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Intervened Exponential Distribution: Properties and Applications

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Abstract

This manuscript aims to study the intervention-based probability model. Statistical and reliability properties such as the expressions for, cumulative density function (cdf), mean deviations about mean and median, r^{th} order central and non-central moments, generation functions for moments have been derived. Moreover, the expression for reliability function, hazard rate function, reverse hazard rate function, aging intensity, mean residual life function, stress-strength reliability, and entropy metrics due to Rényi and Shannon are also derived. Monte Carlo simulation study performance of maximum likelihood estimates (MLE_s) has been carried out, followed by calculations of average bias (Abias), and Mean Square Error (MSE). The applicability of the model in real-life situations has been discussed by analyzing the two real-life data sets.

Key Words: Bias; Entropy; Intervention; Mean Square Error; Monte Carlo Simulation.

Mathematical Subject Classification: 60E05, 62E15.

1. Introduction

In reliability and survival analysis, the lifetime of the system or an individual is viewed as, an essential characteristic. This important feature in reliability theory is investigated with traditional lifetime distributions available in the literature, such as Gamma, Exponential, Weibull, Rayleigh, Normal, Log-normal, etc. For a deep and more detailed summary about the lifetime models, one could refer to Barlow and Proschan (1975), Zacks (1992), Marshall and Olkin (2007), etc. In distribution theory, numerous continuous and discrete lifetime distributions along with modified versions are available in the literature. However, Shanmugam (1985) is the pioneer to develop intervention-based models in distribution theory, due to its impressive applications in several areas of statistics and other applied sciences, very useful publications on intervention-based models of Huang and Fung (1989), Scollnik (1995), Dhanavanthan (1998, 2000), Shanmugam et al. (2002), Scollnik (2006), etc. are available in the statistical literature. Besides these few attempts observed in this direction, reliability theory is yet to explore with the idea.

The most popular and frequently used distribution among all continuous distributions is considered Exponential distribution having a cluster of applications in reliability theory, survival analysis, engineering sciences, economics, physics, business statistics etc. In history, researchers have developed, different types of modified and generalized models of the Exponential distribution. The explanatory summary on Exponential distribution is given by Balakrishnan (2019). However, the recent developments in distribution theory, in the form of discrete intervened probability models motivated us to explore the continuous lifetime intervened model. In this connection, we have attempted to study the new extension of Exponential distribution namely intervened Exponential distribution developed by Shanmugam et al. (2002), and have also given a nice explanation of the intervention parameter (ρ) used in the model. We hope this move

would enlighten distribution theory with new advantageous lifetime probability models such as intervened mixture probability models, generalized intervened probability models, and the intervention-based probability models developed by using different transformations or by adding new parameters are some possible directions. The distribution function (cdf) of intervened Exponential distribution (I_vED) along with probability density function (pdf) are given as

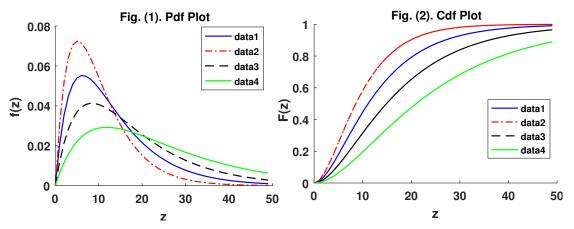
$$F_{I_vED}(z;\xi) = \begin{cases} 1 - \frac{\rho e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta}}{(\rho-1)} & \rho \neq 1. \\ 1 - \left(1 + \frac{z-\alpha}{\beta}\right) e^{-(z-\alpha)/\beta} & \rho = 1. \end{cases}$$
 (1)

and,

$$f_{I_vED}(z;\xi) = \begin{cases} \frac{e^{-(z-\alpha)/\beta\beta} - e^{-(z-\alpha)/\beta}}{(\rho-1)\beta} & \rho \neq 1. \\ \frac{(z-\alpha)}{\beta^2} e^{-(z-\alpha)/\beta} & \rho = 1. \end{cases}$$
 (2)

where $z>\alpha>0$, and the parametric space $\xi=\{(\rho,\alpha,\beta): \rho>0,\alpha>0,\beta>0\}$, where ρ is the intervention parameter, α is the truncation and β is the rate parameter of the distribution. Also, some derived results mentioned in the article of Shanmugam et al. (2002) are noticed as, the mean (μ_z) , $\mu_z=\alpha+(\rho+1)\beta$, variance (σ_z^2) , $\sigma_z^2=(\rho^2+1)\beta^2$ and the mode (M_z) , $M_z=\alpha+(\rho\beta/(\rho-1))\ln|\rho|$ respectively.

The graphical illustration of pdf given in equation (2) is shown in Fig.1, where as for cdf given in equation (1) it is



shown in Fig.2. The set of parametric values for both the plots are taken as, data $1 = (\rho = 2.018, \alpha = 0.010, \beta = 4.500)$, data $2 = (\rho = 1.009, \alpha = 0.050, \beta = 5.055)$, data $3 = (\rho = 2.001, \alpha = 0.020, \beta = 6.050)$, and data $4 = (\rho = 2.018, \alpha = 0.010, \beta = 8.500)$. It is quite visible from the graphical plots the pdf is exhibiting different shapes while changing the parametric values.

2. Inferential Study of I_vED

In this section, some interesting characteristics about I_vED are discussed in detail, these include mean deviations, central and non-central r^{th} order moments, different generating functions for moments, etc. are some of the useful measures having considerable importance in distribution theory. Firstly we begin the section by the median (M_d) derivation. Let z be a continuous and absolutely non-negative random variable (r.v.) possessing $I_vED(\rho,\alpha,\beta)$, then we proceed the steps for derivation as,

$$\begin{split} & \int_{M_d}^{\infty} f_{I_v ED}(z;\xi) dz = 1/2 \\ \Rightarrow & \frac{1}{(\rho-1)\beta} \int_{M_d}^{\infty} \left\{ e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta} \right\} dz = 1/2 \end{split}$$

After solving the integral and simplifications, the final expression obtained is as,

$$\rho e^{-(M_d - \alpha)/\rho\beta} - e^{-(M_d - \alpha)/\beta} = \frac{(\rho - 1)}{2}$$
(3)

This derived expression for M_d does not reveal the explicit solution, but we can find the approximate value for M_d in R-software by using the *uniroot* function.

