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A Generalized Form of Power Transformation on Exponential Family of Distribution with Properties and Application

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Abstract

In this paper, we proposed a new generalized family of distribution namely new alpha power Exponential (NAPE) distribution based on the new alpha power transformation (NAPT) method by Elbatal *et al.* (2019). Various statistical properties of the proposed distribution are obtained including moment, incomplete moment, conditional moment, probability weighted moments (PWMs), quantile function, residual and reversed residual lifetime function, stress-strength parameter, entropy and order statistics. The percentage point of NAPE distribution for some specific values of the parameters is also obtained. The method of maximum likelihood estimation (MLE) has been used for estimating the parameters of NAPE distribution. A simulation study has been performed to evaluate and execute the behavior of the estimated parameters for mean square errors (MSEs) and bias. Finally, the efficiency and flexibility of the new proposed model are illustrated by analyzing three real-life data sets.

Key Words: New Alpha Power Exponential (NAPE) Distribution, Statistical Properties, Parameter Estimation, Simulation.

Mathematical Subject Classification: 60E05, 62E15.

1. Introduction

From the past decades, there has been an increased interest in developing new generalized distributions by adding one or more additional parameters (shape) to an existing family of distributions. Adding extra parameter tends to bring more flexibility in the distribution and it is also useful to incorporate skewness into a family of distribution, Pescim *et al.* (2010).

Since the late 1980s, the method of adding parameters to an existing distribution or combining existing distributions has been used for generating new distribution. For instance, Azzalini (1985) proposed the skew normal distribution by introducing an additional parameter to the normal distribution. This additional parameter incorporates skewness and brings more flexibility to the symmetric normal distribution. Mudholkar & Srivastava (1993) proposed the exponentiated Weibull model with two shape parameters and one scale parameter. Due to the presence of an additional shape parameter, the proposed exponentiated Weibull model is more flexible than the two-parameter Weibull model. Marshall & Olkin (1997) proposed a new method for generating distributions by introducing an additional parameter to any distribution function. Gupta & Kundu (1999) introduce the generalized Exponential distribution and discuss some of its recent developments. Eugene *et al.* (2002) proposed the Beta generated method that uses the Beta distribution to develop the Beta generated distributions. This Beta-generated approach was further generalized by Jones (2004). Alzaatreh *et al.* (2003) introduced a new method for generating families of continuous distributions called the T-X family. A detail regarding the various methods for generating new distributions has been given by Lee *et al.* (2013) and Tahir & Nadarajah (2015).

In the recent past, Mahdavi & Kundu (2017) proposed the alpha power transformation (APT) method. The proposed APT method is quite easy to apply by simply raising the cumulative distribution function (CDF) of an existing distribution to a power of an additional parameter "α". They used the APT method and introduced alpha power Exponential (APE) distribution. Unal *et al.* (2018) proposed alpha power inverted Exponential distribution using the concept of the APT method. Hassan *et al.* (2019) proposed a three-parameter lifetime distribution namely alpha power transformed extended Exponential distribution (APTEE) motivated by the APT method. In recent past, Elbatal *et al.* (2019) used a new scheme to add an extra parameter to introduce a new class of distributions. The proposed method is the new alpha power transformation (NAPT) method.

According to Elbatal *et al.* (2019), the new alpha power transformation method is defined as follows:

Let $F(x)$ be the CDF of a continuous random variable $X \in R$, thus the CDF of the NAPT method is defined as,

$$F(x) = \begin{cases} \dfrac{F(x)\alpha^{F(x)}}{\alpha}; \text{if } \alpha > 0, \alpha \neq 1 \\ F(x); \text{if } \alpha = 1 \end{cases} \tag{1}$$

And the corresponding probability distribution function (PDF) is defined as,

$$f(x) = \frac{f(x)\alpha^{F(x)}}{\alpha}\left[1 + \log(\alpha)F(x)\right]; if\, \alpha > 0, \alpha \neq 1 \tag{2}$$

They have used the proposed model to study a special class of distribution function namely new alpha power transformed Weibull (NAPTW) distribution.

The main aim of this paper is to introduce and study a new lifetime distribution by using the concept of the NAPT method proposed by Elbatal *et al.* (2019). We define the new distribution as new alpha power Exponential (NAPE) distribution. The NAPE distribution is a very versatile distribution which is effective in modeling various lifetime data having monotonic and non-monotonic hazard rate functions. The rest of the paper is organized as follows: In Section 2, we introduce the NAPE distribution and we provide a mixture representation to study the importance of the NAPE distribution. In Section 3, basic statistical properties of NAPE distribution including moments, entropy, order statistics and quantile function are derived. In section 4, the maximum likelihood estimation (MLE) method is applied for estimating the value of the parameters. In section 5, simulation study is performed to outline the performance of the parameters. In section 6, the analyses of three real-life data sets are presented to illustrate the usefulness and flexibility of the NAPE distribution. Finally, in section 7, we conclude the findings of the paper.

2. New Alpha Power Exponential (NAPE) distribution

In this section, we apply the new alpha power transformation (NAPT) method to a specific class of distribution, namely the Exponential distribution and we refer the new distribution as new alpha power Exponential (NAPE) distribution with shape parameter α and scale parameter λ.

A random variable X is said to have a two-parameter NAPE distribution if the CDF of $x > 0$ is,

$$F(x;\alpha,\lambda)=\frac{\left(1-e^{-\lambda x}\right)\alpha^{\left(1-e^{-\lambda x}\right)}}{\alpha};\alpha>0,\alpha\neq1 \tag{3}$$

and the corresponding PDF is,

$$f(x;\alpha,\lambda)=\frac{\lambda e^{-\lambda x}\alpha^{\left(1-e^{-\lambda x}\right)}}{\alpha}\left[1+\log(\alpha)\left(1-e^{-\lambda x}\right)\right];\alpha>0,\alpha\neq1 \tag{4}$$

The survival function $S(x;\alpha,\lambda)$, hazard rate function $h(x;\alpha,\lambda)$, reversed hazard rate function $r(x;\alpha,\lambda)$ and cumulative hazard rate function $H(x;\alpha,\lambda)$ for $x>0$ is given as,

$$S(x;\alpha,\lambda)=\frac{\alpha-\left(1-e^{-\lambda x}\right)\alpha^{\left(1-e^{-\lambda x}\right)}}{\alpha};\alpha>0,\alpha\neq1 \tag{5}$$

$$h(x;\alpha,\lambda)=\frac{\lambda e^{-\lambda x}\alpha^{\left(1-e^{-\lambda x}\right)}\left[1+\log(\alpha)\left(1-e^{-\lambda x}\right)\right]}{\alpha-\left(1-e^{-\lambda x}\right)\alpha^{\left(1-e^{-\lambda x}\right)}};\alpha>0,\alpha\neq1 \tag{6}$$

$$r(x;\alpha,\lambda)=\frac{\lambda e^{-\lambda x}\left[1+\log(\alpha)\left(1-e^{-\lambda x}\right)\right]}{\left(1-e^{-\lambda x}\right)};\alpha>0,\alpha\neq1 \tag{7}$$

$$H(x;\alpha,\lambda)=-\log\left[\frac{\alpha-\left(1-e^{-\lambda x}\right)\alpha^{\left(1-e^{-\lambda x}\right)}}{\alpha}\right];\alpha>0,\alpha\neq1 \tag{8}$$

The main motivation for using the NAPT method on Exponential distribution is as follows:

a) The NAPT method is a very simple and efficient method of introducing only one additional parameter i.e., α.
b) The NAPT method provides greater flexibility to a family of distribution functions.
c) The NAPT method makes the distribution richer and flexible which is capable of modeling monotonically increasing, monotonically decreasing, increasing-decreasing and bathtub shape hazard rate function.
d) The NAPT method provides to be a better fit than other existing models.
e) When the additional parameter $\alpha=1$, we get the original baseline distribution which in this case is the Exponential distribution.

2.1. Graphical representation of NAPE distribution

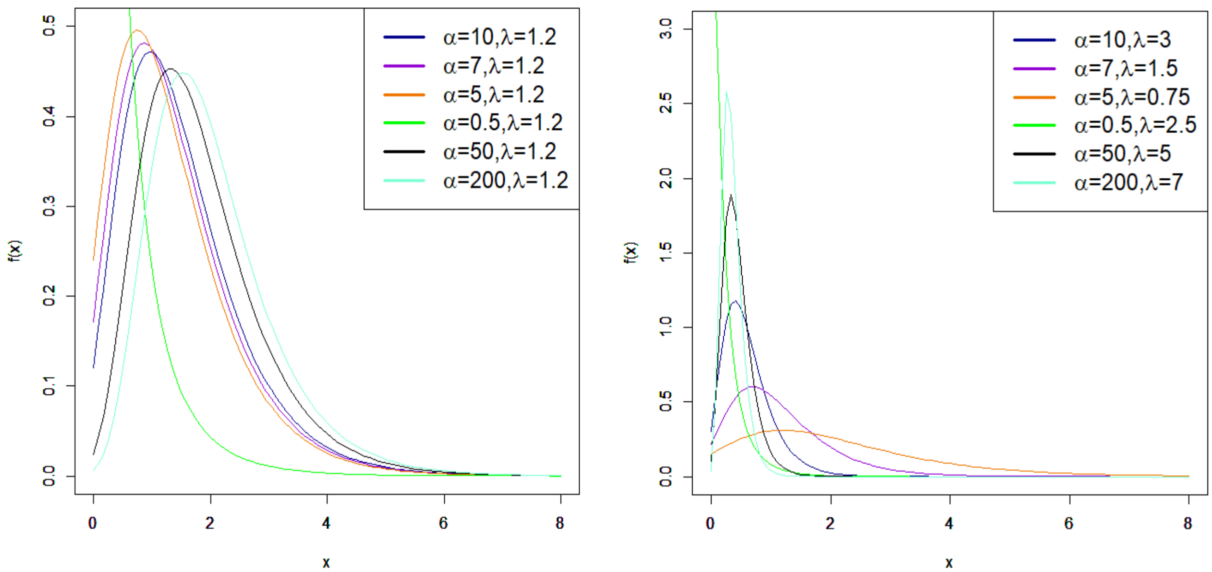

Figure 1: Plot of the density function of NAPE distribution for different values of the parameters

