Type II Exponentiated Half-Logistic-Topp-Leone-G Power Series Class of Distributions with Applications

Thatayaone Moakofi\(^1\), Broderick Oluyede\(^2\)
and Fastel Chipepa\(^3\)

\(^*\)Corresponding author

1. Department of Mathematics and Statistical Sciences, Botswana International University of Science and Technology, Botswana, thatayaone.moakofi@studentmail.biust.ac.bw
2. Department of Mathematics and Statistical Sciences, Botswana International University of Science and Technology, Botswana, oluyedeo@biust.ac.bw
3. Department of Mathematics and Statistical Sciences, Botswana International University of Science and Technology, Botswana, chipepaf@biust.ac.bw

Abstract

This paper aims to develop a new class of distributions, namely, type II exponentiated half-logistic Topp-Leone-G power series (TIIEHL-TL-GPS) class of distributions. Some important properties including moments, quantiles, moment generating function, entropy and maximum likelihood estimates are derived. A simulation study is conducted to evaluate the consistency of the maximum likelihood estimates. We also present three real data examples to illustrate the usefulness of the new class of distributions. Results show that the proposed model performs better than nested and several non-nested models on selected data sets.

Key Words: Topp-Leone; Power Series; Logarithmic Distribution; Maximum Likelihood Estimation.

Mathematical Subject Classification: 60E05, 62E15.

1. Introduction

The limitations of the well-known standard distributions like Weibull distribution, Lindley distribution, Rayleigh distribution and many others have motivated researchers to generalize and extend existing distributions, in order to offer flexible models in terms of data modeling. Several extensions of distributions available in the literature are the beta Marshall-Olkin family of distributions by Alizadeh (3), Topp-Leone generated family of distributions by Rezaei (26), type II power Topp-Leone generated family of distributions by Bantan et al. (5), sine Topp-Leone-G family of distributions by Al-Babtain et al. (1), Burr X exponential-G family of distributions by Sanusi (28), type II half logistic family of distributions by Soliman et al. (32), type II general inverse exponential family of distributions by Jamal et al. (13), the Zografos–Balakrishnan-G family of distributions by Nadarajah et al. (20), beta Weibull-G by Yousof et al. (35), new power generalized Weibull-G by Oluyede et al. (24), Weibull-G by Bourguignon et al. (8) developed, beta-G by Eugene et al. (10).

In this paper, we propose a new class of distributions TIIEHL-TL-GPS class of distributions. An attractive feature about the model is that the extra parameter introduced has the capability to control both the weights at the tails of the
density function. Also, the new class of distributions can model different types of failure rate functions that are available in different areas like reliability, engineering and biological studies. The hazard rate function from the special cases exhibits increasing, decreasing, bathtub and upside bathtub shapes. Of note is the upside bathtub followed by bathtub shape of the hazard rate exhibited in some special cases. The method of estimation used in section 4 was used by Karakaya and Tanis (15), Karakaya and Tanis (16), Tanis and Karakaya (33) and Tanis (34).

The cumulative distribution function (cdf) and probability density function (pdf) of the type II exponentiated half-logistic Topp-Leone-G (TIIEHL-TL-G) family of distributions are given by

\begin{equation}
F(x; a, b, \psi) = 1 - \left[ \frac{1 - \left[ 1 - G^2(x; \psi) \right]^b}{1 + \left[ 1 - G^2(x; \psi) \right]^b} \right]^a
\end{equation}

and

\begin{equation}
f(x; a, b, \psi) = \frac{4abg(x; \psi) \left[ 1 - G^2(x; \psi) \right]^{b-1} G(x; \psi) \left( 1 - \left[ 1 - G^2(x; \psi) \right]^b \right)^{a-1}}{\left( 1 + \left[ 1 - G^2(x; \psi) \right]^b \right)^{a+1}},
\end{equation}

respectively, for \( a, b > 0 \) and parameter vector \( \psi \).

The basic motivations for developing the type II exponentiated half-logistic Topp-Leone-G power series (TIIEHL-TL-GPS) class of distributions are;

- to construct and generate distributions with symmetric, left-skewed, right-skewed, reversed-J shapes;
- to define special models that posseses various types of hazard rate functions including monotonic as well as non-monotonic shapes;
- to provide consistenly better fits than other generated distributions having the same number of parameters;
- to construct heavy-tailed distributions for modeling different real data sets;
- to make the kurtosis more flexible compared to that of the baseline distribution.