2.1. Mean Deviations

In statistics, the two prominent deviations are the mean deviation (D_{μ_z}) and median deviation (D_{M_d}) , which is the average of all the deviations taken from the mean and the respective median. These deviations have the advantage to measure the amount of scatteredness present in the data. Mathematically, the expressions are given by

$$D_{\mu_z} = \int_{-\infty}^{\infty} |z - \mu_z| f(z) dz$$
 and $D_{M_d} = \int_{-\infty}^{\infty} |z - M_d| f(z) dz$

Theorem 2.1. If a continuous and absolutely non-negative $r.v.\ z \sim I_v ED(\rho, \alpha, \beta)$, then the mean deviations about mean (μ_z) and median (M_d) are given by

(i)
$$D_{\mu_z} = 2\left\{ (\mu_z - \alpha) - \beta(\rho + 1) + \frac{\beta}{(\rho - 1)} \left[\rho^2 e^{-(\mu_z - \alpha)/\rho\beta} - e^{-(\mu_z - \alpha)/\beta} \right] \right\}$$
 (4)

(ii)
$$D_{M_d} = (\mu_z - M_d) + 2\left\{ (M_d - \alpha) - \beta(\rho + 1) + \frac{\beta}{(\rho - 1)} \left[\rho^2 e^{-(M_d - \alpha)/\rho\beta} - e^{-(M_d - \alpha)/\beta} \right] \right\}$$
 (5)

Proof. (i) We have a non-negative r.v., $z \sim I_v ED(\rho, \alpha, \beta)$. Then the expression for D_{μ_z} is given by,

$$\begin{split} D_{\mu_z} &= 2 \left\{ \mu_z F_{I_v ED}(\mu_z) - \int_{\alpha}^{\mu_z} z f_{I_v ED}(z; \xi) dz \right\} \\ &= 2 \left\{ \mu_z F_{I_v ED}(\mu_z) - \int_{\alpha}^{\mu_z} z \frac{\left[e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta} \right]}{(\rho - 1)\beta} dz \right\} \\ &= 2 \left\{ (\mu_z - \alpha) - \beta(\rho + 1) + \frac{\beta}{(\rho - 1)} \left[\rho^2 e^{-(\mu_z - \alpha)/\rho\beta} - e^{-(\mu_z - \alpha)/\beta} \right] \right\} \end{split}$$

Hence, completes proof for part first.

(ii) Again, for a continuous, and non-negative r.v., $z \sim I_v ED(\rho, \alpha, \beta)$, we can write the mathematical expression for median deviation as,

$$D_{M_d} = \mu_z - M_d + 2 \left\{ M_d F_{I_v ED}(M_d) - \int_{\alpha}^{M_d} z f_{I_v ED}(z; \xi) dz \right\}$$

$$= (\mu_z - M_d) + 2 \left\{ M_d F_{I_v ED}(M_d) - \int_{\alpha}^{M_d} z \frac{\left[e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta} \right]}{(\rho - 1)\beta} dz \right\}$$

$$= (\mu_z - M_d) + 2 \left\{ (M_d - \alpha) - \beta(\rho + 1) + \frac{\beta}{(\rho - 1)} \left[\rho^2 e^{-(M_d - \alpha)/\rho\beta} - e^{-(M_d - \alpha)/\beta} \right] \right\}$$

This completes proof for part (ii).

2.2. Moments and Generating Functions

In this subsection, we derive the expressions for generating functions, r^{th} order central and non-central moments for $I_v ED(\rho, \alpha, \beta)$, in the following subsequent theorems.

Theorem 2.2. If a continuous r.v., $z \sim I_v ED(\rho, \alpha, \beta)$, then the r^{th} central moment μ_r , and non-central moment μ_r' are given by

(i)
$$\mu_r = \frac{1}{(\rho - 1)} \sum_{n=0}^r {^rC_n(-\mu)^{r-n}} \beta^n \left\{ \rho^{n+1} e^{\alpha/\rho\beta} \Gamma\left(n + 1, \frac{\alpha}{\beta\rho}\right) - e^{\alpha/\beta} \Gamma\left(n + 1, \frac{\alpha}{\beta}\right) \right\}.$$

$$(ii) \quad \mu_{r}^{'} = \frac{\beta^{r+1}}{(\rho-1)} \left[\frac{e^{\alpha/\beta\rho}\rho^{r+1}}{\beta} \Gamma\left(r+1,\frac{\alpha}{\beta\rho}\right) - e^{\alpha/\beta} \Gamma\left(r+1,\frac{\alpha}{\beta}\right) \right].$$

where, r=1,2,...,n and $\Gamma(s,x)=\int_x^\infty t^{s-1}e^{-t}dt$ is the upper incomplete gamma function.

Proof. (i). The expression for moments about mean for a r.v., $z \sim I_v ED(\rho, \alpha, \beta)$ is given by

$$\mu_r = \int_{-\infty}^{\infty} (z - \mu_z)^r f_{I_v ED}(z; \xi) dz$$

$$= \frac{1}{(\rho - 1)\beta} \int_{\alpha}^{\infty} (z - \mu_z)^r \left\{ e^{-(z - \alpha)/\rho\beta} - e^{-(z - \alpha)/\beta} \right\} dz.$$

$$= \frac{1}{(\rho - 1)\beta} \sum_{n=0}^r {^rC_n} (-\mu)^{r-n} \int_{\alpha}^{\infty} z^n \left\{ e^{-(z - \alpha)/\rho\beta} - e^{-(z - \alpha)/\beta} \right\} dz.$$

$$= \frac{1}{(\rho - 1)} \sum_{n=0}^r {^rC_n} (-\mu)^{r-n} \beta^n \left\{ \rho^{n+1} e^{\alpha/\rho\beta} \Gamma\left(n + 1, \frac{\alpha}{\beta\rho}\right) - e^{\alpha/\beta} \Gamma\left(n + 1, \frac{\alpha}{\beta}\right) \right\}$$

Hence, completes proof for part (i).

Proof. (ii) Since, we know the mathematical expression for non-central moments, for a $r.v. z \sim I_v ED(\rho, \alpha, \beta)$, is defined by

$$\begin{array}{lcl} \mu_{r}^{'} & = & \displaystyle \int_{-\infty}^{\infty} z^{r} f_{I_{v}ED}(z,\xi) dz \\ \\ & = & \displaystyle \frac{e^{\alpha/\rho\beta}}{(\rho-1)\beta} \int_{\alpha}^{\infty} z^{r} e^{-z/\rho\beta} dz - \frac{e^{\alpha/\beta}}{(\rho-1)\beta} \int_{\alpha}^{\infty} z^{r} e^{-z/\beta} dz \\ \\ & = & \displaystyle \frac{\beta^{r+1}}{(\rho-1)} \left[\frac{e^{\alpha/\beta\rho} \rho^{r+1}}{\beta} \Gamma\left(r+1,\frac{\alpha}{\beta\rho}\right) - e^{\alpha/\beta} \Gamma\left(r+1,\frac{\alpha}{\beta}\right) \right]; \quad r=1,2,...,n. \end{array}$$

Hence, proved the result.

