An Accelerated Sequential Class to Minimize Combined Risk for Simultaneous Estimation of Parameters of Several Probability Distributions

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Abstract

The fixed sample size procedure sometimes fail to deal with the estimation problem in which it is desired to control the combined risk associated with the estimation of parameters of various probability distributions simultaneously. In this paper for a general probabilistic model is proposed an "accelerated" sequential class to minimize the combined risk for simultaneous estimation of parameters of several probability distributions. The proposed class is shown to provide solutions for many estimation problems under different probabilistic setups for "given precision" problems. Besides, positive and negative moments for the stopping times are obtained and they are used for deriving the asymptotic expression for the "regret" associated with the class of "accelerated" sequential estimation procedure.

Keywords: Loss function, Accelerated sequential procedure, Asymptotic distribution, Regret.

1.1 Introduction

Sometimes, it is desirable to control the combined risk associated with the estimation of parameters of various populations and such problems give rise to the simultaneous estimation problems. The fixed sample size procedures for such problems fail and affirmative solutions are desired. In this direction, Raatikainer(1987) proposed a sequential procedure for simultaneous estimation of percentiles under a general set-up. Then Mukhopadhyay, Hamdy and Darmanto(1988) considered the negative exponential populations and desired the simultaneous confidence interval estimation of the parameters as a follow up of certain selection and ranking problems. Later Mukhopdhyay(1992) provided multi-stage procedures for the simultaneous point estimation of the parameters of several negative exponential populations. For some further work on related simultaneous estimation procedures one can cite the papers of Raatikainer(1993), Mukhopadhyay and Solanky(1998), Aoshima and Mukhopadhyay(1998), Effron(2004) and Ghosh (2005).

In this paper we have considered a generalized problem of simultaneous estimation of parameters of several populations $f(x_i, \theta_i, \psi_i)$, i = 1,2,……,k; where θ_i and ψ_i are the unknown parameters, under a family of general loss function and a linear cost function. The problem of fixed sample size procedure to deal with such problem is established in Section 1.2. A class of "accelerated" sequential procedure to tackle the problem is proposed in Section 1.3 and positive and negative moments for the stopping times are obtained and they are used for deriving the asymptotic expression for the "regret" associated with the class of "accelerated" sequential estimation procedure. Finally in Section 1.4.illustrations of estimation problems are provided which can be dealt with the help of the proposed class.

1.2 The Set-Up of the Problem

Let $(\underline{X}_{ij})$, $j = 1,2,....$ be a sequence of independent and identically distributed (iid) random variables from the $i^{th}, i = 1,2,3,....,k$ t-variate $(t \geq 1)$ population $f(\underline{X}_i; \underline{\theta}_i, \psi_i)$, where $\underline{\theta}_i$ and ψ_i are the unknown parameters. Denoting by R^t and R^+, respectively, the t- dimensional euclidean space and the positive –half of the real line, let $\underline{\theta}_i \in R^t$ and $\psi_i \in R^+$. Having observed a random sample $\underline{X}_{i1},.....,\underline{X}_{in_j}$ of size $n_j (\geq t+1)$ from the ith population let $\hat{\underline{\theta}}_{in_j} = \hat{\underline{\theta}}_i(X_{i1},.....,X_{in_i})$ and $\hat{\underline{\psi}}_{in_i} = \hat{\psi}_i(X_{i1},...,X_{in_i})$ be the estimators of $\underline{\theta}_i$ and ψ_i, respectively, and satisfying the following assumptions:

(A_1): There exist a known txt positive definite matrix Q_i, a number $\delta \in (0,1]$ and an integer

$r(\geq 1)$ such that

$$n_i \psi_i^{-1}\left[\left(\hat{\underline{\theta}}_{in_i} - \hat{\underline{\theta}}_i\right)' Q_i \left(\hat{\underline{\theta}}_{in_i} - \hat{\underline{\theta}}_i\right)\right]^{\delta} \sim \chi^2_r ,$$

where χ^2_r denotes a chi-squarer.v's with r degrees of freedom.

(A_2): For all $n_i \geq m \geq t+1$, $\hat{\underline{\theta}}_{in_i}$ and $\hat{\underline{\psi}}_{in_i}$ are stochastically independent.

