obtain n^{th} order moments as

$$E(X^n) = \begin{cases} \frac{1}{C_1}\left[2(n+4)!\beta^n\left\{1-2^{1-(n+4)}\right\}\zeta(n+4) \right. \\ -2(2(n+2))!\beta^n\left\{1-2^{1-(n+2)}\right\}\zeta(n+2) \\ \left. +3(2n)!\beta^n\left\{1-2^{1-n}\right\}\zeta(n)\right], & when\, n\, is\, even, \\ \\ \frac{1}{C_1}\left[2(n+4)!\beta^n\left\{\zeta(n+4)\left\{1-2^{1-(n+4)}\right\}\right.\right. \\ \left. +\frac{1}{2^{n+4}\lambda^{n+5}}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,n+5)\right\} \\ -4(n+2)!\beta^n\left\{\zeta(n+2)\left\{1-2^{1-(n+2)}\right\}\right. \\ \left. +\frac{1}{2^{n+2}\lambda^{n+3}}\sum_{j=0}^{\infty}(-1)^j j+1\xi(j,n+3)\right\} \\ +6n!\beta^n\left\{\zeta(n)\left\{1-2^{1-n}\right\}\right. \\ \left.\left. +\frac{1}{2^n\lambda^n}\ \sum_{j=0}^{\infty}(-1)^j j+1\xi(j,n+1)\right\}\right], & when\, n\, is\, odd, \end{cases} \tag{12}$$

Proof:

$$E(X^n) = \frac{2}{C_1}\int_{-\infty}^{\infty} X^n \frac{\exp\left(-\frac{x}{\beta}\right)\left\{\left(\left(\frac{x}{\beta}\right)^2 - 1\right)^2 + 2\right\}}{\beta\left\{\exp\left(-\frac{x}{\beta}\right)+1\right\}^2\left\{\exp\left(-\frac{\lambda x}{\beta}\right)+1\right\}}dx$$
$$= \frac{1}{C_1}\left[\frac{1}{\beta^4}\int_{-\infty}^{\infty} x^{n+4}\frac{\left(2\exp\left(-\frac{x}{\beta}\right)\right)}{\beta\left\{\exp\left(-\frac{x}{\beta}\right)+1\right\}^2\left\{\exp\left(-\frac{\lambda x}{\beta}\right)+1\right\}}dx\right.$$
$$- \frac{2}{\beta^2}\int_{-\infty}^{\infty} x^{n+2}\frac{\left(2\exp\left(-\frac{x}{\beta}\right)\right)}{\beta\left\{\exp\left(-\frac{x}{\beta}\right)+1\right\}^2\left\{\exp\left(-\frac{\lambda x}{\beta}\right)+1\right\}}dx$$

$$+3\int_{-\infty}^{\infty}\frac{x^n\left(2\exp\left(-\frac{x}{\beta}\right)\right)}{\beta\left\{\exp\left(-\frac{x}{\beta}\right)+1\right\}^2\left\{\exp\left(-\frac{\lambda x}{\beta}\right)+1\right\}}dx\Bigg]$$

or,

$$E(X^n)=\frac{1}{C_1}[I_4-2I_5+3I_6]. \tag{13}$$

Now the integrals I_4,I_5 and I_6 are same as the $(n+4)^{th}$, $(n+2)^{th}$ and n^{th} order moments of the skew logistic distribution of Nadarajah (2009). Thus from the literature the even order moments of (1) are

$$I_4=2(n+4)!\beta^n 1-2^{1-(n+4)}\zeta(n+4) \tag{14}$$

$$I_5=2(n+2)!\beta^n 1-2^{1-(n+2)}\zeta(n+2) \tag{15}$$

$$I_6=2n!\beta^n 1-2^{1-n}\zeta(n). \tag{16}$$

Thus, by combining (14)-(16) and substituting (13), the even order moment becomes

$$\begin{aligned}E(X^n)=\frac{1}{C_1}&\Big[2\zeta(n+4)(n+4)!\beta^n\left\{1-2^{1-(n+4)}\right\}\\&-2\zeta(n+2)(2(n+2))!\beta^n\left\{1-2^{1-(n+2)}\right\}\\&+3\zeta n(2n)!\beta^n\left\{1-2^{1-n}\right\}\Big].\end{aligned} \tag{17}$$

