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Pareto-Weibull Distribution with Properties and Applications: A Member of Pareto-X Family

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Abstract

In this paper, we have proposed a new family of distributions namely the Pareto-X family of distributions. A sub model of the proposed family called Pareto-Weibull (PW) distribution is discussed. Some important properties of the proposed PW distribution are studied. Parameter estimation of the distribution by using the maximum likelihood method is discussed. The proposed distribution has been fitted on two real data sets about environmental and biological variable. The practical applications shows that the proposed model provides better fit as compared with the other models used in the study.

Key Words: Family of distributions; T-X family; Pareto-X family; Pareto-Weibull distribution; Reliability function; Order statistics; Maximum likelihood estimation.

Mathematical Subject Classification: 60E05, 62E15.

1. Introduction

Probability distributions have widespread applications in almost all areas of life including engineering, medicine, economics and more. In certain situations the standard probability distributions fail to capture underlying behavior of the data and hence extension,and/or generalization of the same is needed. Various families of distributions have been proposed from time to time to handle complex behavior of the data but the room is always there to propose some new families of distributions.

The Pareto distribution, named after a Swiss economist Vilfredo Pareto (1848-1923), is a very familiar probability model for modeling and prediction of various socioeconomic aspects. Many applications of the Pareto distribution in actuarial sciences, economics, finance, life testing, climatology, biology and physics have been studied in the literature. Although the distribution has many uses but one major and perhaps most important use of the distribution is in the studies related of distribution of income. Pareto (1897) initiated this concept in his intimate economic texts. Some applications of the distribution in modeling earthquakes, forest fire areas, oil and gas field sizes are studied by Burroughs and Tebbens (2001).

Several extensions and generalizations of the Pareto distribution have been proposed from time to time, These include the beta-Pareto distribution by Akinsete et al. (2008); Exponentiated Pareto distribution by Shawky and Abu-Zinadah (2009); beta generalized Pareto distribution by Mahmoudi (2011); Gamma-Pareto distribution and its application by Alzaatreh et al. (2012); Kumaraswamy-Pareto distribution by Bourguignon et al. (2013); Weibull-Pareto distribution and its applications by Alzaatreh et al. (2013a); Exponential-Pareto distribution by Al-Kadim and Boshi (2013); transmuted Pareto by Merovci and Puka (2014); A new Weibull-Pareto distribution by Tahir et al. (2016); Kumaraswamy transmuted Pareto distribution by Chhetri et al. (2017b); beta transmuted Pareto distribution by Chhetri et al. (2017a); Pareto-exponential distribution by Waseem and Bashir (2019) and cubic transmuted Pareto distribution by Rahman et al. (2020).

An interesting method to obtain a new probability distributions has been proposed by Alzaatreh et al. (2013b). The method is based upon a "transformer" and a "transformed" distribution and is known as the *T-X* family of distributions. This method is illustrated below.

Let X be a random variable with density function $g(x)$ and distribution function $G(x)$. Also, let T be a continuous random variable with density function $r(t)$ with support on $[a, b]$. The cumulative distribution function (cdf) of the $T - X$ family of distributions is then given as

$$F_{T-X}(x) = \int_a^{W(G(x))} r(t)\, \mathrm{d}t, \tag{1}$$

where $W\left(G(x)\right)$ satisfies following conditions

$$\left.\begin{array}{l} W\left(G(x)\right) \in [a, b], \\ W\left(G(x)\right) \text{ is differentiable and monotonically non-decreasing,} \\ W\left(G(x)\right) \to a \text{ as } x \to -\infty \text{ and } W\left(G(x)\right) \to b \text{ as } x \to \infty. \end{array}\right\}$$

The distribution function $F_{T-X}(x)$ in (1) is a composite function of $(R \cdot W \cdot G)(x)$ and can be written as

$$F_{T-X}(x) = R\left\{W\left(G(x)\right)\right\},$$

where $R(t)$ is the distribution function of random variable T. The corresponding density function is

$$f_{T-X}(x) = \left\{\frac{d}{dx} W\left(G(x)\right)\right\} r\left\{W\left(G(x)\right)\right\}. \tag{2}$$

The density function $r(t)$ in (1) is "transformed" into a new distribution function $F_{T-X}(x)$ through the function $W(G(x))$, which acts as a "transformer". The density function $f_{T-X}(x)$ in (2) is transformed from random variable T through the transformer random variable X and is called "Transformed-Transformer" or "T-X" distribution.

The main focus of this paper is to introduce a new family of distributions called the Pareto-X family of distributions by assuming that the random variable T in (1) has the Pareto distribution.

