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Type II Exponentiated Half-Logistic-Topp-Leone-G Power Series Class of Distributions with Applications

Thatayaone Moakofi[1*], Broderick Oluyede[2] and Fastel Chipepa[3]

*Corresponding author

1. Department of Mathematics and Statistical Sciences, Botswana International University of Science and Technology, Botswana, thatayaone.moakofi@studentmail.biust.ac.bw
2. Department of Mathematics and Statistical Sciences, Botswana International University of Science and Technology, Botswana, oluyedeo@biust.ac.bw
3. Department of Mathematics and Statistical Sciences, Botswana International University of Science and Technology, Botswana, chipepaf@biust.ac.bw

Abstract

This paper aims to develop a new class of distributions, namely, type II exponentiated half-logistic Topp-Leone-G power series (TIIEHL-TL-GPS) class of distributions. Some important properties including moments, quantiles, moment generating function, entropy and maximum likelihood estimates are derived. A simulation study is conducted to evaluate the consistency of the maximum likelihood estimates. We also present three real data examples to illustrate the usefulness of the new class of distributions. Results show that the proposed model performs better than nested and several non-nested models on selected data sets.

Key Words: Topp-Leone; Power Series; Logarithmic Distribution; Maximum Likelihood Estimation.

Mathematical Subject Classification: 60E05, 62E15.

1. Introduction

The limitations of the well-known standard distributions like Weibull distribution, Lindley distribution, Rayleigh distribution and many others have motivated researchers to generalize and extend existing distributions, in order to offer flexible models in terms of data modeling. Several extensions of distributions available in the literature are the beta Marshall-Olkin family of distributions by Alizadeh (3), Topp-Leone generated family of distributions by Rezaei (26), type II power Topp-Leone generated family of distributions by Bantan et al. (5), sine Topp-Leone-G family of distributions by Al-Babtain et al. (1), Burr X exponential-G family of distributions by Sanusi (28), type II half logistic family of distributions by Soliman et al. (32), type II general inverse exponential family of distributions by Jamal et al. (13), the Zografos–Balakrishnan-G family of distributions by Nadarajah et al. (20), beta Weibull-G by Yousof et al. (35), new power generalized Weibull-G by Oluyede et al. (24), Weibull-G by Bourguignon et al. (8) developed, beta-G by Eugene et al. (10).

In this paper, we propose a new class of distributions TIIEHL-TL-GPS class of distributions. An attractive feature about the model is that the extra parameter introduced has the capability to control both the weights at the tails of the density function. Also, the new class of distributions can model different types of failure rate functions that are available in different areas like reliability, engineering and biological studies. The hazard rate function from the special cases exhibits increasing, decreasing, bathtub and upside bathtub shapes. Of note is the upside bathtub followed by bathtub shape of the hazard rate exhibited in some special cases. The method of estimation used in section 4 was used by Karakaya and Tanis (15), Karakaya and Tanis (16), Tanis and Karakaya (33) and Tanis (34).

The cumulative distribution function (cdf) and probability density function (pdf) of the type II exponentiated half-logistic Topp-Leone-G (TIIEHL-TL-G) family of distributions are given by

$$F(x;a,b,\underline{\psi}) = 1-\left[\frac{1-\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}{1+\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}\right]^a \tag{1}$$

and

$$f(x;a,b,\underline{\psi}) = \frac{4abg(x;\underline{\psi})\left[1-\overline{G}^2(x;\underline{\psi})\right]^{b-1}\overline{G}(x;\underline{\psi})\left(1-\left[1-\overline{G}^2(x;\underline{\psi})\right]^b\right)^{a-1}}{\left(1+\left[1-\overline{G}^2(x;\underline{\psi})\right]^b\right)^{a+1}}, \tag{2}$$

respectively, for $a,b>0$ and parameter vector $\underline{\psi}$.
The basic motivations for developing the type II exponentiated half-logistic Topp-Leone-G power series (TIIEHL-TL-GPS) class of distributions are;

- to construct and generate distributions with symmetric, left-skewed, right-skewed, reversed-J shapes;
- to define special models that posseses various types of hazard rate functions including monotonic as well as non-monotonic shapes;
- to provide consistenly better fits than other generated distributions having the same number of parameters;
- to construct heavy-tailed distributions for modeling different real data sets;
- to make the kurtosis more flexible compared to that of the baseline distributon.

Let N be a zero truncated discrete random variable having a power series distribution, whose probability mass function (pmf) is given by

$$P(N=n) = \frac{a_n\theta^n}{C(\theta)}, n = 1,2,3,..., \tag{3}$$

where $C(\theta)=\sum_{n=1}^{\infty} a_n\theta^n$ is finite, $\theta>0$ and $\{a_n\}_{n\geq 1}$ a sequence of positive real numbers. The power series family of distributions include binomial, Poisson, geometric and logarithmic distributions Johnson et al. (14). Several generalized distributions proposed in the literature involving the power series include the exponentiated power generalized Weibull power series family of distributions by Aldahlan et al. (2), the T–R $\{Y\}$ power series family of probability distributions by Osatohanmwen et al. (25), exponentiated generalized power series class of distributions by Oluyede et al. (22), a new generalized Lindley-Weibull class of distributions by Makubate et al. (17), the odd Weibull-Topp-Leone-G power series family of distributions by Oluyede et al. (21), Weibull-power series distributions by Morais and Barreto-Souza (18), complementary exponential power series by Flores et al. (11), complementary extended Weibull-power series by Cordeiro and Silva (9), Burr XII power series by Silva and Cordeiro (31), extended Weibull-power series (EWPS) distribution by Silva et al. (30) and the Burr-Weibull power series class of distributions by Oluyede et al. (23).

The rest of the paper is organized as follows: In Section 2, we present the new model and some of the statistical properties. We present some special cases of the proposed class of distributions in Section 3. A simulation study is presented in Section 4 and applications in Section 5 followed by concluding remarks.

2. The Model, Sub-Classes and Properties

In this section, we develop the new model, referred to as the type II exponentiated half-logistic Topp-Leone-G power series (TIIEHL-TL-GPS) class of distributions. Some statistical properties which include hazard rate function, quantile function, moments and maximum likelihood estimation of model parameters are derived.

2.1. The Model

Let $X_1,X_2,...,X_N$ be N identically and independently distributed (iid) random variables following the TIIEHL-TL-G distribution. Let $X_{(1)}=\min(X_1,X_2,...,X_N)$, then the cdf of $X_{(1)}|N=n$ is given by

$$F_{X_{(1)}|N=n}(x;a,b,\theta,\underline{\psi})=1-\left(\left[\frac{1-\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}{1+\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}\right]^a\right)^n, \tag{4}$$

for $a,b,\theta>0$, $n\geq 1$ and parameter vector $\underline{\psi}$. The type II exponentiated half-logistic Topp-Leone power series (TIIEHL-TL-GPS) class of distributions denoted by TIIEHL-TL-GPS(a,b, θ, $\underline{\psi}$) is defined by the marginal distribution of $X_{(1)}$, that is,

$$F_{X_{(1)}}(x) \quad = \quad 1-\frac{C\left(\theta\left[\frac{1-\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}{1+\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}\right]^a\right)}{C(\theta)}, \tag{5}$$

for a,b, $\theta>0$ and parameter vector $\underline{\psi}$. The corresponding pdf is given by

$$\begin{aligned} f_{X_{(1)}}(x) \quad &= \quad \frac{4ab\theta g(x;\underline{\psi})\left[1-\overline{G}^2(x;\underline{\psi})\right]^{b-1}\overline{G}(x;\underline{\psi})\left(1-\left[1-\overline{G}^2(x;\underline{\psi})\right]^b\right)^{a-1}}{\left(1+\left[1-\overline{G}^2(x;\underline{\psi})\right]^b\right)^{a+1}} \\ &\times \quad \frac{C'\left(\theta\left[\frac{1-\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}{1+\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}\right]^a\right)}{C(\theta)}. \end{aligned} \tag{6}$$

The hazard rate function (hrf) is given by

$$\begin{aligned} h_F(x) \quad &= \quad \frac{4ab\theta g(x;\underline{\psi})\left[1-\overline{G}^2(x;\underline{\psi})\right]^{b-1}\overline{G}(x;\underline{\psi})\left(1-\left[1-\overline{G}^2(x;\underline{\psi})\right]^b\right)^{a-1}}{\left(1+\left[1-\overline{G}^2(x;\underline{\psi})\right]^b\right)^{a+1}} \\ &\times \quad \frac{C'\left(\theta\left[\frac{1-\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}{1+\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}\right]^a\right)}{C\left(\theta\left[\frac{1-\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}{1+\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}\right]^a\right)}. \end{aligned} \tag{7}$$

Table 1 below presents the special classes of TIIEHL-TL-GPS distribution when $C(\theta)$ is specified in equation (5).

Table 1: Special classes of the TIIEHL-TL-GPS Distribution

Distribution	$C(\theta)$	a_n	cdf
TIIEHL-TL-G Poisson	$e^{\theta}-1$	$(n!)^{-1}$	$1-\frac{\exp\left(\theta\left[\frac{1-\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}{1+\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}\right]^a\right)-1}{\exp(\theta)-1}$
TIIEHL-TL-G Geometric	$\theta(1-\theta)^{-1}$	1	$1-\frac{(1-\theta)\left(\left[\frac{1-\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}{1+\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}\right]^a\right)}{\left(1-\theta\left(\left[\frac{1-\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}{1+\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}\right]^a\right)\right)}$
TIIEHL-TL-G Logarithmic	$-\log(1-\theta)$	n^{-1}	$1-\frac{\log\left(1-\theta\left(\left[\frac{1-\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}{1+\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}\right]^a\right)\right)}{\log(1-\theta)}$
TIIEHL-TL-G Binomial	$(1+\theta)^m-1$	$\binom{m}{n}$	$1-\frac{\left(1+\theta\left(\left[\frac{1-\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}{1+\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}\right]^a\right)\right)^m-1}{(1+\theta)^m-1}$

2.2. Sub-classes of TIIEHL-TL-GPS Family of Distributions

- When $a=1$, we obtain the type II half-logistic Topp-Leone-G power series (TIIHL-TL-GPS) class of distributions with the cdf

$$F(x;b,\theta,\underline{\psi}) = 1-\frac{C\left(\theta\left[\frac{1-\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}{1+\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}\right]\right)}{C(\theta)},$$

  for b, $\theta>0$ and parameter vector $\underline{\psi}$. This is a new class of distributions.

- When $b=1$, we obtain the new class of distributions with the cdf

$$F(x;a,\theta,\underline{\psi}) = 1-\frac{C\left(\theta\left[\frac{\overline{G}^2(x;\underline{\psi})}{1+\left[1-\overline{G}^2(x;\underline{\psi})\right]}\right]^a\right)}{C(\theta)},$$

  for a, $\theta>0$ and parameter vector $\underline{\psi}$.

- When $a=b=1$, we obtain the new class of distributions with the cdf

$$F(x;\theta,\underline{\psi}) = 1-\frac{C\left(\theta\left[\frac{\overline{G}^2(x;\underline{\psi})}{1+\left[1-\overline{G}^2(x;\underline{\psi})\right]}\right]\right)}{C(\theta)},$$

  for $\theta>0$ and parameter vector $\underline{\psi}$.