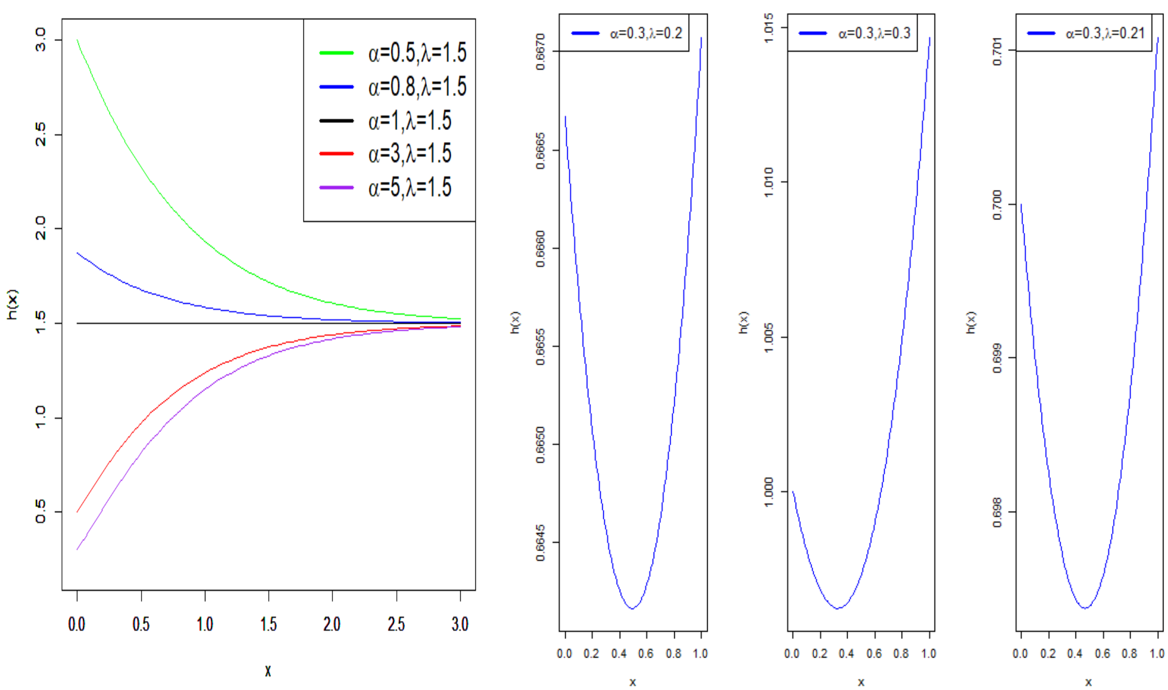

Figure 2: plot of hazard rate function of NAPE distribution for different values of the parameters

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From figure (1), it is observed that the density function of NAPE distribution is log-convex if $\alpha < 1$ and log-concave if $\alpha > 1$. Also, from figure (2), it can be clearly observed that the hazard rate function is increasing, decreasing, constant, upside-down bathtub and bathtub shapes for different values of the parameters.

2.2. Useful expansion of NAPE distribution

In this section, the useful expansion of the mixture representation of the PDF and CDF is presented. Using the series representation,

$$\alpha^{u} = \sum_{k=0}^{\infty} \frac{(\log \alpha)^{k}}{k!} u^{k} \tag{9}$$

The PDF of the NAPE distribution given in equation (4) can be written as,

$$f(x;\alpha,\lambda) = \sum_{k=0}^{\infty} \frac{(\log \alpha)^{k}}{\alpha k!} \lambda e^{-\lambda x} \left(1-e^{-\lambda x}\right)^{k} + \sum_{k=0}^{\infty} \frac{(\log \alpha)^{k+1}}{\alpha k!} \lambda e^{-\lambda x} \left(1-e^{-\lambda x}\right)^{k+1} \tag{10}$$

Furthermore, another form of the PDF given in (10) which provides the following infinite combination

$$f(x;\alpha,\lambda) = \sum_{k=0}^{\infty} \frac{(\log \alpha)^{k}}{\alpha k!} \frac{a}{(k+1)} \lambda e^{-\lambda x} (1-e^{-\lambda x})^{a-1} + \sum_{k=0}^{\infty} \frac{(\log \alpha)^{k+1}}{\alpha k!} \frac{b}{(k+2)} \lambda e^{-\lambda x} (1-\mathrm{e}^{-\lambda x})^{b-1} \tag{11}$$

where, $a = k+1$ and $b = k+2$

Equation (11) can also be written as,

$$f(x;\alpha,\lambda) = \sum_{k=0}^{\infty} \frac{(\log \alpha)^{k}}{\alpha k!} \frac{1}{(k+1)} h_{k+1}(x;\alpha,\lambda) + \sum_{k=0}^{\infty} \frac{(\log \alpha)^{k+1}}{\alpha k!} \frac{1}{(k+2)} h_{k+2}(x;\alpha,\lambda) \tag{12}$$

where,

$h_{(k+1)}(x) = a\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{a-1}$ and $h_{(k+2)}(x) = b\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{b-1}$, are the exponentiated generated (Exp-G) densities with power parameters a and b.

Also, the CDF of the NAPE distribution given in (3) can be written as,

$$F(x;\alpha,\lambda) = \sum_{k=0}^{\infty} \frac{(\log \alpha)^{k}}{\alpha k!} \left(1-e^{-\lambda x}\right)^{k+1} \tag{13}$$

Let u be an integer, then the expression of $f(x;\alpha,\lambda)F(x;\alpha,\lambda)u$ is derived as,

$$f(x;\alpha,\lambda)F(x;\alpha,\lambda)^{u} = \sum_{k=0}^{\infty} \frac{(\log \alpha)^{k} (u+1)^{k}}{a^{u+1} k!} \lambda e^{-\lambda x} \left(1-e^{-\lambda x}\right)^{u+k} + \sum_{k=0}^{\infty} \frac{(\log \alpha)^{k+2} (u+1)^{k}}{\alpha^{u+1} k!} \lambda e^{-\lambda x} \left(1-e^{-\lambda x}\right)^{u+k+1} \tag{14}$$

The different functions derived from (10) to (14) can be used for deriving various statistical properties of the NAPE distribution.

3. Statistical Properties

In this section, some basic statistical properties of the NAPE distribution have been derived and discussed.

3.1. Moment and Moment Generating Function

In statistical probability theory, the moment generating function is used to determine the moments of a distribution, i.e., first moment (mean), second moment (variance), third moment (skewness) and fourth moment (kurtosis).

Theorem 1: Let $X \sim NAPE(\alpha, \lambda)$, then the n^{th} moment of X is

$$\mu_n' = \sum_{l=0}^{k}\sum_{k=0}^{\infty}\frac{(\log\alpha)^k}{\alpha k!}(-1)^l\binom{k}{l}\frac{\Gamma(n+1)}{\lambda^n(k+1)^{n+1}} + \sum_{m=0}^{k+1}\sum_{k=0}^{\infty}\frac{(\log\alpha)^{k+1}}{\alpha k!}(-1)^m\binom{k+1}{m}\frac{\Gamma(n+1)}{\lambda^n(k+2)^{n+1}}$$

and the moment generating function of NAPE distribution is

$$M_X(t) = \sum_{j=0}^{\infty}\frac{t^j}{j!}\left(\sum_{l=0}^{k}\sum_{k=0}^{\infty}\frac{(\log\alpha)^k}{\alpha k!}(-1)^l\binom{k}{l}\frac{\Gamma(j+1)}{\lambda^j(k+1)^{j+1}} + \sum_{m=0}^{k+1}\sum_{k=0}^{\infty}\frac{(\log\alpha)^{k+1}}{\alpha k!}(-1)^m\binom{k+1}{m}\frac{\Gamma(j+1)}{\lambda^j(k+2)^{j+1}}\right)$$

Proof: The n^{th} moment of a random variable X is defined as,

$$E(x^n) = \mu_n' = \int_0^{\infty} x^n f(x;\alpha,\lambda)dx \tag{15}$$

Substituting (4) in (15) we get,

$$\mu_n' = \int_0^{\infty} x^n \frac{\lambda e^{-\lambda x}\alpha^{\left(1-e^{-\lambda x}\right)}}{\alpha}\left[1+\log(\alpha)\left(1-e^{-\lambda x}\right)\right]dx$$

$$\mu_n' = \int_0^{\infty}\frac{x^n\alpha^{1-e^{-\lambda x}}\lambda e^{-\lambda x}}{\alpha}dx + \int_0^{\infty}\frac{x^n\alpha^{1-e^{-\lambda x}}\lambda e^{-\lambda x}\log(\alpha)\left(1-e^{-\lambda x}\right)}{\alpha}dx \tag{16}$$

using the series representation given in equation (9) and binomial expansion in equation (16), the n^{th} moment of NAPE distribution is obtained as,

$$\mu_n' = \sum_{l=0}^{k}\sum_{k=0}^{\infty}\frac{(\log\alpha)^k}{\alpha k!}(-1)^l\binom{k}{l}\frac{\Gamma(n+1)}{\lambda^n(k+1)^{n+1}} + \sum_{m=0}^{k+1}\sum_{k=0}^{\infty}\frac{(\log\alpha)^{k+1}}{\alpha k!}(-1)^m\binom{k+1}{m}\frac{\Gamma(n+1)}{\lambda^n(k+2)^{n+1}} \tag{17}$$

Also, the moment generating function of NAPE distribution can be derived by using

$$M_x(t)=\int_0^\infty e^{tx} f(x;\alpha,\lambda)dx$$

Thus the moment generating function of NAPE distribution is obtained as,

$$M_X(t)=\sum_{j=0}^{\infty}\frac{t^j}{j!}\left(\sum_{l=0}^{k}\sum_{k=0}^{\infty}\frac{(\log\alpha)^k}{\alpha k!}(-1)^l\binom{k}{l}\frac{\Gamma(j+1)}{\lambda^j(k+1)^{j+1}}+\sum_{m=0}^{k+1}\sum_{k=0}^{\infty}\frac{(\log\alpha)^{k+1}}{\alpha k!}(-1)^m\binom{k+1}{m}\frac{\Gamma(j+1)}{\lambda^j(k+2)^{j+1}}\right) \quad (18)$$

Hence proved.