Theorem 2.3. If $z \sim I_v ED(\rho, \alpha, \beta)$, then, we defined the expressions for, moment generating function $M_z(t)$, characteristic function $\phi_z(t)$, and cumulant generating function $K_z(t)$ as

(i)
$$M_z(t) = \frac{e^{\alpha t}}{(1 - \beta t)(1 - \beta \rho t)}$$
 (ii) $\phi_z(t) = \frac{e^{\alpha \iota t}}{(1 - \iota \beta t)(1 - \iota \beta \rho t)}$ (iii) $K_z(t) = \alpha t - \log(1 - \beta t) - \log(1 - \beta \rho t)$

Proof. (i) We know that, if $z \sim I_v ED(\rho, \alpha, \beta)$, then mathematical expression for moment generating function is given by

$$M_{z}(t) = \int_{\alpha}^{\infty} e^{zt} f_{I_{v}ED}(z,\xi) dz = \int_{\alpha}^{\infty} e^{zt} \frac{e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta}}{(\rho-1)\beta} dz$$
$$= \frac{\rho\beta e^{\alpha/\rho\beta}}{\beta(\rho-1)(1-\rho\beta t)} - \frac{\beta e^{\alpha/\beta}}{\beta(\rho-1)(1-\beta t)}$$
$$= e^{\alpha t} [(1-\beta t)(1-\beta\rho t)]^{-1}$$

Hence, proved the result (i).

(ii) To prove part (ii) of the above theorem, similar steps have to be repeated while deriving the expression, but the only difference is that instead of t, we have to proceed with ιt to obtain the resulting equation for characteristic function, i.e

$$\phi_z(t) = e^{\alpha \iota t} [(1 - \iota \beta t)(1 - \iota \beta \rho t)]^{-1}.$$

(iii) Since, we know that, if $z \sim I_v ED(\rho, \alpha, \beta)$, then the cumulant generating function is mathematically defined by:

$$K_z(t) = \log \{M_z(t)\}$$

= $\alpha t - \log(1 - \beta t) - \log(1 - \beta \rho t)$

Hence, proved.

3. Reliability Characterization

Reliability or Survival function is defined as the failure-free operation of a system, during a particular interval of time. For, many decades ago, this function is meant intrinsic features in performance activity measurements of the system, by considering failure data. Mathematically, the reliability $R_{I_vED}(z;\xi)$, for a non-negative, and absolutely continuous $r.v., z \sim I_vED(\rho, \alpha, \beta)$ with $pdf f_{I_vED}(z;\xi)$, is defined as

$$R_{I_n ED}(z; \xi) = Pr(Z > z) = 1 - Pr(Z \le z)$$

$$\Rightarrow R_{I_vED}(z;\xi) = \begin{cases} \frac{\rho e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta}}{(\rho-1)} & \rho \neq 1.\\ \left(1 + \frac{z-\alpha}{\beta}\right) e^{-(z-\alpha)/\beta} & \rho = 1. \end{cases}$$
 (6)

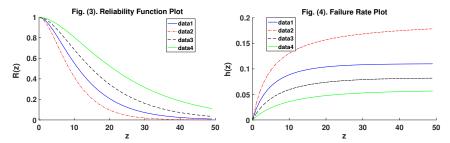
By invariance property of MLE_s , we can write the reliability estimate as given by

$$R_{I_vED}(z;\xi) = \begin{cases} \frac{\hat{\rho}e^{-(z-\hat{\alpha})/\hat{\rho}\hat{\beta}} - e^{-(z-\hat{\alpha})/\hat{\beta}}}{(\hat{\rho}-1)} & \rho \neq 1.\\ \left(1 + \frac{z-\hat{\alpha}}{\hat{\beta}}\right)e^{-(z-\hat{\alpha})/\hat{\beta}} & \rho = 1. \end{cases}$$
(7)

The opposite of the reliability function is hazard rate function, often frequently named failure rate function, maybe constant or the function of the time. In literature, this function is widely acknowledged with alternative sub-titles in demographic and actuarial sciences. Among them, intensity rate, force of mortality, instantaneous force of mortality, and the mortality rate are sharply noticed. For a non-negative continuous r.v., $z \sim I_v ED(\rho, \alpha, \beta)$ with pdf $f_{I_vED}(z;\xi)$ and cdf $F_{I_vED}(z;\xi)$, hazard rate function $h_{I_vED}(z;\xi)$ is as

$$h_{I_vED}(z;\xi) = \frac{e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta}}{\beta \left\{ \rho e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta} \right\}}$$
(8)

The graphical plots of the reliability and hazard rate functions are shown in Fig.(3) and Fig.(4) respectively,



For both the plots the values of parameters are taken as, data $1 = (\rho = 2.018, \alpha = 0.010, \beta = 4.500)$, data $2 = (\rho = 1.009, \beta = 4.500)$

 $\alpha = 0.050$, $\beta = 5.055$), data3 = ($\rho = 2.001$, $\alpha = 0.020$, $\beta = 6.050$), and data4 = ($\rho = 2.018$, $\alpha = 0.010$, $\beta = 8.500$). The graphical illustration of hazard rate function as shown in Fig.(4) is clearly showing the increasing trend, which has useful application while analyzing the failure data in reliability.

In reliability theory, the well-known extended concept reverse hazard rate function is described as a hazard rate, but in a reverse direction of time. The ratio of $pdf(f_{I_vED}(z;\xi))$ upon $cdf(F_{I_vED}(z;\xi))$ is defined as a reverse hazard rate function. For I_vED it is obtained as

$$\lambda_{I_vED}^R(z;\xi) = \frac{e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta}}{\beta \left\{ (\rho - 1) - \left[\rho e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta} \right] \right\}}$$
(9)

Statistical literature revealed, the approximate constant, increasing or decreasing trends of unimodal hazard rate function. Therefore, for the system, aging representation is tedious. Thus, a new reliability function, named aging intensity (A.I) profounded by Jiang et al. (2003) would have the quantitative measurement for aging. Thus A.I for a r.v., $z \sim I_v ED(\rho, \alpha, \beta)$ is denoted by $L_{IvED}(z; \xi)$ and is defined as

$$L_{I_v ED}(z;\xi) = \frac{h_{I_v ED}(z,\xi)}{X_{I_v ED}(z,\xi)} = \frac{-z f_{I_v ED}(z;\xi)}{R_{I_v ED}(z;\xi) \ln |R_{I_v ED}(z;\xi)|}; \quad z > 0.$$

Thus, for $I_v ED$ our derived result for A.I are as

$$L_{I_vED}(z;\xi) = \frac{z \left[e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta} \right]}{\beta \left[\rho e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta} \right] \left[\ln |\rho - 1| - \ln |\rho e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta} \right]}$$
(10)

where, $f_{I_vED}(.)$ and $R_{I_vED}(.)$ are respective pdf and reliability function of model, and the average failure rate function $X_{I_vED}(z;\xi) = (1/z) \int_0^z h(z;\xi) dz$.