Let \( N \) be a zero truncated discrete random variable having a power series distribution, whose probability mass function (pmf) is given by

\begin{equation}
P(N = n) = \frac{a_n \theta^n}{C(\theta)}, \quad n = 1, 2, 3, \ldots,
\end{equation}

where \( C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n \) is finite, \( \theta > 0 \) and \( \{a_n\}_{n \geq 1} \) a sequence of positive real numbers. The power series family of distributions include binomial, Poisson, geometric and logarithmic distributions Johnson et al. (14). Several generalized distributions proposed in the literature involving the power series include the exponentiated power generalized Weibull power series family of distributions by Aldahlan et al. (2), the T–R \{Y\} power series family of probability distributions by Osathamnwen et al. (25), exponentiated generalized power series class of distributions by Oluyede et al. (22), a new generalized Lindley-Weibull class of distributions by Makubate et al. (17), the odd Weibull-Topp-Leone-G power series family of distributions by Oluyede et al. (21), Weibull-power series distributions by Morais and Barreto-Souza (18), complementary exponential power series by Flores et al. (11), complementary extended Weibull-power series by Cordeiro and Silva (9), Burr XII power series by Silva and Cordeiro (31), extended Weibull-power series (EWPS) distribution by Silva et al. (30) and the Burr-Weibull power series class of distributions by Oluyede et al. (23).
The rest of the paper is organized as follows: In Section 2, we present the new model and some of the statistical properties. We present some special cases of the proposed class of distributions in Section 3. A simulation study is presented in Section 4 and applications in Section 5 followed by concluding remarks.

2. The Model, Sub-Classes and Properties

In this section, we develop the new model, referred to as the type II exponentiated half-logistic Topp-Leone-G power series (TIIEHL-TL-GPS) class of distributions. Some statistical properties which include hazard rate function, quantile function, moments and maximum likelihood estimation of model parameters are derived.

2.1. The Model

Let $X_1, X_2, ..., X_N$ be $N$ identically and independently distributed (iid) random variables following the TIIEHL-TL-G distribution. Let $X_{(1)} = \min(X_1, X_2, ..., X_N)$, then the cdf of $X_{(1)}|N=n$ is given by

$$F_{X_{(1)}|N=n}(x; a, b, \theta, \psi) = 1 - \left(\frac{1 - \left[1 - C^2(x; \psi)\right]^{b}}{1 + \left[1 - C^2(x; \psi)\right]^{b}}\right)^n,$$

(4)

for $a, b, \theta > 0$, $n \geq 1$ and parameter vector $\psi$. The type II exponentiated half-logistic Topp-Leone power series (TIIEHL-TL-GPS) class of distributions denoted by TIIEHL-TL-GPS$(a, b, \theta, \psi)$ is defined by the marginal distribution of $X_{(1)}$, that is,

$$F_{X_{(1)}}(x) = 1 - \frac{C(\theta)\left[1 - C^2(x; \psi)\right]^{a}}{C(\theta)},$$

(5)

for $a, b > 0$ and parameter vector $\psi$. The corresponding pdf is given by

$$f_{X_{(1)}}(x) = \frac{4ab\theta g(x; \psi)\left[1 - C^2(x; \psi)\right]^{b-1}C(x; \psi)\left(1 - \left[1 - C^2(x; \psi)\right]^{b}\right)^{a-1}}{\left(1 + \left[1 - C^2(x; \psi)\right]^{b}\right)^{a+1}},$$

(6)

$$\times \frac{C'(\theta)\left[1 - C^2(x; \psi)\right]^{b}}{C(\theta)}.$$

The hazard rate function (hrf) is given by

$$h_{x}(x) = \frac{4ab\theta g(x; \psi)\left[1 - C^2(x; \psi)\right]^{b-1}C(x; \psi)\left(1 - \left[1 - C^2(x; \psi)\right]^{b}\right)^{a-1}}{\left(1 + \left[1 - C^2(x; \psi)\right]^{b}\right)^{a+1}},$$

(7)

$$\times \frac{C'(\theta)\left[1 - C^2(x; \psi)\right]^{b}}{C(\theta)\left[1 + C^2(x; \psi)\right]^{b}}.$$
Table 1 below presents the special classes of TIEHL-TL-GPS distribution when \( C(\theta) \) is specified in equation (5).