If n is odd then, using the definition of zeta function one can obtain the n^{th} the order moment as

$$\begin{aligned}E(X^n)=\frac{1}{C_1}&\Bigg[2(n+4)!\beta^n\Bigg\{\left\{1-2^{1-(n+4)}\right\}\zeta(n+4)\\&+\frac{1}{2^{n+4}\lambda^{n+5}}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,n+5)\Bigg\}\\&-4(n+2)!\beta^n\Bigg\{\left\{1-2^{1-(n+2)}\right\}\zeta(n+2)\\&+\frac{1}{2^{n+2}\lambda^{n+3}}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,n+3)\Bigg\}\\&+6n!\beta^n\Bigg\{\zeta n\left\{1-2^{1-n}\right\}\\&+\frac{1}{2^n\lambda^n}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,n+1)\Bigg\}\Bigg],\end{aligned} \tag{18}$$

where the Riemann's zeta functions are defined by

$$\zeta(a,q)=\sum_{j=0}^{\infty}\frac{1}{(j+q)^a}$$

and

$$\xi(j,k)=\zeta\left(k,\frac{j+2\lambda+1}{2\lambda}\right)-\zeta\left(k,\frac{j+\lambda+1}{2\lambda}\right).$$

One can found these special functions in Prudnikov et al. (1986) and Gradshteyn and Ryzhik (2000).

Remark 1:

Thus, using (17) and (18), the first four raw moments of X can be obtained as

$$E(X) = \frac{1}{C_1}\left[\frac{15\beta^5}{2}\left\{-\frac{5}{4}\psi'''(1) + \frac{1}{\lambda^6}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,6)\right\}\right.$$
$$\left. - 3\beta^3\left\{-3\psi''(1) + \frac{1}{\lambda^4}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,4)\right\} + 3\frac{\beta}{\lambda^2}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,2)\right].$$
$$E(X^2) = \frac{(\pi\beta)^2}{3C_1}\left[\frac{115(\pi\beta)^4 - 98(\pi\beta)^2 + 105}{35}\right].$$
$$E(X^3) = \frac{1}{C_1}\left[\frac{63\beta^5}{4}\left\{-\frac{7}{8}\psi'''(1) + \frac{5}{\lambda^8}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,8)\right\}\right.$$
$$- 15\beta^5\left\{-\frac{15}{4}\psi''(1) + \frac{1}{\lambda^6}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,6)\right\}$$
$$\left. + \frac{9}{2}\beta^3\left\{-3\psi''(1) + \frac{1}{\lambda^4}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,4)\right\}\right].$$
$$E(X^4) = \frac{(\pi\beta)^4}{3C_1}\left[\frac{889(\pi\beta)^4 - 310(\pi\beta)^2 + 147}{35}\right].$$

3.4 Entropy

The amount of information of the distribution, relating to the outcome of an experiment is called the entropy of the distribution Rényi (1961). The Renyi entropy of order γ for a random variable X is defined as

$$H_R(\gamma) = \frac{1}{1-\gamma}\log\int f^{\gamma}(x)dx$$

where, $\gamma > 0$ and $\gamma \neq 1$. Using the single series representation of $TSLG(\beta,\gamma)$, one can write

$$\int_{-\infty}^{\infty} x f^{\gamma}\, dx = \frac{1}{C^{\gamma}}(I_7 + I_8) \quad ; where \tag{19}$$

$$I_7 = \frac{1}{\beta^4}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-\gamma}{j}\binom{-2\gamma}{k}\int_{-\infty}^{\infty} x^4 \exp\left(-\frac{x(\gamma + j\lambda + k)}{\beta}\right)dx$$
$$- \frac{2}{\beta^2}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-\gamma}{j}\binom{-2\gamma}{k}\int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x(\gamma + j\lambda + k)}{\beta}\right)dx$$
$$+ 3\sum_{j=0}^{\infty}\binom{-\gamma}{j}\int_{-\infty}^{\infty}\frac{\exp\left(\left(-\frac{\gamma+\lambda j}{\beta}\right)x\right)}{\left(1+\exp(-\frac{x}{\beta})\right)^{2\gamma}}dx$$

and

$$
\begin{aligned}
I_8 = & \frac{1}{\beta^4}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-\gamma}{j}\binom{-2\gamma}{k}\int_{-\infty}^{\infty} x^4 \exp\left(\frac{x(\gamma\lambda+\gamma+j\lambda+k)}{\beta}\right) dx \\
& - \frac{2}{\beta^2}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-\gamma}{j}\binom{-2\gamma}{k}\int_{-\infty}^{\infty} x^2 \exp\left(\frac{x(\gamma\lambda+\gamma+j\lambda+k)}{\beta}\right) dx \\
& + 3\sum_{j=0}^{\infty}\binom{-\gamma}{j}\int_{-\infty}^{\infty}\frac{\exp\left(\left(-\frac{\gamma-\lambda\gamma-\lambda j}{\beta}\right)x\right)}{\left(1+\exp(-\frac{x}{\beta})\right)^{2\gamma}} dx.
\end{aligned}
$$

