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Bayesian Inference of Triple Seasonal Autoregressive Models

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Abstract

Recently, autoregressive (AR) time-series models have been extended to model time-series with double seasonality. However, in some real applications, high frequency time-series can exhibit triple seasonal patterns. Therefore, in this paper we aim to extend the AR models to fit time-series with three seasonality layers, and accordingly we introduce the Bayesian inference for triple seasonal autoregressive (TSAR) models. In this Bayesian inference, we first assume the normal distribution for the TSAR model errors and employ different priors on the TSAR model parameters, including normal-gamma, g and Jeffreys' priors. Based on the normally distributed errors and employed model parameters' priors, we derive the marginal posterior distributions of different TSAR model parameters in closed forms. Particularly, we show that the marginal posterior of the TSAR model coefficients vector to be a multivariate t distribution and the marginal posterior of the TSAR model precision to be a gamma distribution. We conduct an extensive simulation study aiming to evaluate the efficiency of our proposed Bayesian inference, and also we apply our work to real hourly time-series on electricity load in some European countries.

Key Words: Multiple seasonality; TSAR models; Posterior and predictive analysis; Hourly electricity load.

Mathematical Subject Classification: 60E05, 62E15.

1. Introduction

Time-series with high frequency are observed in many real applications, such as hourly electricity load, half-hourly volumes of call arrivals, and half-hourly access to web sites. These time-series are usually characterized by exhibiting multiple layers of seasonality, such as intraday, intraweek and intrayear seasonal patterns. Since the traditional seasonal autoregressive moving average (SARMA) models can not capture multiple seasonalities, some researchers have extended these models to model and forecast time-series with multiple seasonalities, see for example Amin (2018) in the Bayesian framework, and see also De Livera et al. (2011) and Sulandari et al. (2021) in the non-Bayesian framework.

Modeling time-series with two seasonality layers in the non-Bayesian framework has been the interest of several researchers, see for example Taylor (2008b, 2008a), Ryu et al. (2017), Deb et al. (2017), Taylor and McSharry (2017) and Lago et al. (2018). However, few work have been introduced for the analysis of time-series with three seasonality layers, see for example Taylor (2010b, 2010a) De Livera et al. (2011), Taylor and Snyder (2012) and Dumas and Cornélusse (2018).

As it is known in literature, SARMA models are nonlinear in the coefficients because of the products of non-seasonal and seasonal coefficients, which makes their likelihood function analytically intractable and complicates their posterior and predictive analyses (Amin, 2009). In order to simplify the Bayesian analysis of SARMA models, some approaches have been presented in literature to approximate their posterior and predictive densities. Most of these approaches are based on analytical or Markov-Chain Monte-Carlo (MCMC) approximations. The analytical approximation is simply based on modifying analytically SARMA's posterior and predictive densities to be in closed-form distributions, see for example Shaarawy and Ali (2003), Amin (2018) and Amin (2019a, 2022b). On the other hand, the approximation based on MCMC methods is mainly simulating SARMA's conditional posterior and predictive densities to approximate their intractable posterior and predictive densities, see for example Barnett et al. (1997), Vermaak et al. (1998) and Ismail and Amin (2014).

Bayesian analysis of time-series with single seasonality modeled by SARMA models is rich and well established. For example, Based on MCMC methods, Barnett et al. (1996) introduced the Bayesian analysis of seasonal AR (SAR) models, and also Vermaak et al. (1998) proposed Metropolis within Gibbs sampler to present the Bayesian inference of SAR models. Based on the analytical approximation, Shaarawy and Ali (2003) presented the identification of SAR models by deriving the approximate posterior mass function of the SAR model order. Using Gibbs sampler, Ismail and Amin (2014) presented the Bayesian inference of SARMA models, and also recently Amin (2019b) introduced the Bayesian estimation and prediction of these models.

Bayesian analysis of time-series with double seasonality modeled by double SARMA (DSARMA) models is still in its initial stages. The first work is this direction is introduced by Ismail and Zahran (2013) that used analytical approximations to present the Bayesian inference of double SAR (DSAR) models. In addition, Amin and Ismail (2015) applied Gibbs sampler to present the Bayesian estimation of DSAR models. This work is extended by Amin (2017a, 2017b) to introduce the Bayesian estimation of double seasonal moving average (DSMA) and DSARMA models respectively. Based on analytical approximations, Amin (2018, 2019a) presented the Bayesian estimation of DSARMA models and Bayesian identification of DSAR models respectively. In the same line of work, recently Amin (2020) applied Gibbs sampler to conduct both Bayesian estimation and prediction of DSAR models. However, to the best of our knowledge none has introduced the Bayesian analysis of time-series with triple seasonality, except our recent work of proposing the Bayesian estimation for triple seasonal autoregressive (TSAR) models via Gibbs sampler (Amin, 2022a).

Therefore, in order to enrich the literature of Bayesian analysis of time-series with triple seasonality we introduce in the paper the Bayesian inference of TSAR models based on the analytical approximations, aiming to simplify the analysis without conducting extensive MCMC-based simulations. We first assume the normal distribution for the TSAR model errors and employ different priors on the TSAR model parameters, including normal-gamma, g and Jeffreys' priors, for more details about these priors see for example Amin (2017c, 2019c). Based on the normally distributed errors and employed model parameters' priors, we derive the marginal posterior distributions of different TSAR model parameters in closed forms. Particularly, we show that the marginal posterior of the TSAR model coefficients vector to be a multivariate t distribution and the marginal posterior of the TSAR model precision to be a gamma distribution.

The remainder of this paper is structured as follows: In Section 2 we introduce the TSAR models. We then present the proposed Bayesian inference for TSAR models in Section 3. In Section 4 we introduce the simulation study and discuss results, and then present a real application of our work on hourly time-series of electricity load in six European countries. Finally, we conclude our work in Section 5.

2. Triple Seasonal Autoregressive (TSAR) Models

A time-series $\{u_t\}$ with zero-mean that is generated by a TSAR model of order p, P_1, P_2 and P_3, denoted by TSAR(p)$(\text{P}_1)_{s_1}(\text{P}_2)_{s_2}(\text{P}_3)_{s_3}$, can be written in a compact form as

$$\phi_p(B)\Phi_{P_1}(B^{s_1})\theta_{P_2}(B^{s_2})\Theta_{P_3}(B^{s_3})u_t = w_t \tag{1}$$

where $\{w_t\}$'s are the TSAR model errors that are unobserved and assumed to be independent normal variates with zero-mean and precision τ. The backshift operator B is defined as $B^k u_t = u_{t-k}$, and s_1, s_2 and s_3 are the three seasonal periods. The non seasonal autoregressive polynomial is $\phi(B) = \left(1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p\right)$ with order p. As expected, there are three seasonal autoregressive polynomials in TSAR model, which are:
$\Phi_{P_1}(B^{s_1}) = \left(1 - \Phi_1 B^{s_1} - \Phi_2 B^{2s_1} - \cdots - \Phi_{P_1} B^{P_1 s_1}\right)$ with order P_1,
$\theta_{P_2}(B^{s_2}) = \left(1 - \theta_1 B^{s_2} - \theta_2 B^{2s_2} - \cdots - \theta_{P_2} B^{P_2 s_2}\right)$ with order P_2, and
$\Theta_{P_3}(B^{s_3}) = \left(1 - \Theta_1 B^{s_3} - \Theta_2 B^{2s_3} - \cdots - \Theta_{P_3} B^{P_3 s_3}\right)$ with order P_3.
Finally, the non seasonal and seasonal autoregressive coefficients are $\phi = (\phi_1, \phi_2, \cdots, \phi_p)^T$, $\Phi = (\Phi_1, \Phi_2, \cdots, \Phi_{P_1})^T$, $\theta = (\theta_1, \theta_2, \cdots, \theta_{P_2})^T$ and $\Theta = (\Theta_1, \Theta_2, \cdots, \Theta_{P_3})^T$, respectively.