<table>
<thead>
<tr>
<th>Distribution</th>
<th>( C(\theta) )</th>
<th>( a_n )</th>
<th>cdf</th>
</tr>
</thead>
<tbody>
<tr>
<td>TIEHL-TL-G Poisson</td>
<td>( e^\theta - 1 )</td>
<td>((n!)^{-1})</td>
<td>( 1 - \frac{\exp \left( \theta \left[ \frac{1 - \tau^2(x; \psi)}{1 + \tau^2(x; \psi)} \right]^b \right)^a}{\exp(\theta) - 1} )</td>
</tr>
<tr>
<td>TIEHL-TL-G Geometric</td>
<td>( \theta (1 - \theta)^{-1} )</td>
<td>1</td>
<td>( 1 - \frac{\left( 1 - \frac{1 - \tau^2(x; \psi)}{1 + \tau^2(x; \psi)} \right)^b}{\theta} )</td>
</tr>
<tr>
<td>TIEHL-TL-G Logarithmic</td>
<td>( -\log(1 - \theta) )</td>
<td>( n^{-1} )</td>
<td>( 1 - \frac{\left( 1 - \frac{1 - \tau^2(x; \psi)}{1 + \tau^2(x; \psi)} \right)^b}{\log(1 - \theta)} )</td>
</tr>
<tr>
<td>TIEHL-TL-G Binomial</td>
<td>( (1 + \theta)^m - 1 )</td>
<td>( \binom{m}{n} )</td>
<td>( 1 - \frac{\left[ 1 - \frac{1 - \tau^2(x; \psi)}{1 + \tau^2(x; \psi)} \right]^b}{(1 + \theta)^m - 1} )</td>
</tr>
</tbody>
</table>

2.2. Sub-classes of TIEHL-TL-GPS Family of Distributions

- When \( a = 1 \), we obtain the type II half-logistic Topp-Leone-G power series (TIIHL-TL-GPS) class of distributions with the cdf

\[
F(x; b, \theta, \psi) = 1 - \frac{C \left( \theta \left[ \frac{1 - \tau^2(x; \psi)}{1 + \tau^2(x; \psi)} \right]^b \right)}{C(\theta)},
\]

for \( b, \theta > 0 \) and parameter vector \( \psi \). This is a new class of distributions.

- When \( b = 1 \), we obtain the new class of distributions with the cdf

\[
F(x; a, \theta, \psi) = 1 - \frac{C \left( \theta \left[ \frac{\tau^2(x; \psi)}{1 + \tau^2(x; \psi)} \right]^a \right)}{C(\theta)},
\]

for \( a, \theta > 0 \) and parameter vector \( \psi \).

- When \( a = b = 1 \), we obtain the new class of distributions with the cdf

\[
F(x; \theta, \psi) = 1 - \frac{C \left( \theta \left[ \frac{\tau^2(x; \psi)}{1 + \tau^2(x; \psi)} \right] \right)}{C(\theta)},
\]

for \( \theta > 0 \) and parameter vector \( \psi \).

- When \( \theta \to 0^+ \), we obtain the type II exponentiated half-logistic Topp-Leone-G (TIIHL-TL-G) class of distributions with the cdf

\[
F(x; a, \theta, \psi) = 1 - \frac{C \left( \theta \left[ \frac{\tau^2(x; \psi)}{1 + \tau^2(x; \psi)} \right]^a \right)}{C(\theta)},
\]

for \( a, \theta > 0 \) and parameter vector \( \psi \).
contributions with the cdf

\[ F(x; a, b, \psi) = 1 - \frac{\left[ 1 - \frac{1 - C^2(x; \psi)}{1 - C(x; \psi)} \right]^b}{a}, \]

for \( a, b > 0 \) and parameter vector \( \psi \). This is a new class of distributions.

- When \( a = 1 \) and \( \theta \to 0^+ \), we obtain the type II half-logistic Topp-Leone-G (TIIHL-TL-G) class of distributions with the cdf

\[ F(x; b, \psi) = 1 - \frac{1 - \left[ 1 - \frac{1 - C^2(x; \psi)}{1 - C(x; \psi)} \right]^b}{1 + \left[ 1 - \frac{1 - C^2(x; \psi)}{1 - C(x; \psi)} \right]^b}, \]

for \( b > 0 \) and parameter vector \( \psi \). This is a new class of distributions.

- When \( b = 1 \) and \( \theta \to 0^+ \), we obtain the new class of distributions with the cdf

\[ F(x; a, \psi) = 1 - \frac{\frac{C^2(x; \psi)}{1 + \frac{1 - C^2(x; \psi)}{1 - C(x; \psi)}}}{a}, \]

for \( a > 0 \) and parameter vector \( \psi \).