(A_3): There exist integers q(≥1) and s(≥1) such that, for all $n_i \geq s+1$,

$$q(n_i - s)\underline{\psi}_{in_i} / \psi_i = \sum_{j=1}^{n_i - s} Z_j^q \text{ with } Z_j^q \sim \chi^2_q .$$

Our goal is to estimate the vector $\underline{\theta} = (\underline{\theta}'_1,.....,\underline{\theta}'_k)'$ pointwise. For $n = \sum_{i=1}^{k} n_i$, an obvious estimator of $\underline{\theta}$ is $\hat{\underline{\theta}}_n = (\hat{\underline{\theta}}'_{1n_1},.....,\hat{\underline{\theta}}'_{kn_k})'$. Let the loss incurred in estimating $\underline{\theta}$ by $\hat{\underline{\theta}}_n$ be

$$L(\underline{\theta}, \hat{\underline{\theta}}_n) = A\sum_{i=1}^{k}\left[\left(\hat{\underline{\theta}}_{in_i} - \hat{\underline{\theta}}_i\right)' Q_i \left(\hat{\underline{\theta}}_{in_i} - \hat{\underline{\theta}}_i\right)\right]^u + C(n_1 + ... + n_k), \qquad (1.1)$$

where A (>0) is the known weight and C (>0) is the known cost per unit sample observations from each population.

Utilizing (A_1), the risk corresponding to the loss function (1.1) comes out to be

$$R_n(c) = K(u,r,\delta)\sum_{i=1}^{k}\psi_i^{\,u} n_i^{-u} + C\sum_{i=1}^{k} n_i \text{ , where} \tag{1.2}$$

$$K(u,r,\delta) = 2^{u/\delta} A\Gamma\left(r/2 + u/\delta\right)/\Gamma\left(r/2\right).$$

The value n_i^* of n_i from the ith population, which minimizes the risk (1.2), is given by

$$n_i^* = \left\{\frac{uk(u,r,\delta)}{C}\right\}^{1/(u+1)} \psi_i^{\,u/(u+1)} \tag{1.3}$$

And substituting $n_i = n_i^*$ in (1.2), for $n^* = \sum_{i=1}^{k} n_i^*$, the corresponding minimum risk is

$$R_{n^*}(C) = C(1/u+1)\, n^* \tag{1.4}$$

However, in the absence of any knowledge about ψ_i's, no fixed sample size procedure uniformly minimizes the risk simultaneously for all values of ψ_i's. In such a situation, motivated by (1.3), we propose the following class of 'accelerated' sequential procedure, determining the sample size as a random variable.

1.3. The Class C of 'Accelerated' Sequential Procedure

Take $m \geq \max\{s+1, t+1\}$ to be the initial sample size from each population, where, as in Hall(1983), m is chosen so as to satisfy $m = o(C^{-1/(u+1)})$ as $C \to 0$ and $\limsup_{c\to 0}(m/n_i^*) < 1$. Let $\eta \in (0,1)$ be specified. Start sampling sequentially from the ith population, with the stopping time M_i defined by

$$M_i = \inf\left[n_i \geq m : n_i \geq \eta\left\{\frac{uk(u,r,\delta)}{C}\right\}^{1/(u+1)} \hat{\psi}_{i(n_i)}^{\,u/(u+1)}\right] \tag{1.5}$$

Based on these M_i observations, we compute $\hat{\psi}_{i(M_i)}$. Then we jump ahead and collect $N_i - M_i$ more observations from the ith population,
where

$$M_i = \max\left\{M_i, \left[\left\{\frac{uk(u,r,\delta)}{C}\right\}^{1/(u+1)} \hat{\psi}_{i(n_i)}^{\,u/(u+1)}\right]^{+} + 1\right\} \tag{1.6}$$

After stopping, estimate $\underline{\theta}$ by $\underline{\hat{\theta}}_{(N)}$ where $N = \sum_{i=1}^{k} N_i$ and

$\underline{\hat{\theta}}_{(N)} = \left(\underline{\hat{\theta}}_{(1N_1)} \ldots\ldots \underline{\hat{\theta}}_{k(N_k)}\right)'$.

It can be seen that the risk associated with the class **C** of 'accelerated' sequential estimation procedures (1.5)-(1.6) is

$$R_N(c) = c/u \sum_{i=1}^{k} n_i^{*(u+1)} E\left(N_i^{-u}\right) + c\sum_{i=1}^{k} E(N_i) - c(u+1)n^* \tag{1.7}$$

with N_i determined by present rule.

The "Regret" associated with the estimator $\hat{\underline{\theta}}_{(N)}$ of $\underline{\theta}$ is

$$R_g(c) = R_N(c) - R_{n^*}(c) \tag{1.8}$$

Now we state and prove some lemmas.