Substituting $y = \exp\left(\frac{-x}{\beta}\right)$ in last part of the both the integrals and by applying respectively (3.194.1) – (3.194.2) and (3.351.1) and (3.326.2) in Gradshteyn and Ryzhik (2000) in the remaining part, the integrals I_7 and I_8 reduces to

$$
\begin{aligned}
I_7 = & \frac{24}{\beta^4}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-\gamma}{j}\binom{-2\gamma}{k}\left(\frac{\beta}{\gamma+j\lambda+k}\right)^5 \\
& - \frac{4}{\beta^2}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-\gamma}{j}\binom{-2\gamma}{k}\left(\frac{\beta}{\gamma+j\lambda+k}\right)^3 \\
& + 3\sum_{j=0}^{\infty}\binom{-\gamma}{j}\frac{\beta}{\gamma+j\lambda}\,{}_2F_1(2\gamma,\gamma+\lambda j;1+\gamma+\lambda j;-1)
\end{aligned} \tag{20}
$$

and

$$
\begin{aligned}
I_8 = & \frac{24}{\beta^4}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-\gamma}{j}\binom{-2\gamma}{k}\left(\frac{\beta}{\gamma\lambda+\gamma+j\lambda+k}\right)^5 \\
& - \frac{4}{\beta^2}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-\gamma}{j}\binom{-2\gamma}{k}\left(\frac{\beta}{\gamma\lambda+\gamma+j\lambda+k}\right)^3 \\
& + 3\sum_{j=0}^{\infty}\binom{-\gamma}{j}\frac{\beta}{\gamma\lambda+\gamma+j\lambda}\,{}_2F_1(2\gamma,\gamma+\gamma\lambda+\lambda j;1+\gamma+\gamma\lambda+\lambda j;-1).
\end{aligned} \tag{21}
$$

Substituting (20) and (21) in (19), one obtains the Renyi entropy as

$$
\begin{aligned}
H_r(\gamma) = & \gamma\log(2\beta C_1) + \frac{1}{1-\gamma}\left[\frac{24}{\beta^4}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-\gamma}{j}\binom{-2\gamma}{k}\left\{\left(\frac{\beta}{\gamma+j\lambda+k}\right)^5\right.\right. \\
& \left. + \left(\frac{\beta}{\gamma\lambda+\gamma+j\lambda+k}\right)^5\right\} - \frac{4}{\beta^2}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-\gamma}{j}\binom{-2\gamma}{k}\left\{\left(\frac{\beta}{\gamma+j\lambda+k}\right)^3\right. \\
& \left. + \left(\frac{\beta}{\gamma\lambda+\gamma+j\lambda+k}\right)^3\right\} + 3\sum_{j=0}^{\infty}\binom{-\gamma}{j}\left\{\frac{\beta}{\gamma+j\lambda}\,{}_2F_1(2\gamma,\gamma+\lambda;\gamma+j\lambda+1;\right. \\
& \left.\left. -1) + \frac{\beta}{\gamma\lambda+\gamma+j\lambda}\,{}_2F_1(2\gamma,\gamma\lambda+\gamma+j\lambda;\gamma\lambda+\gamma+j\lambda+1;-1)\right\}\right].
\end{aligned}
$$

4 Characterizations Results

This section considers the characterizations of the TSLG distribution via two truncated moments. For these characterization, the cdf need not to have a closed form.

4.1 Characterizations based on two truncated moments

This subsection deals with the characterizations of TSLG distribution based on a relationship between two truncated moments. The first two characterization apply a theorem of Glänzel (1987) Theorem 3 given below. Clearly, the result holds as well when the H is not a closed interval. This characterization is stable in the sense of weak convergence, please see reference Glanzel (1990).