As compared with the usual single SAR model, the TSAR model (1) has two extra terms, i.e. $\theta_{P_2}(B^{s_2})$ and $\Theta_{P_3}(B^{s_3})$ to accommodate the other two layers of seasonality. This means that the single SAR and double SAR models are special cases of TSAR model. In order to simplify the presentation of the TSAR model structure, we can use the summation notation to expand the compact form of TSAR model (1) as:

$$
\begin{aligned}
u_t \;=\;& \sum_{i=1}^{p}\phi_i u_{t-i} + \sum_{j=1}^{P_1}\Phi_j u_{t-js_1} + \sum_{m=1}^{P_2}\theta_m u_{t-ms_2} + \sum_{k=1}^{P_3}\Theta_k u_{t-ks_3} - \sum_{i=1}^{p}\sum_{j=1}^{P_1}\phi_i\Phi_j u_{t-i-js_1} - \\
& \sum_{i=1}^{p}\sum_{m=1}^{P_2}\phi_i\theta_m u_{t-i-ms_2} - \sum_{i=1}^{p}\sum_{k=1}^{P_3}\phi_i\Theta_k u_{t-i-ks_3} - \sum_{j=1}^{P_1}\sum_{m=1}^{P_2}\Phi_j\theta_m u_{t-js_1-ms_2} - \\
& \sum_{j=1}^{P_1}\sum_{k=1}^{P_3}\Phi_j\Theta_k u_{t-js_1-ks_3} - \sum_{m=1}^{P_2}\sum_{k=1}^{P_3}\theta_m\Theta_k u_{t-ms_2-ks_3} + \sum_{i=1}^{p}\sum_{j=1}^{P_1}\sum_{m=1}^{P_2}\phi_i\Phi_j\theta_m u_{t-i-js_1-ms_2} + \\
& \sum_{i=1}^{p}\sum_{j=1}^{P_1}\sum_{k=1}^{P_3}\phi_i\Phi_j\Theta_k u_{t-i-js_1-ks_3} + \sum_{i=1}^{p}\sum_{m=1}^{P_2}\sum_{k=1}^{P_3}\phi_i\theta_m\Theta_k u_{t-i-ms_2-ks_3} + \\
& \sum_{j=1}^{P_1}\sum_{m=1}^{P_2}\sum_{k=1}^{P_3}\Phi_j\theta_m\Theta_k u_{t-js_1-ms_2-ks_3} - \sum_{i=1}^{p}\sum_{j=1}^{P_1}\sum_{m=1}^{P_2}\sum_{k=1}^{P_3}\phi_i\Phi_j\theta_m\Theta_k u_{t-i-js_1-ms_2-ks_3} + w_t
\end{aligned}
\tag{2}
$$

It is worth noting that form eqn. (2) the TSAR model can be seen as an AR model of large order $(1+p)(1+P_1)(1+P_2)(1+P_3)-1$ but some coefficients are products of non seasonal and seasonal coefficients. It can be shown mathematically that the TSAR model (2) is stationary whenever the roots of $\phi(B)=0$, $\Phi_{P_1}(B^{s_1})=0$, $\theta_{P_2}(B^{s_2})=0$ and $\Theta_{P_3}(B^{s_3})=0$ lie outside the unit circle. It has to be noted here that stationarity and properties of time-series models are discussed in details main time-series textbooks such as Box et al. (2015).

As another way of simplification, we can write the TSAR model in the matrix form as:

$$
\mathbf{u} \;=\; Z\beta + \mathbf{w}, \tag{3}
$$

where $\mathbf{u}=(u_1,u_2,\cdots,u_n)^T$, $\mathbf{w}=(w_1,w_2,\ldots,w_n)^T$, Z is an $n\times p^{\star}$ design matrix, where $p^{\star}=(1+p)(1+P_1)(1+P_2)(1+P_3)-1$, with the t^{th} row:

$$
\begin{aligned}
Z_t \;=\;& \big(u_{t-1},..,u_{t-p},u_{t-s_1},u_{t-s_1-1},..,u_{t-s_1-p},....,u_{t-P_1s_1},u_{t-P_1s_1-1},..,u_{t-P_1s_1-p},u_{t-s_2},\\
& u_{t-s_2-1},..,u_{t-s_2-p},u_{t-s_2-s_1},u_{t-s_2-s_1-1},..,u_{t-s_2-s_1-p},....,u_{t-s_2-P_1s_1},u_{t-s_2-P_1s_1-1},\\
& ..,u_{t-s_2-P_1s_1-p},\cdots\cdots,u_{t-P_2s_2},u_{t-P_2s_2-1},..,u_{t-P_2s_2-p},u_{t-P_2s_2-s_1},u_{t-P_2s_2-s_1-1},..,\\
& u_{t-P_2s_2-s_1-p},....,u_{t-P_2s_2-P_1s_1},u_{t-P_2s_2-P_1s_1-1},..,u_{t-P_2s_2-P_1s_1-p},u_{t-s_3},u_{t-s_3-1},..,\\
& u_{t-s_3-p},u_{t-s_3-s_1},u_{t-s_3-s_1-1},..,u_{t-s_3-s_1-p},....,u_{t-s_3-P_1s_1},u_{t-s_3-P_1s_1-1},..,u_{t-s_3-P_1s_1-p},\\
& u_{t-s_3-s_2},u_{t-s_3-s_2-1},..,u_{t-s_3-s_2-p},u_{t-s_3-s_2-s_1},u_{t-s_3-s_2-s_1-1},..,u_{t-s_3-s_2-s_1-p},....,\\
& u_{t-s_3-P_1s_1-s_2},u_{t-s_3-P_1s_1-s_2-1},..,u_{t-s_3-P_1s_1-s_2-p},\cdots\cdots,u_{t-s_3-P_2s_2},u_{t-s_3-P_2s_2-1},..,\\
& u_{t-s_3-P_2s_2-p},u_{t-s_3-s_1-P_2s_2},u_{t-s_3-s_1-P_2s_2-1},..,u_{t-s_3-s_1-P_2s_2-p},....,u_{t-s_3-P_1s_1-P_2s_2},\\
& u_{t-s_3-P_1s_1-P_2s_2-1},..,u_{t-s_3-P_1s_1-P_2s_2-p},\cdots\cdots\cdots,u_{t-P_3s_3},u_{t-P_3s_3-1},..,u_{t-P_3s_3-p},\\
& u_{t-P_3s_3-s_1},u_{t-P_3s_3-s_1-1},..,u_{t-P_3s_3-s_1-p},....,u_{t-P_3s_3-P_1s_1},u_{t-P_3s_3-P_1s_1-1},..,\\
& u_{t-P_3s_3-P_1s_1-p},u_{t-s_2},u_{t-P_3s_3-s_2-1},..,u_{t-P_3s_3-s_2-p},u_{t-P_3s_3-s_1-s_2},u_{t-P_3s_3-s_1-s_2-1},..,\\
& u_{t-P_3s_3-s_1-s_2-p},....,u_{t-P_3s_3-P_1s_1-s_2},u_{t-P_3s_3-P_1s_1-s_2-1},..,u_{t-P_3s_3-P_1s_1-s_2-p},\cdots\cdots,\\
& u_{t-P_3s_3-P_2s_2},u_{t-P_3s_3-P_2s_2-1},..,u_{t-P_3s_3-P_2s_2-p},u_{t-P_3s_3-s_1-P_2s_2},u_{t-P_3s_3-s_1-P_2s_2-1},..,\\
& u_{t-P_3s_3-s_1-P_2s_2-p},....,u_{t-P_3s_3-P_1s_1-P_2s_2},u_{t-P_3s_3-P_1s_1-P_2s_2-1},..,u_{t-P_3s_3-P_1s_1-P_2s_2-p}\big),
\end{aligned}
\tag{4}
$$

and β is the model coefficients vector written as:

$$\begin{aligned}\beta \;=\; & (\phi_1,..,\phi_p,\Phi_1,-\phi_1\Phi_1,..,-\phi_p\Phi_1,....,\Phi_{P_1},-\phi_1\Phi_{P_1},..,-\phi_p\Phi_{P_1},\theta_1,-\phi_1\theta_1,..,-\phi_p\theta_1,\\ & -\Phi_1\theta_1,\phi_1\Phi_1\theta_1,..,\phi_p\Phi_1\theta_1,....,-\Phi_{P_1}\theta_1,\phi_1\Phi_{P_1}\theta_1,..,\phi_p\Phi_{P_1}\theta_1,\ldots\ldots,\theta_{P_2},-\phi_1\theta_{P_2},\\ & ..,-\phi_p\theta_{P_2},-\Phi_1\theta_{P_2},\phi_1\Phi_1\theta_{P_2},..,\phi_p\Phi_1\theta_{P_2},....,-\Phi_{P_1}\theta_{P_2},\phi_1\Phi_{P_1}\theta_{P_2},..,\phi_p\Phi_{P_1}\theta_{P_2},\\ & \Theta_1,-\phi_1\Theta_1,..,-\phi_p\Theta_1,-\Phi_1\Theta_1,\phi_1\Phi_1\Theta_1,..,\phi_p\Phi_1\Theta_1,....,-\Phi_{P_1}\Theta_1,\phi_1\Phi_{P_1}\Theta_1,..,\phi_p\Phi_{P_1}\Theta_1,\\ & -\theta_1\Theta_1,\phi_1\theta_1\Theta_1,..,\phi_p\theta_1\Theta_1,\Phi_1\theta_1\Theta_1,-\phi_1\Phi_1\theta_1\Theta_1,..,-\phi_p\Phi_1\theta_1\Theta_1,....,\Phi_{P_1}\theta_1\Theta_1,\\ & -\phi_1\Phi_{P_1}\theta_1\Theta_1,..,-\phi_p\Phi_{P_1}\theta_1\Theta_1,\ldots\ldots,-\theta_{P_2}\Theta_1,\phi_1\theta_{P_2}\Theta_1,..,\phi_p\theta_{P_2}\Theta_1,\Phi_1\theta_{P_2}\Theta_1,\\ & -\phi_1\Phi_1\theta_{P_2}\Theta_1,..,-\phi_p\Phi_1\theta_{P_2}\Theta_1,....,\Phi_{P_1}\theta_{P_2}\Theta_1,-\phi_1\Phi_{P_1}\theta_{P_2}\Theta_1,..,-\phi_p\Phi_{P_1}\theta_{P_2}\Theta_1,\\ & \Theta_{P_3},-\phi_1\Theta_{P_3},..,-\phi_p\Theta_{P_3},-\Phi_1\Theta_{P_3},\phi_1\Phi_1\Theta_{P_3},..,\phi_p\Phi_1\Theta_{P_3},....,-\Phi_{P_1}\Theta_{P_3},\phi_1\Phi_{P_1}\Theta_{P_3},..,\\ & \phi_p\Phi_{P_1}\Theta_{P_3},-\theta_1\Theta_{P_3},\phi_1\theta_1\Theta_{P_3},..,\phi_p\theta_1\Theta_{P_3},\Phi_1\theta_1\Theta_{P_3},-\phi_1\Phi_1\theta_1\Theta_{P_3},..,-\phi_p\Phi_1\theta_1\Theta_{P_3},\\ & ....,\Phi_{P_1}\theta_1\Theta_{P_3},-\phi_1\Phi_{P_1}\theta_1\Theta_{P_3},..,-\phi_p\Phi_{P_1}\theta_1\Theta_{P_3},\ldots\ldots,-\theta_{P_2}\Theta_{P_3},\phi_1\theta_{P_2}\Theta_{P_3},..,\\ & \phi_p\theta_{P_2}\Theta_{P_3},\Phi_1\theta_{P_2}\Theta_{P_3},-\phi_1\Phi_1\theta_{P_2}\Theta_{P_3},..,-\phi_p\Phi_1\theta_{P_2}\Theta_{P_3},....,\Phi_{P_1}\theta_{P_2}\Theta_{P_3},\\ & -\phi_1\Phi_{P_1}\theta_{P_2}\Theta_{P_3},..,-\phi_p\Phi_{P_1}\theta_{P_2}\Theta_{P_3})^T \end{aligned} \tag{5}$$

3. Posterior Analysis of TSAR Models

Following the standards in Bayesian statistical modeling, we derive the posterior distribution of TSAR model parameters β and τ by combining the prior information on these parameters, formulated by the prior distribution, with the likelihood function of observed time-series data $\{u_t\}$ (Broemeling, 1985).

Since we assume the TSAR model errors are normally distributed, we apply a straightforward transformation from $\mathbf{w}$ to $\mathbf{u}$ in the TSAR model (3) to write the conditional likelihood function as:

$$\begin{aligned} L(\beta,\tau\mid\mathbf{u}) &\propto \tau^{\frac{n-P^\star}{2}}\exp\left\{-\frac{\tau}{2}\mathbf{w}^T\mathbf{w}\right\},\\ &\propto \tau^{\frac{n-P^\star}{2}}\exp\left\{-\frac{\tau}{2}\left(\mathbf{u}-Z\beta\right)^T\left(\mathbf{u}-Z\beta\right)\right\} \end{aligned} \tag{6}$$

This likelihood function is conditional on the first $P^\star$ initial values, i.e. $(u_0,u_{-1},\cdots,u_{1-P^\star})$, where $P^\star = P_3s_3 + P_2s_2 + P_1s_1 + p$.

In order to ease the derivation of posterior distribution of the TSAR model parameters β and τ, we first assume the products of nonseasonal and seasonal coefficients as free coefficients. We employ the normal-gamma prior for these model parameters β and τ. Let $\tau \sim G(\frac{\nu}{2},\frac{\lambda}{2})$ and $\beta \sim N_{p^\star}(\mu_\beta,\tau^{-1}\Sigma_\beta)$, we can write the normal-gamma prior of β and τ as:

$$\zeta_n\left(\beta,\tau\right) \propto \tau^{\left(\frac{\nu+p^\star}{2}-1\right)}\exp\left\{-\frac{\tau}{2}\left[\lambda+\left(\beta-\mu_\beta\right)^T\Sigma_\beta^{-1}\left(\beta-\mu_\beta\right)\right]\right\}, \tag{7}$$

where $\mu_\beta,\Sigma_\beta,\nu$ and λ are hyper-parameters that have to be specified or estimated.

In addition, to simplify the elicitation of the covariance matrix of coefficients, we can employ the g-prior for β and τ that can be presented in the following form:

$$\zeta_g\left(\beta,\tau\right) \propto \tau^{(\frac{p^\star}{2}-1)}\exp\left\{-\frac{g\tau}{2}\left(\beta-\bar{\beta}\right)^T\left(Z^TZ\right)\left(\beta-\bar{\beta}\right)\right\}, \tag{8}$$

where $\bar{\beta}$ is a prior expected value of β, and g can be specified as a decreasing function of the time-series size n and number of TSAR model coefficients $p^\star$, for more details about setting these hyper-parameters see for example Fernandez et al. (2001) and Amin (2017c).