The organization of the paper follows: The Pareto-X family of distributions is proposed in Section 2. In Section 3, a special sub model of the family is presented and is named as the Pareto-Weibull (*PW*) distribution. In Section 4, some distributional properties of the proposed PW distribution are derived. Distribution of various order statistics are presented in Section 5. Maximum likelihood estimation of the model parameters is discussed in Section 6. Section 7 contains some real data applications of the proposed PW distribution. Finally, some concluding remarks are given.

2. The Pareto-X Family of Distributions

Let T be a Pareto random variable with density function $r(t)$ defined on the support on $[a, \infty)$, $a \geq 0$ and without loss of generality, we assume that $a = x_m$. Also let $W\left(G\left(x\right)\right) = x_m - log\left(1 - G\left(x\right)\right)$, where $x_m \in \mathbb{R}^+$, then the

Table 1: Special cases of the Pareto-X family of distributions

Distribution	Cumulative Distribution Function	Support
Pareto – Uniform	$1 - x_m^{\alpha}\left[x_m - \log(1-x)\right]^{-\alpha}$	$x \in (0,1)$
Pareto – Exponential	$1 - x_m^{\alpha}\left(x_m + \frac{x}{\lambda}\right)^{-\alpha}$	$x \in [0,\infty)$
Pareto – Rayleigh	$1 - x_m^{\alpha}\left(\frac{x^2}{2\sigma^2} + x_m\right)^{-\alpha}$	$x \in [0,\infty)$
Pareto – Lomax	$1 - x_m^{\alpha}\left[\eta(\log(\lambda+x) - \log(\lambda)) + x_m\right]^{-\alpha}$	$x \in [0,\infty)$
Pareto – Gompertz	$1 - x_m^{\alpha}\left(\eta e^{bx} - \eta + x_m\right)^{-\alpha}$	$x \in [0,\infty)$
Pareto – Kumaraswamy	$1 - x_m^{\alpha}\left[x_m - b\log\left(1-x^{a}\right)\right]^{-\alpha}$	$x \in [0,1]$
Pareto – Log-logistic	$1 - x_m^{\alpha}\left[-\beta\log(\eta) + \log\left(\eta^{\beta} + x^{\beta}\right) + x_m\right]^{-\alpha}$	$x \in [0,\infty)$
Pareto – Dagum	$1 - x_m^{\alpha}\left[x_m - \log\left(1 - \left(b^{a}x^{-a}+1\right)^{-p}\right)\right]^{-\alpha}$	$x \in \mathbb{R}^{+}$
Pareto – Burr XII	$1 - x_m^{\alpha}\left[k\log\left(x^{c}+1\right) + x_m\right]^{-\alpha}$	$x \in \mathbb{R}^{+}$
Pareto – Normal	$1 - x_m^{\alpha}\left[x_m - \log(1-\Phi(x))\right]^{-\alpha}$	$x \in \mathbb{R}$

distribution function of the proposed Pareto-X family of distributions is obtained from (1) as

$$F_{P-X}(x) = \int_{x_m}^{x_m - log(1-G(x))} r(t)\,dt = R\{x_m - log(1-G(x))\}. \tag{3}$$

The density function corresponding to (3) is

$$f_{P-X}(x) = \frac{g(x)}{1-G(x)} r\left[x_m - log\left(1-G(x)\right)\right] = h(x)\, r\left[x_m - log\left(1-G(x)\right)\right], \tag{4}$$

where $h(x)$ is the hazard function of X.

The Pareto distribution with parameter (x_m, α) has the cumulative distribution function $R(t) = 1 - \left[\frac{x_m}{t}\right]^{\alpha},\ t \geq x_m$ and density function $r(t) = \frac{\alpha x_m^{\alpha}}{t^{\alpha+1}},\ t \geq x_m$. Using these in (3), the distribution function of the Pareto-X family of distributions is

$$F_{P-X}(x) = 1 - \left[\frac{x_m}{x_m - \log\left(1-G(x)\right)}\right]^{\alpha};\ x \in \mathbb{R}, \tag{5}$$

where $x_m \in \mathbb{R}^{+}$ and $\alpha \in \mathbb{R}^{+}$ are the scale and shape parameters respectively. The density function of the Pareto-X family of distributions is easily obtained from (4) as

$$f_{P-X}(x) = \frac{g(x)}{1-G(x)} \frac{\alpha x_m^{\alpha}}{\left[x_m - \log\left(1-G(x)\right)\right]^{\alpha+1}};\ x \in \mathbb{R}.$$

Various members of the proposed Pareto-X family of distributions can be studied by using different base distributions $G(x)$ in (5). Some special cases of the proposed Pareto-X family of distributions are given in Table 1.

In the following we have obtained a specific member of the Pareto-X family of distributions by using the Weibull baseline distribution.