- When $\theta\rightarrow 0^+$, we obtain the type II exponentiated half-logistic Topp-Leone-G (TIIEHL-TL-G) class of distributions with the cdf

$$F(x;a,b,\underline{\psi}) \quad = \quad 1-\left[\frac{1-\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}{1+\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}\right]^a,$$

for a, $b>0$ and parameter vector $\underline{\psi}$. This is a new class of distributions.

- When $a=1$ and $\theta \to 0^+$, we obtain the type II half-logistic Topp-Leone-G (TIIHL-TL-G) class of distributions with the cdf

$$F(x;b,\underline{\psi}) \quad = \quad 1-\left[\frac{1-\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}{1+\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}\right],$$

for $b>0$ and parameter vector $\underline{\psi}$. This is a new class of distributions.

- When $b=1$ and $\theta \to 0^+$, we obtain the new class of distributions with the cdf

$$F(x;a,\underline{\psi}) \quad = \quad 1-\left[\frac{\overline{G}^2(x;\underline{\psi})}{1+\left[1-\overline{G}^2(x;\underline{\psi})\right]}\right]^a,$$

for $a>0$ and parameter vector $\underline{\psi}$.

- When $a=b=1$ and $\theta \to 0^+$, we obtain the new class of distributions with the cdf

$$F(x;\underline{\psi}) \quad = \quad 1-\left[\frac{\overline{G}^2(x;\underline{\psi})}{1+\left[1-\overline{G}^2(x;\underline{\psi})\right]}\right],$$

for parameter vector $\underline{\psi}$.

2.3. Quantile Function

Let X be a random variable with cdf defined by equation (5). The quantile function $Q_{X_{(1)}}(u)$ is defined by $F_{X_{(1)}}(Q_{X_{(1)}}(u))=u, 0\leq u\leq 1$. Note that

$$1-\frac{C\left(\theta\left[\frac{1-\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}{1+\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}\right]^a\right)}{C(\theta)} \quad = \quad u,$$

so that

$$C\left(\theta\left[\frac{1-\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}{1+\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}\right]^a\right) \quad = \quad C(\theta)\,(1-u).$$

This is equivalent to

$$\left[\frac{1-\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}{1+\left[1-\overline{G}^2(x;\underline{\psi})\right]^b}\right]^a \quad = \quad \frac{C^{-1}\left(C(\theta)\,(1-u)\right)}{\theta},$$

that is,

$$\overline{G}(x;\underline{\psi}) = \left(1-\left(\frac{1-(C(\theta)(1-u))^{\frac{1}{a}}}{1+(C(\theta)(1-u))^{\frac{1}{a}}}\right)^{\frac{1}{b}}\right)^{\frac{1}{2}}.$$

The expression further simplifies to

$$G(x;\underline{\psi}) = 1-\left(1-\left(\frac{1-(C(\theta)(1-u))^{\frac{1}{a}}}{1+(C(\theta)(1-u))^{\frac{1}{a}}}\right)^{\frac{1}{b}}\right)^{\frac{1}{2}}.$$

Therefore, the quantile function of the TIIEHL-TL-GPS class of distributions is given by,

$$Q_{X_{(1)}}(u) = G^{-1}\left[1-\left(1-\left(\frac{1-(C(\theta)(1-u))^{\frac{1}{a}}}{1+(C(\theta)(1-u))^{\frac{1}{a}}}\right)^{\frac{1}{b}}\right)^{\frac{1}{2}}\right]. \quad (8)$$

It follows therefore that random numbers can be generated from the TIIEHL-TL-GPS class of distributions using equation (8) with the aid of statistical software such as R, MATLAB and SAS.

2.4. Expansion of Density

Expansion of the density function of the TIIEHL-TL-GPS class of distributions is presented in this sub-section. The TIIEHL-TL-GPS class of distributions can be expressed as an infinite linear combination of exponentiated-G (Exp-G) densities as

$$f_{X_{(1)}}(x) = \sum_{m=0}^{\infty} \tau_{m+1} g_{m+1}(x;\underline{\psi}), \quad (9)$$

where $g_{m+1}(x;\xi) = (m+1)(G(x;\xi))^m g(x;\xi)$ is the exponentiated-G (Exp-G) distribution with power parameter $m+1$ and

$$\tau_{m+1} = \sum_{n=1}^{\infty}\sum_{j,k,l=0}^{\infty} \frac{4ab\theta n a_n \theta^n}{C(\theta)} \binom{an-1}{j}\binom{an+k}{k}\binom{b(j+k+1)-1}{l} \times \binom{2l+1}{m}\frac{(-1)^{k+j+l+m}}{m+1}. \quad (10)$$

(See Appendix section for details of the derivation)

2.5. Moments and Generating Function

If X follows the TIIEHL-TL-GPS distribution and $Y \sim Exp-G(m+1)$. Then using equation (9), the p^{th} raw moment, μ'_p of the TIIEHL-TL-GPS class of distributions is obtained as

$$\mu'_p = E(X^p) = \int_{-\infty}^{\infty} x^p f(x)dx = \sum_{m=0}^{\infty} \tau_{m+1} E(Y^p),$$

where τ_{m+1} is given by equation (10). The moment generating function (MGF) $M(t) = E(e^{tX})$ is given by:

$$M_X(t) = \sum_{m=0}^{\infty} \tau_{m+1} M_Y(t),$$

where $M_Y(t)$ is the MGF of Y and τ_{m+1} is given by equation (10).

2.6. Order Statistics and Rényi Entropy

In this section, we present the distribution of the i^{th} order statistic and Rényi entropy.

2.6.1. Distribution of Order Statistics

Order statistics are fundamental in many areas of statistical theory and practice. Let $X_1, X_2, ..., X_n$ be a random sample from TIIEHL-TL-GPS class of distributions. Then, the distribution of the k^{th} order statistics from TIIEHL-TL-GPS class of distributions is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\sum_{m=0}^{\infty}\sum_{p=0}^{n-i}\binom{n-i}{p}(-1)^p h_{m+1} g_{m+1}(x;\underline{\psi}), \tag{11}$$

where $g_{m+1}(x;\underline{\psi}) = (m+1)g(x;\underline{\psi})G^m(x;\underline{\psi})$ is an Exp-G with power parameter $m+1$ and the linear component

$$\begin{aligned} h_{m+1} &= \sum_{n,z=1}^{\infty}\sum_{j,k,l=0}^{\infty}\frac{4ab\theta n a_n d_{q,z}\theta^{n+z}(-1)^{q+j+l+m}}{C^{q+1}(\theta)\,(m+1)}\binom{i+p-1}{q} \\ &\times \binom{a(n+z)-1}{j}\binom{a(n+z)+k}{k}\binom{b(j+k+1)-1}{l}\binom{2l+1}{m}. \end{aligned}$$

(See Appendix section for details of the derivation)

2.6.2. Rényi Entropy

In this subsection, Rényi entropy for TIIEHL-TL-GPS class of distributions is derived. An entropy is a measure of uncertainty or variation of a random variable. Rényi entropy (27) is a generalization of Shannon entropy (29). Rényi entropy of the TIIEHL-TL-GPS distribution is defined by

$$I_R(v) = \frac{1}{1-v}\log\left(\sum_{m=0}^{\infty} w^* e^{(1-v)I_{REG}}\right), \tag{12}$$

where $I_{REG} = \int_0^{\infty}[(1+m/v)g(x;\underline{\psi})G^{m/v}]^v dx$ is Rényi entropy for an Exp-G distribution with power parameter $m/\nu+1$ and

$$\begin{aligned} w^* &= \sum_{n=1}^{\infty}\sum_{j,k,l,m=0}^{\infty}(-1)^{k+j+l+m}\frac{n d_{v,n}\theta^{v+n-1}}{(C(\theta))^v}(4ab)^v\binom{b(v+j+k)-v}{l} \\ &\times \binom{a(v+n-1)-v}{j}\binom{a(v+n-1)+v-1+k}{k} \\ &\times \binom{2l+v}{m}\frac{1}{(1+m/\nu)^v}. \end{aligned} \tag{13}$$

Consequently, Rényi entropy for TIIEHL-TL-GPS class of distributions can be obtained from Rényi entropy of the Exp-G distribution (See Appendix section for details of the derivation).

2.7. Maximum Likelihood Estimation

Here we use the maximum likelihood estimation technique to find the maximum likelihood estimates of the parameters of the TIIEHL-TL-GPS class of distributions. Let $X_i \sim TIIEHL-TL-GPS(x;a,b,\theta,\underline{\psi})$ and $\Delta = (a,b,\theta,\underline{\psi})^T$ be the vector of unknown parameters. The total log-likelihood $\ell = \ell(\Delta)$ function is given by

$$\begin{aligned}
\ell(\Delta) &= n\ln(4ab\theta) + (b-1)\sum_{i=1}^{n}\ln\left[1-\overline{G}^2(x_i;\underline{\psi})\right] - n\ln(C(\theta)) \\
&+ (a-1)\sum_{i=1}^{n}\left(1-\left[1-\overline{G}^2(x_i;\underline{\psi})\right]^b\right) - (a+1)\sum_{i=1}^{n}\ln\left(1+\left[1-\overline{G}^2(x_i;\underline{\psi})\right]^b\right) \\
&+ \sum_{i=1}^{n}\ln\left(C'\left(\theta\left[\frac{1-\left[1-\overline{G}^2(x_i;\underline{\psi})\right]^b}{1+\left[1-\overline{G}^2(x_i;\underline{\psi})\right]^b}\right]^a\right)\right) \\
&+ \sum_{i=1}^{n}\ln\left(g(x_i;\underline{\psi})\right) + \sum_{i=1}^{n}\ln\left(\overline{G}(x_i;\underline{\psi})\right).
\end{aligned}$$

The maximum likelihood estimates of the parameters, denoted by $\hat{\boldsymbol{\Delta}}$ is obtained by solving the nonlinear equation $(\frac{\partial \ell_n}{\partial a}, \frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \theta}, \frac{\partial \ell_n}{\partial \underline{\psi}_k})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The multivariate normal distribution $N_{q+3}(\underline{0}, J(\hat{\boldsymbol{\Delta}})^{-1})$, where the mean vector $\underline{\mathbf{0}} = (0,0,0,\underline{0})^T$ and $J(\hat{\boldsymbol{\Delta}})^{-1}$ is the observed Fisher information matrix evaluated at $\hat{\boldsymbol{\Delta}}$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions.

3. Some Special Sub-classes of the TIIEHL-TL-GPS Class of Distributions

In this section, special classes of TIIEHL-TL-GPS class of distributions are presented by specifying the baseline cdf $G(x;\underline{\psi})$ and pdf $g(x;\underline{\psi})$ in equations (5) and (6).