3.2. Incomplete Moment

The incomplete moment is used for measuring inequality, for instance, the Lorentz curve and Gini measures of inequality all rely upon the incomplete moments (Butler & McDonald, (1989)).

Theorem 2: The r^{th} incomplete moment for the density function $f(x;\alpha,\lambda)$ is

$$\gamma_r(x)=\frac{1}{\alpha}\left[\sum_{j=0}^{k}\sum_{k=0}^{\infty}\frac{(\log\alpha)^k}{k!}(-1)^j\binom{k}{j}\frac{1}{\lambda^r(\mathrm{j}+1)^r}\gamma\left((j+1)\lambda t,(r+1)\right)+\sum_{l=0}^{k+1}\sum_{k=0}^{\infty}\frac{(\log\alpha)^{k+1}}{k!}(-1)^l\binom{k+1}{l}\frac{1}{\lambda^r(l+1)^{r+1}}\gamma\left((l+1)\lambda t,(r+1)\right)\right]$$

Proof: The r^{th} incomplete moment of a random variable X is defined as,

$$\gamma_r(x)=\int_o^t x^r f(x;\alpha,\lambda)dx \quad (19)$$

Substituting (4) in (19) we get,

$$\gamma_r(x)=\int_0^\infty x^r\,\frac{\lambda e^{-\lambda x}\alpha^{\left(1-e^{-\lambda x}\right)}}{\alpha}[1+\log(\alpha)(1-\mathrm{e}^{-\lambda x})]\mathrm{d}x$$

$$\gamma_r(x)=\int_o^t x^r\,\frac{\lambda\alpha^{1-e^{-\lambda x}}e^{-\lambda x}}{\alpha}dx+\int_0^t x^r\,\frac{\alpha^{1-e^{-\lambda x}}e^{-\lambda x}}{\alpha}\log(\alpha)(1-\mathrm{e}^{-\lambda x})dx \quad (20)$$

using the series representation given in equation (9) and binomial expansion in equation (20), the r^{th} incomplete moment of NAPE distribution is obtained as,

$$\gamma_r(x)=\frac{1}{\alpha}\left[\sum_{j=0}^{k}\sum_{k=0}^{\infty}\frac{(\log\alpha)^k}{k!}(-1)^j\binom{k}{j}\frac{1}{\lambda^r(\mathrm{j}+1)^r}\gamma\left((j+1)\lambda t,(r+1)\right)+\sum_{l=0}^{k+1}\sum_{k=0}^{\infty}\frac{(\log\alpha)^{k+1}}{k!}(-1)^l\binom{k+1}{l}\frac{1}{\lambda^r(l+1)^{r+1}}\gamma\left((l+1)\lambda t,(r+1)\right)\right] \quad (21)$$

where, $\gamma(a,b)$ is the lower incomplete Gamma function.

Hence proved.

3.3. The Conditional Moment

Theorem 3: Let $X\sim NAPE(\alpha,\lambda)$, then the conditional moments for the random variable X is

$$E(X^n \mid X>t)=\frac{1}{\left(\alpha-\left(1-e^{-\lambda x}\right)\alpha^{1-e^{-\lambda x}}\right)\lambda^n}\left[\begin{array}{l}\sum_{l=0}^{k}\sum_{k=0}^{\infty}\frac{(\log\alpha)^k}{k!}(-1)^l\binom{k}{l}\left(\frac{1-\gamma\left((n+1),\lambda t(l+1)\right)}{(l+1)^{n+1}}\right)+\\ \sum_{m=0}^{k+1}\sum_{k=0}^{\infty}\frac{(\log\alpha)^{k+1}}{k!}(-1)^m\binom{k+1}{m}\left(\frac{1-\gamma\left((n+1),(\lambda t(m+1))\right)}{(m+1)^{n+1}}\right)\end{array}\right]$$

Proof: The conditional moment of a random variable X is defined as,

$$E(X^n \mid x>t)=\frac{1}{S(x;\alpha,\lambda)}\int_t^\infty x^n f(x;\alpha,\lambda)dx \tag{22}$$

Substituting (4) and (5) in (22) we obtained,

$$E(X^n \mid x>t)=\frac{\alpha}{\alpha-\left(1-e^{-\lambda x}\right)\alpha^{\left(1-e^{-\lambda x}\right)}}\int_t^\infty x^n\frac{\lambda e^{-\lambda x}\alpha^{\left(1-e^{-\lambda x}\right)}}{\alpha}\left[1+\log(\alpha)\left(1-e^{-\lambda x}\right)\right]dx$$

$$E(X^n \mid x>t)=\frac{\alpha}{\alpha-\left(1-e^{-\lambda x}\right)\alpha^{\left(1-e^{-\lambda x}\right)}}\left[\int_t^\infty\frac{x^n\lambda\alpha^{1-e^{-\lambda x}}e^{-\lambda x}}{\alpha}dx+\int_t^\infty\frac{x^n\lambda\alpha^{1-e^{-\lambda x}}e^{-\lambda x}}{\alpha}\log(\alpha)\left(1-e^{-\lambda x}\right)dx\right] \tag{23}$$

Using the series representation given in equation (9) and binomial expansion in equation (23), the conditional moment of NAPE distribution is obtained as,

$$E(X^n \mid X>t)=\frac{1}{\left(\alpha-\left(1-e^{-\lambda x}\right)\alpha^{1-e^{-\lambda x}}\right)\lambda^n}\left[\begin{array}{l}\sum_{l=0}^{k}\sum_{k=0}^{\infty}\frac{(\log\alpha)^k}{k!}(-1)^l\binom{k}{l}\left(\frac{1-\gamma\left((n+1),\lambda t(l+1)\right)}{(l+1)^{n+1}}\right)+\\ \sum_{m=0}^{k+1}\sum_{k=0}^{\infty}\frac{(\log\alpha)^{k+1}}{k!}(-1)^m\binom{k+1}{m}\left(\frac{1-\gamma\left((n+1),(\lambda t(m+1))\right)}{(m+1)^{n+1}}\right)\end{array}\right] \tag{24}$$

where $\gamma(a,b)$ is the lower incomplete Gamma function.
Hence proved.

3.4. Probability Weighted Moments(PWMs)

The PWMs are used for estimating the parameters of a probability distribution.

Theorem 4: Let $X \sim NAPE(\alpha,\lambda)$, then the probability weighted moments for the random variable X is

$$\pi_{s,q}=\frac{\lambda}{\alpha^{q+1}}\left[\sum_{m=0}^{k+q}\sum_{k=0}^{\infty}\frac{(\log\alpha)^k(q+1)^k}{k!}\binom{k+q}{m}(-1)^m\frac{\Gamma[s+1]}{\lambda^{s+1}(m+1)^{s+1}}+\sum_{l=0}^{k+q+1}\sum_{k=0}^{\infty}\frac{(\log\alpha)^{k+1}(q+1)^k}{k!}(-1)^l\binom{k+q+1}{l}\frac{\Gamma(s+1)}{\lambda^{s+1}(l+1)^{s+1}}\right]$$

Proof: For a random variable X, the PWMs represented by $\pi_{s,q}$ is given as,

$$\pi_{s,q} = E\left[x^s F(x;\alpha,\lambda)^q\right] = \int_0^\infty x^s f(x;\alpha,\lambda) F(x;\alpha,\lambda)^q dx \tag{25}$$

Substituting equation (3) and (4) in equation (25) we get,

$$\pi_{s,q} = \int_0^\infty x^s \frac{\lambda e^{-\lambda x} \alpha^{\left(1-e^{-\lambda x}\right)}}{\alpha} \left[1+\log(\alpha)(1-e^{-\lambda x})\right] \left\{\frac{\left(1-e^{-\lambda x}\right)\alpha^{\left(1-e^{-\lambda x}\right)}}{\alpha}\right\}^q dx$$

$$\pi_{s,q} = \frac{\lambda}{\alpha^{q+1}} \left[\int_0^\infty x^s \alpha^{\left(1-e^{-\lambda x}\right)} e^{-\lambda x} \alpha^{q\left(1-e^{-\lambda x}\right)} \left(1-e^{-\lambda x}\right)^q + x^s \alpha^{\left(1-e^{-\lambda x}\right)} e^{-\lambda x} \log(\alpha) \left(1-e^{-\lambda x}\right) \alpha^{q\left(1-e^{-\lambda x}\right)} \left(1-e^{-\lambda x}\right)^q\right]$$

Using the series representation given in equation (9) and binomial expansion in the above equation, the PWMs of NAPE distribution is obtained as,

$$\pi_{s,q} = \frac{\lambda}{\alpha^{q+1}} \left[\sum_{m=0}^{k+q} \sum_{k=0}^{\infty} \frac{(\log\alpha)^k (q+1)^k}{k!} \binom{k+q}{m} (-1)^m \frac{\Gamma[s+1]}{\lambda^{s+1} (m+1)^{s+1}} + \sum_{l=0}^{k+q+1} \sum_{k=0}^{\infty} \frac{(\log\alpha)^{k+1} (q+1)^k}{k!} (-1)^l \binom{k+q+1}{l} \frac{\Gamma(s+1)}{\lambda^{s+1} (l+1)^{s+1}}\right] \tag{26}$$

Hence proved.

3.5. Residual and Reversed Residual Life

The residual lifetime of the random variable X denoted by $R_{(t)}$ is given by,

$$R_t(x) = \frac{S(x+t)}{S(t)}$$

Substituting (5) in the above equation, we derived the residual lifetime as,

$$R_t(x) = \frac{\alpha - \left(1-e^{-\lambda(x+t)}\right)\alpha^{\left(1-e^{-\lambda(x+t)}\right)}}{\alpha - \left(1-e^{-\lambda t}\right)\alpha^{\left(1-e^{-\lambda t}\right)}} \tag{27}$$

Also, the reverse residual lifetime of X denoted by $\overline{R}(x)$ is given by,

$$\overline{R}_t(x) = \frac{S(x-t)}{S(t)}$$

Using (5) in the above equation, we derived the reversed residual lifetime as,

$$\overline{R}_t(x) = \frac{\alpha - \left(1-e^{-\lambda(x-t)}\right)\alpha^{\left(1-e^{-\lambda(x-t)}\right)}}{\alpha - \left(1-e^{-\lambda t}\right)\alpha^{\left(1-e^{-\lambda t}\right)}} \tag{28}$$

3.6. Reńyi and q-entropy

In statistical theory, entropy is defined as a statistical tool for measuring the variation of the uncertainty of a random variable X.