3. The Pareto-Weibull Distribution

The Pareto-Weibull (PW) distribution is obtained by using Weibull baseline distribution in (5). For this, suppose that the random variable X follows the Weibull distribution with *cdf*

$$G(x) = 1 - e^{-\left(\frac{x}{\lambda}\right)^k};\ x \in [0, \infty), \tag{6}$$

where $\lambda \in \mathbb{R}^+$ and $k \in \mathbb{R}^+$ are the scale and shape parameters respectively. Now, using (6) in (5) the distribution function of the PW distribution is

$$F(x) = 1 - x_m^{\alpha} \left(x_m + \lambda^{-k} x^k\right)^{-\alpha};\ x \in [0, \infty), \tag{7}$$

where $x_m, \lambda \in \mathbb{R}^+$ are the scale parameters and $\alpha, k \in \mathbb{R}^+$ are the shape parameters of the distribution. The density function of the PW distribution is readily obtained from (7) and is given in the definition below.

Definition: A continuous random variable X is said to have a Pareto-Weibull distribution if its probability density function is

$$f(x) = \alpha k x_m^{\alpha} \lambda^{-k} x^{k-1} \left(x_m + \lambda^{-k} x^k\right)^{-\alpha-1};\ x \in [0, \infty), \tag{8}$$

where $x_m, \lambda \in \mathbb{R}^+$ are the scale parameters and $\alpha, k \in \mathbb{R}^+$ are the shape parameters of the distribution.

Special Cases:

(i) The distribution function of the PW distribution given in (7) reduces to the distribution function of the Pareto-exponential distribution for $k = 1$.

(ii) A Pareto-exponential distribution developed by Waseem and Bashir (2019) is considered as a special case of (7) for $k = 1$ and $x_m = 1$.

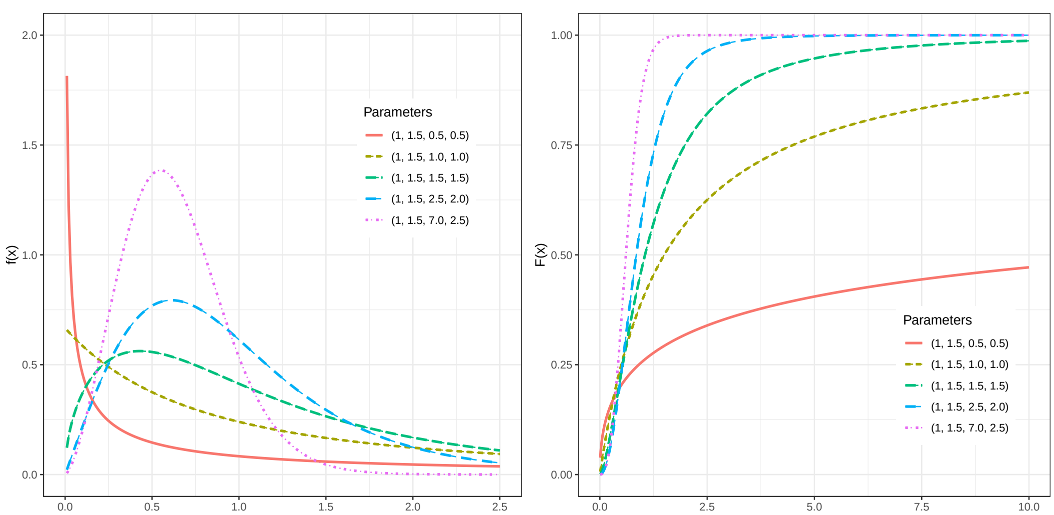

Figure 1: The density function $f(x; x_m, \lambda, \alpha, k)$ and distribution function $F(x; x_m, \lambda, \alpha, k)$ of the PW distribution.

Some of the possible shapes for the density and distribution functions of the proposed PW distribution are given in Figure 1. It has been observed from the Figure that the proposed distribution has the capability to capture different behaviours in datasets.

4. Distributional Properties

In the following we will discuss some important properties of the proposed PW distribution.

4.1. Moments

The moments of random variable are useful in studying various properties of its distribution. In the following we will give the moments of the proposed PW distribution.

Definition: Let X be a random variable having the PW distribution then its rth raw moment is given as

$$\mu_r' = \frac{\lambda^r x_m^{\frac{r}{k}}\, \Gamma\left(1+\frac{r}{k}\right)\Gamma\left(\alpha-\frac{r}{k}\right)}{\Gamma(\alpha)},\ \alpha k > r. \tag{9}$$

Mean can be obtained by setting $r=1$ in (9) and is

$$\text{Mean} = \frac{\lambda x_m^{\frac{1}{k}}\, \Gamma\left(1+\frac{1}{k}\right)\Gamma\left(\alpha-\frac{1}{k}\right)}{\Gamma(\alpha)},\ \alpha k > 1.$$

The variance of the distribution is obtained as

$$\begin{aligned}\text{Variance} &= E(X^2) - E(X)^2 \\ &= \frac{\lambda^2 x_m^{\frac{2}{k}}\left[\Gamma(\alpha)\Gamma\left(1+\frac{2}{k}\right)\Gamma\left(\alpha-\frac{2}{k}\right) - \Gamma\left(1+\frac{1}{k}\right)^2\Gamma\left(\alpha-\frac{1}{k}\right)^2\right]}{\Gamma(\alpha)^2},\ \alpha k > 2.\end{aligned}$$

The higher moments of the distribution can be obtained by using $r > 2$ in (9).