3.1. Shape of Hazard Rate Function

The actual shape behavior of the hazard rate function given in equation (8) is not clear from the graphical plot as shown in Fig.(4). To provide the solid mathematical interpretation regarding the shape of the hazard rate function, the following lemma due to Glaser (1980) is defined as:

Lemma 3.1. Suppose for a continuous r.v. x, the density function f(x) > 0, for all x > 0, f'(x) be its derivative, and $\eta(x) = -f'(x)/f(x)$. Thus, if $\eta'(x) > 0$, for all x > 0. The hazard rate function corresponding to f(x) is increasing function of x.

Proof. For complete proof refer to Glaser (1980).

Theorem 3.2. The hazard rate function of $I_v ED(\alpha, \beta, \rho)$ is increasing, when $\rho \neq 1$ and $(\alpha, \beta, \rho) > 0$.

Proof. The pdf $f(z;\xi)$ of $I_vED(\alpha,\beta,\rho)$, when $\rho \neq 1$, is defined in equation (2). Therefor according to above Lemma 3.1., the function $\eta(z;\xi)$ of a continuous non-negative r.v. z is defined as

$$\eta(z;\xi) = -\frac{f^{'}(z;\xi)}{f(z;\xi)} = \frac{e^{-(z-\alpha)/\rho\beta} - \rho e^{-(z-\alpha)/\beta}}{\rho\beta \left\{e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta}\right\}}$$

After differentiating $\eta(z;\xi)$ with respect to z and on simplifying we get

$$\eta'(z;\xi) = \frac{(\rho - 1)^2}{(\rho \beta)^2} \frac{e^{-(z - \alpha)(\rho + 1)/\rho \beta}}{\left(e^{-(z - \alpha)/\rho \beta} - e^{-(z - \alpha)/\beta}\right)^2} > 0.$$

This is true for $(\alpha, \beta, \rho) > 0$, when $\rho \neq 1$. Thus, we have $\eta'(z; \xi) > 0$, for all z > 0, for given $pdf \ f(z; \xi)$. Therefore according to Lemma 3.1., the statement of Theorem 3.2. is proved, the hazard rate function of $I_v ED(\alpha, \beta, \rho)$, when $\rho \neq 1$, is increasing for all z > 0.

3.2. Mean Residual Life Function

Specifically, to determine the distribution, an alternative approach Mean Residual Life (MRL) function does exist. Suppose a system working up to time $t \geq 0$, then the remaining working hours until it fails i.e beyound time t is termed as residual life, defined by conditional r.v. $Z - t|_{Z>t}$. Precisely, more attractive and interesting story about MRL function being that, this is advantageous in many branches of science, engineering science, economics, survival analysis, and reliability theory for characterizing the lifetime. Although, it is quite difficult to explore failure rate function without undertaking other measures. It is theoretically obvious, the most prominent among them being MRL function as they are complementary functions to each other Finkelstein (2008).

Thus for a r.v. $z \sim I_v ED(\rho, \alpha, \beta)$, the MRL function $(m_{I_v ED}(z; \xi))$ is obtained as,

$$\begin{split} m_{I_v ED}(z;\xi) &= E\left[Z - t|_{Z>t}\right] = \frac{1}{R_{I_v ED}(z;\xi)} \int_z^\infty R_{I_v ED}(z;\xi) dz \\ &= \left[\rho e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta}\right]^{-1} \int_z^\infty \left[\rho e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta}\right] dz \\ &= \beta \left[\rho^2 e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta}\right] \left[\rho e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta}\right]^{-1} \end{split}$$

which is required derived MRL function for I_vED .

4. Mean Inactivity Time

Mean Inactivity Time (MIT) have strong applications in different branches of applied sciences, such as, reliability analysis, survival analysis and economics are some areas. If $z \sim I_v ED(\rho, \alpha, \beta)$, then the mean inactivity time is given by

$$\begin{split} M_{IT}(z;\xi) &= z - \frac{1}{F_{I_v ED}(z;\xi)} \int_{\alpha}^{z} z f_{I_v ED}(z;\xi) dz, \quad z > \alpha. \\ &= z - \frac{1}{(\rho - 1)\beta F_{I_v ED}(z)} \int_{\alpha}^{z} z \left\{ e^{-(z - \alpha)/\rho\beta} - e^{-(z - \alpha)/\beta} \right\} dz. \\ &= z - \frac{1}{(\rho - 1)\beta F_{I_v ED}(z)} \left\{ e^{\alpha/\rho\beta} \int_{\alpha}^{z} z e^{-z/\rho\beta} dz - e^{\alpha/\beta} \int_{\alpha}^{z} z e^{-z/\beta} dz \right\} \end{split}$$

After integration, and simplifications we get the result as

$$\Rightarrow M_{IT}(z;\xi) = \frac{\left\{ (\rho - 1) \left[(z - \alpha) + (\rho + 1)\beta \right] - \beta \left[\rho^2 e^{-(z - \alpha)/\rho\beta} - e^{-(z - \alpha)/\beta} \right] \right\}}{\left\{ (\rho - 1) - \left[\rho e^{-(z - \alpha)/\rho\beta} - e^{-(z - \alpha)/\beta} \right] \right\}}$$
(11)

5. Strong Mean Inactivity Time

This newly developed reliability measure strong mean inactivity time (SMIT) function by Kayid and Izadkhah (2014) is useful in reliability and survival analysis. Let, $z \sim I_v ED(\rho, \alpha, \beta)$, then we derive SMIT function as

$$M_{IT}^{S}(z;\xi) = \frac{1}{F_{I_{v}ED}(z;\xi)} \int_{\alpha}^{z} 2z F_{I_{v}ED}(z;\xi) dz$$

$$= z^{2} - \frac{1}{F_{I_{v}ED}(z;\xi)} \int_{\alpha}^{z} z^{2} f_{I_{v}ED}(z;\xi) dz$$

$$= z^{2} - \frac{1}{F_{I_{v}ED}(z;\xi)} \int_{\alpha}^{z} z^{2} \frac{e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta}}{(\rho-1)\beta} dz$$

$$= z^{2} - \frac{1}{\beta(\rho-1)F(z)} \sum_{r=0}^{\infty} \frac{(-1)^{r} \left[e^{\alpha/\beta\rho} - \rho^{r} e^{\alpha/\beta} \right] \left[z^{r+3} - \alpha^{r+3} \right]}{r!(r+3)\beta^{r}\rho^{r}}$$
(12)

6. Entropy Measures

In this section, we discuss two popular entropy metrics namely Rényi entropy (see Rényi 1961) and Shannon entropy (see Shannon 1948), which are used to measure the amount of information or uncertanity. Here, we derive Rényi and Shannon entropy mathematically.

