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Kumaraswamy Esscher Transformed Laplace Distribution: Properties, Application and Extensions

Dais George[1]*, Rimsha H[2]

*Corresponding author

1. Department of Statistics, Catholicate College, Pathanamthitta, Kerala, India, daiscatholicate@gmail.com
2. Department of Statistics, Devaswom Board Pampa College, Parumala, Kerala, India, rimshahabeeb@gmail.com

Abstract

In this article, we introduce a new generalized family of Esscher transformed Laplace distribution, namely the Kumaraswamy Esscher transformed Laplace distribution. We study the various properties of the distribution including the survival function, hazard rate function, cumulative hazard rate function and reverse hazard rate function. The parameters of the distribution are estimated using the maximum likelihood method of estimation. We coduct a simulation study to establish the performance of the estimators by means of bias and MSE. A real application of this distribution on breaking stress of carbon fibres is also considered. Further, we introduce and study the exponentiated and transmuted exponentiated Kumaraswamy Esscher transformed Laplace distributions.

Key Words: Esscher transformation, Exponentiated Kumaraswamy Esscher transformed Laplace distribution, Hazard rate function, Kumaraswamy Esscher transformed Laplace distribution, Transmuted exponentiated Kumaraswamy Esscher transformed Laplace distribution.

Mathematical Subject Classification: 60E05, 62E10, 62E15, 62P30.

1. Introduction

Now a days, we can see an increased interest in developing generalized families of distributions by introducing one or more additional shape parameter(s) to the base line distribution. The basic motivation behind these works is to make the distribution more flexible in modeling various data sets arise from real life situation. Moreover, in generalized families, the problems related to computing incomplete beta and gamma functions involved can be easily tackled by the researchers through the analytical and computational facilities available in Statistical softwares like MATLAB, MAPLE, MATHEMATICA and R packages. Tahir and Nadarajah (2015) provides a detailed account of various techniques suitable to generate new families of continuous distributions (univariate) through the introduction of additional parameters. Two among them are the Kumaraswamy Marshal-Olkin family [Alizadeh et al. (2015)] and Marshal-Olkin Kumaraswamy-G family [Handique and Chakraborty (2015a,2015b).

In this paper, we introduce a new generalized family of distributions namely, Kumaraswamy Esscher transformed Laplace distribution which is the Kumaraswamy generalization of one parameter Esscher transformed Laplace distribution introduced by Sebastian and Dais (2012) through a concept, namely Esscher transformation, introduced by Esscher (1932). Esscher transformed Laplace distribution being asymmetric and heavy-tailed, is a possible alternative to the distributions with Pareto tails and also to various types of asymmetric Laplace distributions given in Kozubowski and Podgorski (2000).

The Esscher transformed Laplace distribution [ETL(θ)], with parameter θ is a tilted version of the symmetric Laplace distribution with pdf

$$f(x) = \frac{1}{2}\exp(-|x|), \quad -\infty < x < \infty \ .$$

The probability density function and distribution function of ETL(θ) is

$$f(x,\theta) = \frac{(1-\theta^2)}{2}\begin{cases} \exp[x(1+\theta)], & x<0 \\ \exp[-x(1-\theta)], & x \geq 0, \end{cases} \tag{1}$$

and

$$F(x) = \begin{cases} \frac{(1-\theta)}{2}\exp[x(1+\theta)], & x<0 \\ \frac{1-\theta}{2} + \frac{1+\theta}{2}(1-\exp[-x(1-\theta)]), & x \geq 0, \end{cases} \tag{2}$$

respectively, where $\theta \in (-1,1)$.
Graphs of the pdf of ETL(θ) for various values of θ are given in Figure 1.