Lemma 1: For the class **C**of accelerated sequential procedures and all

$$m > \max\{t, s + 2u/(u+1)\}, \text{ as } C \to 0,$$

$$E(N_i) = n_i^* \left[s + \frac{u}{q(u+1)} \{1 + u/(u+1)\} \right] + 1/2 + o(1) \tag{1.9}$$

and

$$Var(N_i) = \frac{2u^2}{\eta q (u+1)^2} n_i^* \, o(C^{-1/(u+1)}) \tag{1.10}$$

Proof : Denoting by

$$U_{M_i} = 1 - \left\{ \left\{ \frac{u\,K(u,r,\delta)}{C} \right\}^{1/(u+1)} \widehat{\Psi}_{M_i}^{u/(u+1)} - \left[\left\{ \frac{u\,K(u,r,\delta)}{C} \right\}^{1/(u+1)} \widehat{\Psi}_{M_i}^{u/(u+1)} \right]^+ \right\}$$

We can write

$$E(N_i) = I + II, \tag{1.11}$$

where

$$I = E\left[N_i \, I\left\{ M_i > \left[\left[\left\{ \frac{u\,K(u,r,\delta)}{C} \right\}^{1/(u+1)} \widehat{\Psi}_{M_i}^{u/(u+1)} \right]^+ + 1 \right] \right\} \right] \tag{1.12}$$

$$II = \left\{ \frac{u\,K(u,r,\delta)}{C} \right\}^{1/(u+1)} E\left[\widehat{\Psi}_{M_i}^{u/(u+1)} \right] + E(U_{M_i}) \tag{1.13}$$

It follows from Hall (1983) that, as $C \to 0$, I= o(1) and U_{M_i} is uniform over (0,1).

Thus, we conclude from (1.11), (1.12) and (1.13) that, as $C \to 0$,

$$E(N_i) = \left\{ \frac{u\,K(u,r,\delta)}{C} \right\}^{1/(u+1)} E\left[\widehat{\Psi}_{M_i}^{u/(u+1)} \right] + 1/2 + o(1) \tag{1.14}$$

Now we evaluate $E\left[\widehat{\Psi}_{M_i}^{u/(u+1)}\right]$. To this end, we rewrite the stopping rule (1.5) as

$$M_i = \inf\left[n_i \geq m - s: \sum_{j=1}^{n_i}\left(q^{-1} Z_j^{(q)}\right) \leq \left(\eta n_i^*\right)^{-(u+)/1} n_i^{1+(u+1)/u} (1+sn_i^{-1})^{(u+1)/u}\right] \quad (1.15)$$

Comparing (1.14) with equation (1.1) ofWoodroofe (1977), we obtain in his notations,

$$S_{n_i} = \sum_{j=1}^{n_i}\left(q^{-1} Z_j^{(q)}\right), \alpha = 1+(u+1)/u, \beta = u/(u+1), \lambda = \eta n_i^*,$$

$L(n_i) = (1+sn_i^{-1})^{(u+1)/u}$, $L_o = s(u+1)/u$, $\mu = 1$, and $\tau^2 = 2q^{-1}$. It follows from his Theorem 2.4 that, for all $m > \{s + 2u/q(u+1)\}$, as $C \to 0$,

$$E(M_i) = \eta n_i^* + \frac{u\upsilon}{(u+1)} - s - \frac{u}{q(u+1)}(1+u/(u+1)) + o(1), \quad (1.16)$$

where υ is specified.

Let us consider the difference

$$D_c = M_i - \eta\left\{\frac{u\,K(u,r,\delta)}{C}\right\}^{1/(u+1)} \widehat{\Psi}_{i(M_i)}^{u/(u+1)} \quad (1.17)$$

It follows from Woodroofe (1977) that the mean of the asymptotic distribution of D_c is $\frac{u\upsilon}{(u+1)}$.

Thus we obtain from (1.15) and (1.16) that, for all $m > \{s + 2u/q(u+1)\}$, as $C \to 0$,

$$\left\{\frac{u\,K(u,r,\delta)}{C}\right\}^{1/(u+1)} E\left(\widehat{\Psi}_{i(M_i)}^{u/(u+1)}\right) = n_i^* - \eta^{-1}\left[s + \frac{u}{q(u+1)}\{1+u/(u+1)\}\right] + o(1). \quad (1.18)$$

Result (1.7) now follows on substituting the value of (1.7) in (1.14).

Let $h(M_i) = \left|M_i - \eta n_i^* / \left(\eta n_i^*\right)^{1/2}\right|$. It follows from a result of Bhattacharya and Mallik (1973) that $h(M_i) \xrightarrow{L} N\left(0, 2u^2/q(u+1)^2\right)$, as $c \to 0$. Moreover, from Theorem1 of Woodroofe(1977), $h^2(M_i)$ is uniformly integrable for all $m > \{s + 2u/q(u+1)\}$.

Hence, for all $m > \{s + 2u/q(u+1)\}$, as C → 0,

$$E\left(h^2(M_i)\right) = \frac{2u^2}{q(u+1)^2} + o(1).$$

And by the definition,

$$Var(N_i) = \eta^{-2} \; Var(M_i).Therefore,$$

$$Var(N_i) = \eta^{-2}\left[\eta\, n_i^*\left\{\frac{2u^2}{\eta q(u+1)^2} + o(1)\right\}\right]$$, and (1.9) holds.