In case of no information is available about β and τ, we employ Jeffreys' prior on β and τ that can be introduced as:

$$\zeta_j\left(\beta,\tau\right) \propto \tau^{-1},\ \tau>0 \tag{9}$$

The prior information on the TSAR model parameters β and τ can be updated by the likelihood function (6) and formulated by the posterior distribution. Therefore, it is required to derive the marginal posteriors of the TSAR model parameters β and τ. We first derive the joint posterior of β and τ, and then we integrate out one of them to obtain the marginal posterior of the other. Accordingly, we multiply the likelihood function in eqn. (6) by each one of the three prior distributions in eqn. (7) - (9) to obtain the joint posterior of β and τ. First, in case of employing the normal-gamma prior, we can obtain the joint posterior of β and τ as:

$$\zeta_n(\beta, \tau \mid \mathbf{u}) \propto \tau^{\left(\frac{n-P^\star+\nu+p^\star}{2}-1\right)} \exp\left\{-\frac{\tau}{2}\left[\lambda + (\beta-\mu_\beta)^T \Sigma_\beta^{-1}(\beta-\mu_\beta) + (\mathbf{u}-Z\beta)^T(\mathbf{u}-Z\beta)\right]\right\}. \tag{10}$$

Second, for employing the g-prior, we can present the joint posterior of β and τ as:

$$\zeta_g(\beta, \tau \mid \mathbf{u}) \propto \tau^{\left(\frac{n-P^\star+p^\star}{2}-1\right)} \exp\left\{-\frac{\tau}{2}\left[(\beta-\bar{\beta})^T (gZ^TZ)(\beta-\bar{\beta}) + (\mathbf{u}-Z\beta)^T(\mathbf{u}-Z\beta)\right]\right\}. \tag{11}$$

Third, for Jeffreys' prior, we can write the joint posterior of β and τ as:

$$\zeta_j(\beta, \tau \mid \mathbf{u}) \propto \tau^{\left(\frac{n-P^\star}{2}-1\right)} \exp\left\{-\frac{\tau}{2}(\mathbf{u}-Z\beta)^T(\mathbf{u}-Z\beta)\right\}. \tag{12}$$

Now, from these joint posteriors (10) - (12), we can derive the marginal posterior of each one of the TSAR model parameters β and τ by integrating out the unwanted parameter. In particular, in the following theorem we show that for employing the normal-gamma prior the resulting marginal posterior of the model coefficients vector β is a multivariate t distribution and also the marginal posterior of the model precision τ is a gamma distribution.

Theorem 3.1. *Using the conditional likelihood function of TSAR model given in eqn. (6) and by employing the normal-gamma prior of TSAR model parameters β and τ given in eqn. (7), the marginal posterior of the TSAR model coefficients vector β is a multivariate t distribution with parameters: degrees of freedom $v_n = (n + \nu - P^\star)$, mean vector $\mu_n = A_n^{-1}B_n$, and covariance matrix $V_n = \frac{C_n}{v_n-2}A_n^{-1}$, and also the marginal posterior of the TSAR model precision τ is a gamma distribution with parameters: $\frac{v_n}{2}$ and $\frac{C_n}{2}$, where:*

$$A_n^{-1} = (Z^TZ + \Sigma_\beta^{-1})^{-1}$$

$$B_n = (Z^T\mathbf{u} + \Sigma_\beta^{-1}\mu_\beta)$$

$$C_n = [\mathbf{u}^T\mathbf{u} + \lambda + \mu_\beta^T\Sigma_\beta^{-1}\mu_\beta - B_n^TA_n^{-1}B_n]$$

Proof. We first multiply the conditional likelihood function of TSAR model given in eqn. (6) by the normal-gamma prior of TSAR model parameters β and τ given in eqn. (7) to obtain their joint posterior that can be written as:

$$\zeta_n(\beta, \tau \mid \mathbf{u}) \propto \tau^{\left(\frac{n-P^\star+\nu+p^\star}{2}-1\right)} \exp\left\{-\frac{\tau}{2}\left[\lambda + (\beta-\mu_\beta)^T \Sigma_\beta^{-1}(\beta-\mu_\beta) + (\mathbf{u}-Z\beta)^T(\mathbf{u}-Z\beta)\right]\right\}. \tag{13}$$

We integrate this joint posterior (13) over the TSAR model precision τ and then complete the square with respect to β results in the marginal posterior of the TSAR model coefficients vector β to be a multivariate t distribution with stated parameters. On the other hand, we complete the square in the exponent of the joint posterior (13) with respect to the TSAR model coefficients vector β and then integrate it out results in the marginal posterior of the TSAR model precision τ to be a gamma distribution with stated parameters.

In addition, in the following two corollaries we show that for employing the g and Jeffreys' priors the resulting marginal posteriors are the same as in Theorem (3.1) but with different parameters: for the TSAR model coefficients vector β is a multivariate t distribution and for the TSAR model precision τ is a gamma distribution.

Lemma 3.1. *Using the conditional likelihood function of TSAR model given in eqn. (6) and by employing the g prior of TSAR model parameters β and τ given in eqn. (8), the marginal posterior of the TSAR model coefficients vector β is a multivariate t distribution with parameters: degrees of freedom $v_g = (n - P^\star)$, mean vector $\mu_g = A_g^{-1}B_g$,* *and covariance matrix* $V_g = \frac{C_g}{v_g - 2} A_g^{-1}$*, and also the marginal posterior of the TSAR model precision* τ *is a gamma distribution with parameters:* $\frac{v_g}{2}$ *and* $\frac{C_g}{2}$*, where:*

$$A_g^{-1} = ((g+1)Z^T Z)^{-1}$$

$$B_g = (Z^T \mathbf{u} + g(Z^T Z)\bar{\beta})$$

$$C_g = [\mathbf{u}^T \mathbf{u} + g\bar{\beta}^T (Z^T Z)\bar{\beta} - B_g^T A_g^{-1} B_g]$$

.

Proof. We first set $\lambda = \nu = 0$, $\mu_\beta = \bar{\beta}$, and $\Sigma_\beta^{-1} = gZ^T Z$ and then we simply get this corollary result directly from Theorem (3.1).

Lemma 3.2. *Using the conditional likelihood function of TSAR model given in eqn. (6) and by employing the Jeffreys' prior of TSAR model parameters* β *and* τ *given in eqn. (9), the marginal posterior of the TSAR model coefficients vector* β *is a multivariate t distribution with parameters: degrees of freedom* $v_j = (n - 2p^\star)$*, mean vector* $\mu_j = A_j^{-1} B_j$*, and covariance matrix* $V_j = \frac{C_j}{v_j - 2} A_j^{-1}$*, and also the marginal posterior of the TSAR model precision* τ *is a gamma distribution with parameters:* $\frac{v_j}{2}$ *and* $\frac{C_j}{2}$*, where:*

$$A_j^{-1} = (Z^T Z)^{-1}$$

$$B_j = Z^T \mathbf{u}$$

$$C_j = [\mathbf{u}^T \mathbf{u} - B_j^T A_j^{-1} B_j]$$

.

Proof. We first set $\lambda = 0$, $\Sigma_\beta^{-1} = \mathbf{0}$, and $\nu = -p^\star$ and then we simply get this corollary result directly from Theorem (3.1).

Using Theorem (3.1) and Corollaries (3.1) and (3.2), we can easily conduct inferential analysis about the TSAR model coefficients and precision. Here, it is worth mentioning an important property of the multivariate t distribution. Let β is a vector that follows a multivariate t distribution with parameters: degrees of freedom v_n, mean vector μ_n, and covariance matrix V_n, then the i^{th} element of β follows a univariate t distribution with parameters: degrees of freedom v_n, mean μ_{n_i}, and variance V_{n_i}, where μ_{n_i} is the i^{th} element in the mean vector μ_n and V_{n_i} is the i^{th} diagonal element in the covariance matrix V_n. The same result is valid for any sub-vector of β. We exploit this property of the multivariate t distribution in our work, and in order to test the significance of any element in the coefficients vector β, say β_i, we can compute an $(1-\alpha)\%$ credible interval as:

$$\mu_{n_i} - t_{\frac{\alpha}{2}, v_n} \sqrt{V_{n_i}} \leq \beta_i \leq \mu_{n_i} + t_{\frac{\alpha}{2}, v_n} \sqrt{V_{n_i}} \tag{14}$$

Note that in this credible interval $t_{\frac{\alpha}{2}, v_n}$ is just the t distribution $(1 - \frac{\alpha}{2})$ percentile with degrees of freedom is v_n. In the same way, we can use the gamma distribution to conduct inferential analysis about the TSAR model precision.

4. Simulations and Applications

We conduct in this section an extensive simulation study aiming to assess the efficiency of our introduced Bayesian inference for the TSAR models, and then we demonstrate the applicability of our work to real time-series with three layers of seasonality using electricity load in some European countries, which is hourly time-series.