Theorem 3 Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ $(d = -\infty,\ e = \infty$ might as well be allowed$)$. Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let k and h be two real functions defined on H such that

$$\mathbf{E}\left[k\left(X\right)\ \mid X \geq x\right] = \mathbf{E}\left[h\left(X\right)\ \mid X \geq x\right]\eta\left(x\right), \quad x \in H,$$

is defined with some real function η. Assume that $k,\ h \in C^1\left(H\right)$, $\eta \in C^2\left(H\right)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta h = k$ has no real solution in the interior of H. Then F is uniquely determined by the functions k, h and η , particularly

$$F\left(x\right) = \int_a^x C\left|\frac{\eta'\left(u\right)}{\eta\left(u\right)h\left(u\right) - k\left(u\right)}\right| \exp\left(-s\left(u\right)\right)\ du\ ,$$

where the function s is a solution of the differential equation $s' = \dfrac{\eta'\ h}{\eta\ h\ -\ k}$ and C is the normalization constant, such that $\int_H dF = 1$."

Proposition 1 Let the random variable $X : \Omega \to \mathbb{R}$ be continuous, and let $h\left(x\right) = \dfrac{1 + e^{-\frac{\lambda x}{\beta}}}{\left[\left(\frac{x}{\beta}\right)^2 - 1\right]^2 + 2}$ and $k\left(x\right) = h\left(x\right)\left(1 + e^{-x/\beta}\right)^{-1}$ for $x \in \mathbb{R}$. Then, the density of X is given in (1) if and only if the function ξ defined in Theorem 3 is

$$\eta\left(x\right) = \frac{1}{2}\left\{1 + \left(1 + e^{-x/\beta}\right)^{-1}\right\}, \qquad x \in \mathbb{R}.$$

Proof. If X has pdf (1), then

$$\left(1 - F\left(x\right)\right)E\left[h\left(X\right)\ \mid X \geq x\right] = \frac{2}{C_1}\left\{1 - \left(1 + e^{-x/\beta}\right)^{-1}\right\}, \qquad x \in \mathbb{R},$$

and

$$\left(1 - F\left(x\right)\right)E\left[k\left(X\right)\ \mid X \geq x\right] = \frac{1}{C_1}\left\{1 - \left(1 + e^{-x/\beta}\right)^{-2}\right\}, \qquad x \in \mathbb{R},$$

and finally

$$\eta\left(x\right)h\left(x\right) - k\left(x\right) = \frac{1}{2}h\left(x\right)\left\{1 + \left(1 + e^{-x/\beta}\right)^{-1}\right\} > 0 \quad for \ \ x \in \mathbb{R}.$$

Conversely, if η has the above form, then

$$s'(x) = \frac{\eta'(x)\,h(x)}{\eta(x)\,h(x) - k(x)} = \frac{\frac{1}{\beta}e^{-x/\beta}\left(1+e^{-x/\beta}\right)^{-2}}{1-\left(1+e^{-x/\beta}\right)^{-1}},$$

and hence

$$s(x) = -\log\left\{1-\left(1+e^{-x/\beta}\right)^{-1}\right\}, \quad x \in \mathbb{R}.$$

In view of Theorem 3, X has pdf (1).

Corollary 1 If $X : \Omega \to \mathbb{R}$ is a continuous random variable and $h(x)$ is as in Proposition 1 Then, X has pdf (1) if and only if there exist functions k and η defined in Theorem 3 satisfying the following first order differential equation

$$\frac{\eta'(x)\,h(x)}{\eta(x)\,h(x) - k(x)} = \frac{\frac{1}{\beta}e^{-x/\beta}\left(1+e^{-x/\beta}\right)^{-2}}{1-\left(1+e^{-x/\beta}\right)^{-1}}.$$

Corollary 2 The general solution of the above differential equation is

$$\eta(x) = \left\{1-\left(1+e^{-x/\beta}\right)^{-1}\right\}\left[-\int \frac{1}{\beta}e^{-x/\beta}\left(1+e^{-x/\beta}\right)^{-2}(h(x))^{-1}k(x) + D\right],$$

where D is a constant. A set of functions satisfying this differential equation is presented in Proposition 1 with $D = \frac{1}{2}$. Clearly, there are other triplets (h, k, ξ) satisfying the conditions of Theorem 3 of which one is given in Proposition 2 below.