4.2. Moment Generating Function

The moment generating function is useful in obtaining moments of a random variable. The MGF for PW distribution is given in the following theorem.

Theorem 4.1. *Let random variable X follows the PW distribution then the moment generating function, $M_X(t)$ is*

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{\lambda^r x_m^{\frac{r}{k}}\, \Gamma\left(1+\frac{r}{k}\right)\Gamma\left(\alpha-\frac{r}{k}\right)}{\Gamma(\alpha)},\ \alpha k > r, \tag{10}$$

where $t \in \mathbb{R}$.

Proof. The moment generating function is defined as

$$M_x(t) = E\left(e^{\text{tx}}\right) = \int_0^\infty e^{\text{tx}} f(x)\, dx,$$

where $f(x)$ is given in (8). Using the series representation of e^{tx} given in Jeffrey and Zwillinger (2007), we have

$$M_x(t) = \int_0^\infty \sum_{r=0}^{\infty} \frac{t^r}{r!} x^r f(x) dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(x^r). \tag{11}$$

Using $E(X^r)$ from (9) in (11), we have (10). □

4.3. Characteristic Function

The characteristic function (CF) plays a vital role both in probability theory and applied statistics. The characteristic function completely describe a probability distribution and this always exist. The CF for PW distribution is given in the following theorem.

Theorem 4.2. *Let X follows the PW distribution then the characteristic function, $\phi_X(t)$ is*

$$\phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \frac{\lambda^r x_m^{\frac{r}{k}} \Gamma\left(1+\frac{r}{k}\right) \Gamma\left(\alpha-\frac{r}{k}\right)}{\Gamma(\alpha)}, \ \alpha k > r,$$

where $i = \sqrt{-1}$ is the imaginary unit and $t \in \mathbb{R}$.

Proof. The proof is simple. □

4.4. Reliability Analysis

The reliability function, used by Ebeling (2004), or survival function, used by Carpenter (1997), is simply the complement of the cumulative distribution function. The reliability function is useful in survival analysis and engineering. The reliability function for the PW distribution is

$$R(t) = 1 - F(t) = x_m^{\alpha} \left(x_m + \lambda^{-k} t^k\right)^{-\alpha}; \ t \in [0, \infty).$$

The hazard function is the ratio of the density function to the reliability function and for the PW distribution, it is given as

$$h(t) = \frac{\alpha k x_m^{\alpha} \lambda^{-k} t^{k-1} \left(x_m + \lambda^{-k} t^k\right)^{-\alpha-1}}{x_m^{\alpha} \left(x_m + \lambda^{-k} t^k\right)^{-\alpha}}; \ t \in [0, \infty),$$

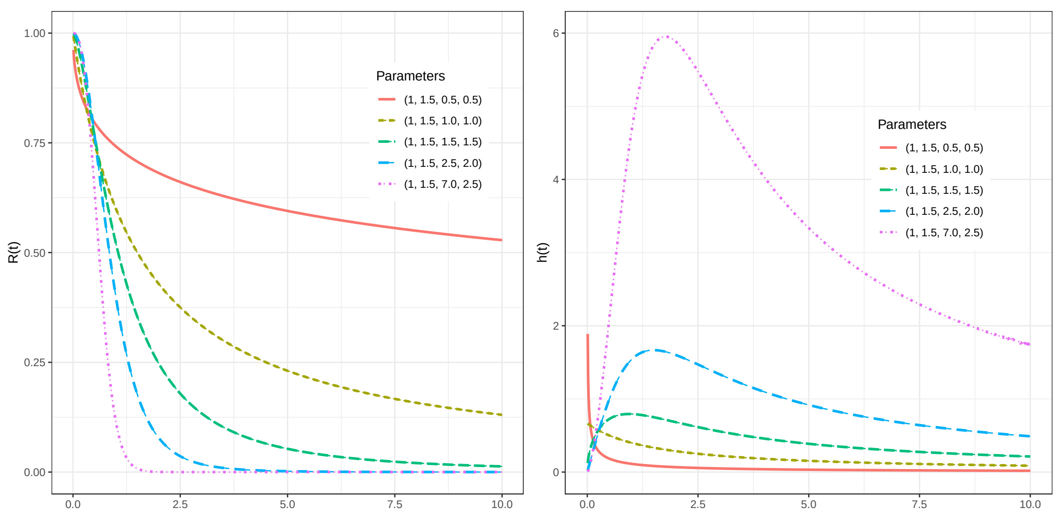

Figure 2: The reliability function $R(t; x_m, \lambda, \alpha, k)$ and hazard function $h(t; x_m, \lambda, \alpha, k)$ of the PW distribution.