Lemma 2: As $C \to 0$, $p\left(N_i \le \eta n_i^*\right) = o\left(C^{q(m-s)/2u}\right)$.

Proof: It is easily proved along the lines of Lemma 3 of Chaturvedi, Pandey and Gupta (1991).

Lemma 3: Let the r.v's W_i's, i=1,2,…,k, be defined by $\left|W_i - 1\right| \le \left|\left(N_i / n_i^*\right) - 1\right|$. Then, for $\delta > 0$, $W_i^{\delta-2}$ is uniformly integrable for all $m > \left\{s + 2u(2-\delta)/q(u+1)\right\}$.

Proof: On the event $N_i > \eta n_i^*, W_i \le 2 - n \;\; and \;\; W_i^{-1} \le \eta^{-1}$, so that, on this event, positive, as well as negative, powers of W_i are uniformly integrable. Furthermore,on the event $N_i > \eta n_i^* \; W_i \le 2 \;\; and \;\; W_i^{-1} \le n_i^* / m$.

Now, for $\delta < 2$, applying Lemma 2, we get from some B (>0),

$$E\left[W_i^{\delta-2}\, I\left(N_i \le \eta n_i^*\right)\right] \le B n_i^{*2-\delta}\; p\left(N_i \le \eta n_i^*\right)$$

$$= o(c^{q(m-s)/2u-(2-\delta)/(u+1)})$$

$$= o\,(1), \text{ for all } m > s + 2u(s+2u)(2-\delta)/q(u+1).$$

Finally, for $\delta \ge 2$, once again applying Lemma 2, we get

$$E\left[W_i^{\delta-2}\, I\left(N_i \le \eta n_i^*\right)\right] \le B\; p\left(N_i \le \eta n_i^*\right)$$

$$= o(1), \text{ for all } m \ge s+1.$$

And the lemma follows.

Lemma 4: For any $\delta > 0$ and all $m > \left\{s + 2u/q(u+1)\right\}, \left\{s + 2u(2-\delta)/q(u+1)\right\}$, as $C \to 0$,

$$E\left(N_i^{\delta}\right) = n_i^{*\delta} + \frac{\delta\, n_i^{*\delta-1}}{\eta q(u+1)^2}\left[(\delta-2)u^2 - (u+1)\left\{qs(u+1) - u - \eta q(n+1)/2\right\}\right] + o(c^{-(\delta-1)/(u+1)}) \tag{1.19}$$

And

$$E\left(N_i^{-\delta}\right) = n_i^{*-\delta} + \frac{\delta\, n_i^{*-(\delta+1)}}{\eta q(u+1)^2}\left[(\delta+2)u^2 - (u+1)\left\{qs(u+1) + u - \eta q(n+1)/2\right\}\right] + o(c^{(\delta+1)/(u+1)}) \tag{1.20}$$

Proof : Using the expression

$$E\left(N_i^{\delta}\right)=\mathrm{n}_{\mathrm{i}}^{*\delta}\left[1+\left(\delta/\mathrm{n}^{*}{}_{\mathrm{i}}\right)\mathrm{E}\left(\mathrm{N}_{\mathrm{i}}-\mathrm{n}_{\mathrm{i}}^{*}\right)+\delta\left(\frac{(\delta-1)}{2}\right)\mathrm{E}\left\{\frac{\left(\mathrm{N}^{*}{}_{\mathrm{i}}-\mathrm{n}^{*}{}_{\mathrm{i}}\right)^2}{\mathrm{n}^{*2}_{\mathrm{i}}}\mathrm{w}_{\mathrm{i}}{}^{\delta-2}\right\}\right]$$

Lemmas 1 and 3 and the fact that $W_i \xrightarrow{a.s} 1$ as $C \to 0$, we obtain for all

$m > \max\{\{s+2u/q(u+1)\},\{s+2u(2-\delta)/q(u+1)\}\}$, as $C \to 0$,

$$E\left(N_i^{\delta}\right)= n_i^{*\delta}+\delta\, n_i^{*\delta-1}\left[-\eta^{-1}\left[s+\frac{u}{q(u+1)}\left(1+u/(u+1)\right)\right]+1/2+o(1)\right]$$

$$+\frac{\delta(\delta-1)}{2}n_i^{*\delta-2}\left[\frac{2u^2}{\eta q(u+1)^2}n_i^{*}+o(c^{-1/(u+1)})\right].$$

And (1.19) follows.

Furthermore, using Taylor's expression for $N_i^{-\delta}$, Lemma 1 and 3, and the fact that $W_i \xrightarrow{a.s} 1$ as $C \to 0$, we obtain for all

$m > \max\{\{s+2u/q(u+1)\},\{s+2u(2-\delta)/q(u+1)\}\}$, as $C \to 0$,

$$E\left(N_i^{-\delta}\right)= n_i^{*-\delta}+\delta\, n_i^{*-(\delta+1)}\left[-\eta^{-1}\left[s+\frac{u}{q(u+1)}\left(1+u/(u+1)\right)\right]+1/2+o(1)\right]$$

$$+\frac{\delta(\delta+1)}{2}n_i^{*-(\delta+2)}\left[\frac{2u^2}{\eta q(u+1)^2}n_i^{*}+o(c^{-1/(u+1)})\right],$$

and (1.20) holds.