3.1. Type II Exponentiated Half-Logistic Topp-Leone-Log-Logistic Power Series (TIIEHL-TL-LLoGPS) Class of Distributions

If the baseline cdf and pdf are given by $G(x;c) = 1-(1+x^c)^{-1}$ and $g(x;c) = cx^{c-1}(1+x^c)^{-2}$, for $c > 0$, and $x > 0$, then the cdf and pdf of the TIIEHL-TL-LLoGPS class of distributions are given by

$$F_{X_{(1)}}(x) = 1 - \frac{C\left(\theta\left[\frac{1-\left[1-(1+x^c)^{-2}\right]^b}{1+\left[1-(1+x^c)^{-2}\right]^b}\right]^a\right)}{C(\theta)}, \tag{14}$$

and

$$\begin{aligned}
f_{X_{(1)}}(x) &= \frac{4ab\theta cx^{c-1}(1+x^c)^{-2}\left[1-(1+x^c)^{-2}\right]^{b-1}(1+x^c)^{-1}}{\left(1+\left[1-(1+x^c)^{-2}\right]^b\right)^{a+1}} \\
&\times \left(1-\left[1-(1+x^c)^{-2}\right]^b\right)^{a-1}\frac{C'\left(\theta\left[\frac{1-\left[1-(1+x^c)^{-2}\right]^b}{1+\left[1-(1+x^c)^{-2}\right]^b}\right]^a\right)}{C(\theta)},
\end{aligned} \tag{15}$$

respectively. The hrf is given by

$$h_F(x) = \frac{4ab\theta cx^{c-1}(1+x^c)^{-2}\left[1-(1+x^c)^{-2})\right]^{b-1}(1+x^c)^{-1}}{\left(1+\left[1-(1+x^c)^{-2}\right]^b\right)^{a+1}} \times \left(1-\left[1-(1+x^c)^{-2}\right]^b\right)^{a-1} \frac{C'\left(\theta\left[\frac{1-\left[1-(1+x^c)^{-2}\right]^b}{1+\left[1-(1+x^c)^{-2}\right]^b}\right]^a\right)}{C\left(\theta\left[\frac{1-\left[1-(1+x^c)^{-2})\right]^b}{1+\left[1-(1+x^c)^{-2})\right]^b}\right]^a\right)}, \quad (16)$$

for a, b, θ, c and $x > 0$.

3.1.1. Type II Exponentiated Half-Logistic Topp-Leone-Log-Logistic Logarithmic (TIIEHL-TL-LLoGL) Distribution

The cdf and pdf of TIIEHL-TL-LLoGL distribution are given by

$$F_{X_{(1)}}(x) = 1 - \frac{-\log\left(1-\theta\left[\frac{1-\left[1-(1+x^c)^{-2})\right]^b}{1+\left[1-(1+x^c)^{-2})\right]^b}\right]^a\right)}{-\log(1-\theta)},$$

and

$$f_{X_{(1)}}(x) = \frac{4ab\theta cx^{c-1}(1+x^c)^{-2}\left[1-(1+x^c)^{-2})\right]^{b-1}(1+x^c)^{-1}}{\left(1+\left[1-(1+x^c)^{-2}\right]^b\right)^{a+1}} \times \left(1-\left[1-(1+x^c)^{-2}\right]^b\right)^{a-1} \frac{\left(1-\theta\left[\frac{1-\left[1-(1+x^c)^{-2}\right]^b}{1+\left[1-(1+x^c)^{-2}\right]^b}\right]^a\right)^{-1}}{-\log(1-\theta)},$$

respectively. The hrf is given by

$$h_F(x) = \frac{4ab\theta cx^{c-1}(1+x^c)^{-2}\left[1-(1+x^c)^{-2})\right]^{b-1}(1+x^c)^{-1}}{\left(1+\left[1-(1+x^c)^{-2}\right]^b\right)^{a+1}} \times \left(1-\left[1-(1+x^c)^{-2}\right]^b\right)^{a-1} \frac{\left(1-\theta\left[\frac{1-\left[1-(1+x^c)^{-2}\right]^b}{1+\left[1-(1+x^c)^{-2}\right]^b}\right]^a\right)^{-1}}{-\log\left(1-\theta\left[\frac{1-\left[1-(1+x^c)^{-2})\right]^b}{1+\left[1-(1+x^c)^{-2})\right]^b}\right]^a\right)},$$

for a, b, θ, c and $x > 0$.

Figure 1 shows the pdfs of the TIIEHL-TL-LLoGL distribution. The pdf can take various shapes that include almost symmetric, reverse-J, left, or right-skewed. Furthermore, the hazard rate functions (hrfs) for the TIIEHL-TL-LLoGL distribution exhibit increasing, reverse-J, bathtub, upside bathtub and upside bathtub followed by bathtub shapes.

We present in Figures 2 and 3, 3D plots of skewness and kurtosis of the TIIEHL-TL-LLoGL distribution. We observe that

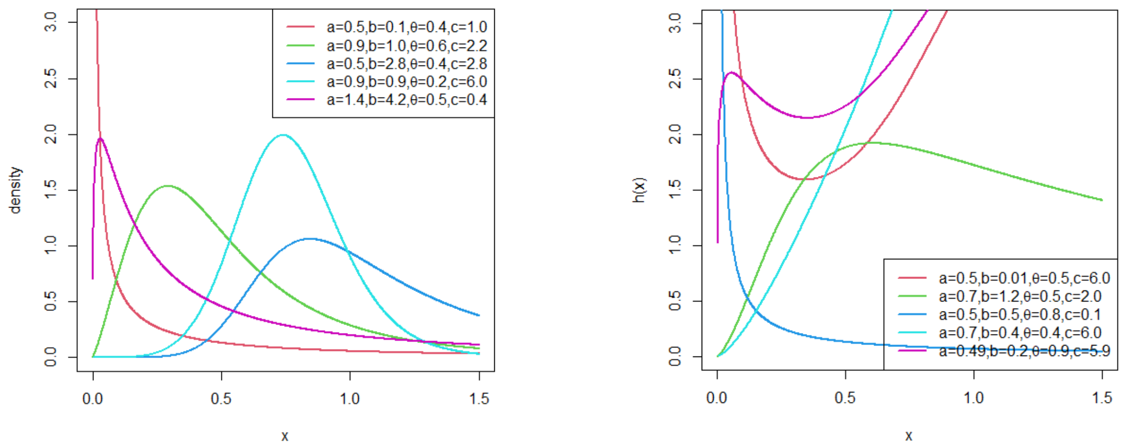

Figure 1: Plots of the pdf and hrf for the TIIEHL-TL-LLoGL distribution

- When we fix the parameters θ and a, the skewness and kurtosis of the TIIEHL-TL-LLoGL distribution decreases as b and λ increase.

- When we fix the parameters θ and λ, the skewness and kurtosis of the TIIEHL-TL-LLoGL distribution increases as a and b increase.

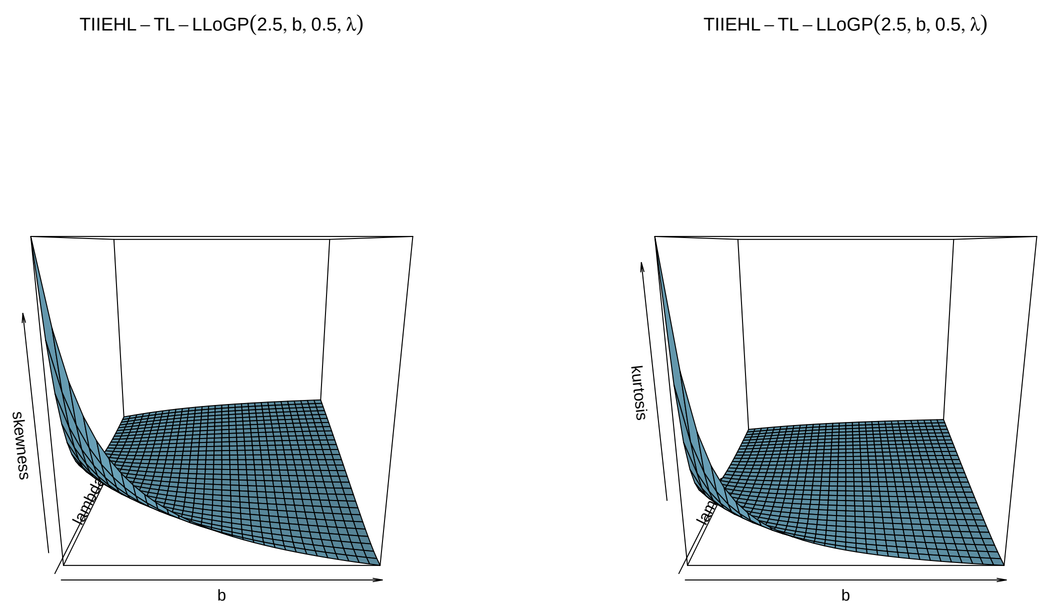

Figure 2: 3 D Plots of skewness and kurtosis for TIIEHL-TL-LLoGL distribution

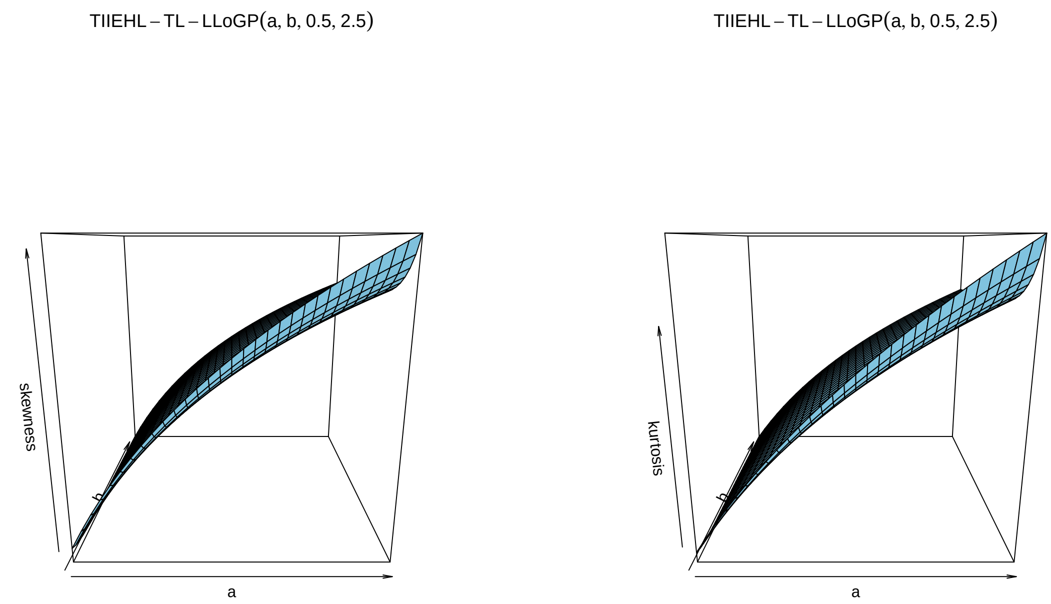

Figure 3: 3 D Plots of skewness and kurtosis for TIIEHL-TL-LLoGL distribution

3.1.2. Type II Exponentiated Half-Logistic Topp-Leone-Log-Logistic Poisson (TIIEHL-TL-LLoGP) Distribution

The cdf and pdf of TIIEHL-TL-LLoGLP distribution are given by

$$F_{X_{(1)}}(x) \quad = \quad 1-\frac{\exp\left(\theta\left[\frac{1-\left[1-(1+x^{c})^{-2})\right]^{b}}{1+\left[1-(1+x^{c})^{-2})\right]^{b}}\right]^{a}-1\right)}{\exp(\theta-1)},$$

and

$$\begin{aligned} f_{X_{(1)}}(x) \quad &= \quad \frac{4ab\theta cx^{c-1}\left(1+x^{c}\right)^{-2}\left[1-\left(1+x^{c}\right)^{-2})\right]^{b-1}\left(1+x^{c}\right)^{-1}}{\left(1+\left[1-\left(1+x^{c}\right)^{-2}\right]^{b}\right)^{a+1}} \\ &\times \quad \left(1-\left[1-\left(1+x^{c}\right)^{-2}\right]^{b}\right)^{a-1}\frac{\exp\left(\theta\left[\frac{1-\left[1-(1+x^{c})^{-2}\right]^{b}}{1+\left[1-(1+x^{c})^{-2}\right]^{b}}\right]^{a}\right)}{\exp(\theta-1)}, \end{aligned}$$

respectively. The hrf is given by

$$
\begin{aligned}
h_F(x) &= \frac{4ab\theta c x^{c-1}(1+x^c)^{-2}\left[1-(1+x^c)^{-2}\right]^{b-1}(1+x^c)^{-1}}{\left(1+\left[1-(1+x^c)^{-2}\right]^b\right)^{a+1}} \\
&\times \left(1-\left[1-(1+x^c)^{-2}\right]^b\right)^{a-1} \frac{\exp\left(\theta\left[\frac{1-\left[1-(1+x^c)^{-2}\right]^b}{1+\left[1-(1+x^c)^{-2}\right]^b}\right]^a\right)}{\exp\left(\theta\left[\frac{1-\left[1-(1+x^c)^{-2}\right]^b}{1+\left[1-(1+x^c)^{-2}\right]^b}\right]^a-1\right)},
\end{aligned}
$$

for a, b, θ, c and $x > 0$.