The reliability and hazard rate functions of the PW distribution are given in Figure 2 above. We can see that the hazard rate function shows both increasing and decreasing behavior.

4.5. Quantile Function and Median

The quantile function, denoted by x_q, is useful in obtaining quantiles of a distribution. This function is also useful in generating the random sample from any distribution. The quantile function is obtained by solving $F(x) = q$ for x, see for example Rahman et al. (2020). The quantile function for the PW distribution is readily obtained by solving (7) for x and is

$$x_q = \left[x_m \lambda^k \left\{(1-q)^{-\frac{1}{\alpha}} - 1\right\}\right]^{\frac{1}{k}}. \quad (12)$$

The meadian is obtained by using $q = 0.5$ in above equation and is

$$\text{Median} = x_{0.5} = \left[x_m \lambda^k \left\{2^{\frac{1}{\alpha}} - 1\right\}\right]^{\frac{1}{k}}.$$

The lower and upper quartiles can also be obtained by using $q = 0.25, 0.75$ in (12) respectively.

4.6. Generating Random Sample

Random sample is often required in simulation studies. The random sample from PW distribution can be generated by using the following expression, see Rahman et al. (2020),

$$X = \left[x_m \lambda^k \left\{(1-u)^{-\frac{1}{\alpha}} - 1\right\}\right]^{\frac{1}{k}}, \tag{13}$$

where $u \sim U(0,1)$. One can generate random sample from PW distribution by using (13) for various values of the model parameters.

5. Order Statistics

Order statistics are widely used in many fields like economics, geology etc. The distribution of order statistics is given below.

Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote the order statistic of a random sample $X_1, X_2, \cdots, X_n$ from a continuous distribution $F_X(x)$. The density function of $X_{r:n}$ is given as

$$f_{X_{r:n}}(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r} f(x), \tag{14}$$

where $f_X(x)$ is density function corresponding to $F_X(x)$.

The density function of the rth order statistic for the PW distribution is obtained by using its density and distribution function in (14) and is

$$\begin{aligned} f_{X_{r:n}}(x) &= \frac{n!}{(r-1)!(n-r)!}\left[1 - x_m^{\alpha}\left(x_m + \lambda^{-k}x^k\right)^{-\alpha}\right]^{r-1} \\ &\times \left[x_m^{\alpha}\left(x_m + \lambda^{-k}x^k\right)^{-\alpha}\right]^{n-r} \alpha k x_m^{\alpha} \lambda^{-k} x^{k-1}\left(x_m + \lambda^{-k}x^k\right)^{-\alpha-1}, \end{aligned} \tag{15}$$

where $r = 1, 2, \cdots, n$. Using $r = 1$ in (15), the density function of the smallest order statistic $X_{1:n}$ for PW distribution is

$$f_{X_{1:n}}(x) = n\alpha k x_m^{\alpha} \lambda^{-k} x^{k-1}\left[x_m^{\alpha}\left(x_m + \lambda^{-k}x^k\right)^{-\alpha}\right]^{n-1}\left(x_m + \lambda^{-k}x^k\right)^{-\alpha-1}.$$

Again, using $r = n$ in (15), the density function of the largest order statistic $X_{n:n}$, is

$$f_{X_{n:n}}(x) = n\alpha k x_m^{\alpha} \lambda^{-k} x^{k-1}\left[1 - x_m^{\alpha}\left(x_m + \lambda^{-k}x^k\right)^{-\alpha}\right]^{n-1}\left(x_m + \lambda^{-k}x^k\right)^{-\alpha-1}.$$

6. Estimation and Inference

The parameter estimation is an essential step in fitting distribution to some real data. The maximum likelihood method of estimation is, perhaps, the most popular method to estimate the parameters of a distribution. In this section, we have discussed the maximum likelihood estimation for parameters of the PW distribution. For this, suppose $x_1, x_2, \cdots, x_n$ is a random sample of size n from the PW distribution. The likelihood function is

$$\mathcal{L}(x) = \alpha^n k^n x_m^{n\alpha} \lambda^{-nk} \cdot \left(\prod_{i=1}^{n} x_i^{k-1}\right) \cdot \left[\prod_{i=1}^{n}\left(x_m + \lambda^{-k}x_i^k\right)^{-\alpha-1}\right].$$