Now we prove the main theorem of this section, which provides asymptotic expression for the 'regret' corresponding to the class **C** of 'accelerated' sequential estimation procedures.

Theorem: For the class **C** of 'accelerated' sequential procedures and for all $m > \max\{\{t, s+2u/q(u+1)\},\{s+2u(2-\delta)/q(u+1)\}\}$, as $C \to 0$,

$$R_g(c)=\frac{Cku^2}{\eta q(u+1)}+o(C).$$

Proof: Substituting the values of $R_n(C)$ and $R_N(C)$ from (1.4) and (1.7) in (1.8) respectively, we obtain

$$R_g(C)=(C/u)\sum_{i=1}^{k}n_i^{*(u+1)}E(N_i^{-u})+C\sum_{i=1}^{k}E(N_i)-C(1/u+1)n^{*}. \qquad (1.21)$$

Now, applying Lemma 4, we obtain for any $\delta > 0$ and all $m > \max\{\{t, s+2u/q(u+1)\}, \{s+2u(2-\delta)/q(u+1)\}\}$, as $C \to 0$, we get

$$R_g(c) = (C/u)\sum_{i=1}^{k} n_i^{*(u+1)}\left[n_i^{*-u} + \frac{u}{\eta q(u+1)^2} n_i^{*-(u+1)}\left\{(u+2)u^2 + (u+1)(qs(u+1)) + u - \frac{\eta q(u+1)}{2} + o(C)\right\}\right]$$
$$+ C\sum_{i=1}^{k} n_i^* \eta^{-1}\left[n_i^* - \eta^{-1}\left\{s + \frac{u}{q(u+1)^2}(1+u/(u+1))\right\} - 1/2 + o(1)\right] - C(1/u+n)n^*,$$

and the theorem follows .

1.4 Estimation Problems Having Solutions Provided by Class C of"Accelerated" Sequential Procedure

1.4.1 Simultaneous estimation of the means of several normal populations

Let us consider a sequence $(\underline{X_{ij}})$, j= 1,2,…. of iidr.v's from the ith,i=1,2,……k univariate normal population $N(x_{i,}\mu_i, \sigma_i^2)$, where $\mu_i \in (-\infty, \infty)$ and $\sigma_i^2 \in (\theta, \infty)$ are the unknown mean and variance, respectively. Having recorded a random sample $X_{i1}, \ldots\ldots, X_{in_i}$ of size $n_i (\geq 2)$ from the ith population, let $\overline{X}_{i(n_i)} = n_i^{-1}\sum_{j=1}^{n_i} X_{ij}$ and $s_{i(n_i)}^2 = (n_i - 1)^{-1}\sum_{j=1}^{n_i}(X_{ij} - \overline{X}_{i(n_i)})^2$ .It can be verified (see Starr(1966)] that $(A_1) - (A_3)$ are satisfied for $\theta_i = \mu_i, \psi_i = \sigma_i^2$, $\hat{\underline{\theta}}_{in_i} = \overline{X}_{i(n_i)}$, $\hat{\Psi}_{i(n_i)}^2 = s_{i(n_i)}^2$, r=q=s=t=1, $Q_i = I_{1x1} = 1$ and $\delta = 1$.

1.4.2. Simultaneous estimation of the mean vectors of several multinormal populations

Let us consider a sequence $(\underline{X}_{ij})$, j= 1,2,…. of iidr.v's from the ith,i=1,2,……k univariate normal population $N_p(x_{i,}\underline{\mu}_i, \sigma_i^2 \Sigma_i)$, where $\underline{\mu}_i$ is the px1 unknown mean vector, $\sigma_i^2 \in (0, \infty)$ are the unknown scalar and Σ_i is a known pxp positive definite matrix. Given a random sample $\underline{X}_{i1}, \ldots\ldots, \underline{X}_{in_i}$ of size $n_i (\geq p+1)$ from the ith population, let $\overline{X}_{i(n_i)} = n_i^{-1}\sum_{j=1}^{n_i}\underline{X}_{ij}$ and $\hat{\sigma}_{i(n_i)}^2 = (p(n_i - 1))^{-1}\sum_{j=1}^{n_i}(\underline{X}_{ij} - \overline{\underline{X}}_{i(n_i)})^2$ .It can be verified (see, Wang, 1980) that $(A_1) - (A_3)$ hold for $\theta_i = \mu_i, \psi_i = \sigma_i^2$, $\hat{\underline{\theta}}_{in_i} = \overline{\underline{X}}_{i(n_i)}$, $\hat{\Psi}_{i(n_i)}^2 = \hat{\sigma}_{i(n_i)}^2$, r=t=p, $Q_i = \Sigma_i^{-1}$ and $\delta = 1$.