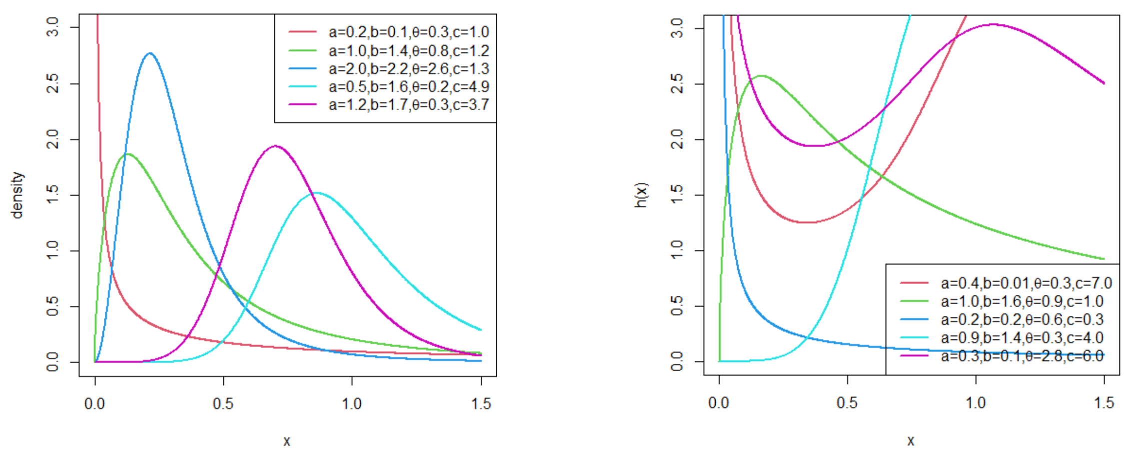

Figure 4: Plots of the pdf and hrf for the TIIEHL-TL-LLoGP distribution

Figure 4 shows the pdfs of the TIIEHL-TL-LLoGP distribution. The pdf can take various shapes that include almost symmetric, reverse-J, left, or right-skewed. Furthermore, the hazard rate functions (hrfs) for the TIIEHL-TL-LLoGP distribution exhibit increasing, reverse-J, bathtub, upside bathtub and bathtub followed by upside bathtub shapes.

We present in Figures 5 and 6, 3D plots of skewness and kurtosis of the TIIEHL-TL-LLoGP distribution. We observe that

- When we fix the parameters θ and a, the skewness and kurtosis of the TIIEHL-TL-LLoGP distribution decreases as b and λ increase.
- When we fix the parameters θ and λ, the skewness and kurtosis of the TIIEHL-TL-LLoGP distribution increases as a and b increase.

3.2. Type II Exponentiated Half-Logistic Topp-Leone-Weibull Power Series (TIIEHL-TL-WPS) Class of Distributions

Suppose the cdf and pdf of the Weibull distribution are given by $G(x;\lambda) = 1-\exp\left(-x^{\lambda}\right)$, for $x \geq 0, \lambda > 0$ and $g(x;\lambda) = \lambda x^{\lambda-1}\exp\left(-x^{\lambda}\right)$, for $\lambda > 0$, and $x > 0$, then, the cdf and pdf of the TIIEHL-TL-WGPS class of distributions are given by

$$
F_{X_{(1)}}(x) = 1-\frac{C\left(\theta\left[\frac{1-\left[1-\exp\left(-2x^{\lambda}\right)\right]^b}{1+\left[1-\exp\left(-2x^{\lambda}\right)\right]^b}\right]^a\right)}{C(\theta)},
$$

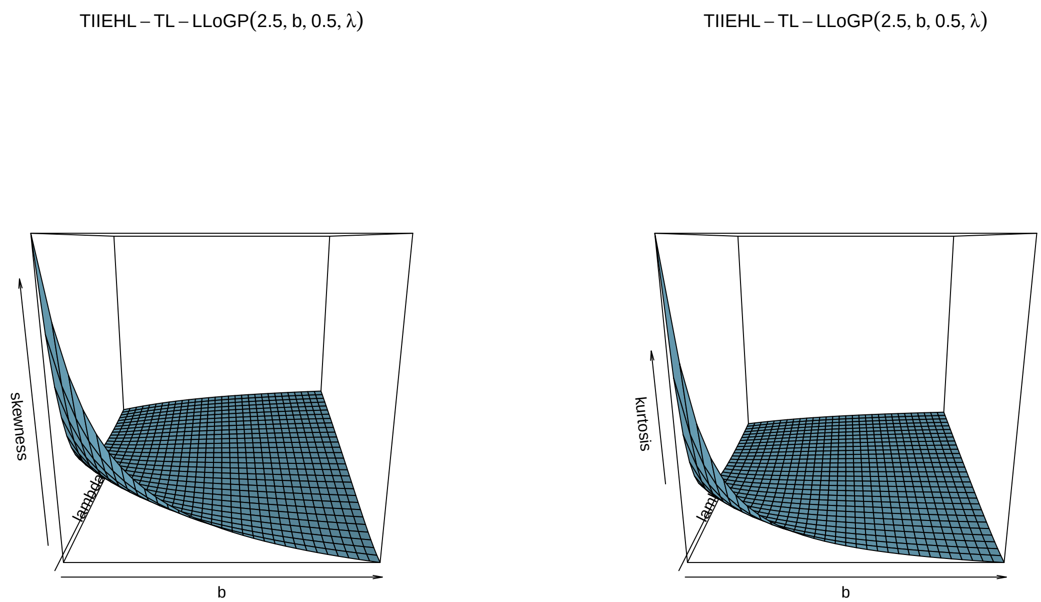

Figure 5: 3 D Plots of skewness and kurtosis for TIIEHL-TL-LLoGP distribution

and

$$f_{X_{(1)}}(x) = \frac{4ab\theta\lambda x^{\lambda-1}\exp\left(-x^{\lambda}\right)\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b-1}\exp\left(-x^{\lambda}\right)}{\left(1+\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}\right)^{a+1}} \times \left(1-\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}\right)^{a-1}\frac{C'\left(\theta\left[\frac{1-\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}{1+\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}\right]^{a}\right)}{C(\theta)},$$

respectively. The hrf is given by

$$h_F(x) = \frac{4ab\theta\lambda x^{\lambda-1}\exp\left(-x^{\lambda}\right)\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b-1}\exp\left(-x^{\lambda}\right)}{\left(1+\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}\right)^{a+1}} \times \left(1-\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}\right)^{a-1}\frac{C'\left(\theta\left[\frac{1-\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}{1+\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}\right]^{a}\right)}{C\left(\theta\left[\frac{1-\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}{1+\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}\right]^{a}\right)},$$

for a, b, θ, λ and $x > 0$.

3.2.1. Type II Exponentiated Half-Logistic Topp-Leone-Weibull Logarithmic (TIIEHL-TL-WL) Distribution

The cdf and pdf of TIIEHL-TL-WL distribution are given by

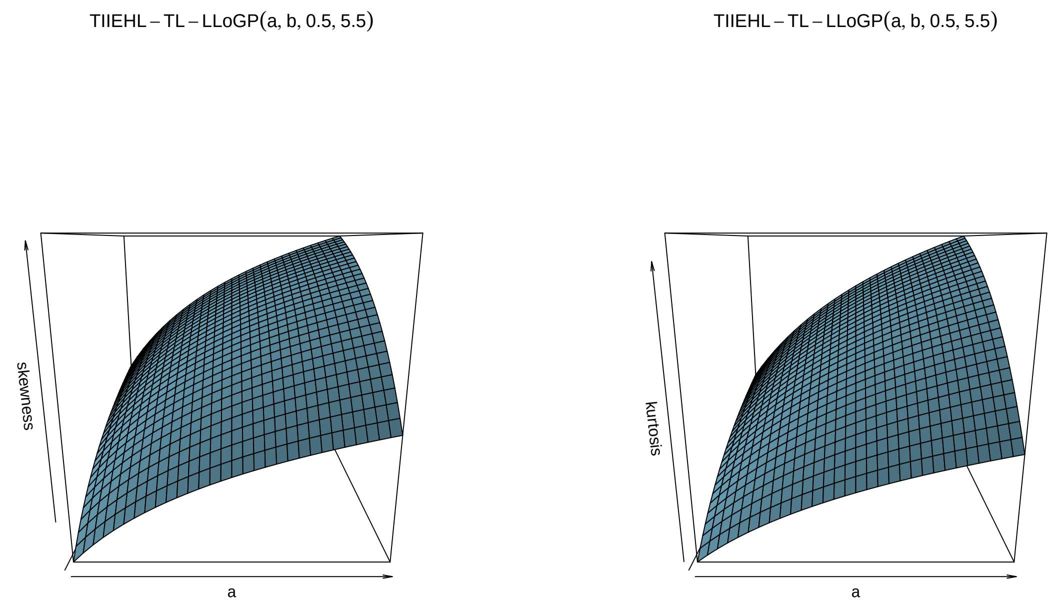

Figure 6: 3 D Plots of skewness and kurtosis for TIIEHL-TL-LLoGP distribution

$$F_{X_{(1)}}(x) = 1 - \frac{-\log\left(1-\theta\left[\frac{1-\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}{1+\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}\right]^{a}\right)}{-\log(1-\theta)},$$

and

$$\begin{aligned} f_{X_{(1)}}(x) &= \frac{4ab\theta\lambda x^{\lambda-1}\exp\left(-x^{\lambda}\right)\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b-1}\exp\left(-x^{\lambda}\right)}{\left(1+\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}\right)^{a+1}} \\ &\times \left(1-\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}\right)^{a-1}\frac{\left(1-\theta\left[\frac{1-\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}{1+\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}\right]^{a}\right)^{-1}}{-\log(1-\theta)}, \end{aligned}$$

respectively. The hrf is given by

$$\begin{aligned} h_{F}(x) &= \frac{4ab\theta\lambda x^{\lambda-1}\exp\left(-x^{\lambda}\right)\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b-1}\exp\left(-x^{\lambda}\right)}{\left(1+\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}\right)^{a+1}} \\ &\times \left(1-\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}\right)^{a-1}\frac{\left(1-\theta\left[\frac{1-\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}{1+\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}\right]^{a}\right)^{-1}}{-\log\left(1-\theta\left[\frac{1-\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}{1+\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}\right]^{a}\right)}, \end{aligned}$$

for a, b, θ, λ and $x > 0$. Figure 7 shows the pdfs of the TIIEHL-TL-WL distribution. The pdf can take various shapes