The log-likelihood function is

$$\ell(x) = n \cdot \ln(\alpha) + n \cdot \ln(k) + n\alpha \cdot \ln(x_m) - nk \cdot \ln(\lambda) + (k-1)\sum_{i=1}^{n} \ln(x_i) - (\alpha+1)\sum_{i=1}^{n} \ln\left[x_m + \left(\frac{x_i}{\lambda}\right)^k\right]. \quad (16)$$

The maximum likelihood estimator of x_m is the first-order statistic $x_{(1)}$. The derivatives of (16) with respect to λ, α and k are

$$\frac{\delta\ell}{\delta\lambda} = (\alpha+1)\sum_{i=1}^{n} \frac{kx_i\left(\frac{x_i}{\lambda}\right)^{k-1}}{\lambda^2\left[\left(\frac{x_i}{\lambda}\right)^k + x_m\right]} - \frac{nk}{\lambda},$$

$$\frac{\delta\ell}{\delta\alpha} = n\left[\frac{1}{\alpha} + \ln(x_m)\right] - \sum_{i=1}^{n} \ln\left[\left(\frac{x_i}{\lambda}\right)^k + x_m\right],$$

and

$$\frac{\delta\ell}{\delta k} = \sum_{i=1}^{n} \ln(x_i) + \frac{n}{k} - n\ln(\lambda) - \ln(\lambda) - (\alpha+1)\sum_{i=1}^{n} \frac{\left(\frac{x_i}{\lambda}\right)^k \ln\left(\frac{x_i}{\lambda}\right)}{\left(\frac{x_i}{\lambda}\right)^k + x_m}.$$

Now setting, $\frac{\partial l}{\partial\lambda} = 0$, $\frac{\partial l}{\partial\alpha} = 0$ and $\frac{\partial l}{\partial k} = 0$ and solving the resulting nonlinear system of equations the maximum likelihood estimate $\hat{\Theta} = \left(\hat{\lambda}, \hat{\alpha}, \hat{k}\right)'$ of $\Theta = (\lambda, \alpha, k)'$ can be obtained. Also, as $n \to \infty$, the asymptotic distribution of the $MLEs$ $\left(\hat{\lambda}, \hat{\alpha}, \hat{k}\right)$ are given by, see for example Aryal and Tsokos (2011),

$$\begin{pmatrix} \hat{\lambda} \\ \hat{\alpha} \\ \hat{k} \end{pmatrix} \sim N_3\left[\begin{pmatrix} \lambda \\ \alpha \\ k \end{pmatrix}, \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} \\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} \\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} \end{pmatrix}\right],$$

where $\hat{V}_{ij} = V_{ij}|_{\Theta=\hat{\Theta}}$. The asymptotic variance-covariance matrix V, of the estimates $\hat{\lambda}$, $\hat{\alpha}$ and $\hat{k}$ is obtained by inverting Hessian matrix; see Appendix. An approximate $100(1-\alpha)\%$ two sided confidence intervals for λ, α and k are given by:

$$\hat{\lambda} \pm Z_{\alpha/2}\sqrt{\hat{V}_{11}}, \quad \hat{\alpha} \pm Z_{\alpha/2}\sqrt{\hat{V}_{22}} \quad \text{and} \quad \hat{k} \pm Z_{\alpha/2}\sqrt{\hat{V}_{33}}$$

where Z_α is the αth percentile of the standard normal distribution.

7. Real-life Applications

In this section we have given two real data applications of the PW distribution.

7.1. Floyd River Data

We considered this dataset for the Floyd River, located in James, Iowa, USA, which provides the consecutive annual flood discharge rates for the year $1935 - 1973$. The dataset has been previously used by Akinsete et al. (2008). For the source and details of the data, see Mudholkar and Hutson (1996). The dataset is: 1460, 4050, 3570, 2060, 1300, 1390, 1720, 6280, 1360, 7440, 5320, 1400, 3240, 2710, 4520, 4840, 8320, 13900, 71500, 6250, 2260, 318, 1330, 970, 1920, 15100, 2870, 20600, 3810, 726, 7500, 7170, 2000, 829, 17300, 4740, 13400, 2940, and 5660.

Table 2: Summary statistics for the selected datasets.