- When \( a = b = 1 \) and \( \theta \to 0^+ \), we obtain the new class of distributions with the cdf

\[ F(x; \psi) = 1 - \frac{\frac{C^2(x; \psi)}{1 + \frac{1 - C^2(x; \psi)}{1 - C(x; \psi)}}}{1 - \frac{1 - C^2(x; \psi)}{1 - C(x; \psi)}}, \]

for parameter vector \( \psi \).

### 2.3. Quantile Function

Let \( X \) be a random variable with cdf defined by equation (5). The quantile function \( Q_{X(t)}(u) \) is defined by \( F_{X(t)}(Q_{X(t)}(u)) = u, 0 \leq u \leq 1 \). Note that

\[ C(\theta) \left( \frac{1 - \left[ 1 - \frac{1 - C^2(x; \psi)}{1 + \frac{1 - C^2(x; \psi)}{1 - C(x; \psi)}} \right]^b}{a} \right) = u, \]

so that

\[ C(\theta) \left( \frac{1 - \left[ 1 - \frac{1 - C^2(x; \psi)}{1 + \frac{1 - C^2(x; \psi)}{1 - C(x; \psi)}} \right]^b}{a} \right) = C(\theta)(1 - u). \]

This is equivalent to

\[ \frac{1 - \left[ 1 - \frac{1 - C^2(x; \psi)}{1 + \frac{1 - C^2(x; \psi)}{1 - C(x; \psi)}} \right]^b}{a} = \frac{C^{-1}(C(\theta)(1 - u))}{\theta}, \]
that is,

\[ G(x; \psi) = \left( 1 - \left( \frac{1 - (C(\theta)(1-u))^\frac{1}{2}}{1 + (C(\theta)(1-u))^\frac{1}{2}} \right)^a \right)^\frac{1}{b}. \]

The expression further simplifies to

\[ G(x; \psi) = 1 - \left( 1 - \left( \frac{1 - (C(\theta)(1-u))^\frac{1}{2}}{1 + (C(\theta)(1-u))^\frac{1}{2}} \right)^a \right)^\frac{1}{b}. \]

Therefore, the quantile function of the TIIEHL-TL-GPS class of distributions is given by,

\[ Q_{X(1)}(u) = G^{-1} \left[ 1 - \left( 1 - \left( \frac{1 - (C(\theta)(1-u))^\frac{1}{2}}{1 + (C(\theta)(1-u))^\frac{1}{2}} \right)^a \right)^\frac{1}{b} \right]. \]  

(8)

It follows therefore that random numbers can be generated from the TIIEHL-TL-GPS class of distributions using equation (8) with the aid of statistical software such as R, MATLAB and SAS.

2.4. Expansion of Density

Expansion of the density function of the TIIEHL-TL-GPS class of distributions is presented in this sub-section. The TIIEHL-TL-GPS class of distributions can be expressed as an infinite linear combination of exponentiated-G (Exp-G) densities as

\[ f_{X(1)}(x) = \sum_{m=0}^{\infty} \tau_{m+1} g_{m+1}(x; \psi), \]  

where

\[ g_{m+1}(x; \xi) = (m+1)(G(x; \xi))^m g(x; \xi) \] is the exponentiated-G (Exp-G) distribution with power parameter \( m+1 \) and

\[ \tau_{m+1} = \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \frac{4ab\theta na_n \theta^n}{C(\theta)} \binom{an-1}{j} \binom{an+k}{k} \binom{b(j+k+1)-1}{l} \times \binom{2l+1}{m} \binom{-1}{m}^{j+k+l+m} n \]  

(10)

(See Appendix section for details of the derivation)

2.5. Moments and Generating Function

If \( X \) follows the TIIEHL-TL-GPS distribution and \( Y \sim \text{Exp} - G(m+1) \). Then using equation (9), the \( p^{th} \) raw moment, \( \mu'_p \) of the TIIEHL-TL-GPS class of distributions is obtained as

\[ \mu'_p = E(X^p) = \int_{-\infty}^{\infty} x^p f(x) dx = \sum_{m=0}^{\infty} \tau_{m+1} E(Y^p), \]

where \( \tau_{m+1} \) is given by equation (10). The moment generating function (MGF) \( M(t) = E(e^{tX}) \) is given by:

\[ M_X(t) = \sum_{m=0}^{\infty} \tau_{m+1} M_Y(t), \]
where $M_Y(t)$ is the MGF of $Y$ and $\tau_{m+1}$ is given by equation (10).