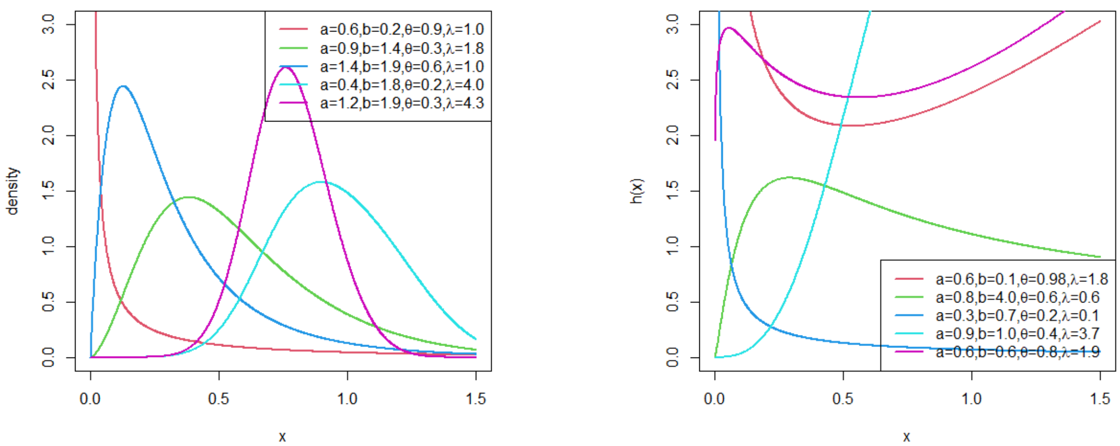

Figure 7: Plots of the pdf and hrf for the TIIEHL-TL-WL distribution

that include almost symmetric, reverse-J, left, or right-skewed. Furthermore, the hrfs for the TIIEHL-TL-WL distribution exhibit increasing, reverse-J, bathtub, upside bathtub and upside bathtub followed by bathtub shapes.

We present in Figures 8 and 9, 3D plots of skewness and kurtosis of the TIIEHL-TL-WL distribution. We observe that

- When we fix the parameters θ and λ, the skewness and kurtosis of the TIIEHL-TL-WL distribution increases as a and b increase.

- When we fix the parameters θ and b, the skewness and kurtosis of the TIIEHL-TL-WL distribution increases as a and λ increase.

3.2.2. Type II Exponentiated Half-Logistic Topp-Leone-Weibull Poisson (TIIEHL-TL-WP) Distribution

The cdf and pdf of TIIEHL-TL-WP distribution are given by

$$F_{X_{(1)}}(x) = 1 - \frac{\exp\left(\theta\left[\frac{1-\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}{1+\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}\right]^{a} - 1\right)}{\exp(\theta - 1)},$$

and

$$\begin{aligned} f_{X_{(1)}}(x) &= \frac{4ab\theta\lambda x^{\lambda-1}\exp\left(-x^{\lambda}\right)\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b-1}\exp\left(-x^{\lambda}\right)}{\left(1+\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}\right)^{a+1}} \\ &\times \left(1-\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}\right)^{a-1}\frac{\exp\left(\theta\left[\frac{1-\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}{1+\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}\right]^{a}\right)}{\exp(\theta-1)}, \end{aligned}$$

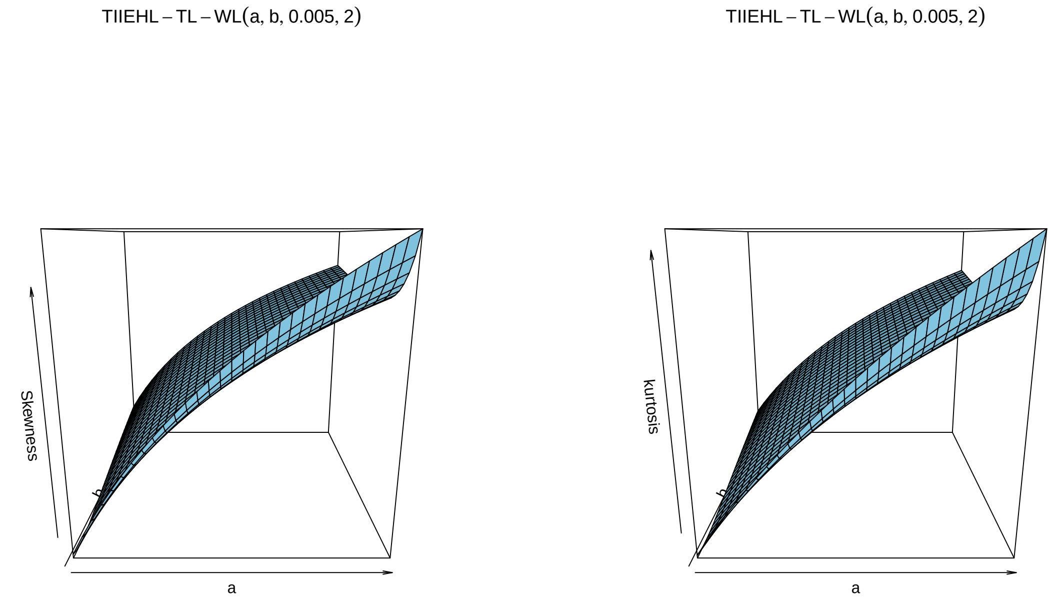

Figure 8: 3 D Plots of skewness and kurtosis for TIIEHL-TL-WL distribution

respectively. The hrf is given by

$$
\begin{aligned}
h_F(x) &= \frac{4ab\theta\lambda x^{\lambda-1}\exp\left(-x^{\lambda}\right)\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b-1}\exp\left(-x^{\lambda}\right)}{\left(1+\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}\right)^{a+1}} \\
&\times \left(1-\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}\right)^{a-1} \frac{\exp\left(\theta\left[\frac{1-\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}{1+\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}\right]^{a}\right)}{\exp\left(\theta\left[\frac{1-\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}{1+\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}\right]^{a}-1\right)},
\end{aligned}
$$

for a, b, θ, λ and $x > 0$. Figure 10 shows the pdfs of the TIIEHL-TL-WP distribution. The pdf can take various shapes that include almost symmetric, reverse-J, left, or right-skewed. Furthermore, the hrfs for the TIIEHL-TL-WP distribution exhibit increasing, reverse-J, bathtub, upside bathtub and upside bathtub followed by bathtub shapes.

We present in Figures 11 and 12, 3D plots of skewness and kurtosis of the TIIEHL-TL-WP distribution. We observe that

- When we fix the parameters θ and a, the skewness and kurtosis of the TIIEHL-TL-WP distribution decreases as b and λ increase.

- When we fix the parameters θ and λ, the skewness and kurtosis of the TIIEHL-TL-WP distribution increases as a and b increase.

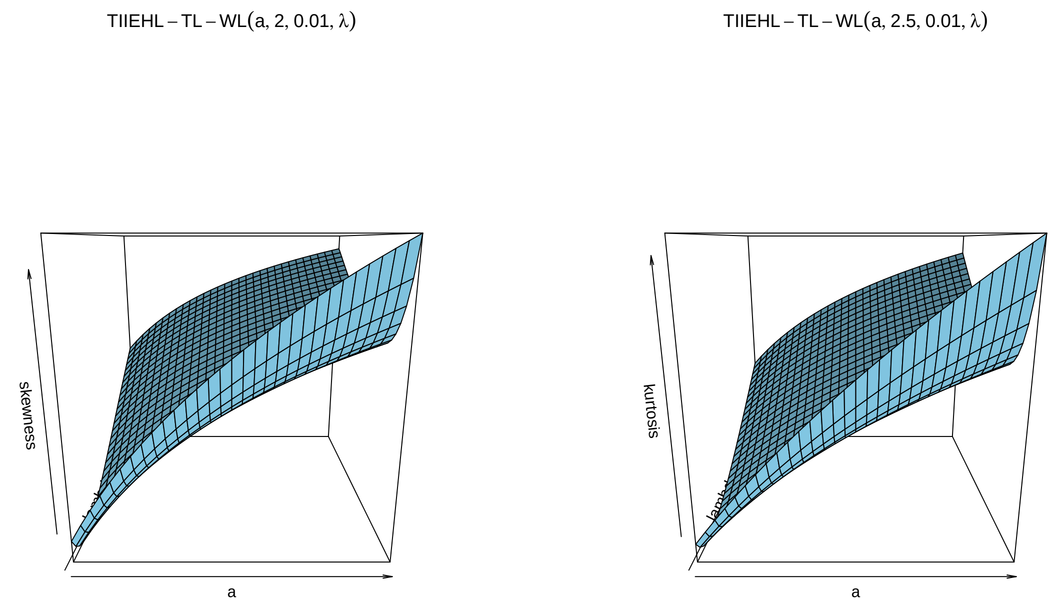

Figure 9: 3 D Plots of skewness and kurtosis for TIIEHL-TL-WL distribution

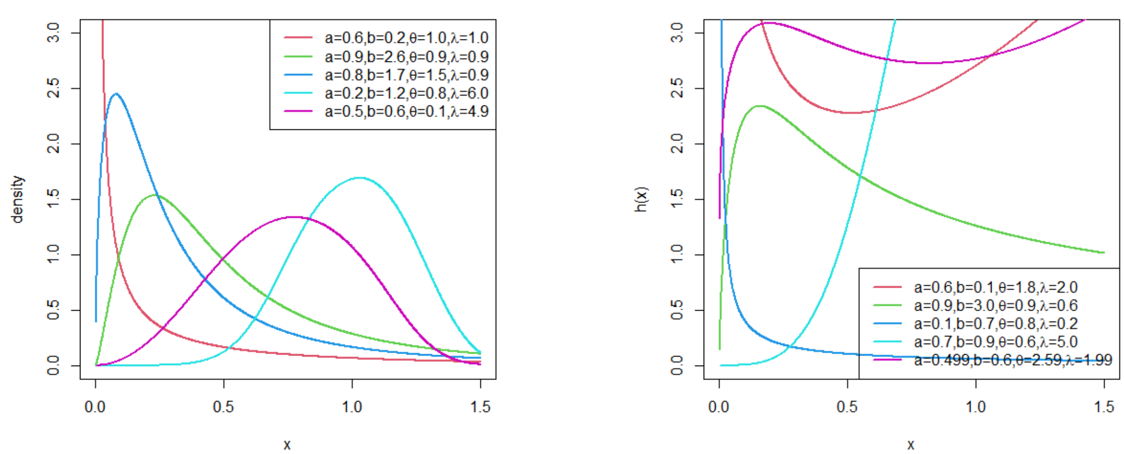

Figure 10: Plots of the pdf and hrf for the TIIEHL-TL-WP distribution

4. Simulation Study

In this section, the performance of the TIIEHL-TL-WL distribution is examined by conducting various simulations for different sizes (n=25, 50, 100, 200, 400 and 800). We simulate $N = 1000$ samples for the true parameters values given in Table 2. The table lists the mean MLEs of the model parameters along with the respective bias and root mean squared errors (RMSEs). The precision of the MLEs is discussed by means of the following measures: mean, mean square error (MSE) and average bias.