4.1. Simulation Study

In order to evaluate the efficiency of the introduced Bayesian inference for TSAR models, in this simulation study we try to simulate different seasonality patterns with different time-series sample sizes. Particularly, we generate 1,000 time-series of size n (from 1,000 to 3,000 with an increment of 1,000 observations) from four TSAR models. The design of this simulation study for these TSAR models is presented in Table 1, including true parameters values of the four TSAR models.

Table 1: Design of simulation study.

TSAR Model	ϕ_1	ϕ_2	Φ_1	Φ_2	θ_1	θ_2	Θ_1	Θ_2	τ
I. $(1)(1)_3(1)_{21}(1)_{210}$	0.6		-0.5		-0.3		0.4		1.0
II. $(1)(1)_4(1)_{20}(1)_{240}$	-0.2		0.6		-0.4		0.3		1.0
III. $(2)(2)_4(1)_{20}(1)_{240}$	-0.2	0.3	0.6	-0.4	-0.4		0.3		1.0
IV. $(2)(2)_3(2)_{21}(2)_{210}$	0.3	-0.4	-0.6	0.3	0.2	-0.4	-0.4	0.3	1.0

Once we generate these time-series datasets from these specified four TSAR models, we conduct our introduced Bayesian inference - by employing the three priors Jeffreys', g and normal-gamma priors for the TSAR model parameters - for each time-series dataset and then we compute the Bayesian estimates for the TSAR model coefficients and precision, including mean, standard deviation and 95% credible interval.

Before we discuss our simulation results there are some important remarks we should to highlight. First, Bayesian estimates resulting from employing Jeffreys' prior are theoretically identical to those estimates can be obtained from the classical approach, such as ordinary least square (OLS) or maximum likelihood (ML) estimates. This way can be used to compare our Bayesian estimates with employing different priors for the TSAR model parameters to those estimates results from the classical approach. Second, in our previous work Amin (2019c) we evaluated the sensitivity of the posterior distribution to the prior selection, and based on this work results we set $g = 1/n$, where n is the time-series size. Third, we follow the empirical Bayesian approach to estimate the hyper-parameters for the normal-gamma prior of the TSAR model parameters, and for more details about this empirical Bayesian approach see for example Berger (1985).

We present the simulation results for TSAR Model-I in Table 2, including the mean, standard deviation and 95% credible interval of the posterior means assuming three priors Jeffreys', g and normal-gamma priors. These simulation results for TSAR Model-I show that the Bayesian estimates of the TSAR model parameters obtained from the three posteriors are close to each other and also close to their true values. Also, each 95% credible interval contains the parameter's true value, which confirms the accuracy of the proposed Bayesian estimation. Whenever the sample size grows, these estimates become much closer to the true values, which highlights the consistency of these Bayesian estimates. However, the Bayesian estimates of the parameters' standard deviation result from employing the normal-gamma prior are highly different from (i.e. much smaller than) those result from employing Jeffreys' and g priors, which are very close to each other. For instance, from Table 2, for n = 2,000, different posteriors provide a ϕ_1's estimate is about 0.6, and the Bayesian estimates of its standard deviation obtained from the three posteriors $\zeta_j(\beta \mid \mathbf{y})$, $\zeta_g(\beta \mid \mathbf{y})$ and $\zeta_n(\beta \mid \mathbf{y})$ are about 0.020, 0.020 and 0.014, respectively. This confirms that the Bayesian estimates result from employing the normal-gamma prior have higher precision compared to those result from employing Jeffreys' and g priors. We present the simulation results for TSAR Model-II to Model-IV in Tables 3 to 5 and we get similar conclusions to those of TSAR Model-I.

Table 2: Bayesian results for TSAR Model-I.

		True	$\zeta_j(\beta \mid \mathbf{u})^*$				$\zeta_g(\beta \mid \mathbf{u})$				$\zeta_n(\beta \mid \mathbf{u})$			
n	β_i	Value	$\hat{\mu}$	$\hat{\sigma}$	L	U	$\hat{\mu}$	$\hat{\sigma}$	L	U	$\hat{\mu}$	$\hat{\sigma}$	L	U
1,000	ϕ_1	0.6	0.595	0.032	0.532	0.659	0.595	0.032	0.532	0.659	0.595	0.023	0.550	0.640
	Φ_1	-0.5	-0.494	0.033	-0.559	-0.428	-0.494	0.033	-0.559	-0.429	-0.494	0.023	-0.540	-0.448
	θ_1	-0.3	-0.295	0.035	-0.366	-0.224	-0.295	0.035	-0.366	-0.224	-0.295	0.025	-0.345	-0.245
	Θ_1	0.4	0.396	0.033	0.330	0.462	0.396	0.033	0.330	0.461	0.396	0.023	0.349	0.442
	τ	1.0	1.000	0.051	0.897	1.103	1.000	0.051	0.898	1.103	1.000	0.036	0.927	1.073
2,000	ϕ_1	0.6	0.598	0.020	0.557	0.638	0.598	0.020	0.557	0.638	0.598	0.014	0.569	0.626
	Φ_1	-0.5	-0.497	0.022	-0.541	-0.453	-0.497	0.022	-0.541	-0.453	-0.497	0.016	-0.528	-0.466
	θ_1	-0.3	-0.297	0.023	-0.342	-0.251	-0.297	0.023	-0.342	-0.251	-0.297	0.016	-0.329	-0.265
	Θ_1	0.4	0.398	0.022	0.354	0.443	0.398	0.022	0.354	0.443	0.398	0.016	0.367	0.430
	τ	1.0	1.001	0.034	0.932	1.070	1.001	0.034	0.933	1.070	1.001	0.024	0.953	1.050
3,000	ϕ_1	0.6	0.598	0.016	0.566	0.630	0.598	0.016	0.566	0.630	0.598	0.011	0.576	0.621
	Φ_1	-0.5	-0.498	0.017	-0.533	-0.464	-0.498	0.017	-0.533	-0.464	-0.498	0.012	-0.523	-0.474
	θ_1	-0.3	-0.298	0.018	-0.335	-0.261	-0.298	0.018	-0.335	-0.261	-0.298	0.013	-0.324	-0.272
	Θ_1	0.4	0.399	0.018	0.363	0.435	0.399	0.018	0.363	0.435	0.399	0.013	0.374	0.424
	τ	1.0	1.001	0.027	0.947	1.055	1.001	0.027	0.947	1.055	1.001	0.019	0.963	1.039

$^*\zeta_j(\beta \mid \mathbf{u})$, $\zeta_g(\beta \mid \mathbf{u})$ and $\zeta_n(\beta \mid \mathbf{u})$ are posteriors result from employing Jeffreys', g, and normal-gamma priors respectively; and L and U are the lower and the upper limits of an 95% credible interval respectively.

From all these results, we can highlight some general conclusions. First, the simulation results confirm the efficiency of the proposed Bayesian inference of TSAR models; since all the TSAR model parameters' Bayesian estimates are on average very close to their true values that are included in the constructed credible intervals. Second, the key hyper-parameter in the g prior, i.e. g, as we stated above, is usually an decreasing function of the time-series size, and accordingly for a large time-series size, as in our case of triple seasonal time-series, the g parameter value is very small and closes to zero, which makes the Bayesian estimates obtained in case of employing g prior are very

Table 3: Bayesian results for TSAR Model-II.