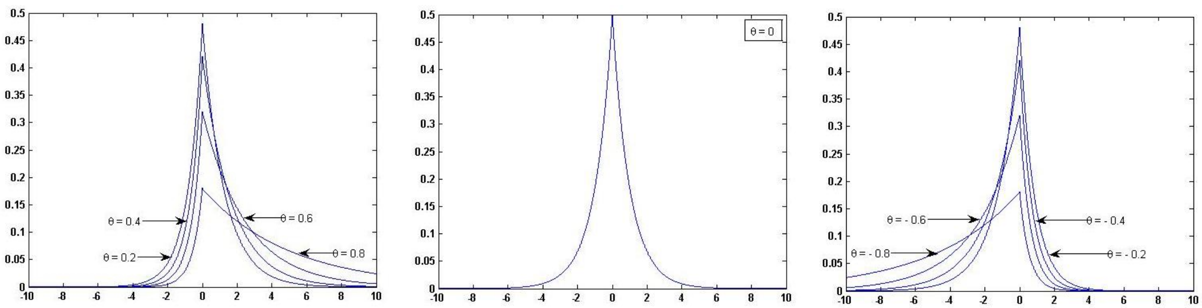

Figure 1: Densities of Esscher transformed Laplace distribution for (a) $\theta \in (0,1)$, (b) $\theta = 0$ (classical Laplace) and (c) $\theta \in (-1,0)$

The distribution is unimodal, positively skewed and leptokurtic. Again ETL(θ) distribution is infinitely divisible, geometric infinite divisible and self-decomposable. It can be considered as the mixtures of normal distributions, convolution of Exponential distributions and the log ratio of two independent random variables with Pareto type I distributions with density $f(x) = \frac{1}{x^2},\ x \geq 1$, (for details, see Sebastian and Dais (2012)). The application of this distribution in Web server data, Marshall-Olkin extension with its application in financial data, application in time series, pathway generalization, multivariate extension and discretization are also considered respectively by Dais and Sebastian (2011, 2013), Dais et al. (2016) Rimsha and Dais (2019), Rimsha and Dais (2020) and Krishnakumari and Dais (2020). Again Sebastian et al. (2016) introduced Log-Esscher transformed Laplace distribution and studied its applications.
The rest of the research paper is organized in four Sections. In Section 2, we introduce the Kumaraswamy Esscher transformed Laplace distribution and study some of its properties. The parameters of the distribution are estimated.A simulation study as well as a real data application of the distribution are also considered in this section. Later, some extensions viz. the exponentiated and transmuted exponentiated forms of the new family of distributions are introduced in Section 3. This paper is concluded by Section 4.

2. Kumaraswamy Esscher Transformed Laplace Distribution

Beta distribution is one of the most basic distributions supported on finite range (0, 1) and has been used widely in both practical and theoretical generalization aspects of statistics (see, Nadarajah and Kotz (2004), Akinsete et al. (2008), Nassar and Elmasry (2012) and Nassar and Nada (2011, 2012 & 2013)). An alternative distribution like the beta distribution, which is easier to work with, is the Kumaraswamy distribution proposed by Kumaraswamy (1980). The Kumaraswamy distribution is like the beta distribution in many ways. Kumaraswamy's densities are also unimodal, uniantimodal, increasing, decreasing or constant depending in the same way as the beta distribution on the values of its parameters. In addition, one can easily show that the Kumaraswamy distribution has the same basic shape properties of the beta distribution. In the literature we can see Kumaraswamy generalizations of a lot many distributions. Among them to know about the recently developed ones, see Cordeiro (2010), Shuaib et al. (2016), Chhetri et al. (2017), Ahmad et al. (2018), Elgarhy et al. (2018), Zohdy et al. (2019) and Tahir et al. (2020). Still only few Kumaraswamy

generalized asymmetric and heavy tailed distributions are developed and hence this study has much importance.
Here we consider the Kumaraswamy generalization of one parameter Esscher transformed Laplace Distribution. If $X \sim ETL(\theta)$ distribution with probability density function (1) and distribution function (2), the probability density function and distribution function of the Kumaraswamy Esscher transformed Laplace (KETL) distribution are,

$$\begin{aligned} F^{K_{wGETL}}(x) &= 1-[1-F(x)^a]^b \\ &= \begin{cases} 1-[1-\left(\frac{1-\theta}{2}e^{x(1+\theta)}\right)^a]^b, & x<0 \\ 1-[1-\left(\frac{1-\theta}{2}+\frac{1+\theta}{2}(1-\exp[-x(1-\theta)])\right)^a]^b, & x\geq 0 \end{cases} \end{aligned} \tag{3}$$

and

$$\begin{aligned} f^{K_{wGETL}}(x) &= abf(x)F(x)^{(a-1)}1-[1-F(x)^a]^{b-1} \\ &= ab\frac{1-\theta^2}{2}\begin{cases} (\frac{1-\theta}{2})^{a-1}(e^{x(1+\theta)})^a\left[1-\left(\frac{1-\theta}{2}e^{x(1+\theta)}\right)^a\right]^{b-1}, & x<0 \\ e^{-x(1-\theta)}\left(1-\frac{1+\theta}{2}e^{-x(1-\theta)}\right)^{a-1} \\ \left[1-\left(1-\frac{1+\theta}{2}e^{-x(1-\theta)}\right)^a\right]^{b-1}, & x\geq 0. \end{cases} \end{aligned} \tag{4}$$