6.1. Rényi Entropy

Let a r.v. $z \sim I_v ED(\rho, \alpha, \beta)$, then the Rényi entropy of order κ is given by

$$H_{R}(\kappa) = \frac{1}{(1-\kappa)} \log \left\{ \int_{\alpha}^{\infty} (f_{I_{v}ED}(z;\xi))^{\kappa} dz \right\}; \quad \kappa \geq 0, \kappa \neq 1$$

$$= \frac{1}{(1-\kappa)} \log \left\{ \int_{\alpha}^{\infty} \left(\frac{e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta}}{(\rho-1)\beta} \right)^{\kappa} dz \right\}$$

$$= \frac{1}{(1-\kappa)} \log \left\{ \frac{\rho}{(\rho-1)^{2}} \int_{0}^{1} z^{\kappa} \left[1-z \right]^{\frac{\rho-\kappa\rho\beta-1}{\rho-1}} dz \right\}$$

$$= \frac{1}{(1-\kappa)} \log \left| \frac{\rho}{(\rho-1)^{2}} B\left(2 - \frac{\beta\rho\kappa}{(\rho-1)}, \kappa + 1 \right) \right|$$

which is required expression of Rényi entropy of order κ .

Note: $B(a,b) = \int_0^1 z^{a-1} (1-z)^{b-1} dz$ is the beta function.

6.2. Shannon Entropy

Shannon entropy is the extended result of Rényi entropy, thus for a continuous $r.v.\ z \sim I_v ED(\rho, \alpha, \beta)$ this measure is obtained as

$$H_{I_v ED}(z) = -\int_0^\infty f_{I_v ED}(z, \xi) \log (f_{I_v ED}(z, \xi)) dz.$$

After substituting the equation (2) of pdf, the simplified result we obtained for Shannon measure of entropy are given by

$$H_{I_vED}(z) = \rho \sum_{r=1}^{\infty} \left\{ r \left[\rho \beta + (r+1)(\rho-1) \right] \left[\beta \rho + r(\rho-1) \right] \right\}^{-1} + \frac{2\beta \rho + \rho - 1}{\beta (\beta \rho + \rho - 1)^2}$$

7. Order Statistics of I_vED

Order statistics are counted among important branches of statistics, having numerous applications in reliability analysis and life testing modeling used to study the system reliability characteristics. Let $z=(z_1,z_2,z_3,...,z_n)$, be a random sample from $I_vED(\rho,\alpha,\beta)$ with cdf and its pdf defined in equations (1) and (2) respectively. Let us consider the ordered random sample as $z_{(1:n)} \leq z_{(2:n)} \leq z_{(3:n)} \leq ... \leq z_{(n:n)}$, so that $z_{(i:n)}$; [i=1,2,...,n] having the life time of [n-i+1] out-of-n system consisting n i.i.d components. The pdf of i^{th} order statistic $z_{(i:n)}$; $1 \leq i \leq n$, is given by:

$$f_{i:n}(z) = K_1 \left[F_{I_v ED}(z;\xi) \right]^{i-1} \left[1 - F_{I_v ED}(z;\xi) \right]^{n-i} f_{I_v ED}(z;\xi). \tag{13}$$

The joint (i, j)th order density function of $(z_{(i:n)}, z_{(j:n)})$ for $1 \le i \le j \le n$ is given below:

$$f_{i:j:n}(z_i, z_j) = K_2 \left[F_{I_v ED}(z_i) \right]^{i-1} \left[F_{I_v ED}(z_j) - F_{I_v ED}(z_i) \right]^{j-i-1} \left[1 - F_{I_v ED}(z_j) \right]^{n-j} f_{I_v ED}(z_i) f_{I_v ED}(z_j). \tag{14}$$

where,

$$K_1 = \frac{n!}{(i-1)!(n-i)!}$$
, and $K_2 = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}$.

Thus, for an ordered random sample of n observations, 1^{st} order statistic is the smallest observation given by $z_{(1)} = min.\{z_{(1)}, z_{(2)}, z_{(3)}, ..., z_{(n)}\}$, while the n^{th} order statistic is the largest sample observation given by $z_{(n)} = max.\{z_{(1)}, z_{(2)}, z_{(3)}, ..., z_{(n)}\}$, and the median order statistics is the middle observation given by z_{m+1} .

7.1. Density Function of 1^{st} , n^{th} and Median Order Statistic

Let $z_1, z_2, z_3, ..., z_n$ be *i.i.d* r.v. from I_vED . Then, the density functions for first $(f_{1:n}(.))$, last $(f_{n:n}(.))$ and the median order $(f_{m+1:n}(.))$; $[m=\frac{n}{2}]$ are given below:

$$f_{1:n}(z) = n \left[1 - F_{I_v ED}(z; \xi) \right]^{n-1} f_{I_v ED}(z_{(1)}; \xi).$$

$$f_{1:n}(z) = \frac{n}{\beta(\rho - 1)^2} \left[\rho e^{-(z_{(1)} - \alpha)/\rho\beta} - e^{-(z_{(1)} - \alpha)/\beta} \right]^{n-1} \left[e^{-(z_{(1)} - \alpha)/\rho\beta} - e^{-(z_{(1)} - \alpha)/\beta} \right]. \tag{15}$$

Similarly,

$$f_{n:n}(z) = n \left[F_{I_v ED}(z; \xi) \right]^{n-1} f_{I_v ED}(z_{(n)}; \xi).$$

$$f_{n:n}(z) = \frac{n}{\beta(\rho - 1)^2} \left[(\rho - 1) - \left\{ \rho e^{-(z_{(n)} - \alpha)/\rho\beta} - e^{-(z_{(n)} - \alpha)/\beta} \right\} \right]^{n-1} \left[e^{-(z_{(n)} - \alpha)/\rho\beta} - e^{-(z_{(1)} - \alpha)/\beta} \right]. \quad (16)$$

and,

$$f_{m+1:n}(z) = \frac{(2m+1!)}{(m!)^2} \left[F_{I_v ED}(\bar{z}, \xi) \right]^m \left[1 - F_{I_v ED}(\bar{z}, \xi) \right]^m f_{I_v ED}(\bar{z}, \xi). \tag{17}$$