1.4.3. Simultaneous estimation of the regression parameters of several linear models

Consider the ith linear model $\underline{Y}_{i(n_i)} = \underline{X}_{i(n_i)} \underline{\beta}_i + \underline{\Sigma}_{i(n_i)}$, i=1,2,…..k.,
where $\underline{Y}_{i(n_i)}$ is an observed $n_i * 1$ random vector, $\underline{X}_{i(n_i)}$ is a $n_i * p$

Matrix of rank p, $\underline{\beta}_i$ is the $p*1$ vector of unknown regression parameters, and $\underline{\Sigma}_{i(n_i)}$ is the disturbance term following $N_{n_i}(\underline{0}, \sigma_i^2 I_{n_i})$ distribution. The ordinary least squares estimator of $\underline{\beta}_i$ is $\hat{\underline{\beta}}_{i(n_i)} = (\underline{X}'_{i(n_i)} \underline{X}_{i(n_i)})^{-1} \underline{X}_{i(n_i)} \underline{Y}_{i(n_i)}$ and we use $\hat{\sigma}^2_{i(n_i)} = (n_i - p)^{-1} \underline{Y}'_{i(n_i)} [I_{n_i} - \underline{X}_{i(n_i)} (\underline{X}'_{i(n_i)} \underline{X}_{i(n_i)})^{-1} \underline{X}'_{i(n_i)}] \underline{Y}_{i(n_i)}$ to estimate σ_i^2. It can be seen [see, Judge and Bock (1978, P.20)] that $(A_1) - (A_3)$ are satisfied for r = t = p, $\underline{\theta}_i = \underline{\beta}_i$, $\psi_i = \sigma_i^2$, $\hat{\underline{\theta}}_{in_i} = \hat{\underline{\beta}}_{i(n_i)}$,
$\hat{\Psi}^2_{i(n_i)} = \hat{\sigma}^2_{i(n_i)}$, q=1,s=p, $Q_i = n_i^{-1} (\underline{X}'_{i(n_i)} \underline{X}_{i(n_i)})^{-1}$ and $\delta = 1$.

1.4.4. Simultaneous estimation of the location parameters of several negative exponential populations

Let $(\underline{X}_{ij})$, j=1,2,….. be a sequence of the iidr.v's from the ith negative exponential populations,

$f(x_i; \underline{\mu}_i, \sigma_i) = \sigma_i^{-1} \exp[-{(x_i - \mu_i)}/{\sigma_i}]$; $x_i \geq \mu_i$, i=1,2,…..,k, where $\mu_i \in (-\infty, \infty)$ and $\sigma_i \in (0, \infty)$ are the unknown location and scale parameters, respectively. For a random sample X_{i1},....., X_{in_i} of size $n_i (\geq 2)$ from the ith population, we define $X^*_{i(n_i)} = \min(X_{i1}, ..., X_{in_i})$ and $\hat{\sigma}_{i(n_i)} = (n_i - 1)^{-1} \sum_{j=1}^{n_i} (\underline{X}_{ij} - \underline{X}^*_{i(n_i)})$.

Assumptions $(A_1) - (A_3)$ are satisfied [see Basu (1971)] for r=2, t=1, $\underline{\theta}_i = \mu_i$, $\psi_i = \sigma_i$, $\hat{\underline{\theta}}_{in_i} = X^*_{i(n_i)}$, $\hat{\Psi}_{i(n_i)} = \sigma^2_{i(n_i)}$, q=2, s=1, $Q_i = I_{1\times1} = 1$ and $\delta = 1/2$.

1.4.5. Simultaneous estimation of the parameters of several Pareto distribution

Let $(\underline{X}_{ij})$, j=1,2,….. be a sequence of the iidr.v's from the ith (i= 1,2,…..,k) first kind of Pareto distribution

$f(x_i;\underline{\mu}_i,\sigma_i)= \sigma_i^{-1}\mu_i^{-1/\sigma_i}\, x_i^{-1/\sigma_i-1}$; $x_i \geq \underline{\mu}_i > 0,\ \sigma_i > 0$, where $\underline{\mu}_i$ and σ_i are respectively , unknown scale and shape parameters. Based on random sample $\underline{X}_{i1},.....,\underline{X}_{in_i}$ of size, for $n_i(\geq 2)$ $X^*_{i(n_i)} = \min(X_{i1},.....,X_{in_i})$, we use

$$\mu^*_{i(n_i)} = \log X^*_{i(n_i)} \text{ and } \hat{\sigma}_{i(n_i)} = (n_i-1)^{-1}\sum_{j=1}^{n_i}\log X_{ij}/X^*_{i(n_i)}$$

as the estimators of $\log\underline{\mu}_i$ and σ_i are respectively.We observe that [see, Wang (1973)] $(A_1)-(A_3)$ are satisfied for r = 2, t = 1, $\underline{\theta}_i = \log\mu_i$, $\psi_i = \sigma_i$, $\hat{\underline{\theta}}_{in_i} = \mu^*_{i(n_i)}$, $\hat{\Psi}_{i(n_i)} = \hat{\sigma}_{i(n_i)}$, q=2, s=1, $Q_i = I_{1\times1} = 1$ and $\delta = 1/2$.