Dataset	Min.	Q_1	Median	Mean	Q_3	Max.
Floyd River	318	1590	3570	6771	6725	71500
Bladder Cancer	0.080	3.348	6.395	9.366	11.838	79.050

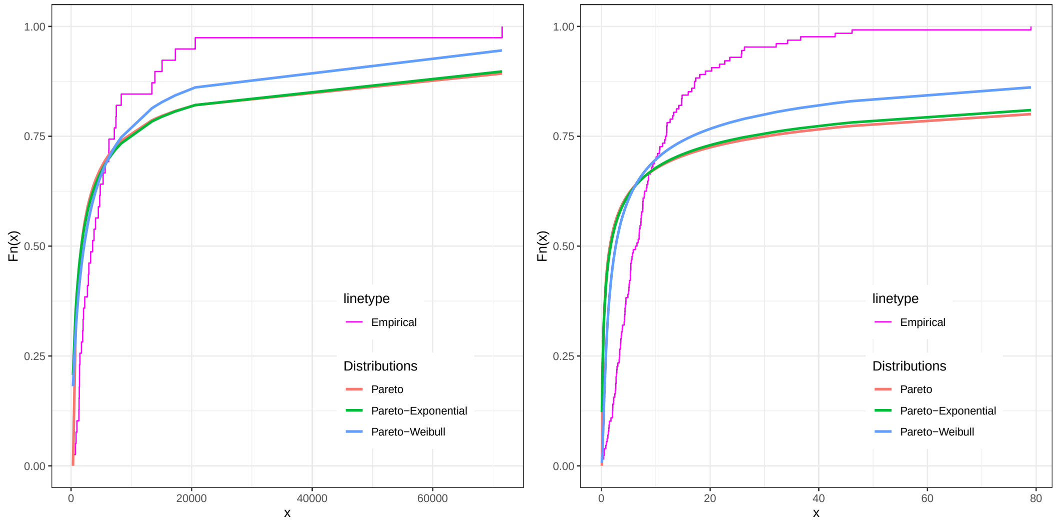

Figure 3: Estimated distribution function plotted over empirical distribution function for the selected datasets.

Table 2 represents the summary statistics for the dataset. We have considered Pareto-exponential distribution proposed by Waseem and Bashir (2019) and Pareto distribution developed by Pareto (1897) to examine the performance of the proposed PW distribution.

Table 3: MLE of parameters and respective SE for the selected models.

Distribution	Parameter	Estimate	SE
Pareto-Weibull	x_m	318	–
	λ	1.500	0.375
	α	1.083	0.563
	k	0.778	0.117
Pareto-Exponential	x_m	318	–
	λ	1.500	0.217
	α	0.454	0.077
Pareto	x_m	318	–
	α	0.412	0.066

Table 4: Selection criteria estimated for the selected models.

Distribution	logLike	AIC	AICc	BIC
Pareto-Weibull	-377.763	761.526	762.212	766.517
Pareto-Exponential	-380.745	765.491	765.824	768.818
Pareto	-392.810	787.620	787.728	789.284

Table 3 contains the estimated values of the model parameters alongside the standard errors. The estimated distribution function of the PW distribution is plotted alongside the empirical distribution, for the Floyd river data, in the left panel of Figure 3 below. This figure also contains the fitted distribution function of the competing models. Various model selection criteria like Log-likelihood, Akaike's information criterion (AIC), corrected Akaike's information criterion (AICc), Bayesian information criterion (BIC) are shown in Table 4. The results of this table shows that the PW distribution is the best fit for this data.

7.2. Bladder Cancer Data

The dataset consist of a set of remission times collected from cancer patients in a bladder cancer study, see Lee and Wang (2003). The dataset is: 4.50, 19.13, 14.24, 7.87, 5.49, 2.02, 9.22, 3.82, 26.31, 2.62, 0.90, 21.73, 0.51, 3.36, 43.01, 0.81, 3.36, 1.46, 17.14, 15.96, 7.28, 4.33, 22.69, 2.46, 3.48, 4.23, 6.54, 8.65, 5.41, 2.23, 4.34, 32.15, 4.87, 5.71, 7.59, 3.02, 4.51, 1.05, 9.47, 79.05, 2.02, 4.26, 11.25, 10.34, 10.66, 12.03, 2.64, 14.76, 1.19, 8.66, 14.83, 5.62, 18.10, 25.74, 17.36, 1.35, 9.02, 6.94, 7.26, 3.70, 3.64, 3.57, 11.64, 6.25, 25.82, 3.88, 20.28, 46.12, 5.17, 0.20, 36.66, 10.06, 4.98, 5.06, 16.62, 12.07, 6.97, 0.08, 1.40, 2.75, 7.32, 1.26, 6.76, 7.62, 3.52, 9.74, 0.40, 5.41, 2.54, 2.69, 8.26, 0.50, 5.32, 5.09, 2.09, 7.93, 12.02, 13.80, 5.85, 7.09, 5.32, 2.83, 8.37, 14.77, 8.53, 11.98, 1.76, 4.40, 34.26, 2.07, 17.12, 12.63, 7.66, 4.18, 13.29, 23.63, 3.25, 7.63, 2.87, 3.31, 2.26, 2.69, 11.79, 5.34, 6.93, 10.75, 13.11, and 7.39.

Table 5: MLE of parameters and respective SE for the selected models.