7.2. Joint Order Density

The joint $(i, j)^{th}$ order statistic pdf of I_vED is given by:

$$f_{i:j:n}(z_{(i)}, z_{(j)}) = \frac{K_2}{\beta^2 (\rho - 1)^{n - i + 2}} \left[(\rho - 1) - \left\{ \rho e^{-(z_{(i)} - \alpha)/\rho\beta} - e^{-(z_{(i)} - \alpha)/\beta} \right\} \right]^{i - 1}$$

$$\cdot \left[\left\{ \rho e^{-(z_{(i)} - \alpha)/\rho\beta} - e^{-(z_{(i)} - \alpha)/\beta} \right\} - \left\{ \rho e^{-(z_{(j)} - \alpha)/\rho\beta} - e^{-(z_{(j)} - \alpha)/\beta} \right\} \right]^{j - i - 1}$$

$$\cdot \left[\rho e^{-(z_{(j)} - \alpha)/\rho\beta} - e^{-(z_{(j)} - \alpha)/\beta} \right]^{n - j} \left[e^{-(z_{(i)} - \alpha)/\rho\beta} - e^{-(z_{(i)} - \alpha)/\beta} \right]$$

$$\cdot \left[e^{-(z_{(j)} - \alpha)/\rho\beta} - e^{-(z_{(j)} - \alpha)/\beta} \right]$$

8. Stress-Strength Reliability

Stress-strength modeling, is another famous system reliability measurement technique, having a wide variety of applications in different areas of statistical sciences and engineering. Suppose a system, having specific strength Z_1 subjected to common stress Z_2 . Whenever stress exceeds the strength, failure will occur and functions smoothly when $Z_1 > Z_2$. Since, $R = Pr.(Z_1 > Z_2)$ is the desired model for system reliability under stress strength modeling. Now, for independent r.v.'s $Z_1 \sim I_vED(\alpha, \rho_1, \beta_1)$ and $Z_2 \sim I_vED(\alpha, \rho_2, \beta_2)$, having the same parameter (α) , with pdf of Z_1 and cdf of Z_2 , when $\rho_i \neq 1$, (i = 1, 2), the derivation for R is given below:

$$f_{Z_1}(z) = \frac{e^{-(z-\alpha)/\rho_1\beta_1} - e^{-(z-\alpha)/\beta_1}}{(\rho_1 - 1)\beta_1} \qquad \rho_1 \neq 1$$
(18)

and,

$$F_{Z_2}(z) = 1 - \frac{\{\rho_2 e^{-(z-\alpha)/\rho_2 \beta_2} - e^{-(z-\alpha)/\beta_2}\}}{(\rho_2 - 1)} \quad \rho_2 \neq 1$$
(19)

We have,

$$R = \int_{\alpha}^{\infty} \left\{ \int_{\alpha}^{z} f_{z_{2}}(z) dz \right\} f_{z_{1}}(z) dz = \int_{\alpha}^{\infty} F_{Z_{2}}(z) f_{Z_{1}}(z) dz$$

$$= \int_{\alpha}^{\infty} \left\{ 1 - \frac{\{\rho_{2}e^{-(z-\alpha)/\rho_{2}\beta_{2}} - e^{-(z-\alpha)/\beta_{2}}\}}{(\rho_{2}-1)\beta_{2}} \right\} \left\{ \frac{e^{-(z-\alpha)/\rho_{1}\beta_{1}} - e^{-(z-\alpha)/\beta_{1}}}{(\rho_{1}-1)\beta_{1}} \right\} dz$$

$$= 1 - \beta_{2}^{2} \left\{ \frac{\rho_{1}\beta_{1}(1+\rho_{2}+\rho_{2}^{2}) + \rho_{2}\beta_{1}\beta_{2}(\rho_{1}+1)(\rho_{2}+1) + \rho_{2}^{2}\beta_{2}^{2}}{(\beta_{1}+\beta_{2})(\beta_{1}+\rho_{2}\beta_{2})(\beta_{2}+\rho_{1}\beta_{1})(\rho_{1}\beta_{1}+\rho_{2}\beta_{2})} \right\}$$

Now, let us consider $\rho_1 = 1$ and $\rho_2 \neq 1$. Then, in this situation, we have derived the expression for stress-strength reliability, which is given by

$$R = 1 - \frac{\beta_2^2}{(\rho_2 - 1)} \left\{ \frac{1}{(\beta_1 + \beta_2)^2} + \frac{\rho_2^3}{(\beta_1 + \rho_2 \beta_2)} \right\}$$

The next possibility is if $\rho_1 \neq 1$ but $\rho_2 = 1$. Then, in this case, the obtained stress-strength reliability is given by

$$R = 1 - \frac{\beta_2^2}{(\beta_1 + \beta_2)(\rho_1 \beta_1 + \beta_2)} - \frac{\beta_1 \beta_2^2 \{(\rho_1 + 1)\beta_2 + 2\rho_1 \beta_1\}}{(\beta_1 + \beta_2)^2 (\rho_1 \beta_1 + \beta_2)}$$

The last possibility is when both ρ_1 and ρ_2 are equal to 1. Then, in this case, we have derived the stress-strength reliability given by

$$R = 1 - \frac{\beta_2^2}{(\beta_1 + \beta_2)^2} - \frac{2\beta_2}{(\beta_1 + \beta_2)^3}$$

9. Stochastic Ordering

To judge the performances of lifetime distributions stochastic ordering is considered to be the critical and essential technique briefly discussed by Shaked and Shanthikumar (2007). Let Z_1 and Z_2 , be two r.v.'s each following I_vED having $cdf's\ F_1(z)$ and $F_2(z)$, with respective $pdf's\ f_1(z)$ and $f_2(z)$ respectively. Then, we say Z_1 is smaller than Z_2 , according to the below-mentioned orderings:

- [a_1] Stochastic order $(Z_1 \leq_{st} Z_2)$, if $F_1(z) \geq F_2(z)$ for all z.
- [a_2] Hazard rate order $(Z_1 \leq_{hr} Z_2)$, if $H_{Z_1}(z) \geq H_{Z_2}(z)$ for all z.
- $[a_3]$ Mean residual life order $(Z_1 \leq_{MRL} Z_2)$, if $m_{Z_1}(z) \geq m_{Z_2}(z)$ for all z.
- [a_4] Likelihood ratio order ($Z_1 \leq_{LR} Z_2$), if $\frac{f_1(z)}{f_2(z)}$ decreasing in z.

Hence, the above stochastic ordering reveal the following implications.

$$Z_1 \leq_{LR} Z_2 \Rightarrow Z_1 \leq_{hr} Z_2 \Rightarrow Z_1 \leq_{MRL} Z_2$$
 and $Z_1 \leq_{hr} Z_2 \Rightarrow Z_1 \leq_{st} Z_2$

The following, theorem defines likelihood ratio ordering of I_vED with respect to their strongest likelihood.

Theorem 9.1. Let $Z_1 \sim I_v ED(\alpha, \rho_1, \beta_1)$ and $Z_2 \sim I_v ED(\alpha, \rho_2, \beta_2)$. If $(\rho_1, \beta_1) > (\rho_2, \beta_2)$, then $Z_1 \leq_{st} Z_2$.