### 2.6. Order Statistics and Rényi Entropy

In this section, we present the distribution of the $i$th order statistic and Rényi entropy.

#### 2.6.1. Distribution of Order Statistics

Order statistics are fundamental in many areas of statistical theory and practice. Let $X_1, X_2, \ldots, X_n$ be a random sample from TIEHL-TL-GPS class of distributions. Then, the distribution of the $k$th order statistic from TIEHL-TL-GPS class of distributions is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)! (n-i)!} \sum_{m=0}^{\infty} \sum_{p=0}^{n-i} \binom{n-i}{p} (-1)^p h_{m+1} g_{m+1}(x; \psi),$$

(11)

where $g_{m+1}(x; \psi) = (m+1) g(x; \psi) G^m(x; \psi)$ is an Exp-G with power parameter $m+1$ and the linear component

$$h_{m+1} = \sum_{n=1}^{\infty} \sum_{j,k,l=0}^{\infty} \frac{4ab\theta n d_d d_e}{C^q(\theta)} \frac{\theta^{q+i} (-1)^{q+j+l+m}}{(m+1)} \binom{i+p-1}{q} \left( \binom{a(n+z)-1}{j} \binom{a(n+z)+k}{l} \binom{b(j+k+1)-1}{m} \right).$$

(See Appendix section for details of the derivation)

#### 2.6.2. Rényi Entropy

In this subsection, Rényi entropy for TIEHL-TL-GPS class of distributions is derived. An entropy is a measure of uncertainty or variation of a random variable. Rényi entropy (27) is a generalization of Shannon entropy (29). Rényi entropy of the TIEHL-TL-GPS distribution is defined by

$$I_R(\nu) = \frac{1}{1-\nu} \log \left( \sum_{m=0}^{\infty} w^* e^{(1-v) I_{REG}} \right),$$

(12)

where $I_{REG} = \int_0^\infty \left( (1+m/\nu) g(x; \psi) G^m(x; \psi) \right)^{\nu} dx$ is Rényi entropy for an Exp-G distribution with power parameter $m/\nu + 1$ and

$$w^* = \sum_{n=1}^{\infty} \sum_{j,k,l,m=0}^{\infty} (-1)^{k+j+l+m} \frac{nd_d d_e \theta^{n} \nu^{n-1}}{(C^{q+1}(\theta))^{\nu}} \binom{b(v+j+k) - v}{l} \left( \binom{a(v+n-1) - v}{j} \binom{a(v+n+1)+v-1+k}{k} \right) \left( \binom{2l+v}{m} \frac{1}{(1+m/\nu)^{\nu}} \right).$$

(13)

Consequently, Rényi entropy for TIEHL-TL-GPS class of distributions can be obtained from Rényi entropy of the Exp-G distribution (See Appendix section for details of the derivation).

### 2.7. Maximum Likelihood Estimation

Here we use the maximum likelihood estimation technique to find the maximum likelihood estimates of the parameters of the TIEHL-TL-GPS class of distributions. Let $X_i \sim TIEHL - TL - GPS(\alpha, \beta, \psi)$ and $\Delta = (\alpha, \beta, \psi)^T$ be the vector of unknown
parameters. The total log-likelihood \( \ell = \ell(\Delta) \) function is given by

\[
\ell(\Delta) = n \ln(4ab\theta) + (b - 1) \sum_{i=1}^{n} \ln \left[ 1 - G^2(x; \psi) \right] - n \ln(C(\theta)) \\
+ (a - 1) \sum_{i=1}^{n} \left( 1 - \left[ 1 - G^2(x; \psi) \right]^b \right) - (a + 1) \sum_{i=1}^{n} \ln \left( 1 + \left[ 1 - G^2(x; \psi) \right]^b \right) \\
+ \sum_{i=1}^{n} \ln \left( C' \left( \frac{1 - \left[ 1 - G^2(x; \psi) \right]^b}{1 + \left[ 1 - G^2(x; \psi) \right]^b} \right) \right) \\
+ \sum_{i=1}^{n} \ln \left( g(x; \psi) \right) + \sum_{i=1}^{n} \ln \left( \tilde{G}(x; \psi) \right).
\]

The maximum likelihood estimates of the parameters, denoted by \( \hat{\Delta} \), is obtained by solving the nonlinear equation \( (\frac{\partial \ell}{\partial a}, \frac{\partial \ell}{\partial b}, \frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial \psi})^T = 0 \), using a numerical method such as Newton-Raphson procedure. The multivariate normal distribution \( N_{q+3}(0, J(\hat{\Delta})^{-1}) \), where the mean vector \( \theta = (0, 0, 0, 0)^T \) and \( J(\hat{\Delta})^{-1} \) is the observed Fisher information matrix evaluated at \( \hat{\Delta} \), can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions.