Distribution	Parameter	Estimate	SE
Pareto-Weibull	x_m	0.080	–
	λ	1.500	0.017
	α	0.189	0.079
	k	2.000	0.173
Pareto-Exponential	x_m	0.080	–
	λ	1.500	0.122
	α	0.255	0.023
Pareto	x_m	0.080	–
	α	0.234	0.021

Table 6: Selection criteria estimated for the selected models.

Distribution	logLike	AIC	AICc	BIC
Pareto-Weibull	-382.951	771.903	772.096	780.459
Pareto-Exponential	-480.947	965.894	965.990	971.599
Pareto	-538.523	1079.046	1079.078	1081.898

The summary statistics of the data are presented in Table 2. The estimated values of the model parameters alongside the standard errors are given in Table 5. The estimated distribution function of the PW distribution alongside the empirical distribution function are given in the right panel of Figure 3. Table 6 presents the computed values for different selection criterion. The results of the selection criterion indicate that the PW distribution is most suitable fit for this data.

8. Concluding Remarks

In this paper, a new Pareto-X family of distributions has been introduced. A four-parameters sub-model of the proposed family called the Pareto-Weibull distribution is studied in detail. The distributional properties of the proposed PW distribution including moments, moment generating function, characteristics function, quantile function, random number generation, reliability functions and the distribution of order statistics are discussed. The maximum likelihood estimation of the parameters is done. Finally, two applications of the proposed PW distribution are given by using real data sets. We have found that the proposed PW distribution is a suitable model for modeling of the data sets.

Appendix

The Hessian matrix for the Pareto-Weibull distribution is given as

$$H = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix},$$

where the variance-covariance matrix V is obtained by

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix}^{-1},$$

with the elements of Hessian matrix are obtained as

$$H_{11} = -\frac{\delta^2\ell}{\delta\lambda^2} = (\alpha+1)\sum_{i=1}^{n}\left[-\frac{k^2x_i^2\left(\frac{x_i}{\lambda}\right)^{2k-2}}{\lambda^4\left(\left(\frac{x_i}{\lambda}\right)^k + x_m\right)^2} + \frac{(k-1)kx_i^2\left(\frac{x_i}{\lambda}\right)^{k-2}}{\lambda^4\left(\left(\frac{x_i}{\lambda}\right)^k + x_m\right)} + \frac{2kx_i\left(\frac{x_i}{\lambda}\right)^{k-1}}{\lambda^3\left(\left(\frac{x_i}{\lambda}\right)^k + x_m\right)}\right] - \frac{nk}{\lambda^2},$$

$$H_{12} = -\frac{\delta^2\ell}{\delta\lambda\,\delta\alpha} = -\sum_{i=1}^{n}\frac{kx_i\left(\frac{x_i}{\lambda}\right)^{k-1}}{\lambda^2\left(\left(\frac{x_i}{\lambda}\right)^k + x_m\right)},$$

$$H_{13} = -\frac{\delta^2\ell}{\delta\lambda\,\delta k} = (\alpha+1)\sum_{i=1}^{n}\left[-\frac{x_i\left(\frac{x_i}{\lambda}\right)^{k-1}}{\lambda^2\left(\left(\frac{x_i}{\lambda}\right)^k + x_m\right)} - \frac{kx_i\left(\frac{x_i}{\lambda}\right)^{k-1}\ln\left(\frac{x_i}{\lambda}\right)}{\lambda^2\left(\left(\frac{x_i}{\lambda}\right)^k + x_m\right)} + \frac{kx_i\left(\frac{x_i}{\lambda}\right)^{2k-1}\ln\left(\frac{x_i}{\lambda}\right)}{\lambda^2\left(\left(\frac{x_i}{\lambda}\right)^k + x_m\right)^2}\right] + \frac{n}{\lambda},$$

$$H_{22} = -\frac{\delta^2\ell}{\delta\alpha^2} = \frac{n}{\alpha^2},$$

$$H_{23} = -\frac{\delta^2\ell}{\delta\alpha\,\delta k} = \sum_{i=1}^{n}\frac{\left(\frac{x_i}{\lambda}\right)^k\ln\left(\frac{x_i}{\lambda}\right)}{\left(\frac{x_i}{\lambda}\right)^k + x_m},$$

and

$$H_{33} = -\frac{\delta^2\ell}{\delta k^2} = (\alpha+1)\sum_{i=1}^{n}\left[\frac{\left(\frac{x_i}{\lambda}\right)^k\ln^2\left(\frac{x_i}{\lambda}\right)}{\left(\frac{x_i}{\lambda}\right)^k + x_m} - \frac{\left(\frac{x_i}{\lambda}\right)^{2k}\ln^2\left(\frac{x_i}{\lambda}\right)}{\left(\left(\frac{x_i}{\lambda}\right)^k + x_m\right)^2}\right] + \frac{n}{k^2}.$$

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