If a = b = 1, the pdf reduces to the pdf of Esscher transformed Laplace distribution. The survival function, hazard rate function (hrf), reverse hazard rate function (rhrf) and cumulative hazard rate function of the distribution are respectively.

$$\begin{aligned} \bar{F}^{K_{wGETL}}(x) &= \begin{cases} [1-\left(\frac{1-\theta}{2}e^{x(1+\theta)}\right)^a]^b, & x<0 \\ [1-\left(\frac{1-\theta}{2}+\frac{1+\theta}{2}(1-\exp[-x(1-\theta)])\right)^a]^b, & x\geq 0, \end{cases} \\ h^{K_{wGETL}}(x) &= ab\frac{1-\theta^2}{2}\begin{cases} \frac{(\frac{1-\theta}{2})^{a-1}(e^{x(1+\theta)})^a}{1-(\frac{1-\theta}{2}e^{x(1+\theta)})^a}, & x<0 \\ \frac{e^{-x(1-\theta)}\left(1-\frac{1+\theta}{2}e^{-x(1-\theta)}\right)^{a-1}}{1-\frac{1+\theta}{2}\left(1-\frac{1+\theta}{2}e^{-x(1-\theta)}\right)^a} & x\geq 0, \end{cases} \end{aligned}$$

$$r^{K_{wG}}(x) = ab\left(\frac{1-\theta^2}{2}\right)\begin{cases} \frac{\left(\frac{1-\theta}{2}\right)^{a-1}\left(e^{x(1+\theta)}\right)^a\left[1-\left(\frac{1-\theta}{2}e^{x(1+\theta)}\right)^a\right]^{b-1}}{1-[1-\left(\frac{1-\theta}{2}e^{x(1+\theta)}\right)^a]^b}, & x<0 \\ \frac{e^{-x(1-\theta)}\left(1-\frac{1+\theta}{2}e^{-x(1-\theta)}\right)^{a-1}[1-\left(1+\frac{1+\theta}{2}(1-\exp[-x(1-\theta)])\right)^a]^{b-1}}{[1-\left(1+\frac{1+\theta}{2}(1-\exp[-x(1-\theta)])\right)^a]^b}, & x\geq 0 \end{cases}$$

and

$$H^{K_{wGETL}}(x) = \begin{cases} -blog\left[1-\left(\left(\frac{1-\theta}{2}\right)e^{x(1+\theta)}\right)^a\right], & x<0 \\ -\,blog\left[1-\left(\left(\frac{1-\theta}{2}\right)(1-e^{x(1+\theta)})\right)^a\right], & x\geq 0. \end{cases}$$

The probability density plots of KETL distribution are given in figure 2.

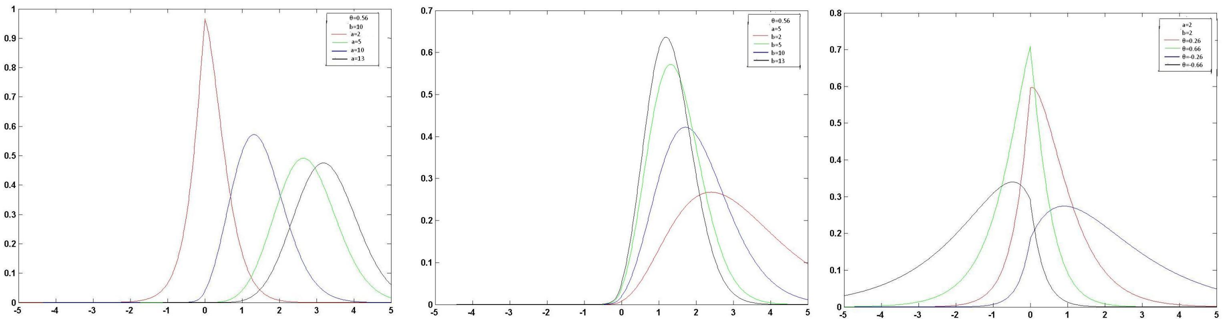

Figure 2: Densities of Kumaraswamy Esscher Transformed Laplace Distribution for (a) $\theta = 0.56, b = 10$ and $a = 2, 5, 10, 13$ (b) $\theta = 0.56, a = 5$ and $b = 2, 5, 10, 13$ (c) $a = 5, b = 5$ and $\theta = 0.26, 0.63, -0.26, -0.66$

As the parameter a changes the location changes whereas as b changes, the tail heaviness of the distribution increases to the left side if θ is negative and to the right side if θ is positive.