3. Some Special Sub-classes of the TIIIEHL-TL-GPS Class of Distributions

In this section, special classes of TIIIEHL-TL-GPS class of distributions are presented by specifying the baseline cdf \( G(x; \psi) \) and pdf \( g(x; \psi) \) in equations (5) and (6).


If the baseline cdf and pdf are given by \( G(x; c) = 1 - (1 + x^c)^{-1} \) and \( g(x; c) = c x^{c-1} (1 + x^c)^{-2} \), for \( c > 0 \), and \( x > 0 \), then the cdf and pdf of the TIIIEHL-TL-LLoGPS class of distributions are given by

\[
F_{X_{(1)}}(x) = 1 - \frac{C \left( \theta \left[ 1 - \frac{1 - (1 + x^c)^{-2}}{1 + (1 + x^c)^{-2}} \right]^{b/a} \right)}{C(\theta)},
\]

and

\[
f_{X_{(1)}}(x) = \frac{4ab\theta c x^{c-1} (1 + x^c)^{-2} \left[ 1 - (1 + x^c)^{-2} \right]^{b-1} (1 + x^c)^{-1}}{\left( 1 + \left[ 1 - (1 + x^c)^{-2} \right]^{b} \right)^{a+1}} \\
\times \left( 1 - \left[ 1 - (1 + x^c)^{-2} \right]^{b} \right)^{a-1} \frac{C \left( \theta \left[ 1 - \frac{1 - (1 + x^c)^{-2}}{1 + (1 + x^c)^{-2}} \right]^{b/a} \right)}{C(\theta)},
\]

(15)
We present in Figures 2 and 3, 3D plots of skewness and kurtosis of the TIIEHL-TL-LLoGL distribution. We observe that the distribution exhibit increasing, reverse-J, bathtub, upside bathtub and upside bathtub followed by bathtub shapes.

Figure 1 shows the pdfs of the TIIEHL-TL-LLoGL distribution. The pdf can take various shapes that include almost symmetric, reverse-J, left, or right-skewed. Furthermore, the hazard rate functions (hrfs) for the TIIEHL-TL-LLoGL distribution are given by

\[
h_f(x) = \frac{4ab\theta cx^{c-1}(1+x^c)^{-2}\left[1-(1+x^c)^{-2}\right]^{b-1}(1+x^c)^{-1}}{\left(1+\left[1-(1+x^c)^{-2}\right]^b\right)^{a+1}} \\
\times \left(1-\left[1-(1+x^c)^{-2}\right]^b\right)^{a-1} \frac{C\left(\theta \left[\frac{1-\left[1-(1+x^c)^{-2}\right]^b}{1+\left[1-(1+x^c)^{-2}\right]^b}\right]^a\right)}{C\left(\theta \left[\frac{1-\left[1-(1+x^c)^{-2}\right]^b}{1+\left[1-(1+x^c)^{-2}\right]^b}\right]^b\right)^a},
\]

for \( a, b, \theta, c \) and \( x > 0 \).

### 3.1.1. Type II Exponentiated Half-Logistic Topp-Leone-Log-Logistic Logarithmic (TIIEHL-TL-LLoGL) Distribution

The cdf and pdf of TIIEHL-TL-LLoGL distribution are given by

\[
F_{X(1)}(x) = 1 - \frac{-\log \left(1 - \theta \left[\frac{1-\left[1-(1+x^c)^{-2}\right]^b}{1+\left[1-(1+x^c)^{-2}\right]^b}\right]^a \right)}{-\log(1 - \theta)},
\]

and

\[
f_{X(1)}(x) = \frac{4ab\theta cx^{c-1}(1+x^c)^{-2}\left[1-(1+x^c)^{-2}\right]^{b-1}(1+x^c)^{-1}}{\left(1+\left[1-(1+x^c)^{-2}\right]^b\right)^{a+1}} \\
\times \left(1-\left[1-(1+x^c)^{-2}\right]^b\right)^{a-1} \frac{\left(1-\theta \left[\frac{1-\left[1-(1+x^c)^{-2}\right]^b}{1+\left[1-(1+x^c)^{-2}\right]^b}\right]^a \right)^{-1}}{-\log(1 - \theta)},
\]