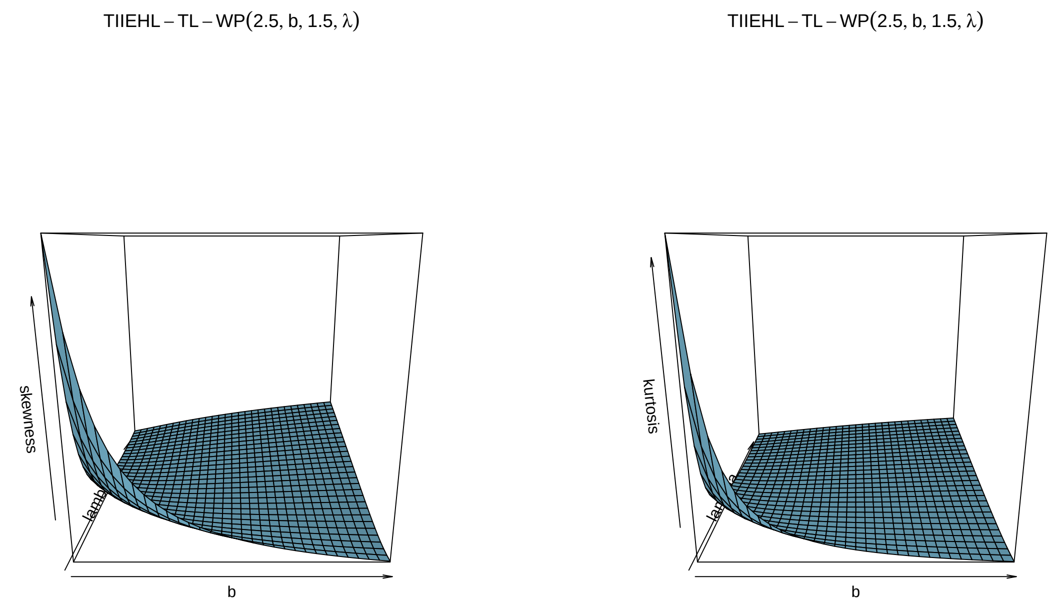

Figure 11: 3 D Plots of skewness and kurtosis for TIIEHL-TL-WL distribution

The bias and RMSE for the estimated parameter, say, $\hat{\theta}$, are given by:

$$Bias(\hat{\theta}) = \frac{\sum_{i=1}^{N} \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2}{N}},$$

respectively. The results show that the estimation method used is appropriate for estimating the TIIEHL-TL-WL model parameters as the means of the parameters tend to be closer to the true parameter values when n increases.

5. Application

In this section, we present two real data examples to demonstrate the importance and applicability of the TIIEHL-TL-WL distribution. The R software was used for data fitting and model diagnostics. The following goodness-of-fit statistics Cramer-von-Mises (W^*) and Andersen-Darling (A^*), -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Kolmogorov-Smirnov (K-S) statistic (and it's p-value), and sum of squares (SS) are used to assess the performance of the model. The model with the smallest values of the goodness-of-fit statistics and a bigger p-value for the K-S statistic is regarded as the best model.

The TIIEHL-TL-WL distribution was compared to its nested models and to the following non-nested models:
type II exponentiated half logistic Weibull (TIIEHLW) distribution by Al-Mofleh et al. (4) with the pdf

$$\begin{aligned} f_{TIIEHLW}(x;a,\lambda,\delta,\gamma,) &= 2a\lambda\delta\gamma x^{\gamma-1}\exp(-\delta x^{\gamma})\left[1-\exp(-\delta x^{\gamma})\right]^{\lambda-1} \\ &\times \frac{\left[1-\left[1-\exp(-\delta x^{\gamma})\right]^{\lambda}\right]^{a-1}}{\left[1+\left[1-\exp(-\delta x^{\gamma})\right]^{\lambda}\right]^{a+1}}, \end{aligned}$$

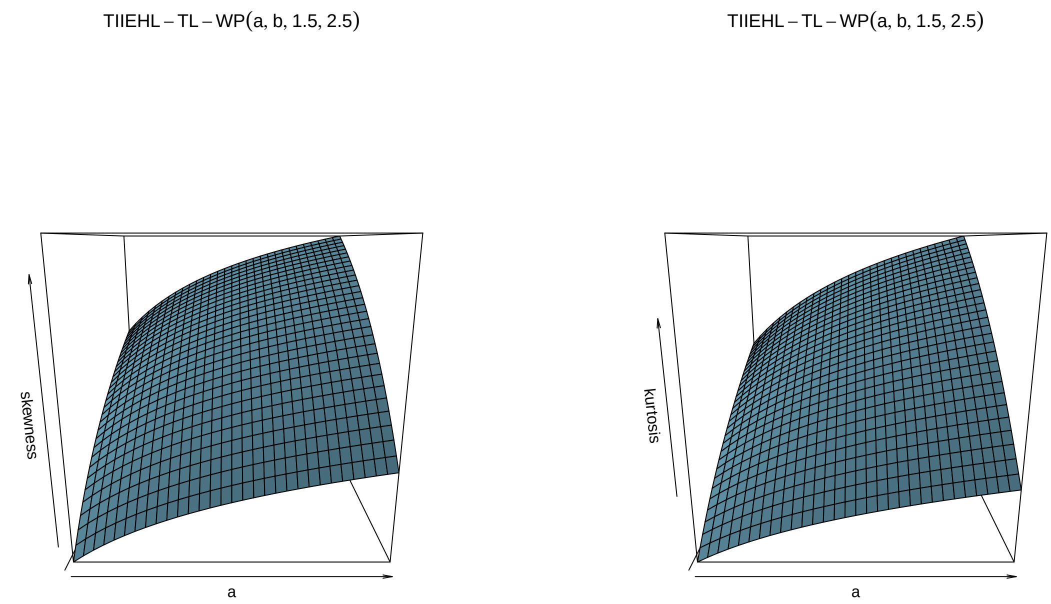

Figure 12: 3 D Plots of skewness and kurtosis for TIIEHL-TL-WL distribution

Table 2: Monte Carlo Simulation Results

		(1.5, 1.0, 0.9, 0.6)			(1.0, 0.5, 0.5, 1.0)			(0.7, 0.4, 0.4, 1.0)		
parameter	Sample Size	Mean	RMSE	Bias	Mean	RMSE	Bias	Mean	RMSE	Bias
a	25	1.9519	1.8074	0.4519	1.5811	0.9439	0.5811	0.7142	0.5022	0.0142
	50	1.9491	1.4855	0.4491	1.3136	0.6695	0.3136	0.8110	1.0306	0.1110
	100	1.7883	1.2965	0.2883	1.2610	0.5273	0.2610	0.7431	0.2660	0.0431
	200	1.6299	0.8137	0.1299	1.1773	0.3629	0.1773	0.7219	0.1962	0.0219
	400	1.5915	0.6830	0.0915	1.1275	0.2876	0.1275	0.7045	0.1395	0.0045
	800	1.5061	0.3518	0.0061	1.0643	0.1726	0.0643	0.7030	0.0990	0.0030
b	25	1.3639	1.0016	0.3639	0.4056	0.2439	-0.0943	0.3518	0.3813	-0.0481
	50	1.2581	0.7241	0.2581	0.4474	0.2104	-0.0525	0.3186	0.1624	-0.0813
	100	1.1998	0.7133	0.1998	0.4516	0.1688	-0.0483	0.3188	0.1349	-0.0811
	200	1.0563	0.4182	0.0563	0.4577	0.1595	-0.0422	0.3220	0.1230	-0.0779
	400	1.0169	0.3898	0.0169	0.4723	0.1321	-0.0276	0.3305	0.1077	-0.0694
	800	0.9833	0.2126	-0.0166	0.5074	0.1090	0.0074	0.3527	0.0827	-0.0472
θ	25	0.7043	0.3329	-0.1956	0.7530	0.3573	0.2530	0.7075	0.4034	0.3075
	50	0.7716	0.2709	-0.1283	0.7483	0.3547	0.2483	0.7115	0.4024	0.3115
	100	0.7890	0.2659	-0.1109	0.7229	0.3299	0.2229	0.6827	0.3751	0.2827
	200	0.8380	0.2141	-0.0619	0.6908	0.3039	0.1908	0.6611	0.3553	0.2611
	400	0.8810	0.1298	-0.0189	0.6606	0.2835	0.1606	0.6409	0.3310	0.2409
	800	0.8972	0.1220	-0.0027	0.5940	0.2620	0.0940	0.5683	0.2905	0.1683
λ	25	0.8853	0.8499	0.2853	1.5311	0.9999	0.5311	1.2960	0.5125	0.2960
	50	0.7385	0.4875	0.1385	1.3954	0.7584	0.3954	1.2479	0.4104	0.2479
	100	0.7017	0.3609	0.1017	1.3538	0.5857	0.3538	1.1968	0.3186	0.1968
	200	0.6890	0.2650	0.0890	1.2874	0.4607	0.2874	1.1580	0.2651	0.1580
	400	0.6571	0.1780	0.0571	1.2136	0.3445	0.2136	1.1145	0.1936	0.1145
	800	0.6535	0.1411	0.0535	1.0962	0.1973	0.0962	1.0716	0.1362	0.0716

for $a, \lambda, \delta, \gamma > 0$ and $x > 0$,
type II general inverse exponential Burr III (TIIGIE-BIII) distribution by Jamal et al. (13) with the pdf

$$\begin{aligned} f_{TIIGIE-BIII}(x;\lambda,\theta,c,k) &= \frac{\lambda\theta ckx^{-c-1}\left(1+x^{-c}\right)^{-k-1}\left[1-\left(1+x^{-c}\right)^{-k}\right]^{\theta-1}}{\left(1-\left[1-\left(1+x^{-c}\right)^{-k}\right]^{\theta}\right)^{2}} \\ &\times \exp\left(-\lambda\frac{\left[1-\left(1+x^{-c}\right)^{-k}\right]^{\theta}}{1-\left[1-\left(1+x^{-c}\right)^{-k}\right]^{\theta}}\right), \end{aligned}$$

for $\lambda, \theta, c, k > 0$ and $x > 0$, and
type II general inverse exponential Lomax (TIIGIE-Lx) distribution by Hamedani et al. (12) with the pdf

$$\begin{aligned} f_{TIIGIE-Lx}(x;\lambda,\alpha,a,b) &= \lambda\alpha\frac{a}{b}\left(1+\frac{x}{b}\right)^{-(a+1)}\left(1+\frac{x}{b}\right)^{a(\alpha+1)} \\ &\times \exp\left(\lambda\left(1-\left(1+\frac{x}{b}\right)^{a\alpha}\right)\right), \end{aligned}$$

for $\lambda, \alpha, a, b > 0$ and $x > 0$.
Application results are shown in Tables 3 and 4. Histogram of data, fitted densities and probability plots are shown in Figures 13 and 14.