Let the random variable $X \sim NAPE(\alpha,\lambda)$, then the Reńyi entropy of X is define as,

$$I_R(\delta)=\frac{1}{(1-\delta)}\log\left[\int_0^\infty\left[f(x)\right]^\delta\right];\delta>0,\delta\neq1 \tag{29}$$

Using the generalized binomial expansion series in the following form

$$\left[1+\log(\alpha)\left(1-e^{-\lambda x}\right)\right]^\delta=\sum_{i=0}^\infty(-1)^i\binom{\delta+i-1}{i}(\log\alpha)^i$$

we get,

$$I_R(\delta)=\frac{\lambda^\delta}{(1-\delta)}\log\sum_{i,j=0}^\infty\frac{(-1)^{i+1}\binom{\delta+i-1}{i}\binom{i+j}{l}(\log\alpha)^{i+j}\delta^j}{\alpha^\delta j!}\int_0^\infty e^{-\delta\lambda x}\left(1-e^{-\lambda x}\right)^{i+j}dx$$

Solving the above equation, the Reńyi entropy is derived as,

$$I_R(\delta)=\frac{\lambda^\delta}{(1-\delta)\lambda}\log\left[\sum_{i,j=0}^\infty\sum_{l=0}^{i+j}\frac{(-1)^{i+l}\binom{\delta+i-1}{i}\binom{i+j}{l}(\log\alpha)^{i+j}\delta^j}{\alpha^\delta j!(\delta+l)}\right] \tag{30}$$

Furthermore, the q-entropy say $H_q(f)$ is defined by,

$$H_q(f)=\frac{1}{(1-\delta)}\log\left[1-\int_0^\infty\left[f(x)\right]^\delta\right],q>0,q\neq1 \tag{31}$$

Using the generalized binomial expansion and solving the above expression, the q- entropy is derived as,

$$H_q(f)=\frac{\lambda^q}{(1-\delta)\lambda}\log\left[1-\sum_{i,j=0}^\infty\sum_{m=0}^{i+j}\frac{(-1)^{i+m}\binom{q+i-1}{i}\binom{i+j}{m}(\log\alpha)^{i+j}q^j}{\alpha^q j!(q+m)}\right] \tag{32}$$

3.7. Order Statistics

Let $X_{1:n},X_{2:n},....,X_{n:n}$ be the order statistics of the random variable $X_1,X_2,....,X_n$, then the density function of $X_{i:n}$ is given by

$$f_{i:n}(x)=\frac{f(x;\alpha,\lambda)}{B(i,i-n+1)}\sum_{\nu=0}^{n-i}\binom{n-i}{\nu}(-1)^\nu\left[F(x;\alpha,\lambda)\right]^u \tag{33}$$

where $B(.,.)$ denotes the Beta function.

The PDF of the i^{th} order statistics of NAPE distribution is derived by substituting (3) and (4) in (33) by replacing u with $n+\nu-1$, we have

$$f_{i:n}(x)=\frac{\sum_{k,v=0}^{n-i}\binom{n-i}{v}(-1)^{v}}{B(i,i-n+1)}\left[\begin{array}{l}\frac{\lambda}{\alpha^{n+v}}\frac{(\log\alpha)^{k}}{k!}(n+v)^{k}e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{n+v+k-1}+\\ \frac{\lambda}{\alpha^{n+v}}\frac{(\log\alpha)^{k+2}}{k!}(n+v)^{k}e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{n+v+k}\end{array}\right] \tag{34}$$

Furthermore, s^{th} ordinary moments of the i^{th} order statistics for NAPE distribution is derived as,

$$E\left(x_{i:n}^{s}\right)=\int_{-\infty}^{\infty}x_{i:n}^{s}f_{i:n}(x)dx$$

Using (34) in the above equation and solving the above equation we derive the s^{th} moment of the i^{th} order statistics as,

$$E\left(x_{i:n}^{s}\right)=\frac{\sum_{k,v=0}^{n-i}\binom{n-i}{v}(-1)^{v}}{B(i,i-n+1)}\left[\begin{array}{l}\frac{(\log(\alpha))^{k}(n+v)^{k}}{\alpha^{n+v}k!}\sum_{m=0}^{n+v+k-1}(-1)^{m}\binom{n+v+k-1}{m}\frac{\Gamma(s+1)}{\lambda^{s}(n+v+k)^{s+1}}+\\ \frac{(\log(\alpha))^{k+2}(n+v)^{k}}{\alpha^{n+v}k!}\sum_{l=0}^{n+v+k}(-1)^{l}\binom{n+v+k}{l}\frac{\Gamma(s+1)}{\lambda^{s}(n+v+k+1)^{s+1}}\end{array}\right] \tag{35}$$

3.8. Quantile function

The quantile function is used for simulation study and to measure the percentile. The quantile function is defined as the inverse of the cumulative distribution function F(x) for a random variable X.

Theorem 5: The p^{th} quantile function x_p of NAPE distribution, for $\alpha \neq 1$ is

$$x_p=\left[-\frac{1}{\lambda}\log\left(\frac{1}{\alpha p+1}\right)\right] \qquad \text{for} \quad 0\le p\le 1$$

Proof: Let us consider the identity,

$$Q(u)=\inf\{x\in\Re, u\le F(x)\}=F^{-1}(x), u\in[0,1]$$

Now, let $F(x)=u$ then by using equation (3), we obtained

$$\log\left(1-e^{-\lambda x}\right)+\left(1-e^{-\lambda x}\right)\log(\alpha)-\log(\alpha u)=0$$

Solving the above equation numerically, the quantile function of NAPE distribution is defined as,

$$X = \left[-\frac{1}{\lambda} \log\left(\frac{1}{\alpha u + 1} \right) \right]$$

Thus the p^{th} quantile function x_p of NAPE distribution is obtained as,

$$x_p = \left[-\frac{1}{\lambda} \log\left(\frac{1}{\alpha p + 1} \right) \right]$$

Hence proved.

Remark 1: The first three quantiles of NAPE distribution can be obtained by setting $p = \frac{1}{4} = 0.25$ (25th percentile), $p = \frac{1}{2} = 0.50$ (50th percentile or median) and $p = \frac{3}{4} = 0.75$ (75th percentile) in the given equation, i.e.

$$x_p = \left[-\frac{1}{\lambda} \log\left(\frac{1}{\alpha p + 1} \right) \right]$$

Therefore the median of NAPE distribution is as follows:

$$x_{\frac{1}{2}} = \left[\frac{1}{\lambda} \log\left(\frac{2}{\alpha + 2} \right) \right]$$

Also, the 25^{th} and 75^{th} can be obtained as,

$$x_{\frac{1}{4}} = \left[\frac{1}{\lambda} \log\left(\frac{4}{\alpha + 4} \right) \right]$$

and

$$x_{\frac{3}{4}} = \left[\frac{1}{\lambda} \log\left(\frac{4}{3\alpha + 4} \right) \right]$$

Table 1 displays the percentage point of NAPE distribution for some specific values of the scale parameter $\lambda > 0$ and shape parameter $\alpha > 0$. It contains the first quartile (25%), median (50%) and third quartile (75%).

Table 1: The following table displays the percentage point for different values of the parameters

λ	α	25%	50%	75%
	2	0.394679	0.681924	0.916422
1	5	0.792747	1.23669	1.558278
	7	0.990686	1.486563	1.832712
	2	0.197339	0.340962	0.458211
2	5	0.396373	0.618345	0.779139
	7	0.495343	0.743282	0.916356
	2	0.078936	0.136385	0.183284
5	5	0.158549	0.247338	0.311656
	7	0.198137	0.297313	0.366542
	2	0.056383	0.097418	0.130917
7	5	0.11325	0.17667	0.222611
	7	0.141527	0.212366	0.261816

From table 1, it is observed that as the value of α increases, for a fixed value of λ, the values of the percentage point increase. Also, as the value of λ increases, for the fixed value of α, the value of the percentage point decreases.

4. Parameter Estimation

The method of maximum likelihood estimation method has been used for estimating the parameters of NAPE distribution.

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from the NAPE distribution with PDF given in (4), then the log-likelihood function is

$$\log L = -n\log(\alpha) + \sum_{i=1}^{n}\log\left(\lambda e^{-\lambda x}\right) + \sum_{i=1}^{n}\left(1-e^{-\lambda x}\right)\log(\alpha) + \sum_{i=1}^{n}\log\left(1+\log(\alpha)\left(1-e^{-\lambda x}\right)\right) \tag{36}$$

For obtaining the partial derivatives, differentiating (36) for α and λ we get,

$$\frac{\partial}{\partial\alpha}(\text{logL}) = -\frac{n}{\alpha} + \frac{1}{\alpha}\sum_{i=1}^{n}\left(1-e^{-\lambda x}\right) + \frac{1}{\alpha}\sum_{i=1}^{n}\frac{\left(1-e^{-\lambda x}\right)}{\left(1+\log(\alpha)\left(1-e^{-\lambda x}\right)\right)} \tag{37}$$

$$\frac{\delta}{\delta\lambda}(\text{logL}) = \sum_{i=1}^{n}\left(\frac{1-\lambda x}{\lambda}\right) + \sum_{i=1}^{n}\log(\alpha)xe^{-\lambda x} + \sum_{i=1}^{n}\frac{\log(\alpha)xe^{-\lambda x}}{\left(1+\log(\alpha)\left(1-e^{-\lambda x}\right)\right)} \tag{38}$$

Setting (37) and (38) to zero and solving these equations simultaneously gives the MLE of α and λ i.e., $\hat{\alpha}$ and $\hat{\lambda}$. However, solving these equations to get the estimates of the unknown parameter is quite difficult. Therefore, a numerical technique such as the newton-raphson method may be used to solve these non-linear equations.

5. Simulation Study

In this section, a simulation study has been performed to illustrate the behaviour of the estimates $\hat{\alpha}$ and $\hat{\lambda}$ in terms of the sample size n. We generate 1000 random sample $x_1, x_2, \ldots, x_n$ of sizes n= (30, 50, 100, 150, 200) from NAPE distribution using theorem 5. Then, considering the initial values of the parameters $\alpha = (1.5, 3)$ and $\lambda = (3, 5)$, we generate the bias and MSE from NAPE distribution.