n	β_i	True Value	$\zeta_j(\beta \mid \mathbf{u})$				$\zeta_g(\beta \mid \mathbf{u})$				$\zeta_n(\beta \mid \mathbf{u})$			
			$\hat{\mu}$	$\hat{\sigma}$	L	U	$\hat{\mu}$	$\hat{\sigma}$	L	U	$\hat{\mu}$	$\hat{\sigma}$	L	U
1,000	ϕ_1	-0.2	-0.198	0.037	-0.273	-0.124	-0.198	0.037	-0.273	-0.124	-0.198	0.026	-0.251	-0.145
	Φ_1	0.6	0.594	0.030	0.534	0.654	0.594	0.030	0.534	0.654	0.594	0.021	0.551	0.637
	θ_1	-0.4	-0.396	0.036	-0.468	-0.323	-0.396	0.036	-0.467	-0.324	-0.396	0.025	-0.446	-0.345
	Θ_1	0.3	0.301	0.034	0.232	0.369	0.301	0.034	0.232	0.369	0.301	0.024	0.252	0.349
	τ	1.0	0.998	0.051	0.895	1.101	0.998	0.051	0.896	1.101	0.998	0.036	0.926	1.071
2,000	ϕ_1	-0.2	-0.198	0.023	-0.244	-0.152	-0.198	0.023	-0.244	-0.152	-0.198	0.016	-0.231	-0.166
	Φ_1	0.6	0.597	0.020	0.558	0.637	0.597	0.020	0.558	0.636	0.597	0.014	0.569	0.625
	θ_1	-0.4	-0.398	0.022	-0.443	-0.353	-0.398	0.022	-0.442	-0.353	-0.398	0.016	-0.430	-0.366
	Θ_1	0.3	0.299	0.023	0.254	0.345	0.299	0.023	0.254	0.344	0.299	0.016	0.267	0.331
	τ	1.0	0.999	0.034	0.931	1.068	0.999	0.034	0.931	1.067	0.999	0.024	0.951	1.048
3,000	ϕ_1	-0.2	-0.199	0.018	-0.236	-0.162	-0.199	0.018	-0.236	-0.162	-0.199	0.013	-0.225	-0.173
	Φ_1	0.6	0.598	0.015	0.567	0.628	0.598	0.015	0.567	0.628	0.598	0.011	0.576	0.619
	θ_1	-0.4	-0.399	0.017	-0.434	-0.364	-0.399	0.017	-0.433	-0.364	-0.399	0.012	-0.423	-0.374
	Θ_1	0.3	0.299	0.018	0.263	0.335	0.299	0.018	0.263	0.335	0.299	0.013	0.273	0.324
	τ	1.0	1.000	0.027	0.945	1.054	1.000	0.027	0.945	1.054	1.000	0.019	0.961	1.038

Table 4: Bayesian results for TSAR Model-III.

n	β_i	True Value	$\zeta_j(\beta \mid \mathbf{u})$				$\zeta_g(\beta \mid \mathbf{u})$				$\zeta_n(\beta \mid \mathbf{u})$			
			$\hat{\mu}$	$\hat{\sigma}$	L	U	$\hat{\mu}$	$\hat{\sigma}$	L	U	$\hat{\mu}$	$\hat{\sigma}$	L	U
1,000	ϕ_1	-0.2	-0.196	0.036	-0.268	-0.123	-0.196	0.036	-0.268	-0.123	-0.196	0.026	-0.247	-0.144
	ϕ_2	0.3	0.291	0.037	0.217	0.365	0.291	0.037	0.218	0.365	0.291	0.026	0.239	0.344
	Φ_1	0.6	0.591	0.034	0.523	0.660	0.591	0.034	0.523	0.660	0.591	0.024	0.543	0.640
	Φ_2	-0.4	-0.393	0.037	-0.467	-0.319	-0.393	0.037	-0.467	-0.319	-0.393	0.026	-0.445	-0.340
	θ_1	-0.4	-0.390	0.036	-0.462	-0.319	-0.390	0.036	-0.461	-0.319	-0.390	0.025	-0.441	-0.339
	Θ_1	0.3	0.296	0.035	0.225	0.367	0.296	0.035	0.226	0.366	0.296	0.025	0.246	0.346
	τ	1.0	1.003	0.051	0.900	1.106	1.003	0.051	0.901	1.105	1.003	0.036	0.930	1.076
2,000	ϕ_1	-0.2	-0.198	0.023	-0.244	-0.152	-0.198	0.023	-0.244	-0.153	-0.198	0.016	-0.231	-0.166
	ϕ_2	0.3	0.294	0.024	0.246	0.342	0.294	0.024	0.247	0.342	0.294	0.017	0.261	0.328
	Φ_1	0.6	0.596	0.023	0.550	0.643	0.596	0.023	0.550	0.643	0.596	0.016	0.563	0.629
	Φ_2	-0.4	-0.397	0.024	-0.445	-0.349	-0.397	0.024	-0.445	-0.350	-0.397	0.017	-0.431	-0.363
	θ_1	-0.4	-0.396	0.023	-0.441	-0.351	-0.396	0.022	-0.441	-0.351	-0.396	0.016	-0.428	-0.364
	Θ_1	0.3	0.297	0.022	0.252	0.342	0.297	0.022	0.253	0.341	0.297	0.016	0.265	0.329
	τ	1.0	1.001	0.034	0.933	1.070	1.001	0.034	0.934	1.069	1.001	0.024	0.953	1.050
3,000	ϕ_1	-0.2	-0.198	0.018	-0.235	-0.162	-0.198	0.018	-0.234	-0.163	-0.198	0.013	-0.224	-0.173
	ϕ_2	0.3	0.297	0.019	0.259	0.334	0.297	0.018	0.260	0.334	0.297	0.013	0.270	0.323
	Φ_1	0.6	0.598	0.019	0.560	0.635	0.598	0.019	0.561	0.635	0.598	0.013	0.571	0.624
	Φ_2	-0.4	-0.398	0.019	-0.436	-0.359	-0.398	0.019	-0.436	-0.359	-0.398	0.014	-0.425	-0.370
	θ_1	-0.4	-0.397	0.018	-0.433	-0.362	-0.397	0.018	-0.432	-0.362	-0.397	0.013	-0.422	-0.372
	Θ_1	0.3	0.297	0.019	0.260	0.335	0.297	0.019	0.260	0.335	0.297	0.013	0.271	0.324
	τ	1.0	1.002	0.027	0.948	1.055	1.002	0.027	0.949	1.055	1.002	0.019	0.964	1.040

Table 5: Bayesian results for TSAR Model-IV.