5.1. Survival Times (in years)

The first real data set is a subset of data reported by Bekker et al. (7) which corresponds to the survival times (in years) of a group of patients given chemotherapy treatment alone. The data consisting of survival times (in years) for 46 patients are:
0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863,
1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

From the results in Table 3, we conclude that the TIIEHL-TL-WL distribution is the "best" model compared to the

Table 3: MLEs and goodness-of-fit statistics

	Estimates				Statistics							
Model	a	b	θ	λ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
TIIEHL-TL-WL	3.2823×10^{-01}	1.4055	1.0674×10^{-06}	1.0753	114.412	122.412	123.412	129.6387	0.0556	0.3916	0.0960	0.7647
	(1.1778×10^{-01})	(7.1513×10^{-10})	(1.8747×10^{-01})	(2.6972×10^{-01})								
TIIEHL-TL-WL(1, b, θ, λ)	1	1.1021	1.3610×10^{-10}	6.8916×10^{-01}	196.9675	202.9643	203.5497	208.3843	0.0613	0.4242	0.5308	2.321×10^{-12}
	-	(1.5849×10^{-01})	(1.8371×10^{-03})	(7.1734×10^{-02})								
TIIEHL-TL-WL(a, 1, θ, λ)	9.7192×10^{-01}	1	1.9653×10^{-10}	2.4731×10^{-01}	268.0722	274.073	274.6584	279.493	0.0571	0.4111	0.7854	2.2×10^{-16}
	(1.4490×10^{-01})	-	(2.2104×10^{-03})	(3.4972×10^{-02})								
TIIEHL-TL-WL(a, b, θ, 1)	9.9680×10^{-01}	1.9004	2.7632×10^{-09}	1	174.8767	180.8763	181.4617	186.2963	0.0596	0.4164	0.3571	1.21×10^{-05}
	(2.2545×10^{-01})	(3.8360×10^{-10})	(8.1565×10^{-03})	-								
TIIEHL-TL-WL(1, 1, θ, λ)	1	1	16.3267×10^{-10}	7.0003×10^{-02}	385.1095	389.1099	389.3956	392.7232	0.0598	0.4362	0.8895	2.2×10^{-16}
	-	-	(4.1664×10^{-03})	(1.0384×10^{-02})								
TIIEHL-TL-WL(1, b, θ, 1)	1	2.8752	1.4472×10^{-08}	1	166.6634	170.6634	170.9491	174.2767	0.0565	0.4008	0.3138	0.0002
	-	(3.7626×10^{-01})	(2.1368×10^{-02})	-								
TIIEHL-TL-WL(a, 1, θ, 1)	9.9617×10^{-01}	1	4.3253×10^{-10}	1	211.5891	215.5888	215.8745	219.2021	0.0785	0.5277	0.4819	3.87×10^{-10}
	(1.4850×10^{-01})	-	(3.4541×10^{-03})	-								
	λ	θ	c	k								
TIIGIE-BIII	17.0303	1.1525	1.3717	0.0720	119.7969	127.7969	128.7969	135.0236	15.8450	90.2978	0.9901	2.2×10^{-16}
	(0.0038)	(0.5246)	(0.6776)	(0.0967)								
	a	λ	δ	γ								
TIIEHLW	3.5837×10^{02}	1.1514	12.4872×10^{-03}	9.1589×10^{-01}	116.2558	124.2558	125.2558	131.4824	0.0811	0.5427	0.1094	0.6152
	(1.6288×10^{-03})	(1.7314)	(2.2467×10^{-02})	(1.3684)								
	λ	α	a	b								
TIIGIE-Lx	6.8295	3.7824×10^{04}	9.1447×10^{-02}	3.5879×10^{04}	115.9789	123.9789	124.9789	131.2055	0.0882	0.5867	0.1164	0.5365
	(1.1302×10^{01})	(1.9479×10^{-07})	(1.3345×10^{-01})	(4.0338×10^{-07})								

selected models since it has the lowest values for the goodness-of-fit statistics $-2\log L$, AIC, $AICC$, BIC, A^*, W^* and K-S (and the largest p-value for the K-S statistic). Also, the plots in figure 13 show that the TIIEHL-TL-WL fitted the data better than the other models of comparison and also has the smallest SS value, hence termed the better model.

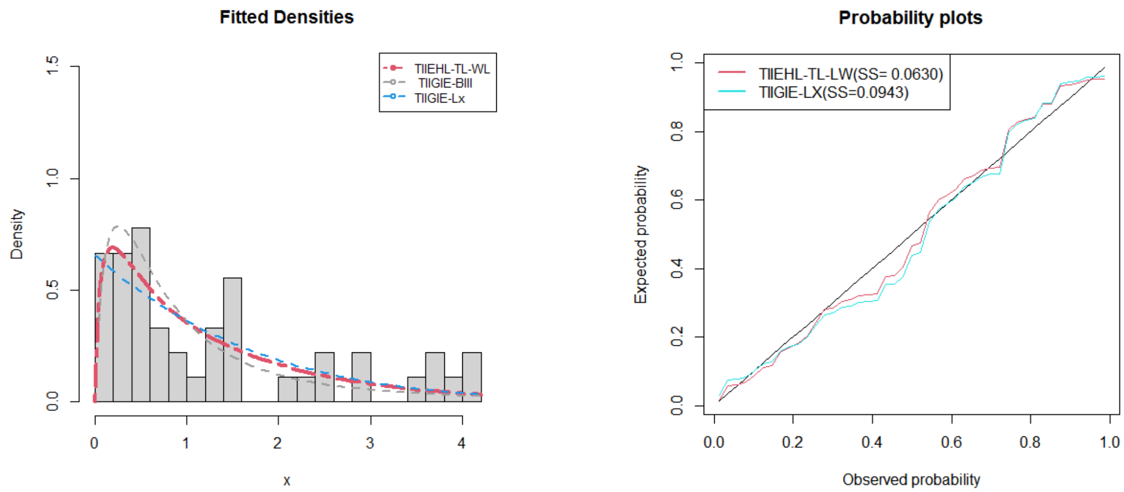

Figure 13: Fitted pdfs and probability plots for survival times (in years) data set

5.2. Time to Failure of Kevlar 49/epoxy Data

The second data is concerned with the study of the lifetimes of kevlar 49/epoxy spherical pressure vessels that are subjected to a constant sustained pressure until vessel failure, commonly known as static fatigue or stress-rupture. The data set consists of 101 observations of stress-rupture life of kevlar 49/epoxy strands which are subjected to constant sustained pressure at the 90% stress level until all have failed, so that the complete data set with the exact times of failure is recorded. These failure times in hours, are originally given by Barlow et al. (6). The data are
0.01,0.01,0.02,0.02,0.02,0.03,0.03,0.04,0.05,0.06,0.07,0.07,0.08,0.09,0.09, 0.10,
0.10,0.11,0.11,0.12,0.13,0.18,0.19,0.20,0.23,0.24,0.24,0.29,0.34,0.35, 0.36,0.38,
0.40,0.42,0.43,0.52,0.54,0.56,0.60,0.60,0.63,0.65,0.67,0.68,0.72, 0.72,0.72, 0.73,
0.79,0.79,0.80,0.80,0.83,0.85,0.90,0.92,0.95,0.99,1.00,1.01, 1.02,1.03, 1.05,1.10,
1.10,1.11,1.15,1.18,1.20,1.29,1.31,1.33,1.34,1.40,1.43, 1.45,1.50, 1.51,1.52,1.53,
1.54,1.54,1.55,1.58,1.60,1.63,1.64,1.80,1.80,1.81, 2.02,2.05, 2.14, 2.17,2.33,3.03,
3.03,3.34,4.20,4.69,7.89.

Table 4: MLEs and goodness-of-fit statistics

	Estimates				Statistics							
Model	a	b	θ	λ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
TIIEHL-TL-WL	0.3180	0.7098	0.0494	1.2281	206.5335	214.5335	214.9502	224.994	0.1581	0.9407	0.0850	0.4584
	(0.1586)	(0.2100)	(1.2284)	(0.2595)								
TIIEHL-TL-WL(1, b, θ, λ)	1	4.3039	3.3319×10^{-05}	4.3119×10^{-01}	224.1607	230.1607	230.4081	238.006	0.5829	3.1213	0.1752	0.0040
	-	(5.6589×10^{-01})	(1.9637×10^{-02})	(4.8213×10^{-02})								
TIIEHL-TL-WL(a, 1, θ, λ)	9.9909×10^{-01}	1	1.8028×10^{-09}	9.7888×10^{-02}	700.859	706.8599	707.1073	714.7052	0.5696	3.0477	0.8371	2.2×10^{-16}
	(9.9890×10^{-02})	-	(4.4638×10^{-03})	(9.6435×10^{-03})								
TIIEHL-TL-WL(a, b, θ, 1)	9.9787×10^{-01}	9.0097×10^{-01}	1.1956×10^{-08}	1	341.6717	347.6722	347.9196	355.5176	0.1802	1.0344	0.4630	2.2×10^{-16}
	(1.6895×10^{-01})	(1.3937×10^{-01})	(1.4043×10^{-02})	-								
TIIEHL-TL-WL(1, 1, θ, λ)	1	1	1.4105×10^{-09}	7.0003×10^{-02}	771.8714	775.8723	775.9948	781.1026	0.5885	3.1505	0.8669	2.2×10^{-16}
	-	-	(3.9364×10^{-03})	(6.9093×10^{-03})								
TIIEHL-TL-WL(1, b, θ, 1)	1	9.0000×10^{-01}	1.3713×10^{-08}	1	342.503	346.5031	346.6255	351.7333	0.1801	1.0333	0.4638	2.2×10^{-16}
	-	(8.1773×10^{-02})	(1.4051×10^{-02})	-								
TIIEHL-TL-WL(a, 1, θ, 1)	4.9999×10^{-01}	1	3.4527×10^{-09}	1	214.2777	218.2774	218.3998	223.5076	0.2330	1.2794	0.2026	0.0005
	(4.9759×10^{-02})	-	(6.9473×10^{-03})	-								
	λ	θ	c	k								
TIIGIE-BIII	5.1829	3.3191×10^{03}	5.9244×10^{-02}	1.0274×10^{01}	233.0062	241.0063	241.4229	251.4667	32.0684	197.6112	0.9981	2.2×10^{-16}
	(3.4987)	(3.2371×10^{-04})	(9.7386×10^{-03})	(3.3749×10^{-01})								
	a	λ	δ	γ								
TIIEHLW	0.3523	0.7289	1.8270	1.2123	206.6925	214.6925	215.1092	225.153	0.1673	0.9811	0.0891	0.3978
	(0.4424)	(0.2131)	(3.1301)	(0.2386)								
	λ	α	a	b								
TIIGIE-Lx	1.7010×10^{02}	1.0186×10^{-03}	9.0000	9.1000×10^{-01}	221.7345	229.7345	230.1511	240.1949	0.4843	2.6166	0.1706	0.0056
	(7.5375×10^{-07})	(2.1156×10^{-04})	(1.0787×10^{-04})	(2.5260×10^{-01})								

from the results shown in Table 4, we conclude that the TIIEHL-TL-WL distribution is indeed the "best" model compared to several selected models since it is associated with the lowest values for all the goodness-of-fit statistics (and the largest p-value for the K-S statistic). Also, the plots in figure 14 show that the TIIEHL-TL-WL fitted the data better than the other models of comparison and also has the smallest SS value.