The bias and MSE are calculated by

$$Bias(\alpha) = \frac{1}{W}\sum_{i=1}^{w}\left(\hat{\alpha}_i - \alpha\right) \text{ and } Bias(\lambda) = \frac{1}{W}\sum_{i=1}^{w}\left(\hat{\lambda}_i - \lambda\right)$$

$$MSE(\alpha) = \frac{1}{W}\sum_{i=1}^{w}\left(\hat{\alpha}_i - \alpha\right)^2 \text{ and } MSE(\lambda) = \frac{1}{W}\sum_{i=1}^{w}\left(\hat{\lambda}_i - \lambda\right)^2$$

The average values of MSEs and Bias from NAPE distribution for different values of n are displayed in table 2. From table 2, it can be observed that as the value of the sample size n increases i.e., n = (30, 50, 100, 150, 200), the MSEs and Bias decreases indicating the reliability and accuracy of the estimates.

Table 2: The average values of Bias and MSEs of NAPE distribution for different values of n.

	Parameter		MSE		Bias	
n	α	λ	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$
30	1.5	3	0.03718	1.26063	0.0352	0.20499
50	1.5	3	0.00625	1.22657	0.01118	0.15663
100	1.5	3	0.00461	0.61209	0.00679	0.07824
150	1.5	3	0.00372	0.50503	0.00498	0.05803
200	1.5	3	0.00142	0.45721	0.00411	0.04781
500	1.5	3	0.00003	0.01329	0.00025	-0.0052
30	3	5	0.19454	1.76047	0.08053	0.24225
50	3	5	0.14793	0.68568	0.05439	0.11711
100	3	5	0.06947	0.28125	0.02636	0.05303
150	3	5	0.03175	0.24348	0.01455	0.04029
200	3	5	0.03163	0.23242	0.01263	0.03409
500	3	5	0.01653	0.01273	0.00575	-0.0051

6. Application

In this sub-section, we analyzed three real life data sets to illustrate the performance of the proposed NAPE distribution. The first data set is the uncensored data set corresponding to intervals in days between 109 successive coal-mining disasters in Great Britain, for the period 1875-1951, published by Maguire *et al*. (1952). The second data set is the survival times of 72 guinea pigs which are observed and reported by Elbatal *et al.* (2013). The third data sets represent the COVID-19 data set from Italy, recorded between 13 June and 12 August 2021, studied by Almetwally et al. [2022]. The observed data sets are presented below:

First Data set:

1 ,4 ,4 ,7, 11, 13, 15, 15, 17, 18, 19, 19, 20, 20, 22, 23, 28, 29, 31, 32, 36, 37, 47, 48, 49, 50, 54, 54, 55, 59, 59, 61, 61, 66, 72, 72, 75, 78, 78, 81, 93, 96, 99, 108, 113, 114 ,120, 120, 120, 123, 124, 129, 131, 137, 145, 151, 156, 171, 176, 182, 188, 189, 195, 203, 208, 215, 217, 217, 217, 224, 228, 233, 255, 271, 275, 275, 275, 286, 291, 312, 312, 312, 315, 326, 326, 329, 330, 336, 338 ,345, 348, 354, 361, 364, 369, 378, 390, 457, 467, 498, 517,566, 644, 745, 871, 1312, 1357, 1613, 1630

Second Data set:

0.1, 0.33 ,0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92,0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08,1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21,1.22, 1.22, 1.24, 1.3 ,1.34, 1.36, 1.39, 1.44, 1.46,1.53, 1.59, 1.6 ,1.63, 1.63, 1.68 ,1.71, 1.72, 1.76,1.83, 1.95, 1.96 ,1.97, 2.02, 2.13, 2.15, 2.16, 2.22,2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78,2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55

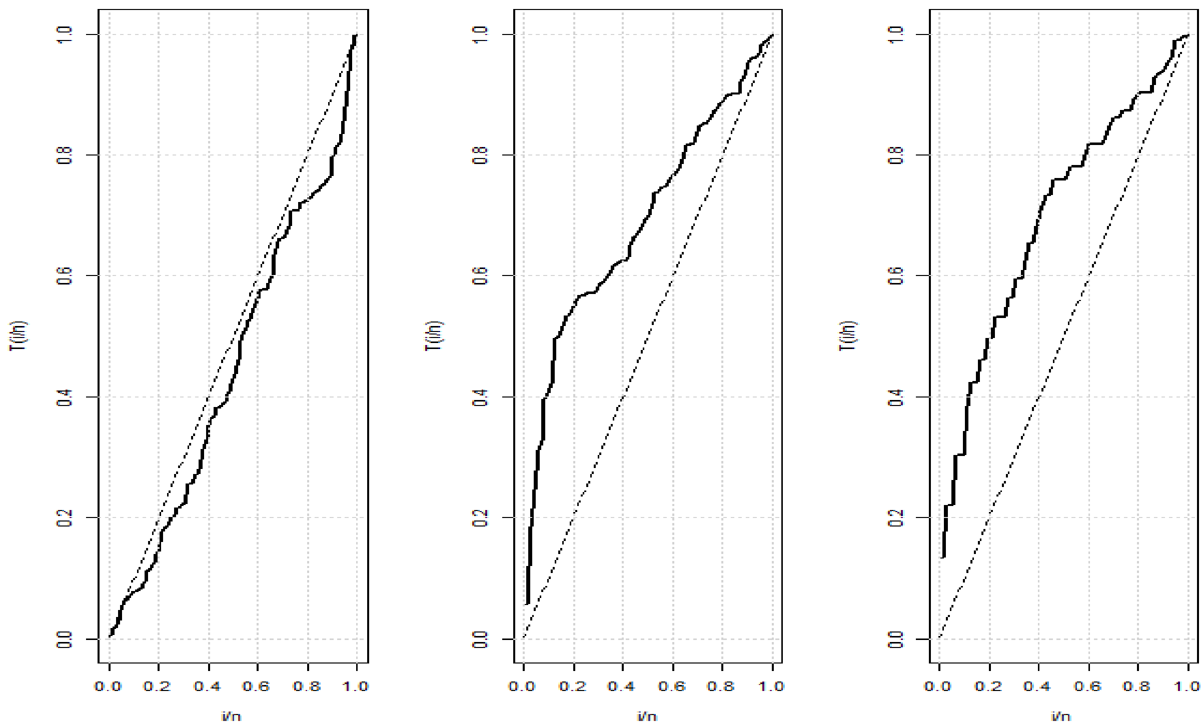
Third Data set:

52, 26, 36, 63, 52, 37, 35, 28, 17, 21, 31, 30, 10, 56, 40, 14, 28, 42, 24, 21, 28, 22, 12, 31, 24, 14, 13, 25, 12, 7, 13, 20, 23, 9, 11, 13, 3, 7, 10, 21, 15, 17, 5, 7, 22, 24, 15, 19, 18, 16, 5, 20, 27, 21, 27, 24, 22, 11, 22, 31, 31

Table 3: Descriptive statistic of the data sets

First data set						
minimum	maximum	first quartile	median	third quartile	skewness	kurtosis
1	1630	54	145	312	2.998619	13.5256
Second data set						
minimum	maximum	first quartile	median	third quartile	skewness	kurtosis
0.1	5.55	1.080	1.495	2.240	1.37059	5.22477
Third data set						
minimum	maximum	first quartile	median	third quartile	skewness	kurtosis
3	63	13	21	28	1.096814	4.472707

The descriptive statistic of the observed real life data sets is presented in table (3). From table (3) it is observed that the first data set are skewed and leptokurtic, the second data set are right-skewed and leptokurtic and the third data set are highly skewed and leptokurtic (kurtosis>3). Hence, the NAPE distribution is a reasonable choice for fitting these data sets.



3(a) First data set 3(b) Second data set 3(c) third data set

Figure 3: TTT plots of the first, second and third data sets

The TTT plot of the first, second and third data sets are shown in figure 3. The TTT plot shown in figure 3(a) represents a decreasing hazard rate function, 3(b) represents an increasing hazard rate function and 3(c) represents an increasing hazard rate function.

We fit the proposed NAPE distribution to the above three data sets along with other competing distribution namely; alpha power Exponential (APE) (by Mahdavi & Kundu, 2017), alpha power inverted Exponential (APIE) (by Unal et al., 2018), Exponential (E) (by Gupta et al., 2010), Exponentiated Exponential (EE) (by Gupta & Kundu, 2001) and Generalized Inverted Exponential (GIE) (by Abouammoh & Alshangiti, 2009) distribution respectively. We have computed the maximum likelihood estimates (MLEs) along with its standard error (SE) and the associated log likelihood(-LogL) in all the cases.

We consider the analytical measures to verify which distribution has the best fits for the observed data sets. The adequacy and efficiency of the distribution are checked by considering the goodness of fit measures such as Akaike Information Criterion (AIC), corrected Akaike information criterion (CAIC), Bayesian Information Criterion (BIC), Hannan-Quinn information criterion (HQIC), Kolmogorov-Smirnov (KS) test statistic and its *p*-value, Cramer-von-Misses (CM) test statistic and Anderson-Darling (AD) test statistic. From tables 4 and 5, it is clear that based on the *p*-value, -LogL and analytical measures of the NAPE distribution along with the other competing distribution, the proposed NAPE distribution provides the overall best fit as compared to the other well-known probability distribution. Hence, we can presume that NAPE distribution may be chosen as a suitable distribution as compared to the other competing distributions for explaining the first data set. Figure 4 and figure 5, represents the estimated densities and CDFs of the fitted distribution and estimated density and CDF of NAPE distribution for the first data set. From figure 4 and figure 5, it is evident that the NAPE distribution provides to be a better fit for the first data set. Also, the pp-plot of the fitted distributions is provided in figure 6.

Table 4, table 6 and table 8 provides the maximum likelihood estimates (MLEs) along with the standard error (SE), minus log-likelihood and *p*-values for the first, second and third data sets respectively. The analytical measures for the first, second and third data set are provided in table 5, table 7 and table 9 respectively.