1.4.6. Simultaneous estimation of the means of the several inverse Gaussian populations

Let $(\underline{X}_{ij})$, j=1,2,….. be a sequence of the iidr.v's from an inverse Gaussian distribution

$$f(x_i;\underline{\mu}_i,\lambda_i) = \left\{\lambda_i/2\pi x_i^3\right\}^{1/2}\exp\left\{-\frac{\lambda_i}{2\mu_i^2}\frac{(x_i-\mu)^2}{x_i}\right\};\ x_i>0,$$

where $\underline{\mu}_i \in (-\infty,\infty)$ and $\lambda_i^{-1} \in (0,\infty)$ are the unknown mean and scalar parameters respectively. Given a random sample $\underline{X}_{i1},.....,\underline{X}_{in_i}$ of size, $n_i(\geq 2)$ from the ith population, we use $\overline{X}_{i(n_i)} = n_i^{-1}\sum_{j=1}^{n_i}X_{ij}$, $\hat{\lambda}_{i(n_i)} = (n_i-1)^{-1}\sum_{j=1}^{n_i}(X_{ij}^{-1} - \overline{X}^{-1}_{i(n_i)})$

as the estimators of μ_i and are λ_i^{-1} respectively.We observe that (see, Chaturvedi,1985) $(A_1)-(A_3)$ are satisfied for r = t = 1, $\underline{\theta}_i = \mu_i$, $\psi_i = \lambda_i^{-1}$, $\hat{\underline{\theta}}_{in_i} = \overline{X}^*_{i(n_i)}$, $\hat{\Psi}_{i(n_i)} = \hat{\lambda}_{i(n_i)}$, q = s = δ =1 and $Q_{i(n_i)} = I_{1\times1} = 1$.

1.4.7.Simultaneous estimation problem related to multiple comparison procedures

Let us consider the linear regression models of sections 3. In multiple comparison procedures [see Hochberg and Tamhane(1987)] for a brief discussion], one may be interested in estimating the parametric functions of the components of a lx1 $(1 \leq l \leq p)$ subvector $\underline{\beta}_i^*$ of $\underline{\beta}_i$ .Let $\hat{\underline{\beta}}^*_{i(n_i)}$ be the corresponding subvector of $\hat{\underline{\beta}}_{i(n_i)}$ and V_i is 1x1submatrix of $(X'_{i(n_i)}X_{i(n_i)})^{-1}$ corresponding to the $\hat{\underline{\beta}}^*_{i(n_i)}$ part of $\underline{\beta}_{i(n_i)}$. For a known

p x1 matrix $G_i = (\underline{g'_{i1}},.....,\underline{g'_{ip}})'$, $\underline{\gamma_i} = (\underline{g'}_{i1}\underline{\beta}_i^*,.....,\underline{g'}_{ip}\underline{\beta}_i^*) = G_i\,\underline{\beta}_i^*$. We use the least-squares estimator $\underline{\gamma_i} = G_i\,\underline{\hat{\beta}}^*_{i(n_i)}$ to estimated by $s^2_{i(n_i)} = \nu_i^{-1}\left\|\underline{Y_i} - \underline{\hat{Y}}_{(n_i)}\right\|^2$, where $\underline{\hat{Y}}_{i(n_i)} = \underline{X}_{i(n_i)}\underline{\hat{\beta}}_{i(n_i)}, \nu = n_i - s_{io}$ is the error degree of freedom and s_{io} = rank$(X_{i(n_i)})$ .Assumptions $(A_1)-(A_3)$ are satisfied for r = t = p, $\underline{\theta}_i = \underline{\gamma}_i$, $\psi_i = \sigma_i^2$, $\underline{\hat{\theta}}_{in_i} = \underline{\hat{Y}}_{i(n_i)}$, $\hat{\Psi}_{i(n_i)} = s^2_{i(n_i)}$, q = 1, s = s_{io} , δ =1 and $Q_i = (G_iV_iG_i')^{-1}$.