n	β_i	True Value	$\zeta_j(\beta \mid \mathbf{u})$				$\zeta_g(\beta \mid \mathbf{u})$				$\zeta_n(\beta \mid \mathbf{u})$			
			$\hat{\mu}$	$\hat{\sigma}$	L	U	$\hat{\mu}$	$\hat{\sigma}$	L	U	$\hat{\mu}$	$\hat{\sigma}$	L	U
1,000	ϕ_1	0.3	0.296	0.046	0.204	0.388	0.296	0.045	0.206	0.387	0.296	0.032	0.231	0.361
	ϕ_2	-0.4	-0.389	0.049	-0.487	-0.290	-0.389	0.048	-0.486	-0.292	-0.389	0.035	-0.458	-0.319
	Φ_1	-0.6	-0.571	0.047	-0.665	-0.476	-0.571	0.046	-0.663	-0.478	-0.571	0.033	-0.637	-0.504
	Φ_2	0.3	0.280	0.049	0.181	0.379	0.280	0.049	0.183	0.377	0.280	0.035	0.210	0.350
	θ_1	0.2	0.184	0.044	0.096	0.272	0.184	0.043	0.097	0.271	0.184	0.031	0.122	0.247
	θ_2	-0.4	-0.365	0.046	-0.456	-0.273	-0.365	0.045	-0.455	-0.275	-0.365	0.032	-0.430	-0.300
	Θ_1	-0.4	-0.389	0.046	-0.481	-0.298	-0.389	0.045	-0.479	-0.300	-0.389	0.032	-0.454	-0.325
	Θ_2	0.3	0.302	0.045	0.213	0.391	0.302	0.044	0.214	0.390	0.302	0.032	0.239	0.365
	τ	1.0	1.000	0.066	0.867	1.133	1.000	0.065	0.870	1.131	1.000	0.047	0.906	1.094
2,000	ϕ_1	0.3	0.299	0.026	0.247	0.351	0.299	0.026	0.247	0.350	0.299	0.018	0.262	0.336
	ϕ_2	-0.4	-0.397	0.027	-0.452	-0.343	-0.397	0.027	-0.451	-0.344	-0.397	0.019	-0.436	-0.359
	Φ_1	-0.6	-0.591	0.027	-0.646	-0.537	-0.591	0.027	-0.645	-0.538	-0.591	0.019	-0.630	-0.553
	Φ_2	0.3	0.295	0.028	0.239	0.350	0.295	0.027	0.240	0.349	0.295	0.020	0.255	0.334
	θ_1	0.2	0.196	0.025	0.146	0.246	0.196	0.025	0.147	0.245	0.196	0.018	0.161	0.231
	θ_2	-0.4	-0.388	0.025	-0.439	-0.337	-0.388	0.025	-0.438	-0.338	-0.388	0.018	-0.424	-0.352
	Θ_1	-0.4	-0.396	0.025	-0.446	-0.347	-0.396	0.024	-0.445	-0.348	-0.396	0.017	-0.431	-0.361
	Θ_2	0.3	0.294	0.024	0.246	0.342	0.294	0.024	0.247	0.341	0.294	0.017	0.260	0.328
	τ	1.0	1.001	0.035	0.931	1.072	1.001	0.035	0.932	1.071	1.001	0.025	0.951	1.051
3,000	ϕ_1	0.3	0.300	0.020	0.260	0.340	0.300	0.020	0.261	0.339	0.300	0.014	0.272	0.328
	ϕ_2	-0.4	-0.399	0.020	-0.439	-0.359	-0.399	0.020	-0.438	-0.359	-0.399	0.014	-0.427	-0.370
	Φ_1	-0.6	-0.595	0.021	-0.636	-0.553	-0.595	0.020	-0.635	-0.554	-0.595	0.015	-0.624	-0.566
	Φ_2	0.3	0.297	0.022	0.254	0.341	0.297	0.021	0.255	0.340	0.297	0.015	0.267	0.328
	θ_1	0.2	0.198	0.020	0.159	0.237	0.198	0.019	0.159	0.237	0.198	0.014	0.170	0.226
	θ_2	-0.4	-0.393	0.019	-0.431	-0.354	-0.393	0.019	-0.431	-0.355	-0.393	0.014	-0.420	-0.365
	Θ_1	-0.4	-0.397	0.019	-0.436	-0.359	-0.397	0.019	-0.435	-0.359	-0.397	0.014	-0.425	-0.370
	Θ_2	0.3	0.295	0.019	0.257	0.334	0.295	0.019	0.257	0.333	0.295	0.014	0.268	0.322
	τ	1.0	1.000	0.028	0.944	1.056	1.000	0.027	0.945	1.055	1.000	0.020	0.961	1.040

close to those obtained by employing Jeffreys' prior. Third, employing the normal-gamma prior reduces the posterior estimates of parameters' standard deviations compared to employing the g and Jeffreys' priors, as it can be clearly seen in simulation results. Therefore, by assuming the normal-gamma prior we get $(1-\alpha)\%$ credible intervals of the

TSAR model parameters that are more precise (shorter) than those resulted from assuming the g and Jeffreys' prior.

4.2. Real Application on Hourly Time-Series of Electricity Load

With the objective of demonstrating the applicability of our proposed Bayesian inference to real time-series with three seasonality layers, we conduct our Bayesian inference of TSAR models to hourly time-series on of electricity load in some European countries. These electricity load time-series exhibit three seasonal patterns: intraday, intraweek and intrayear. In particular, these electricity load time-series are collected during four years, starting from 1st January 2006 till 31st December 2009, in the European countries: Germany, France, Austria, Belgium, Spain and Czech Republic. Aiming to visualize the three seasonality layers that are exhibited by these electricity load time-series, we first display in Figure (1) the time line of hourly electricity load during three periods in Austria : (a) hourly electricity load of only one week to show the first seasonality layer with s_1 = 24, (b) hourly electricity load of only four weeks to show the second seasonality layer with s_2 = 168, and (c) hourly electricity load during four years to show the third seasonality layer with s_3 = 8,736. We visualize hourly electricity load time-series of different European countries in Figure (2). It is clear from this figure that all hourly electricity load time-series exhibit three seasonality layers.

As it is well-known in time-series analysis, the first step in analyzing real time-series data before estimating the time-series model it is required to identify (specify) the best order for this model. Following the same standard, we need first to specify the best suitable TSAR model order for each one of these datasets. Accordingly, we apply the mostly-used Akaike's information criterion (AIC) to estimate the suitable order of the TSAR model with setting the maximum order value of nonseasonal and seasonal polynomials to be three in the TSAR model (2), i.e. $p = P_1 = P_2 = P_3 = 3$. Thus, we estimate all TSAR models with different orders up to three, compute their corresponding AIC$_c$ values, and finally select the best TSAR model that has the smallest AIC$_c$ value. We present the identified TSAR model for each electricity load time-series in Table (6).

After we identified the suitable order of the TSAR model for the underlying hourly electricity time-series, we employ the the normal-gamma prior for the TSAR model parameters and also we use the same setting of our simulation study in the previous subsection estimate the hyper-parameters. We present our Bayesian estimates of the identified TSAR models in Table (7). We use Theorem (3.1) to test the significance of each estimated TSAR model coefficient by simply dividing the coefficient's posterior estimate by its posterior standard deviation estimate and then comparing the result to $t_{\frac{\alpha}{2},v_n} \approx 2$. Based on our testing results from Table (7), we conclude that estimates of all TSAR models coefficients are significant.

Table 6: Identified models for hourly time-series on electricity load.

Time-series	Country	TSAR model order			
		p	P_1	P_2	P_3
1	Austria	3	3	3	1
2	Belgium	3	3	2	1
3	Czech Republic	2	3	3	1
4	France	2	3	3	1
5	Germany	3	3	3	1
6	Spain	3	3	3	1

Table 7: Bayesian estimates of identified models for hourly time-series on electricity load.

Dataset	Austria		Belgium		Czech		France		Germany		Spain	
Parameter	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\sigma}$
ϕ_1	1.33	0.006	1.20	0.006	0.90	0.006	1.44	0.006	1.53	0.006	0.67	0.006
ϕ_2	-0.40	0.010	-0.20	0.010	0.06	0.006	-0.46	0.006	-0.65	0.011	0.14	0.008
ϕ_3	0.03	0.006	-0.05	0.006	-	-	-	-	0.09	0.006	0.06	0.006
Φ_1	0.24	0.006	0.27	0.006	0.22	0.006	0.34	0.006	0.27	0.006	0.19	0.006
Φ_2	0.09	0.006	0.10	0.006	0.10	0.006	0.08	0.007	0.10	0.007	0.07	0.006
Φ_3	0.09	0.006	0.08	0.006	0.09	0.006	0.09	0.006	0.11	0.006	0.09	0.006
θ_1	0.20	0.006	0.24	0.006	0.25	0.006	0.21	0.006	0.19	0.006	0.17	0.006
θ_2	0.08	0.006	0.15	0.006	0.13	0.006	0.07	0.007	0.09	0.006	0.10	0.007
θ_3	0.13	0.006	-	-	0.14	0.006	0.08	0.006	0.15	0.006	0.10	0.006
Θ_1	0.23	0.006	0.28	0.006	0.25	0.006	0.36	0.006	0.25	0.006	0.13	0.007

Figure 1: Hourly time-series on electricity load in Austria during different periods