Table 4: The MLEs (SE) of the parameter fitted to the first data set

Models	MLEs (SE)	-LogL	*p*-value
NAPE	$\hat{\alpha}$=0.487 (0.072) $\hat{\lambda}$= 0.003(0.0003)	700.763	0.832
APE	$\hat{\alpha}$=0.277 (0.187) $\hat{\lambda}$= 0.003 (0.0005)	701.204	0.575
APIE	$\hat{\alpha}$= 147.517 (94.348) $\hat{\lambda}$= 13.404 (2.138)	721.551	0.005
E	$\hat{\lambda}$= 0.004 (0.0004)	703.316	0.547
EE	$\hat{\alpha}$= 0.874 (0.106) $\hat{\lambda}$= 0.004 (0.0005)	702.562	0.476
GIE	$\hat{\alpha}$=0.513 (0.061) $\hat{\lambda}$= 20.387 (3.307)	743.644	0.0001

Table 5: Analytical measures of the NAPE distribution and other competing distributions for the first data set

Models	W	A	KS	AIC	CAIC	BIC	HQIC
NAPE	0.066	0.479	0.059	1405.525	1405.639	1410.908	1407.708
APE	0.067	0.516	0.075	1406.408	1406.522	1411.791	1408.591
APIE	0.562	3.191	0.165	1447.101	1447.214	1452.484	1449.284
E	0.067	0.608	0.076	1408.632	1408.669	1411.323	1409.723
EE	0.067	0.621	0.081	1409.125	1409.238	1414.508	1411.308
GIE	1.059	6.120	0.213	1491.287	1491.400	1496.670	1493.470

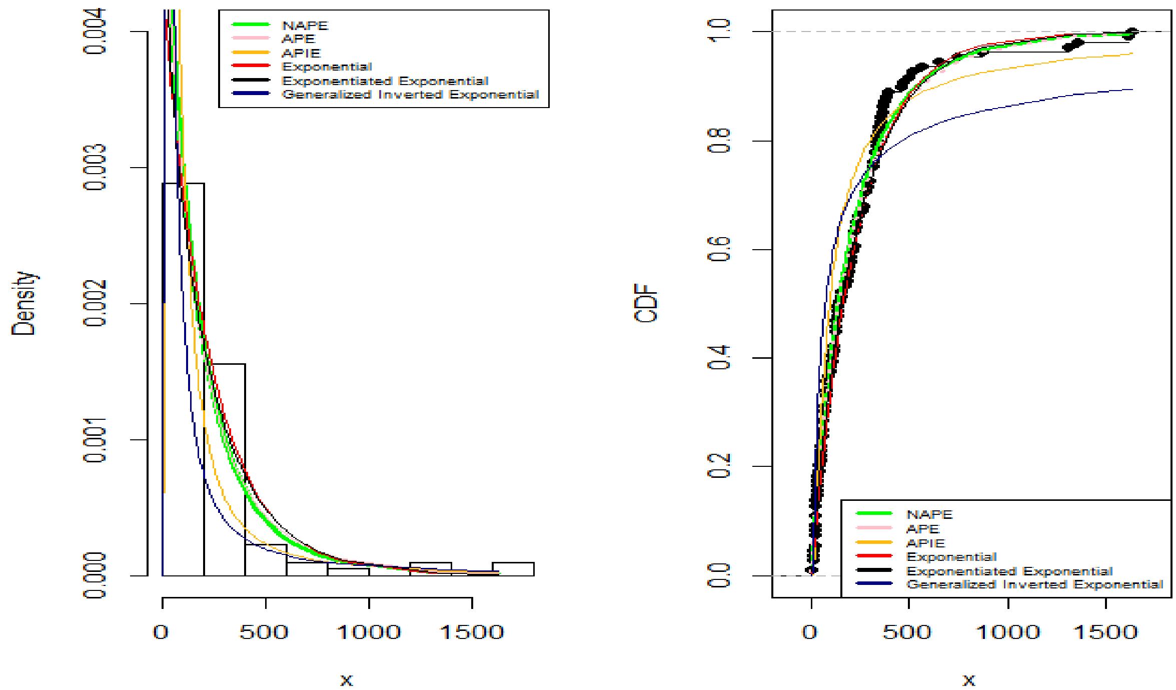


Figure 4: Plot of the estimated densities and CDFs of the fitted distributions for the first data set

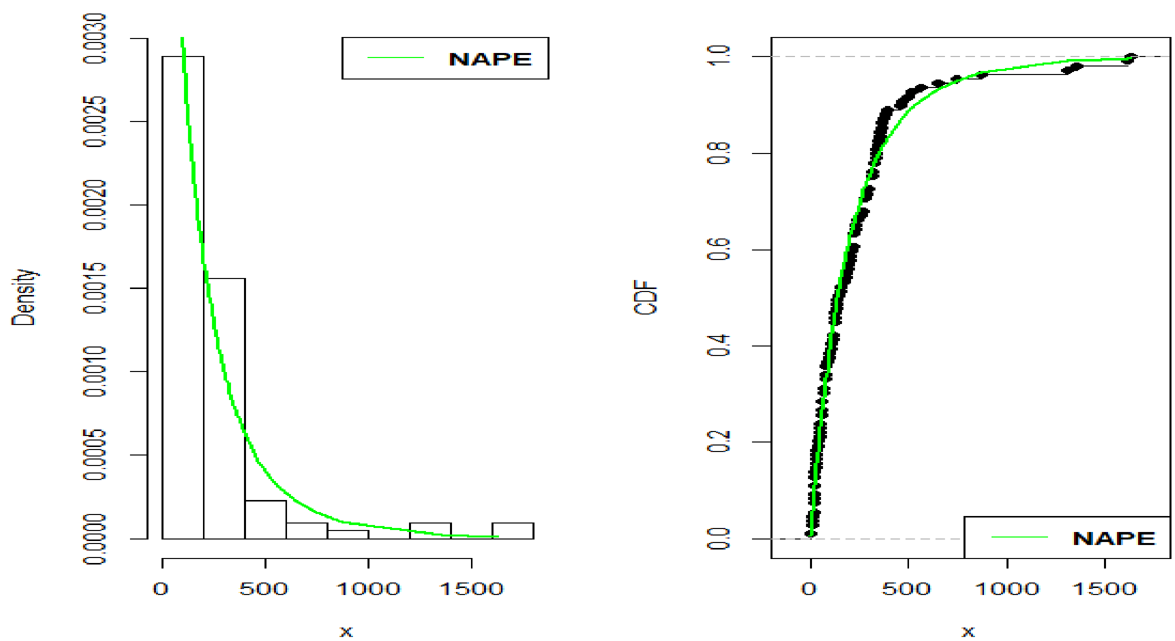


Figure 5: Plot of the estimated density and CDF of NAPE distribution for the first data set

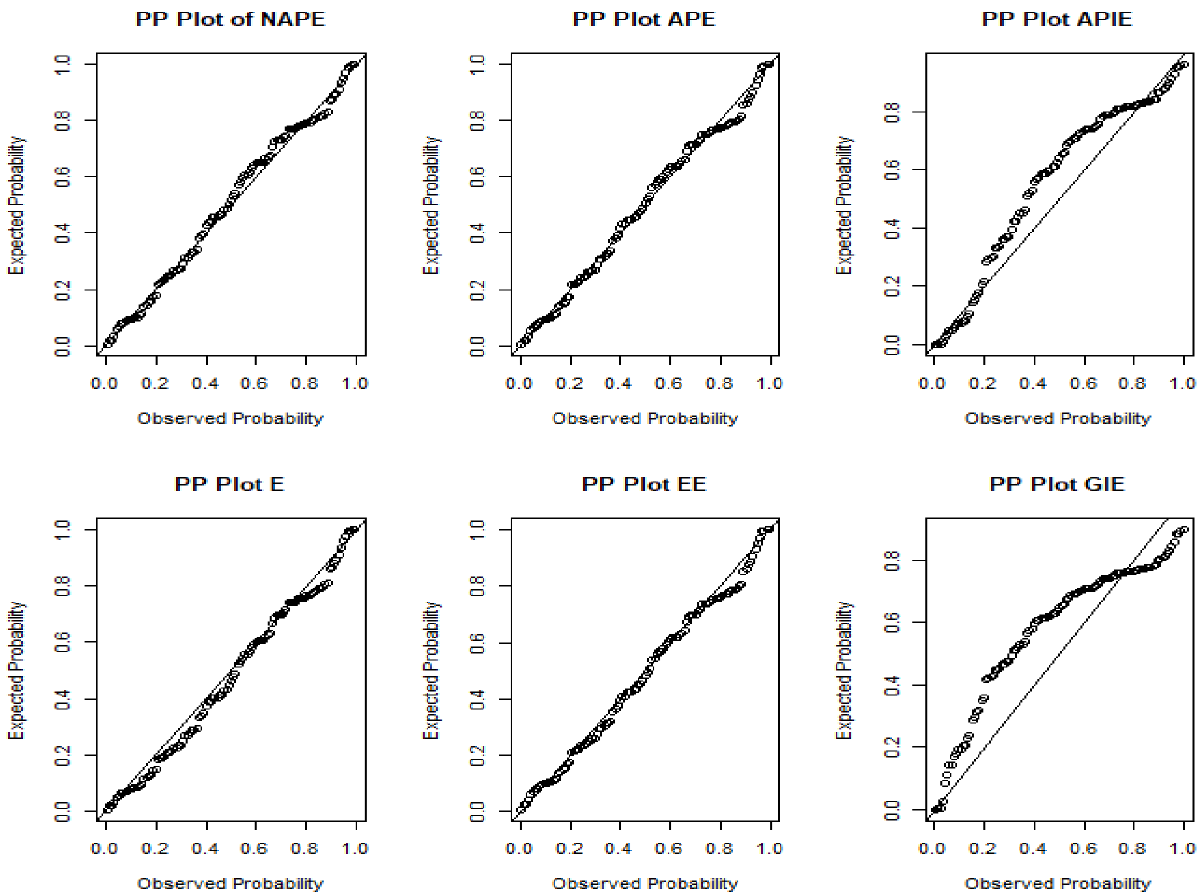


Figure 6: PP-plots of the fitted distribution to first data set

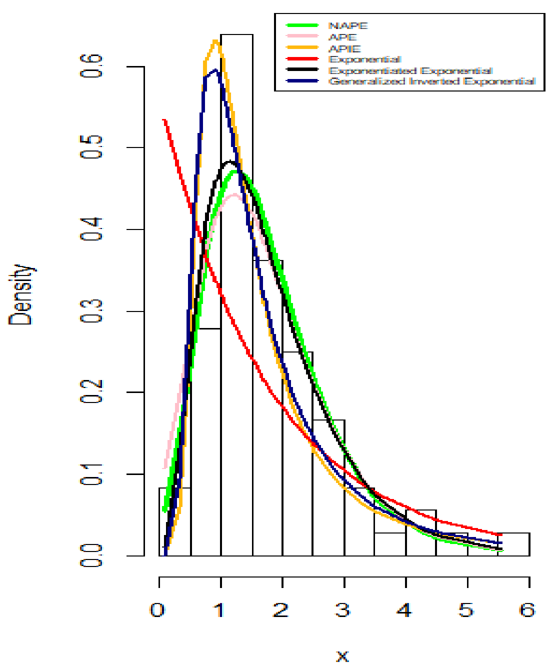
Table 6: The MLEs (SE) of the parameter fitted to the second data set