1.4.8. Simultaneous estimation of the means of several random one-way models

Let us consider the ith (i=1,2,……,k) random one way model:

$$Y_{ijl} = \mu_i + \tau_{ij} + \varepsilon_{ijl},\ \text{j=1,2,…...,}n_i\text{; l= 1,……..,}r_i,$$

where τ_{ij}'s are independently distributed as $N(\theta,\sigma^2_{il})$, ε_{ijl}'s are independently distributed as $N(\theta,\sigma^2_{ie})$, τ_{ij} and ε_{ijl} are independent for all i ,j and l and $\sigma_{ij} \in (0,\infty)$ and $\sigma_{ie} \in (0,\infty)$ are unknown. We assume that r_i-the number of samples per treatment is known. Let $\overline{Y}_i = (n_ir_i)^{-1}\sum_{j=1}^{n_i}\sum_{l=1}^{r_i}Y_{ijl}$ be the over all sample mean from the ith population and MST and MSE denotes, respectively, the usual ANOVA mean squares for the treatments and errors. We notice that the assumptions $(A_1)-(A_3)$ are valid for r = t = 1, $\underline{\theta}_i = \underline{\mu}_i$, $\psi_i = (r_i\sigma^2_{il} + \sigma^2_{ie})$, $\underline{\hat{\theta}}_{in_i} = \underline{\hat{Y}}_i$,

$\hat{\Psi}_{i(n_i)}$=MST = $r_i\sum_{j=1}^{n_i}\frac{(\overline{Y}_{ij.} - \overline{Y}_i)^2}{(n_i-1)}$, q = n_i , s = δ =1 and $Q_i = r_i\,I_{1x1} = r_i$.

References

1. Basu, A.P. (1971). On a sequential rule for estimating the location parameter of an exponential distribution. *Nav.Res.Logist.Quart.,*18,329-337.
2. Bhattacharya, P.K. and Mallik, A. (1973).Asymptotic normality of the stopping times of some sequential procedures. *Ann. Statist.,* 1, 1203-1211.
3. Chaturvedi, A. (1985). Sequential estimation of an inverse Gaussian distribution with prescribed proportional closeness. *Cal. Statist. Assoc. Bull.,* 35, 215-219.
4. Chaturvedi, A. Pandey, S.K. and Gupta, M.(1991).On a class of asymptotically risk-efficient sequential procedures. *Scand, Actur. Jour.,* 1:87-96.
5. Ghosh, D. (2005). Simultaneous estimation procedures and multiple testing: a decision- theoretic framework. *Working Paper Series- Working Paper 54. The University of Michigan- Department of Statistics.*

6. Efron, B. (2004). Selection and estimation for large scale simultaneous inference. estimation of the mean. *Jour. Amer. Statist. Asso.,*96, 96-104.
7. Hall, P. (1983). Sequential estimation saving sampling operations. *Jour.Roy.Statist.Soc.* B 45, 303-310.
8. Hochberg, Y. and Tamhane, A.C. (1987). Multiple comparison procedures. *John Wiley and Sons Inc., U.S.A.*
9. Judge, George G. and Bock, M.E. (1978). The Statistical Economics. *North-HillPublishing Company, New York, U.S.A.*
10. Raatikainer, K.E.E. (1987). Sequential procedure for simultaneous estimation of several percentiles. *Simulation,* Vol. 49, 4,159-164.
11. Raatikainer, K.E.E. (1993). A sequential procedure for simultaneous estimation of several means. *ACM Trans. On Modelling and Comp. Simulation.,* 3,2,108-133.
12. Aoshima, M. and Mukhopdhyay, N. (1998). Fixed width simultaneous confidence intervals for multinormal means in several intraclass correlation models. *Jour.Multiv.Anal.,*66,1, 46-63.
13. Mukhopdhyay, N. (1992). Simultaneous point estimation problems for two parameter negative exponential population. *Jour.Ind.Statist.Assoc.,*30, 33-41.
14. Mukhopdhyay.N, Hamdy, H.I and Darmanto,S. (1988).Simultaneous estimation after selection ranking and other procedures: the negative exponential case. *Metrika,* 35, 275-286.
15. Mukhopdhyay, N. and Solanky, T.K.S. (1998). Multi-stage methodologies for fixed with simultaneous confidence intervals for all pairwise comparisons. *Jour. Stat. Plan. Infer.,*73, 1-2, 163-176.
16. Starr, N. (1966). The performance of a sequential procedure for the fixed width interval estimation of the mean. *Ann.Math.Statist.,*37, 36-50.
17. Wang, Y.H. (1973).Sequential estimation of the scale parameters of the Paretodistribution. *Commun. Statist.-Theo. Math.,* A2, 145-154.
18. Wang, Y.H. (1980). Sequential estimation of the mean of a multinormal population. *Jour. Amer. Statist. Assoc.,* 75, 977-983.
19. Woodroofe, M. (1977). Second order approximations for sequential point and interval *Ann.Statist.,*5, 984-995.