respectively. The hrfs is given by

\[
h_f(x) = \frac{4ab\theta cx^{c-1}(1+x^c)^{-2}\left[1-(1+x^c)^{-2}\right]^{b-1}(1+x^c)^{-1}}{\left(1+\left[1-(1+x^c)^{-2}\right]^b\right)^{a+1}} \\
\times \left(1-\left[1-(1+x^c)^{-2}\right]^b\right)^{a-1} \frac{\left(1-\theta \left[\frac{1-\left[1-(1+x^c)^{-2}\right]^b}{1+\left[1-(1+x^c)^{-2}\right]^b}\right]^a \right)^{-1}}{-\log \left(1 - \theta \left[\frac{1-\left[1-(1+x^c)^{-2}\right]^b}{1+\left[1-(1+x^c)^{-2}\right]^b}\right]^a \right)},
\]

for \( a, b, \theta, c \) and \( x > 0 \).

Figure 1 shows the pdfs of the TIIEHL-TL-LLoGL distribution. The pdf can take various shapes that include almost symmetric, reverse-J, left, or right-skewed. Furthermore, the hazard rate functions (hrfs) for the TIIEHL-TL-LLoGL distribution exhibit increasing, reverse-J, bathtub, upside bathtub and upside bathtub followed by bathtub shapes.

We present in Figures 2 and 3, 3D plots of skewness and kurtosis of the TIIEHL-TL-LLoGL distribution. We observe that
Figure 1: Plots of the pdf and hrf for the TIIEHL-TL-LLoGL distribution

- When we fix the parameters $\theta$ and $a$, the skewness and kurtosis of the TIIEHL-TL-LLoGL distribution decreases as $b$ and $\lambda$ increase.

- When we fix the parameters $\theta$ and $\lambda$, the skewness and kurtosis of the TIIEHL-TL-LLoGL distribution increases as $a$ and $b$ increase.

Figure 2: 3D Plots of skewness and kurtosis for TIIEHL-TL-LLoGL distribution
3.1.2. Type II Exponentiated Half-Logistic Topp-Leone-Log-Logistic Poisson (TIIEHL-TL-LLoGP) Distribution

The cdf and pdf of TIIEHL-TL-LLoGLP distribution are given by

\[
F_{X(t)}(x) = 1 - \frac{\exp \left( \theta \left[ \frac{1 - (1 + x^c)^{-2}}{1 + (1 + x^c)^{-2}} \right]^a \right) - 1}{\exp(\theta - 1)},
\]

and

\[
f_{X(t)}(x) = \frac{4ab\theta cx^{c-1}(1 + x^c)^{-2} \left[ 1 - (1 + x^c)^{-2} \right]^{b-1} (1 + x^c)^{-1}}{(1 + \left[ 1 - (1 + x^c)^{-2} \right]^{k})^{a+1}}
\]

\[
\times \left( 1 - \left[ 1 - (1 + x^c)^{-2} \right]^{k} \right)^{a-1} \exp \left( \theta \left[ \frac{1 - (1 + x^c)^{-2}}{1 + (1 + x^c)^{-2}} \right]^a \right) \frac{\exp(\theta - 1)}{\exp(\theta - 1)},
\]
respectively. The hrf is given by

\[
h_r(x) = \frac{4ab\theta cx^{-1}(1+x)^{-2}\left[1-(1+x)^{-2}\right]^{b-1}(1+x)^{-1}}{\left(1+\left[1-(1+x)^{-2}\right]^{b}\right)^{a+1}}
\times \left(1-\left[1-(1+x)^{-2}\right]^{b}\right)^{a-1} \exp\left(\theta \frac{\left[1-(1+x)^{-2}\right]^{b}}{1+\left[1-(1+x)^{-2}\right]^{b}}\right)^{a-1} \exp\left(\theta \frac{\left[1-(1+x)^{-2}\right]^{b}}{1+\left[1-(1+x)^{-2}\right]^{b}}\right) - 1,
\]

for \(a, b, \theta, c\) and \(x > 0\).

Figure 4: Plots of the pdf and hrf for the TIIEHL-TL-LLoGP distribution

Figure 4 shows the pdfs of the TIIEHL-TL-LLoGP distribution. The pdf can take various shapes that include almost symmetric, reverse-J, left, or right-skewed. Furthermore, the hazard rate functions (hrfs) for the TIIEHL-TL-LLoGP distribution exhibit increasing, reverse-J, bathtub, upside bathtub and bathtub followed by upside bathtub shapes.