2.1 Estimation

The parameters of the KETL distribution are estimated by the maximum likelihood method. Let $X_1, X_2, ..., X_n$ be a random sample from KETL distribution given by equation (4) and $\beta = (a, b, \theta)^T$ be the unknown parameter vector. The log-likelihood function is given by

$$\begin{aligned} l(\beta) &= \sum_{i=1}^{n} \delta_i log\left(\frac{ab(1+\theta)(1-\theta)^a}{2^a}\right) + \sum_{i=1}^{n} \delta_i a x_i (1+\theta) \\ &+ (b-1)log\left[1 - \left(\left(\frac{1-\theta}{2}\right) e^{x(1+\theta)}\right)^a\right] + \sum_{i=1}^{n}(1-\delta_i) log\left[ab\left(\frac{1-\theta^2}{2}\right)\right] \\ &+ \sum_{i=1}^{n}(1-\delta_i)(-x_i(1-\theta) + (a-1)log\left[1 - \left(\frac{1+\theta}{2}\right) e^{-x(1-\theta)}\right] \\ &+ (b-1)log\left[1 - \left(1 - \left(\frac{1+\theta}{2}\right) e^{-x(1-\theta)}\right)\right] \end{aligned} \tag{5}$$

where

$$\delta_i = \begin{cases} 1\,, & if \;\; x_i < 0 \\ 0\,, & if \;\; x_i \geq 0. \end{cases} \tag{6}$$

This log-likelihood function can not be solved analytically because of its complex form but it can be maximized numerically by employing global optimization methods available with software's like R, SAS, Mathematica or by solving the nonlinear likelihood equations obtained by differentiating (5). By taking the partial derivatives of the log-likelihood function with respect to a, b and θ, we obtain the components of the score vector as $U(\beta) = (U_a, U_b, U_\theta)^T$.

Here,

$$\begin{aligned}
U_a &= \sum_{i=1}^{n}\frac{(1-\delta_i)}{a}+\sum_{i=1}^{n}(1-\delta_i)log\left(1-\left(\frac{1+\theta}{2}\right)e^{-x(1-\theta)}\right) \\
&\quad -\frac{(b-1)\left[1-\left(\frac{1+\theta}{2}\right)e^{-x(1-\theta)}\right]^a log\left[1-\left(\frac{1+\theta}{2}\right)e^{-x(1-\theta)}\right]}{1-\left[1-\left(\frac{1+\theta}{2}\right)e^{-x(1-\theta)}\right]^a} \\
&\quad +\sum_{i=1}^{n}\delta_i\frac{(1+\theta)2^a(2^{-a}b-(1+\theta)^{-1+a}-ablog2)}{ab}+\sum_{i=1}^{n}\delta_i \\
&\quad \left(\frac{(b-1)(2^{-a}(1+\theta e^{x(1+\theta)}))^a log(2)-2^{-a}(1+\theta e^{x(1+\theta)})^a log(1+\theta)e^{x(1+\theta)}}{1-2^{-a}((1-\theta)e^{x(1+\theta)})^a}+x_i(1-\theta^2)\right),
\end{aligned} \tag{7}$$

$$\begin{aligned}
U_b &= \sum_{i=1}^{n}(1-\delta_i)\left[\frac{1}{b}+log\left[1-\left(1-\left(\frac{1+\theta}{2}\right)e^{-x(1-\theta)}\right)^a\right]\right] \\
&\quad +\delta_i\left[\frac{1}{b}+log\left(1-\left[2^{-a}(1-\theta)e^{x(1+\theta)}\right]\right)^a\right]
\end{aligned}$$

and

$$\begin{aligned}
U_\theta &= \sum_{i=1}^{n}(1-\delta_i)\left[\frac{-2\theta}{1-\theta^2}+x_i\right] \\
&\quad +(a-1)\left[\frac{-1}{1+\theta}+x_i\right]+(b-1)a\left[\frac{1}{1+\theta}-x_i\right] \\
&\quad +\sum_{i=1}^{n}\delta_i\left[\frac{1}{1+\theta}-\frac{a}{1-\theta}\right]+\sum_{i=1}^{n}\delta_i a x_i \\
&\quad -(b-1)\left[\frac{1}{1-\theta}-x_i\right].
\end{aligned} \tag{8}$$