Proposition 2 Let the random variable $X : \Omega \to \mathbb{R}$ be continuous, and let $h(x) = \dfrac{\left\{1+e^{-\frac{x}{\beta}}\right\}^2\left\{1+e^{-\frac{\lambda x}{\beta}}\right\}}{\left[\left(\frac{x}{\beta}\right)^2-1\right]^2+2}$ and $k(x) = h(x)\,e^{-x/\beta}$ for $x \in \mathbb{R}$. Then, the density of X is given in (1) if and only if the function ξ defined in Theorem 3 is

$$\eta(x) = \frac{1}{2}e^{-x/\beta}, \qquad x \in \mathbb{R}.$$

Remark 2. Similar Corollaries can be stated for the Proposition 2 as well.

5 Parameter Estimation

5.1 Location and Scale Extension

We consider an extension of tri-modal skew logistic distribution by introducing location and scale extension parameters. Using the transformation of $Z = \mu + \beta X$, where $X \sim TSLG(\lambda)$, the corresponding density of$Z$

can be written as

$$f(z;\mu,\beta,\lambda)=\frac{2\left\{\left(\left(\frac{z_i-\mu}{\beta}\right)^2-1\right)^2+2\right\}\exp\left(-\frac{z_i-\mu}{\beta}\right)}{\beta\mathrm{C}_1\left\{\exp\left(-\frac{z_i-\mu}{\beta}\right)+1\right\}^2\left\{\exp\left(-\frac{\lambda\left(z_i-\mu\right)}{\beta}\right)+1\right\}}. \tag{22}$$

5.2 Maximum Likelihood Estimation

This section concern the inference process about the parameters $\theta=(\mu,\beta,\lambda)$ of the location-scale family introduced above. If z_1, z_2,..., z_n are independently and identically distributed random variables drawn from the tri-modal skew logistic distribution, then the log-likelihood function for $\theta=(\mu,\beta,\lambda)$ is expressed as

$$\begin{aligned}l(\theta)=&\,n\log 2+\sum_{i=1}^{n}\log\left(\left\{\left(\left(\frac{z_i-\mu}{\beta}\right)^2-1\right)^2+2\right\}\right)-\frac{1}{\beta}\sum_{i=1}^{n}(z_i-\mu)\\&-n\log(\beta)-n\log\left(C_1\right)-2\sum_{i=1}^{n}\log\left(\left\{\exp\left(-\frac{z_i-\mu}{\beta}\right)+1\right\}\right)\\&-\sum_{i=1}^{n}\log\left(\left\{\exp\left(-\frac{\lambda\left(z_i-\mu\right)}{\beta}\right)+1\right\}\right).\end{aligned} \tag{23}$$

Differentiate equation (23) with respect to the parameters $\theta=(\mu,\beta,\lambda)$, the likelihood equation becomes

$$\begin{aligned}\frac{\partial l(\theta)}{\partial\mu}=&\,n-4\sum_{i=1}^{n}\frac{(z_i-\mu)\left(\left(\frac{z_i-\mu}{\beta}\right)^2-1\right)}{\beta\left\{\left(\left(\frac{z_i-\mu}{\beta}\right)^2-1\right)^2+2\right\}}-2\sum_{i=1}^{n}\frac{\exp\left(-\frac{z_i-\mu}{\beta}\right)}{\left\{\exp\left(-\frac{z_i-\mu}{\beta}\right)+1\right\}}\\&-\lambda\sum_{i=1}^{n}\frac{\exp\left(-\frac{\lambda\left(z_i-\mu\right)}{\beta}\right)}{\left\{\exp\left(-\frac{\lambda\left(z_i-\mu\right)}{\beta}\right)+1\right\}}.\end{aligned} \tag{24}$$

$$\begin{aligned}\frac{\partial l(\theta)}{\partial\beta}=&-4\sum_{i=1}^{n}\frac{(z_i-\mu)\left(\left(\frac{z_i-\mu}{\beta}\right)^2-1\right)}{\beta\left\{\left(\left(\frac{z_i-\mu}{\beta}\right)^2-1\right)^2+2\right\}}+\beta\left(\sum_{i=1}^{n}(z_i-\mu)\right)+\beta^2(-n)\\&-2\beta\sum_{i=1}^{n}\frac{(z_i-\mu)\exp\left(-\frac{z_i-\mu}{\beta}\right)}{\left\{\exp\left(-\frac{z_i-\mu}{\beta}\right)+1\right\}}\beta(-\lambda)\sum_{i=1}^{n}\frac{(z_i-\mu)\exp\left(-\frac{\lambda\left(z_i-\mu\right)}{\beta}\right)}{\left\{\exp\left(-\frac{\lambda\left(z_i-\mu\right)}{\beta}\right)+1\right\}}.\end{aligned} \tag{25}$$

$$\frac{\partial l(\theta)}{\partial\lambda}=\sum_{i=1}^{n}\frac{(z_i-\mu)\exp\left(-\frac{\lambda\left(z_i-\mu\right)}{\beta}\right)}{\left\{\exp\left(-\frac{\lambda\left(z_i-\mu\right)}{\beta}\right)+1\right\}}. \tag{26}$$