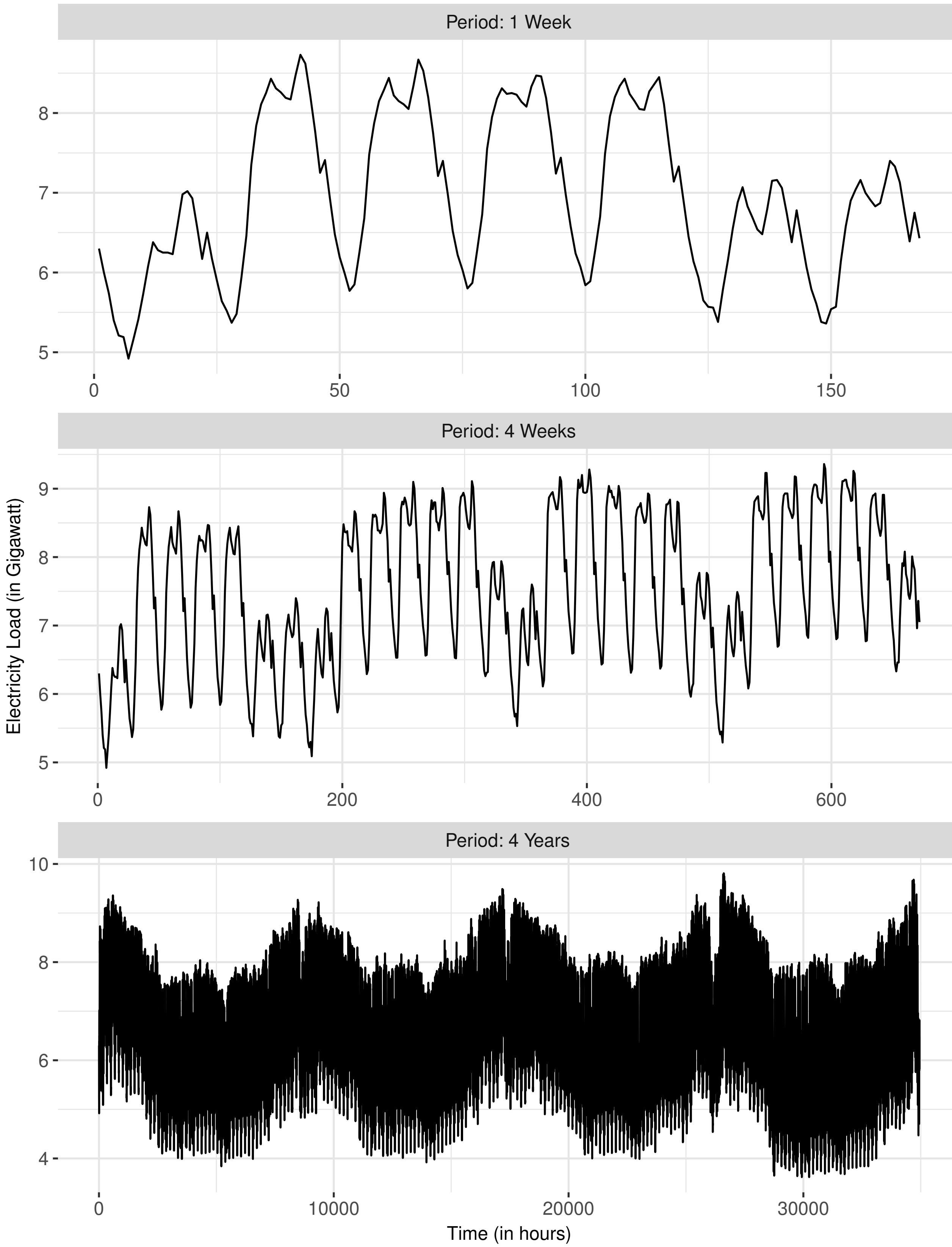
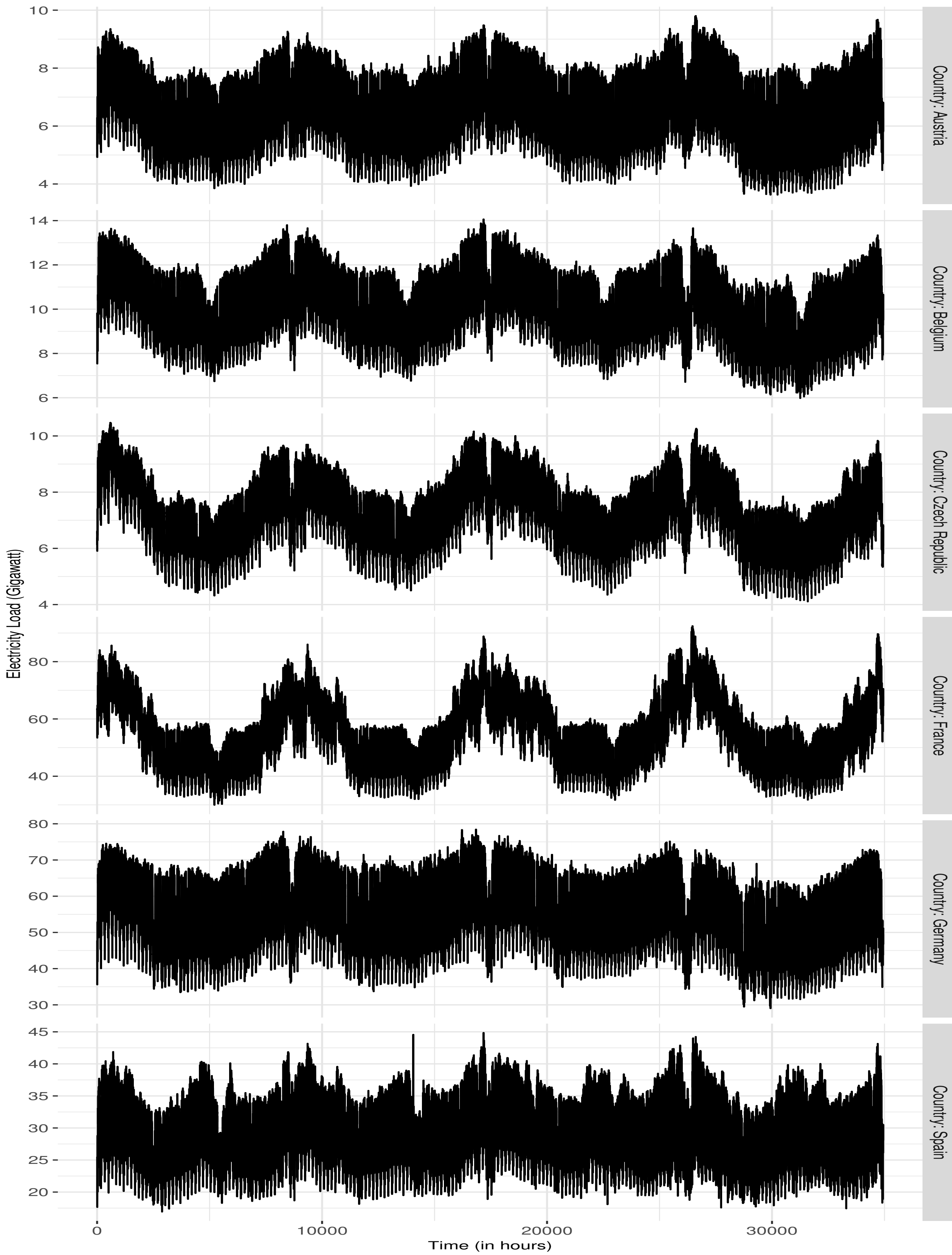


Figure 2: Hourly time-series on electricity load



5. Conclusions

In this work, we introduced Bayesian inference for TSAR models to fit and model time-series with three seasonality layers. In order to ease the derivation of the parameters posterior distribution of TSAR model, we mainly assumed the normal distribution for the TSAR model errors, and to consider different situations about the prior information, we employed three prior distributions on the TSAR model coefficients and precision, mainly Jeffreys' prior as a non-informative prior and g and normal-gamma priors. Accordingly, we derived the joint posterior resulting from each employed prior, and we approximated the marginal posterior of the TSAR model coefficients vector to be a multivariate t distribution, and the marginal posterior of the TSAR model precision to be a gamma distribution. Since these derived marginal posteriors of the TSAR model coefficients and precision are standard probability distributions, we straightforwardly conducted the Bayesian inference for TSAR models. We executed a large simulation study and its results confirmed the efficiency of our proposed Bayesian inference, and also we applied our work to real hourly time-series on electricity load in some European countries. As a future work, we plan to conduct a comparison study of our current work and our previous work in Amin (2022a), and the comparison has to evaluate both accuracy and computational cost. Our plan for future work also includes a Bayesian identification of TSAR models and an extension to multivariate time-series models.

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