Setting $U(\hat{\beta})=0$ and solving them simultaneously, we obtain the maximum likelihood estimate of $\beta=(a,b,\theta)^T$ as $\hat{\beta}=(\hat{a},\hat{b},\hat{\theta})^T$. The (3×3) Fisher information matrix is

$$\Sigma(\beta) = \begin{pmatrix} U_{aa} & U_{ab} & U_{a\theta} \\ U_{ba} & U_{bb} & U_{b\theta} \\ U_{\theta a} & U_{\theta b} & U_{\theta\theta} \end{pmatrix}$$

where the diagonal elements

$$U_{bb}=-\frac{n}{b^2} \tag{9}$$

$$\begin{aligned}
U_{aa} &= \sum_{i=1}^{n}(1-\delta_i)\left[-\frac{1}{a^2}-(b-1)\frac{\left[1-\left(\frac{1+\theta}{2}\right)e^{-x(1-\theta)}\right]^a 2log\left[1-\left(\frac{1+\theta}{2}\right)e^{-x(1-\theta)}\right]}{1-\left[1-\left(\frac{1+\theta}{2}\right)e^{-x(1-\theta)}\right]^a}\right. \\
&\quad \left.\left[\frac{\left(1-\left(\frac{1+\theta}{2}\right)e^{-x(1-\theta)}\right)^a}{1-\left(1-\left(\frac{1+\theta}{2}\right)e^{-x(1-\theta)}\right)^a}+1\right]\right] \\
&+ \sum_{i=1}^{n}\delta_i\left[\frac{-(b-1)\left(\left(\frac{1-\theta}{2}\right)e^{x(1+\theta)}\right)^{2a}}{\left[1-\left(\left(\frac{1-\theta}{2}\right)e^{x(1+\theta)}\right)^a\right]^2}\left[log2-loge^{x(1+\theta)}\right]^2\right. \\
&+ \frac{2log2(b-1)\left(\left(\frac{1-\theta}{2}\right)e^{x(1+\theta)}\right)^a}{1-\left(\left(\frac{1-\theta}{2}\right)e^{x(1+\theta)}\right)^a}\left[1+log(1+\theta)e^{x(1+\theta)}\right] \\
&- \left.\frac{log(1-\theta)e^{x(1+\theta)}}{log2}-\left(\frac{1}{a^2}+(log2)^2+2log2\right)\right]
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
U_{\theta\theta} &= \sum_{i=1}^{n}-(1-\delta_i)\left[\frac{4\theta^2(1-\theta^2)^{-3}+2(1-\theta^2)}{(1-\theta^2)^2}\right] \\
&+ (a-1)\frac{1}{(1+\theta)^2}+(b-1)a\frac{1}{(1+\theta)^2} \\
&+ \sum_{i=1}^{n}\delta_i\left[\frac{1}{(1+\theta)^2}+\frac{a}{(1+\theta)^2}\right]+\frac{(b-1)a}{(1-\theta)^2}.
\end{aligned} \tag{11}$$

2.2 Simulation Study

In this section, we conduct a simulation study to illustrate the performance of the parameters. Using this simulated data from kETL distribution for different sample sizes (n= 15, 20, 30, 50, 100), we use the acceptance rejection method suggested by Arnold et al. (1999). The acceptance rejection method is a way to simulate random samples from one unknown (for difficult to sample from) distribution by using random samples from a similar, more convenient probability distributions with the property $f(x) \leq Cg(x)$. Here g(x) is the pdf KETL distribution, f(x) the pdf of ETL distribution and

$$C = ab\begin{cases} \left(\frac{1-\theta}{2}\right)^{a-1}\left(e^{x(1+\theta)}\right)^{a-1}\left[1-\left(\frac{1-\theta}{2}e^{x(1+\theta)}\right)^a\right]^{b-1}, & x<0 \\ \left(1-\frac{1+\theta}{2}e^{-x(1-\theta)}\right)^{a-1} & \\ \left[1-\left(1-\frac{1+\theta}{2}e^{-x(1-\theta)}\right)^a\right]^{b-1}, & x\geq 0 \end{cases} \tag{12}$$