More preciously, by solving the above simultaneous equations, one can obtain the desired estimate of the parameters. However, the direct solution of the above normal equation is not mathematically sound. Therefore, we implement a numerical procedure using the GenSA package at R software.

6 Simulation Study

A simulation study has been carried out to evaluate the performance of the maximum likelihood estimates of the parameters of the $TSL(\lambda)$ model. To generate the set of random numbers we have used the Metropolis-Hastings (M-H) algorithm, with twenty-seven combinations of parameters. The process is replicated 1000 times along with the three different generated samples of size n = 100, 300 and 500 and finally the MLE are estimated for each generated sample using the $GenSA$ packagee (GenSA-package,Version − 1.0.3) in R software. The estimated statistics are presented in terms of biases and mean square errors (MSEs) of the estimates and the formula are given by

$$Bias(\hat{\theta}) = E(\hat{\theta}) - \theta \ \ and \ \ MSE(\hat{\theta}) = V(\hat{\theta}) + Bias(\hat{\theta})^2 \ \ Where, \ \hat{\theta} = (\hat{\mu}, \hat{\sigma}, \hat{\lambda})$$

Table 1: Results of Simulation

$\mu = 0$, $\sigma = 1$							
		μ		σ		λ	
λ	n	Bias	MSE	Bias	MSE	Bias	MSE
	100	0.090	2.879	0.709	1.486	1.000	2.318
-1	300	0.0212	0.759	0.286	0.594	0.932	2.197
	500	-0.0072	0.755	0.240	0.570	0.891	2.147
	100	0.00036	0.749	0.0556	0.555	0.229	0.974
-0.5	300	1.86E-10	0.766	0.0585	0.574	0.278	1.016
	500	4.61E-05	0.745	0.0605	0.577	0.258	0.984
	100	-0.039	1.354	-0.012	0.329	0.083	1.357
0	300	0.013	1.343	-0.022	0.329	-0.002	1.329
	500	-0.567	0.989	0.514	1.039	-0.223	1.089
	100	-0.547	1.503	0.459	0.972	-0.511	1.034
0.5	300	-0.541	1.039	0.531	1.066	-0.496	0.981
	500	-0.507	1.010	0.467	0.989	-0.502	0.983
	100	-0.512	1.005	0.532	1.298	-0.235	1.696
1	300	-0.494	1.223	0.458	1.159	-0.254	1.823
	500	4.59E-05	0.328	0.001	0.339	1.071	2.422

Table 2: Results of Simulation

$\mu = 1$, $\sigma = 2$							
		μ		σ		λ	
λ	n	Bias	MSE	Bias	MSE	Bias	MSE
	100	-0.973	3.912	-0.249	1.101	0.967	2.326
-1	300	-0.999	3.996	-0.194	1.1	0.998	2.389
	500	-1.039	4.099	-0.254	1.081	0.973	2.3
	100	-1.005	4.119	-0.186	1.091	0.516	1.026
-0.5	300	-0.968	3.918	-0.251	1.087	0.463	0.944
	500	-1.127	4.269	-0.22	1.108	0.481	1.005
	100	-0.954	3.977	-0.273	1.092	-0.029	0.758
0	300	-0.934	3.932	-0.287	1.112	-0.007	0.765
	500	-1.026	4.091	-0.259	1.123	-0.012	0.765
	100	-0.959	3.923	-0.243	1.066	-0.493	0.998
0.5	300	-1.005	4.052	-0.268	1.145	-0.535	1.046
	500	-1.035	4.221	-0.205	1.017	-0.471	1.003
	100	-0.957	3.839	-0.221	1.117	0.916	2.308
1	300	-1.066	4.084	-0.256	1.076	-0.991	2.307
	500	-1.04	4.202	-0.282	1.104	1.082	2.327

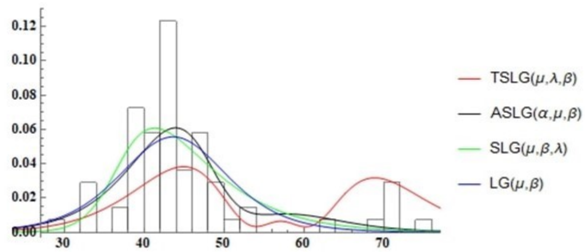

Figure 2: The histogram of N latitude degrees of 69 samples from world lakes data and fitted densities.