Models	MLEs (SE)	-LogL	*p*-value
NAPE	$\hat{\alpha}$=45.905 (34.515) $\hat{\lambda}$= 1.246 (0.116)	93.471	0.589
APE	$\hat{\alpha}$=66.769 (38.924) $\hat{\lambda}$= 1.184 (0.105)	94.472	0.246
APIE	$\hat{\alpha}$=0.018 (0.018) $\hat{\lambda}$= 2.420 (0.304)	112.41	0.067
E	$\hat{\lambda}$=0.566 (0.067)	113.037	7.503e-06
EE	$\hat{\alpha}$= 3.629 (0.721) $\hat{\lambda}$= 1.127 (0.132)	94.236	0.561
GIE	$\hat{\alpha}$=2.889 (0.603) $\hat{\lambda}$= 2.105 (0.278)	108.013	0.049

Table 7: The analytical measures of the NAPED and other competing distribution for the second data set

Models	W	A	KS	AIC	CAIC	BIC	HQIC
NAPE	0.094	0.543	0.091	190.941	191.115	195.495	192.754
APE	0.106	0.612	0.121	192.944	193.118	197.497	194.757
APIE	0.395	2.619	0.154	228.82	228.994	233.373	230.633
E	0.097	0.598	0.295	228.074	228.131	230.351	228.980
EE	0.097	0.552	0.093	192.472	192.646	197.025	194.285
GIE	0.303	2.045	0.160	220.026	220.199	224.579	221.838

From tables 6 and 7, it is clear that NAPE distribution provides the overall best fit as compared to the other well-known probability distribution. Hence, we can say that NAPE distribution is more adequate as compared to the other competing distributions like Alpha Power Exponential (APE), Alpha Power Inverted Exponential (APIE), New Alpha Power Transformed Exponential (NAPTE), Exponential (E), Exponentiated Exponential (EE) and Generalized Inverted exponential (GIE) distribution for explaining the second data set. Figure 7 and figure 8 shows the estimated densities and CDFs of the fitted distribution and estimated density and CDF of NAPE distribution for the second data set. Also, the pp-plot of the fitted distributions for the second data set is provided in figure 9.



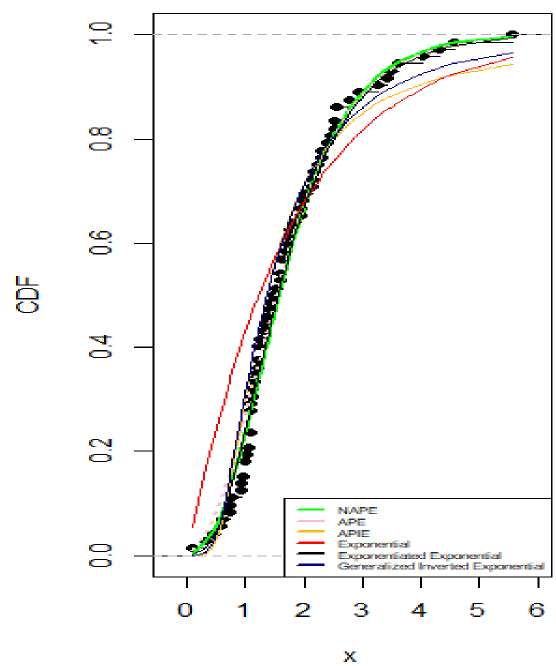


Figure 7: Plot of the estimated densities and CDFs of the fitted distributions for the second data set

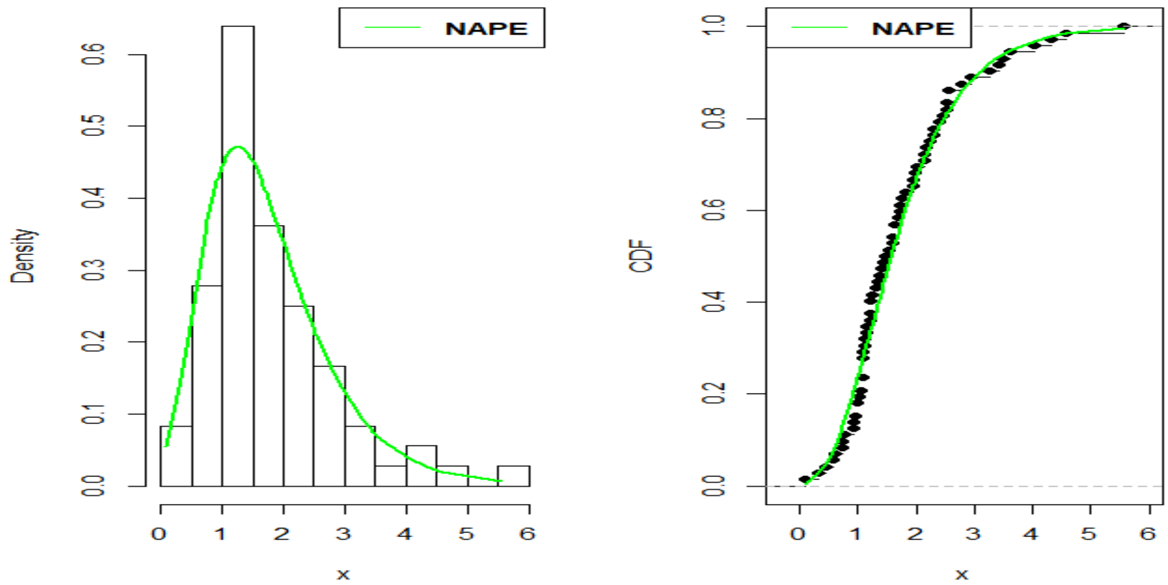


Figure 8: Plot of the estimated density and CDF of NAPE distribution for the second data set

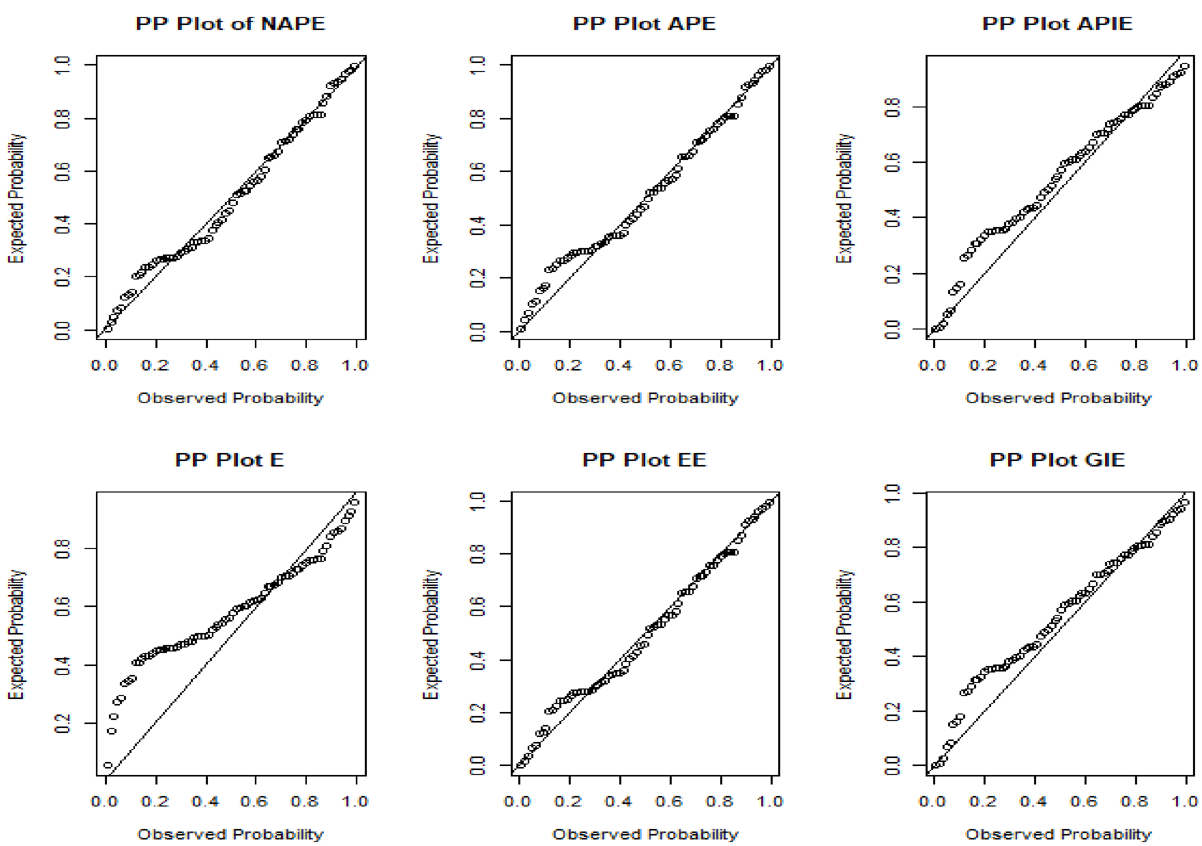


Figure 9: PP-plots of the fitted distribution to second data set

Table 8: The MLEs (SE) of the parameter fitted to the third data set

Models	MLEs (SE)	-LogL	p-value
NAPE	$\hat{\alpha}$= 41.683 (33.320) $\hat{\lambda}$= 0.096 (0.009)	234.736	0.779
APE	$\hat{\alpha}$= 61.050 (38.212) $\hat{\lambda}$= 0.091 (0.009)	235.579	0.625
APIE	$\hat{\alpha}$= 0.008 (0.009) $\hat{\lambda}$= 35.202 (4.318)	243.443	0.091
E	$\hat{\lambda}$= 0.044 (0.006)	251.257	0.002
EE	$\hat{\alpha}$= 3.721 (0.804) $\hat{\lambda}$= 0.089 (0.011)	235.758	0.671
GIE	$\hat{\alpha}$=3.463 (0.804) $\hat{\lambda}$= 31.033 (4.284)	240.341	0.121

Table 9: The analytical measures of the NAPED and other competing distribution for the third data set

Models	W	A	KS	AIC	CAIC	BIC	HQIC
NAPE	0.038	0.230	0.084	473.472	473.679	477.694	475.126
APE	0.039	0.231	0.096	475.159	475.366	479.381	476.813
APIE	0.289	1.686	0.159	490.885	491.092	495.107	492.539
E	0.047	0.277	0.242	504.514	504.582	506.625	505.341
EE	0.059	0.326	0.093	474.517	474.724	478.738	476.171
GIE	0.216	1.244	0.152	484.681	484.888	488.903	486.336

From table 8 and table 9, it is evident that NAPE distribution provides the overall best fit as compared to the other well-known probability distribution. Hence, we can say that NAPE distribution is more suitable for explaining the third data set as compared to the other competing distributions. Figure 10 and 11 shows the estimated densities and CDFs of the fitted distribution and estimated density and CDF of NAPE distribution for the third data set. Also, the pp-plot of the fitted distributions for the third data set is provided in figure 12.