where $C \leq \infty$ and a constant. Obivisiouly it is preferable to have C close to one which in turn means that g(x) should look as much as possible alike f(x). Using this simulated data (1000 random samples and hence 1000 estimates for each parameter) with help of the nlm() package in R, we obtain the estimates for θ, a and b. We study the performance of the estimators, using average Bias and average mean squared error (MSE). Table 1 shows the estimated values of the parameters for different sample sizes along with its average Bias and average MSE.

Table 1: Average Bias and Average MSE of the Simulated Estimates for θ, a and b.

n	$\theta = 0.0466$			$a = 0.5213$			$b = 0.5962$		
n	$\hat{\theta}$	Bias	M.S.E	$\hat{a}$	Bias	M.S.E	$\hat{b}$	Bias	M.S.E
15	0.0424	0.00644	0.0069	0.4881	0.0075	0.0063	0.5783	0.0028	0.0041
20	0.0433	0.00413	0.0057	0.4904	0.0068	0.0049	0.5798	0.0027	0.0030
30	0.0439	0.00368	0.0038	0.4962	0.0054	0.0036	0.5816	0.0017	0.0024
50	0.0446	-0.0008	0.0014	0.5066	-0.0038	0.0027	0.5898	0.0013	0.0021
100	0.0453	-0.0010	0.0009	0.5160	-0.0043	0.0036	0.5902	-0.00029	0.0016

From the table we can see that as the sample size increases the estimates become more and more close to the actual value of the parameters θ, a, b. Also the average Bias and average MSE of the estimators are reasonably small for various choices of the sample sizes.

2.3 Real Data Analysis

For data analysis, we consider the data set consisting of 380 observations which represents the breaking stress of carbon fibres starting from 07/11/2015 to 06/11/2017. We collect the secondary data from the Steel Plant, Visakhapatnam, Andhra Pradesh and its descriptive statistics are given in Table 2.

Table 2 Descriptive Statistics of 380 Breaking Stress of Carbon Fibres.

Min.	Q1	Median	Mean	Q3	Max.	Var.
0.39	1.73	2.5	2.53	3.12	5.56	1.089.

Figure 3 represents the histogram of this data.

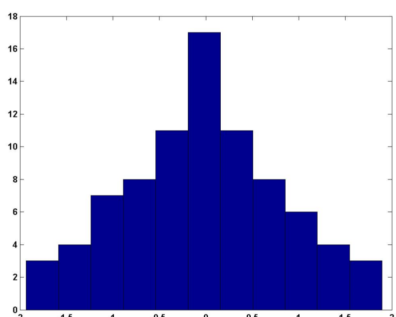

Figure 3: Histogram of the Observed Data

This resembles the shape of the graph given in Figure 1. We fit the data to the new model and compare the result with the Kumaraswamy Laplace model by estimating the parameters of both the distributions using nlm method and the estimated parameters are displayed in Table 3.

Table 3 MLE of the Model Parameters

Distribution	Estimates			
	a	b	θ	κ
KETL	0.533167	0.456913	0.0489	
KL	0.653167	0.578913		0.8962

The frequency curves of the distribution are superimposed in the histogram and are presented in Figure 4.

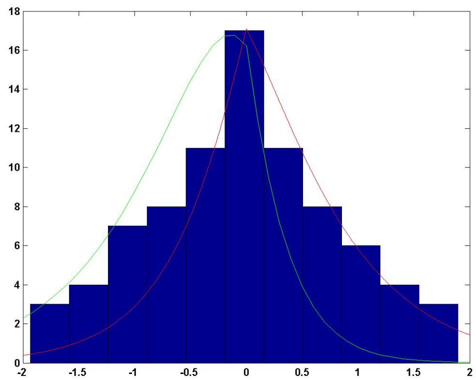

Figure 4: Embedded Frequency Polygon of the Observed Data

From the figure it is clear that the kumaraswamy Esscher transformed Laplace distribution is a better model than the Kumaraswamy Laplace distribution for the considered data.
The K-S statistic and numerical values of the $-log\hat{\ell}$, AIC and BIC are displayed in Table 4.