Proof. The likelihood ratio is given by

$$\phi = \frac{f_{Z_1}(z)}{f_{Z_2}(z)} = \frac{\beta_1(\rho_1 - 1)}{\beta_2(\rho_2 - 1)} \frac{\left\{ e^{-(z-\alpha)/\rho_1\beta_1} - e^{-(z-\alpha)/\beta_1} \right\}}{\left\{ e^{-(z-\alpha)/\rho_2\beta_2} - e^{-(z-\alpha)/\beta_2} \right\}}$$

$$\Rightarrow \phi' = (C/K) \left\{ (\rho_1\beta_1) \left(e^{-(z-\alpha)/\rho_1\beta_1} - e^{-(z-\alpha)/\beta_1} \right) \left(\beta_2 e^{-(z-\alpha)/\rho_2\beta_2} - \rho_2\beta_2 e^{-(z-\alpha)/\beta_2} \right) - (\rho_2\beta_2) \left(e^{-(z-\alpha)/\rho_2\beta_2} - e^{-(z-\alpha)/\beta_2} \right) \left(\beta_1 e^{-(z-\alpha)/\rho_1\beta_1} - \rho_1\beta_1 e^{-(z-\alpha)/\beta_1} \right) \right\}$$

where, $C = \{(\rho_2 - 1)\}\{\beta_1^2 \rho_1 \rho_2 (\rho_1 - 1)\}^{-1}$ and $K = (e^{-(z-\alpha)/\rho_2\beta_2} - e^{-(z-\alpha)/\beta_2})^2$. Hence it is clear form ϕ' , if $(\rho_1, \beta_1) > (\rho_2, \beta_2) \Rightarrow \phi' > 0$. Therefore, $(Z_1 \leq_{LR} Z_2)$. By above relationship other statements also hold.

10. Estimation Procedure for Parameters

Let $z=(z_1,z_2,z_3,...,z_n)$ be a sample of random observations drawn form I_vED with some desired parameters ρ , α and β . Let us consider $\omega=(\rho,\alpha,\beta)^T$ be a $k\times 1$ vector of parameters. Then the complete sample log likelihood function when $\rho\neq 1$ is defined below:

$$\log L = \sum_{i=1}^{n} \log \left\{ e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta} \right\} - n \log(\rho - 1) - n \log \beta$$
 (20)

Let us consider, $G_1 = e^{-(z-\alpha)/\rho\beta}$ and $G_2 = e^{-(z-\alpha)/\beta}$. Therefore the above equation can be written as,

$$\log L = \sum_{i=1}^{n} \log \{G_1 - G_2\} - n \log(\rho - 1) - n \log \beta$$
 (21)

Now differentiate above equation partially with respect to given parameters ρ , α and β we get.

$$\frac{\partial \log L}{\partial \rho} = \sum_{i=1}^{n} \frac{(z_i - \alpha)G_1}{\beta \rho^2 \left[G_1 - G_2\right]} - \frac{n}{(\rho - 1)}$$
(22)

$$\frac{\partial \log L}{\partial \alpha} = \sum_{i=1}^{n} \frac{G_1 - \rho G_2}{\beta \rho \left[G_1 - G_2 \right]} - 0 - 0 \tag{23}$$

$$\frac{\partial \log L}{\partial \beta} = \sum_{i=1}^{n} \frac{(z_i - \alpha) \left[G_1 - \rho G_2 \right]}{\beta \rho^2 \left[G_1 - G_2 \right]} - \frac{n}{\beta}$$
(24)

On equating the above equations to zero $i, e \frac{\partial \log L}{\partial \rho} = 0$, $\frac{\partial \log L}{\partial \alpha} = 0$ and $\frac{\partial \log L}{\partial \beta} = 0$, the MLE_s of parameters (ρ, α, β) are obtained say $(\hat{\rho}, \hat{\alpha}, \hat{\beta})$. Since the equations (22), (23) and (24) are not in closed form, so alternative approach such as Newton-Raphson technique can be employed to obtain the MLE_s . However log-likelihood equation may be maximized more conveniently with the help of R-software by using the *optim* or *nlm* functions. For all the three parameters of $I_vED(\rho, \alpha, \beta)$, the second order partial derivatives of log-likelihood function exists.

For all the three parameters of $I_v ED(\rho, \alpha, \beta)$, the second order partial derivatives of log-likelihood function exists Hence the obtained Inversion dispersion matrix is given below:

$$\begin{pmatrix} \hat{\rho} \\ \hat{\alpha} \\ \hat{\beta} \end{pmatrix} \sim N \begin{pmatrix} \begin{bmatrix} \rho \\ \alpha \\ \beta \end{bmatrix}, \begin{bmatrix} \hat{Z}_{\rho\rho} & \hat{Z}_{\rho\alpha} & \hat{Z}_{\rho\beta} \\ \hat{Z}_{\alpha\rho} & \hat{Z}_{\alpha\alpha} & \hat{Z}_{\alpha\beta} \\ \hat{Z}_{\beta\rho} & \hat{Z}_{\beta\alpha} & \hat{Z}_{\beta\beta} \end{bmatrix} \end{pmatrix}$$
(25)

Therefore,

$$I^{-1} = -E \begin{bmatrix} Z_{\rho\rho} & Z_{\rho\alpha} & Z_{\rho\beta} \\ Z_{\alpha\rho} & Z_{\alpha\alpha} & Z_{\alpha\beta} \\ Z_{\beta\rho} & Z_{\beta\alpha} & Z_{\beta\beta} \end{bmatrix}$$

where
$$Z_{\rho\rho}=\frac{\partial^2 \log L}{\partial^2 \rho}, Z_{\alpha\alpha}=\frac{\partial^2 \log L}{\partial^2 \alpha}, Z_{\beta\beta}=\frac{\partial^2 \log L}{\partial^2 \beta}, Z_{\rho\alpha}=\frac{\partial^2 \log L}{\partial \rho \partial \alpha}, Z_{\alpha\beta}=\frac{\partial^2 \log L}{\partial \alpha \partial \beta}$$
 and $Z_{\beta\rho}=\frac{\partial^2 \log L}{\partial \beta \partial \rho}$.

After finding the Inverse dispersion matrix, we can obtain easily asymptotic variances and co-variances for $MLE_s(\rho, \alpha, \beta)$. Now, to determine the approximate confidence interval of $100(1 - \delta)\%$ by using (25) as given below:

$$\hat{\rho} \pm y_{\frac{\delta}{2}} \sqrt{\hat{Z_{\rho\rho}}}, \hat{\alpha} \pm y_{\frac{\delta}{2}} \sqrt{\hat{Z_{\alpha\alpha}}} \text{ and } \hat{\beta} \pm y_{\frac{\delta}{2}} \sqrt{\hat{Z_{\beta\beta}}} \; .$$

where y_{δ} is defined as upper $100\delta th$ quantile of standard normal distribution.