From $Table\ 1\ \ and\ \ 2$, it can be seen that the MLEs perform well in estimating the parameters of the model. Also, as the sample size increases, the bias and mean-square error of the MLEs decrease as expected. Then, it follows that the estimation of MLEs was asymptotically consistent for moderate and large sample sizes.

7 Real Life Applications

This section illustrates the application of the new distribution using two real-life data sets. We fit the proposed distribution with other distributions like the logistic, the skew logistic, and the alpha skew logistic distribution.

With the help of the GenSA package in R software, we obtained the maximum likelihood estimate of the parameters. Also, we compare these fitted models by using some analytical measures like the Akaike information criterion (AIC) and Bayesian information criterion (BIC).

Illustration I:

The data set considered here is the N latitude degrees of 69 samples from world lakes; we have received this data set from the website http://users.stat.umn.edu/sandy/ courses/8061/datasets/lakes.lsp. Table 1 shows the MLEs, log-likelihood, AIC and BIC of the distributions mentioned above.

Table 3: MLE's, log-likelihood, AIC and BIC for the data set of N latitude degrees of 69 samples from world lakes

$Distributions$	μ	λ	α	β	$\log l$	AIC	BIC
$LG(\mu,\beta)$	43.64	–	–	4.49	-246.65	497.29	501.86
$SLG(\mu,\beta,\lambda)$	36.79	2.83	–	6.42	-239.05	484.11	490.81
$ASLG(\mu,\beta,\alpha)$	49.09	–	0.86	3.45	-237.35	480.70	487.40
$TSLG(\mu,\beta,\lambda)$	57.13	-0.43	–	2.78	**-232.71**	471.41	478.12

It is found from the Table 3 and figure 2 that TSLG distribution is better fitted to the data rather than the other three distributions in terms of AIC and BIC.

Illustration II:

The data set is heterodatatrain$V4, obtained from R software using "Rmixmod" package packages. Vila et al. (2022) analyzed these data sets to show the modelling ability of tri-modal distribution. Table 4 shows the MLEs and log-likelihood AIC and BIC of the earlier distributions.

It is found from the Table 4 and figure 3 that TSLG distribution is better fitted to the data rather than the other three distributions in terms of AIC and BIC.

Table 4: MLE's, log-likelihood AIC and BIC of the heterodatatrain$V4 data of 300 individuals

Distributions	μ	λ	α	β	*logL*	*AIC*	*BIC*
$LG(\mu,\beta)$	-0.76	–	–	3.56	-957.87	1925.27	1936.38
$SLG(\mu,\beta,\lambda)$	-7.01	32.99	–	4.94	-874.27	1754.55	1765.66
$ASLG(\mu,\beta,\alpha)$	-0.49	–	21.08	1.68	-811.18	1628.37	1639.48
$TSLG(\mu,\beta,\lambda)$	-0.34	-0.02	–	1.05	**-755.29**	1516.59	1527.70

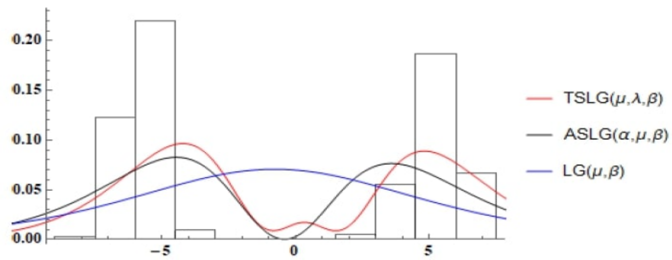

Figure 3: The histogram of heterodatatrain$V4 data and fitted densities

8 Conclusion

A new form of tri-modal logistic distribution was studied. The main statistical properties and the parameter estimation with maximum likelihood method are investigated. The characterizations of the TSLG distribution via two truncated moments have studied. The new family is found to fit unimodal as well as tri-modal data very well considering two well known multimodal data sets.

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