Table 4 Log-likelihood Functions $[-log\ell(\theta)]$, AIC, BIC and K-S Statistics

Distribution	$-log\ell(\hat{\theta})$	AIC	BIC	K-S
KETL	268.129	544.25	553	0.04968.
KL	548.129	634.25	663	0.07898.

Since the values of K-S statistic, $-log\hat{\ell}$, AIC, BIC are smaller for the KETL distribution compared with those values of KL distribution, the new distribution seems to be a competitive model for the breaking stress data.

3. Some Extensions of Kumaraswamy Esscher transformed Laplace Distribution

In this section, we introduce some new extensions of Kumaraswamy Esscher transformed Laplace Distribution namely, exponentiated Kumaraswamy Esscher transformed Lapalce distribution and transmuted exponentiated Kumaraswamy Esscher transformed Lapalce distribution.

3.1 Exponentiated Kumaraswamy Esscher Transformed Laplace Distribution

Exponentiated family of distribution was introduced by Gompertz (1825) and Verhulst (1838, 1845, 1847) by the first half of the nineteenth centuary and Gompertz and Verhulst cumulative distribution function was the first member of the exponentiated family of distributions. Later several exponentiated distributions are developed and for the recently introduced ones available in the literature, see Hassan and Elgarhy (2016), Amer Ibrahim et al. (2019), Ahmad et al. (2019), Suleman Nasirua et al. (2019), Dawlah Al-Sulami (2020), Francisco Louzada et al. (2020), Abdulkabir and Ipinyomi (2020), Badr and Ijaz (2021) and Ali et al. (2021). Four different methods are seen in the literature to obtain the exponentiated family of distributions. Among them we use the method of Lehmann alternative 1 (LA1) due to Lehmann (1953).

According to Lehmann alternative method, if G(z) is the cdf of the baseline distribution, then an exponentiated family of distributions is defined by

$$F(z) = G(z)^a, \tag{13}$$

where $a > 0$ is a positive real parameter.

If Kumaraswamy Esscher transformed Laplace distribution with distribution function F(x) given in (3) as the baseline distribution, the cumulative distribution function and probability density function of the exponentiated Kumaraswamy

Esscher transformed Laplace distribution denoted by EKETL distribution are

$$F(x) = \begin{cases} \left[1 - \left[1 - \left(\left(\frac{1-\theta}{2}\right) e^{x(1+\theta)}\right)^a\right]^b\right]^\alpha, & x < 0 \\ \left[1 - \left[1 - \left(\frac{1-\theta}{2} + \frac{1+\theta}{2}(1 - \exp[-x(1-\theta)])\right)^a\right]^b\right]^\alpha, & x \geq 0 \end{cases}$$

and

$$f(x) = \alpha ab\left(\frac{1-\theta^2}{2}\right) \begin{cases} \left(\frac{1-\theta}{2}\right)^{a-1} \left(e^{x(1+\theta)}\right)^a \left[1 - \left(\left(\frac{1-\theta}{2}\right) e^{x(1+\theta)}\right)^a\right]^{b-1} \\ \left[1 - \left[1 - \left(\left(\frac{1-\theta}{2}\right) e^{x(1+\theta)}\right)^a\right]^b\right]^\alpha, \; x < 0 \\ e^{-x(1-\theta)} \left(1 - \frac{1+\theta}{2} e^{-x(1-\theta)}\right)^{a-1} \left[1 - \left(1 - \frac{1+\theta}{2} e^{-x(1-\theta)}\right)^a\right]^{b-1} \\ \left[1 - \left[1 - \left(\frac{1-\theta}{2} + \left(\frac{1+\theta}{2}\right)(1 - \exp[-x(1-\theta)])\right)^a\right]^b\right]^\alpha, \; x \geq 0 \end{cases}$$

respectively.