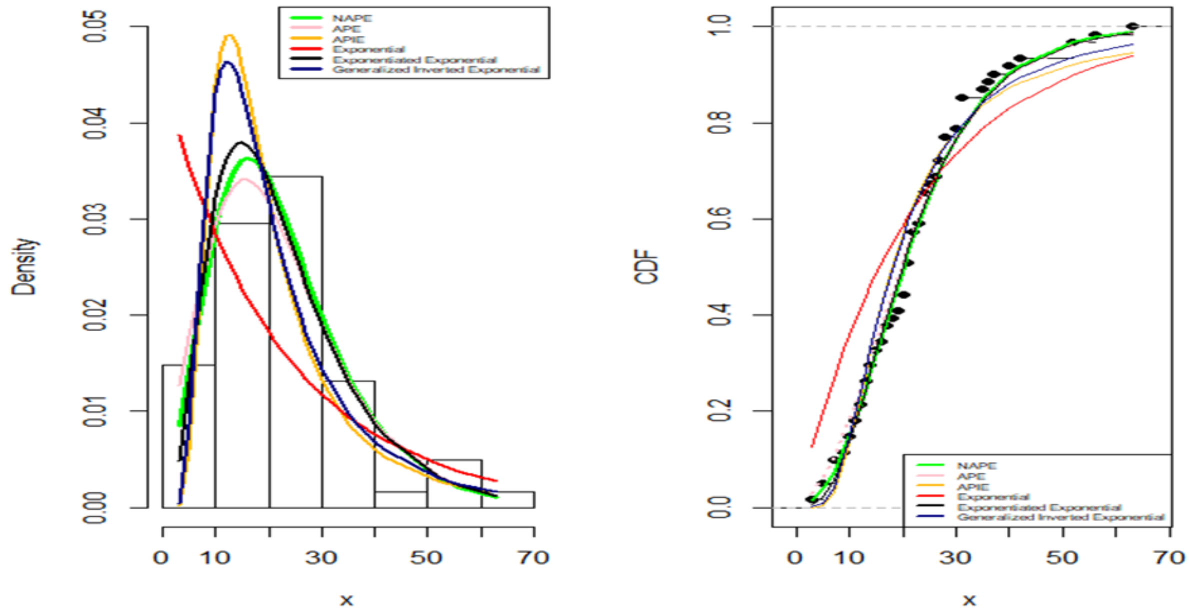


Figure 10: Plot of the estimated densities and CDFs of the fitted distributions for the third data set

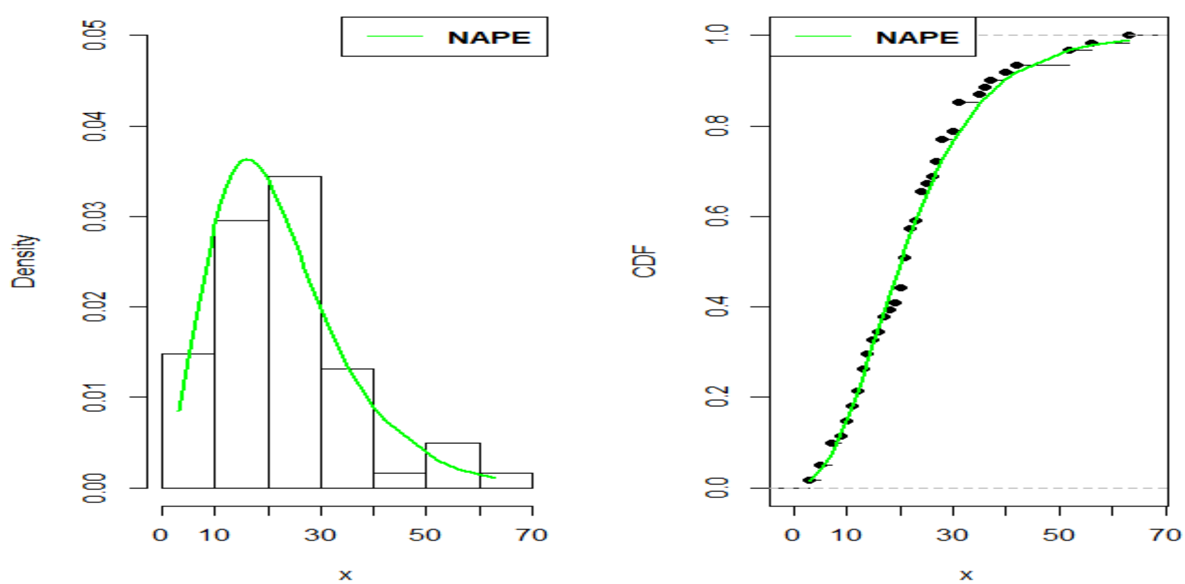


Figure 11: Plot of the estimated density and CDF of NAPE distribution for the second data set

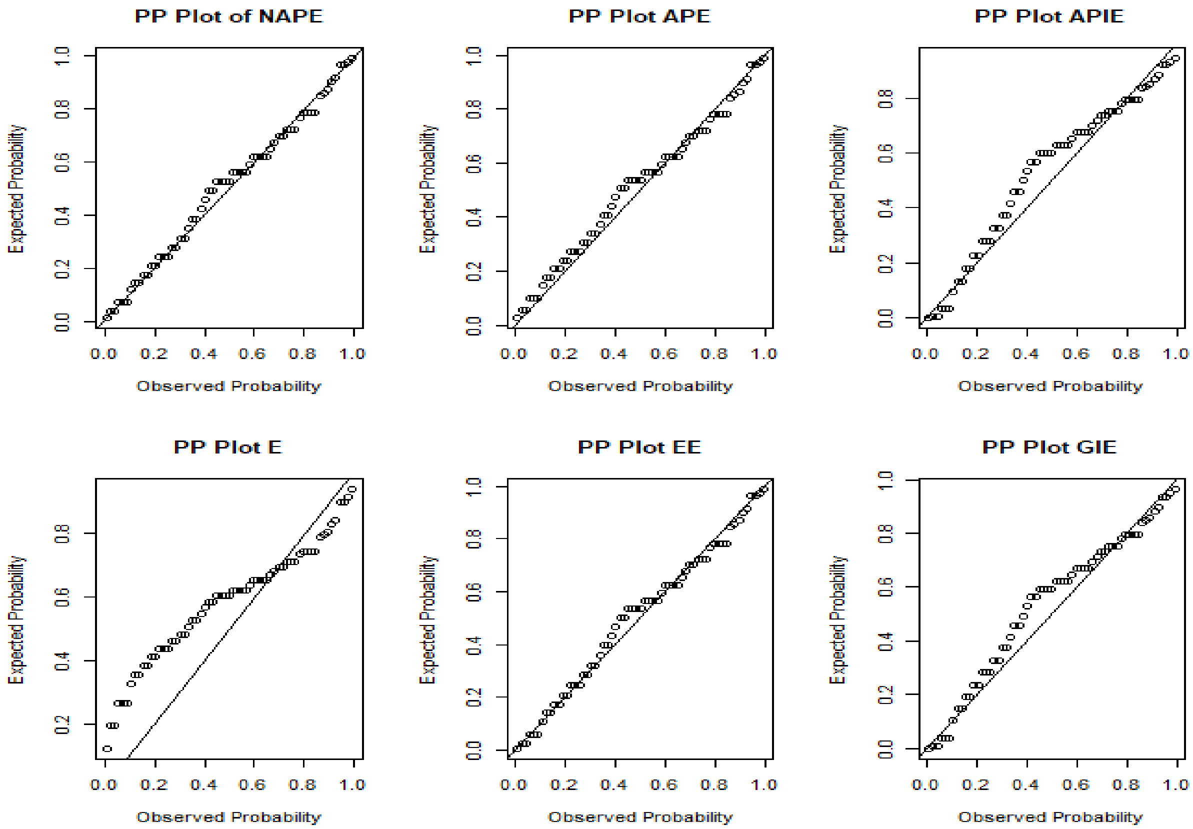


Figure 12: PP-plots of the fitted distribution to second data set

7. Conclusion

In this paper, we have proposed a new generalized family of Exponential distribution namely new alpha power Exponential (NAPE) distribution based on the NAPT method proposed by Elbatal *et al.* (2019). The density function of the proposed distribution can be increasing-decreasing, right-skewed or symmetrical depending upon the values of the parameters. The hazard rate function can be increasing, decreasing and bathtub shape. Various statistical properties of the NAPE distribution have been discussed along with the method of maximum likelihood estimation (MLE) for estimating the unknown parameters of the NAPE distribution. To show the capability and effectiveness of the NAPE distribution we have considered three real life data sets and it is shown that the proposed NAPE distribution tends to provide a very good fit to all the considered data sets as compared to other competing distributions. Hence, for modeling monotonic and non-monotonic functions the NAPE distribution can be used more effectively.

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