10.1. Simulation Study Performance

For $I_v ED(\rho,\alpha,\beta)$, a Monte Carlo simulation study by repeating the process 1000 times has been carried out for different sample sizes, to check the performances of MLE_s . Since the quantile function of the proposed model is not in closed form. Thus inverse transformation method is not applicable to generate the data from IvED. However, in such a situation other data generation techniques namely, the acceptance-rejection algorithm is used to simulate the data. The results are shown in table 1.

(α, β, ρ)	n	Abias				MSE		
		\hat{lpha}	\hat{eta}	$\hat{ ho}$	\hat{lpha}	\hat{eta}	$\hat{ ho}$	
(0.02, 0.97, 1.21)	025	0.16432	0.32000	41.8401	1 0.02700	0.10240	1750.595	
	075	0.05933	0.18554	02.5004	7 0.00352	0.03442	06.25237	
	125	0.03919	0.12989	00.7705	0.00154	0.01687	00.59368	
	175	0.03062	0.10273	00.5909	5 0.00094	0.01055	00.34922	
	250	0.02590	0.08546	00.3666	9 0.00067	0.00730	00.13446	
	400	0.01624	0.05426	00.2141	4 0.00026	0.00294	00.04586	
(0.02, 1.27, 1.14)	025	0.21289	0.42164	39.4459	3 0.04532	0.17778	1555.982	
	075	0.07938	0.26857	03.8653	8 0.00630	0.07213	14.94114	
	125	0.05215	0.19681	00.9598	1 0.00272	0.03873	00.92124	
	175	0.03794	0.17211	00.6108	4 0.00144	0.02962	00.37313	
	250	0.03146	0.15040	00.4386	1 0.00099	0.02262	00.19238	
	400	0.02107	0.10770	00.2914	9 0.00044	0.01160	00.08497	
(1.02, 0.97, 1.21)	025	0.16654	0.28572	28.1959	2 0.02773	0.08164	795.0101	
	075	0.06501	0.17070	02.9590	7 0.00423	0.02914	08.75612	
	125	0.03997	0.12275	01.1499	7 0.00160	0.01507	01.32244	
	175	0.03377	0.11341	00.5941	1 0.00114	0.01286	00.35298	
	250	0.02215	0.07431	00.3247	3 0.00049	0.00552	00.10545	
	400	0.01646	0.06040	00.2467	5 0.00027	0.00365	00.06089	

Table 1. Simulated results of parameters for I_vED based on different sample sizes

We have obtained the estimates for all parameters with the help of R-software by using the nlm package, corresponding Abias and MSE for parameters are also calculated with different sample sizes (n=25, 75, 125, 175, 250, 400) respectively. The simulation study reveals that, the decrease of absolute Abias and MSE while increasing the sample size. Hence consistency property for I_vED are satisfied.

11. Applications

This section illustrates, model applicability based on two data sets. The comparison of I_vED with other pre-existing models available in the literature, such as Exponential distribution (ED) and generalized Exponential distribution (GED) has been performed. In Table 2, and Table 3, the $Comparision\ Criteria-I$ will provide the key information regarding Akaike, Corrected Akaike, Bayesian and Hannon-Quinn Information Criteria abbreviated as AIC, CAIC, BIC, and HQIC. Whereas the $Comparision\ Criteria-II$ will provide the information about Cramer-Von Mises (W), Anderson Darling (A), Kolmogorov-Smirnov (KS) statistic and p-value. Thus both the comparisons together will fulfill the model adequacy requirement.

The first data set given by Fuller Jr contains window strength data. By using this data set, the lifespan of the glass-airplane window was predicted by Fuller Jr et al. (1994). The collected data are: 18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.5, 25.52, 25.8, 26.96, 26.77, 26.78, 27.05, 27.67, 29.9, 31.11, 33.2, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.09, 39.58, 44.045, 45.29, 45.381.

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 $Comparison\ Criteria-I$ Models. $\hat{\alpha}$ β $\hat{\rho}$ AICCAICBICHQIC $I_v ED$ 18.2147 06.1514 01.0419 215.1429 216.0318 219,4448 216.5452 277.0139 ED30.8192 276.5464 276.6843 277.9804 GED00.0927 10.6835 225.7815 226.2101 228.6495 226.7164 $Comparison\ Criteria-II$ Models. $\hat{\alpha}$ Ĝ WKSA $p ext{-}value$ $I_v ED$ 18.2147 0.41225 06.1514 01.0419 0.06367 0.14016 0.53060 1.7×10^{-6} ED30.8192 0.07574 0.41521 0.45854 GED00.0927 10.6835 0.07391 0.39461 0.16570 0.32550

Table 2. The output results based on data set first is given below:

The data set second contains the survival lifetimes of forty-four patients, suffering from the Head and Neck cancer disease given by Efron (1988). The data is, 12.20, 23.56, 23.74, 25.87, 31.98, 37, 41.35, 47.38, 55.46, 58.36, 63.47, 68.46, 78.26, 74.47, 81.43, 84,92, 94, 110, 112, 119, 127, 130, 133, 140, 146, 155, 159, 173, 179, 194, 195, 209, 249, 281, 319, 339, 432, 469, 519, 633, 725, 817, 1776.

 $Comparison\ Criteria-I$ Models. $\hat{\alpha}$ β AICCAICBICHQIC13.4493 $I_v ED$ 04.2968 15.0007 567.1858 567.7858 527.5384 569.1708 591.2208 590.0983 ED116.593 589.4366 589.5319 GED0.008882.39893 586.1392 586.4319 589.7076 587.4625 $Comparison\ Criteria-II$ Models. \overline{W} \overline{KS} $\hat{\alpha}$ β Ap-value $I_v ED$ 04.2968 15.0007 13.4493 0.09111 0.52060 0.14707 0.26970 ED116.593 0.18146 1.05015 0.20163 0.04809 GED0.00888 2.39893 0.71889 0.15369 0.12254 0.22530

Table 3. The output results based on data set second is given below:

From the results shown in Table 2 and Table 3 it is clear that I_vED performs better than ED and GED respectively. Hence, this confirms the model applicability for future data analysis purposes.

12. Conclusion

In this article, we introduced intervened Exponential distribution (I_vED) as a lifetime model in reliability and survival analysis. The derived statistical and reliability properties has been presented in this study. We have plotted graphs of different functions, for pdf it exhibits different shapes, whereas the hazard rate function plot shows this could be useful to model the increasing failure rate data sets. Simulation study of I_vED based on MLE_s were also discussed to judge the performance of the parameters. Finally, we have done the comparison study with the pre-exists models by analyzing two real data sets, and this study revealed that the I_vED outperformed among the two well-known models, namely ED and GED available in statistical literature.

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