3.2 Transmuted Exponentiated Kumaraswamy Esscher Transformed Laplace Distribution

Shaw and Buckley (2007) introduced transmuted family of distributions. Let $P(t)$ be the probability density function of a random variable $T \in [a, b]$ for $-\infty < a < b < \infty$ and let $W[G(x)]$ be a function of the cumulative distribution function of a random variable X such that $W[G(x)]$ satisfies the following conditions:

1. $W[G(x)] \in [a, b]$
2. $W[G(x)]$ is differentiable and monotonically nondecreasing and
3. $W[G(x)] \longrightarrow a$ as $x \longrightarrow -\infty$ and $W[G(x)] \longrightarrow b$ as $x \longrightarrow \infty$

Later, Alzaghal. et al. (2013) defined the T-X family of distributions by

$$F(x) = \int_a^{W[G(x)]} P(t)dt, \tag{14}$$

where $W[G(x)]$ satisfies the above conditions. The corresponding probability density function is

$$f(x) = \left\{\frac{d}{dx} W[G(x)]\right\} p\left\{W[G(x)]\right\}. \tag{15}$$

Based on the T-X family, we construct a new generator by taking $W[G(x)] = \frac{\theta G(x)}{1+(\theta-1)G(x)}$ and $P(t) = 1+\lambda-2\lambda t, 0 < t < 1$ and there by indroduce a new family of distributions, namely transmuted exponentiated Kumaraswamy Esscher transformed Laplace distribution (TEKETL). The transmuted exponentiated Kumaraswamy Esscher transformed Laplace distribution is the transmuted version of the exponentiated Kumaraswamy Esscher transformed Laplace distribution with cumulative distribution function (14). The cumulative distribution function and probability density function of the transmuted exponentiated Kumaraswamy Esscher transformed Laplace distribution (TEKETL) are obtained respectively as

$$F(x,\lambda,a,b,\theta) = \begin{cases} \left[1-\left(1-\left(\frac{1-\theta}{2}\right)e^{x(1+\theta)}\right)^a\right]^b (1+\lambda \\ -\lambda\left[1-\left(1-\left(\frac{1-\theta}{2}\right)e^{x(1+\theta)}\right)^a\right]^b\right), \ x<0 \\ \left[1-\left(1-\left(\frac{1+\theta}{2}\right)(2-\exp[-x(1-\theta)])\right)^a\right]^b \\ \left(1+\lambda-\lambda\left[1-\left(1-\left(\frac{1+\theta}{2}\right)(2-\exp[-x(1-\theta)])\right)^a\right]^b\right), \ x\geq 0 \end{cases}$$

and

$$f(x,\lambda,a,b,\theta) = ab\frac{1-\theta^2}{2}\begin{cases} \left(\frac{1-\theta}{2}\right)^{a-1}\left(e^{x(1+\theta)}\right)^a\left[1-\left(\left(\frac{1-\theta}{2}\right)e^{x(1+\theta)}\right)^a\right]^{b-1} \\ \left(1+\lambda-2\lambda\left[1-\left(1-\left(\frac{1-\theta}{2}\right)e^{x(1+\theta)}\right)^a\right]^b\right), \ x<0 \\ e^{-x(1-\theta)}\left(1-\left(\frac{1+\theta}{2}\right)e^{-x(1-\theta)}\right)^{a-1}\left[1-\left(1-\left(\frac{1+\theta}{2}\right)e^{-x(1-\theta)}\right)^a\right]^{b-1} \\ \left(1+\lambda-2\lambda\left[1-\left(1-\left(\frac{1+\theta}{2}\right)(2-\exp[-x(1-\theta)])\right)^a\right]^b\right), \ x\geq 0. \end{cases}$$

Here the parameters a >0 and b>0 introduce asymmetry and heavier tails in the baseline distribution. If a = b = 1 and $\lambda=0$, the probability density function reduces to the Esscher transformed Laplace distribution and $\lambda=0$, the probability density function reduces to the kumaraswamy Esscher transformed Laplace distribution.

4. Conclusion

In this paper, we introduced the Kumaraswamy generalization of the Esscher transformed Laplace distribution namely, Kumaraswamy Esscher transformed Laplace distribution and studied their survival function, hazard rate function, cumulative hazard rate function and reverse hazard rate function. The parameters of the distribution were estimated using the maximum likelihood method of estimation. The simulation study established the performance of the estimators. A real application of this distribution on breaking stress of carbon fibres was also considered here. Later we introduced the exponentiated and transmuted forms of this new family of distributions namely, Exponentiated Kumaraswamy Esscher transformed Laplace distribution and Transmuted exponentiated Kumaraswamy Esscher transformed Laplace distribution. Being asymmetric and heavy-tailed and since tail heaviness (left tail heaviness and right tail heavyness) increases according as the change in the values of the parameters, these newly proposed flexible models will be very helpful for modeling such data sets generated from different fields.

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