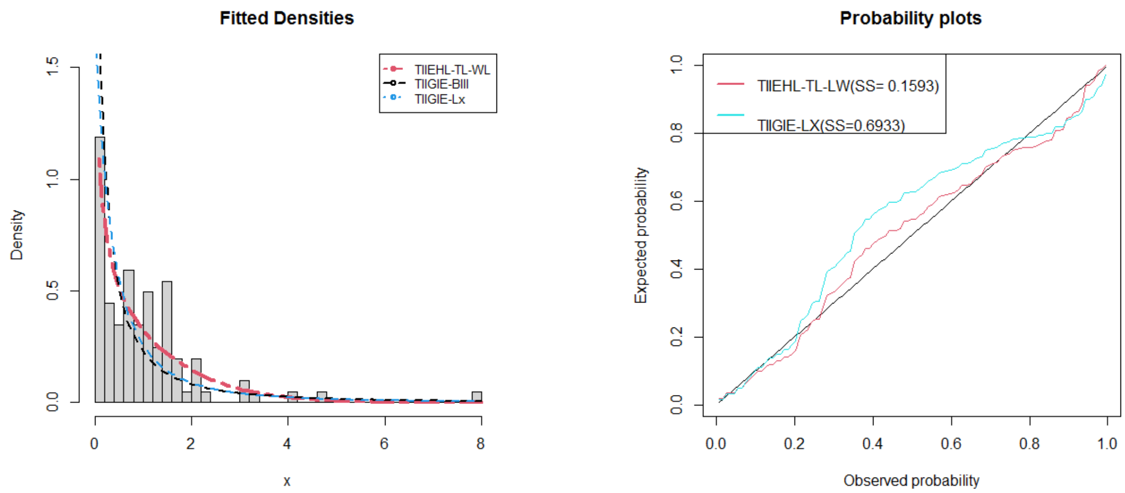

Figure 14: Fitted pdfs and probability plots for time to failure of kevlar 49/epoxy data set

5.3. Failure Times of 50 Components Data Set

The third data set taken from Murthy et al. (19) represents the failure times of 50 components (per 1000 hours). The data are
$0.036, 0.148, 0.590, 3.076, 6.816, 0.058, 0.183, 0.618, 3.147, 7.896, 0.061, 0.192, 0.645, 3.625, 7.904, 0.074, 0.254, 0.961, 3.704, 8.022, 0.078, 0.262, 1.228, 3.931, 9.337, 0.086, 0.379, 1.600, 4.073, 10.940, 0.102, 0.381, 2.006, 4.393, 11.020, 0.103, 0.538, 2.054, 4.534, 13.880, 0.114, 0.570, 2.804, 4.893, 14.73, 0.116, 0.574, 3.058, 6.274, 15.08.$

Table 5: MLEs and goodness-of-fit statistics

Model	Estimates				Statistics							
	a	b	θ	λ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
TIIEHL-TL-WL	0.0896	1.0133	0.9023	0.9389	200.1913	208.1913	209.0802	215.8394	0.1179	0.7430	0.1157	0.479
	(0.1290)	(0.3564)	(0.3241)	(0.3977)								
TIIEHL-TL-WL(1, b, θ, λ)	1	1.1021	3.7550×10^{-10}	3.9163×10^{-01}	323.7181	329.7176	330.2394	335.4537	0.1505	0.9358	0.5971	8.882×10^{-16}
	-	(1.5167×10^{-01})	(2.8821×10^{-03})	(3.6788×10^{-02})								
TIIEHL-TL-WL(a, 1, θ, λ)	9.4878×10^{-01}	1	3.8253×10^{-09}	2.4777×10^{-01}	340.7781	346.778	347.2997	352.514	0.1606	0.9952	0.7332	8.882×10^{-16}
	(1.4217×10^{-01})	-	(9.7940×10^{-03})	(3.1658×10^{-02})								
TIIEHL-TL-WL(a, b, θ, 1)	8.5440×10^{-01}	9.2572×10^{-01}	3.0344×10^{-09}	1	546.6781	552.6787	553.2005	558.4148	0.3004	2.3604	0.4657	1.849×10^{-10}
	(1.9039×10^{-01})	(2.0649×10^{-01})	(9.2514×10^{-03})	-								
TIIEHL-TL-WL(1, 1, θ, λ)	1	1	1.2491×10^{-09}	7.0003×10^{-02}	459.5549	463.5551	463.8104	467.3792	0.1758	1.0886	0.8858	8.882×10^{-16}
	-	-	(7.2458×10^{-03})	(9.7717×10^{-03})								
TIIEHL-TL-WL(1, b, θ, 1)	1	2.0371	0.0001	1	606.0939	610.0939	610.3492	613.9179	0.1176	0.8470	0.4614	2.909×10^{-10}
	-	(0.2601)	(0.1831)	-								
TIIEHL-TL-WL(a, 1, θ, 1)	4.7096×10^{-01}	1	4.9115×10^{-09}	1	326.3046	330.307	330.5623	334.131	0.1504	0.9845	0.3885	2.628×10^{-07}
	(6.6607×10^{-02})	-	(1.0310×10^{-02})	-								
	λ	θ	c	k								
TIIGIE-BIII	1.4492	1.1742×10^{03}	7.4801×10^{-02}	9.9836	204.9782	212.9781	213.8679	220.6262	17.9579	99.9618	0.9813	8.882×10^{-16}
	(2.4554)	(1.9884×10^{-03})	(2.3451×10^{-02})	(1.3601)								
	a	λ	δ	γ								
TIIEHLW	1.0704	1.5713×10^{-01}	8.9786×10^{-05}	3.5666	201.4852	209.4852	210.3741	217.1333	0.1193	0.8021	0.1360	0.2865
	(7.5006×10^{-04})	(1.8898×10^{-02})	(4.5116×10^{-05})	(8.8732×10^{-04})								
	λ	α	a	b								
TIIGIE-Lx	1.4663×10^{-03}	1.6231×10^{01}	4.0485×10^{-02}	1.2245×10^{-04}	204.6059	212.6059	213.4948	2 220.254	0.1517	0.9504	0.1252	0.3812
	(8.1400×10^{-03})	(1.2061×10^{-05})	(4.8355×10^{-03})	(9.8404×10^{-04})								

Furthermore, from the results shown in Table 5, we conclude that the TIIEHL-TL-WL distribution is indeed the "best" model compared to several selected models since it is associated with the lowest values for all the goodness-of-fit statistics (and the largest p-value for the K-S statistic). Also, the plots in figure 15 show that the TIIEHL-TL-WL fitted the data better than the other models of comparison and also has the smallest SS value.

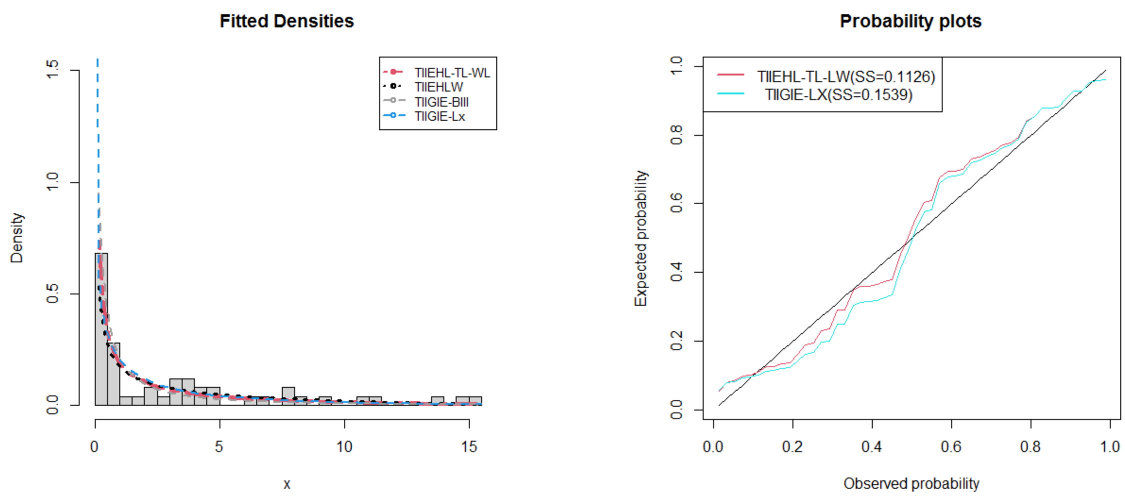

Figure 15: Fitted pdfs and probability plots for failure times of 50 components data set

Table 6: Likelihood ratio test results

		Survival Times (in years)	Time to failure of kevlar 49/epoxy
Model	df	$\chi^2(p-value)$	$\chi^2(p-value)$
TIIEHL-TL-WL(1, b, θ, λ)	1	82.5525(<0.00001)	17.6272(0.00003)
TIIEHL-TL-WL(a, 1, θ, λ)	1	153.6602(<0.00001)	494.3255(<0.00001)
TIIEHL-TL-WL(a, b, θ, 1)	1	60.4647(<0.00001)	135.1382(<0.00001)
TIIEHL-TL-WL(1, 1, θ, λ)	2	270.6975(<0.00001)	565.3379(<0.00001)
TIIEHL-TL-WL(1, b, θ, 1)	2	52.2514(<0.00001)	135.9695(<0.00001)
TIIEHL-TL-WL(a, 1, θ, 1)	2	97.1771(<0.00001)	7.7442(0.0208)

Table 7: Likelihood ratio test results

		Failure Times of 50 Components Data Set
Model	df	$\chi^2(p-value)$
TIIEHL-TL-WL(1, b, θ, λ)	1	123.5628(<0.00001)
TIIEHL-TL-WL(a, 1, θ, λ)	1	140.5868(<0.00001)
TIIEHL-TL-WL(a, b, θ, 1)	1	346.4868(<0.00001)
TIIEHL-TL-WL(1, 1, θ, λ)	2	259.3636(<0.00001)
TIIEHL-TL-WL(1, b, θ, 1)	2	405.9026(<0.00001)
TIIEHL-TL-WL(a, 1, θ, 1)	2	126.1133(<0.00001)

5.4. Likelihood Ratio Test

The likelihood ratio test results in Table 6 and 7 indicates that the TIIEHL-TL-WL performs better than its nested models at 5% level of significance, since all p-values are less than 0.05 among all the data sets considered.

6. Conclusion

A new class of distributions called the type II exponentiated half-logistic Topp-Leone-G power series (TIIEHL-TL-GPS) class of distributions is introduced. Some mathematical properties including moments and moment generating function, order statistics, entropy and quantiles are provided. Model parameters are estimated using the maximum likelihood method and the performance of the estimates is assessed by means of a simulation study. The potentiality of the new model is demonstrated by means of three real data sets.

Appendix

The following URL contains the appendix material https://drive.google.com/file/d/1TTrilDRb5cFul49QiVl7XwTM
view?usp=sharing

7. Declarations

The authors have nothing to declare.

8. Funding

The research is not a funded research.

9. Conflict of Interest

Authors declare no conflict of interest.

10. Availability of data and material

Not applicable

11. Code availability

R codes provided in the appendix section.

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