We present in Figures 5 and 6, 3D plots of skewness and kurtosis of the TIIEHL-TL-LLoGP distribution. We observe that

- When we fix the parameters \(\theta\) and \(a\), the skewness and kurtosis of the TIIEHL-TL-LLoGP distribution decreases as \(b\) and \(\lambda\) increase.
- When we fix the parameters \(\theta\) and \(\lambda\), the skewness and kurtosis of the TIIEHL-TL-LLoGP distribution increases as \(a\) and \(b\) increase.

3.2. Type II Exponentiated Half-Logistic Topp-Leone-Weibull Power Series (TIIEHL-TL-WPS) Class of Distributions

Suppose the cdf and pdf of the Weibull distribution are given by \(G(x; \lambda) = 1 - \exp\left(-x^\lambda\right)\), for \(x \geq 0, \lambda > 0\) and \(g(x; \lambda) = \lambda x^{\lambda-1} \exp\left(-x^\lambda\right)\), for \(\lambda > 0, x > 0\), then, the cdf and pdf of the TIIEHL-TL-WGPS class of distributions are given by

\[
F_{X_{(1)}}(x) = 1 - \frac{C\left(\theta \left[\frac{1-\exp(-2x)}{1+\exp(-2x)}\right]^{b}\right)^{a}}{C(\theta)},
\]

\[
g(x; \lambda) = \lambda x^{\lambda-1} \exp\left(-x^\lambda\right),
\]

for \(a, b, \theta, c\) and \(x > 0\).
The cdf and pdf of TIIEHL-TL-LLoGP distribution are given by

\[
f_X(x) = \frac{4a b \theta \lambda x^{\lambda-1} \exp(-x^\lambda) \left[1 - \exp(-2x^\lambda)\right]^{b-1} \exp(-x^2)}{(1 + \left[1 - \exp(-2x^\lambda)\right]^b)^{a+1}}
\]

\[
\times \left(1 - \left[1 - \exp(-2x^\lambda)\right]^b\right)^{a-1} \frac{C' \left(\theta \left[\frac{1 - \exp(-2x^\lambda)}{1 + \exp(-2x^\lambda)}\right]^b\right)^a}{C(\theta)},
\]

respectively. The hrf is given by

\[
h_f(x) = \frac{4a b \theta \lambda x^{\lambda-1} \exp(-x^\lambda) \left[1 - \exp(-2x^\lambda)\right]^{b-1} \exp(-x^2)}{(1 + \left[1 - \exp(-2x^\lambda)\right]^b)^{a+1}}
\]

\[
\times \left(1 - \left[1 - \exp(-2x^\lambda)\right]^b\right)^{a-1} \frac{C' \left(\theta \left[\frac{1 - \exp(-2x^\lambda)}{1 + \exp(-2x^\lambda)}\right]^b\right)^a}{C(\theta)},
\]

for \(a, b, \theta, \lambda\) and \(x > 0\).

### 3.2.1. Type II Exponentiated Half-Logistic Topp-Leone-Weibull Logarithmic (TIIEHL-TL-WL) Distribution

The cdf and pdf of TIIEHL-TL-WL distribution are given by
Figure 6: 3D Plots of skewness and kurtosis for TIIEHL-TL-LLoGP distribution

\[ F_{X_{(1)}}(x) = 1 - \frac{-\log \left( 1 - \theta \left[ \frac{1-\exp(-2x^\lambda)}{1+\exp(-2x^\lambda)} \right]^b \right)^a}{-\log(1 - \theta)} , \]

and

\[ f_{X_{(1)}}(x) = \frac{4ab\theta \lambda x^{\lambda-1} \exp(-x^\lambda) \left[ 1 - \exp(-2x^\lambda) \right]^{b-1} \exp(-x^\lambda)}{\left( 1 + \left[ 1 - \exp(-2x^\lambda) \right]^{b}\right)^{a+1}} \times \left( 1 - \left[ 1 - \exp(-2x^\lambda) \right]^{b} \right)^{a-1} \frac{1 - \theta \left[ \frac{1-\exp(-2x^\lambda)}{1+\exp(-2x^\lambda)} \right]^b}{-\log(1 - \theta)} , \]

respectively. The hrf is given by

\[ h_{x}(x) = \frac{4ab\theta \lambda x^{\lambda-1} \exp(-x^\lambda) \left[ 1 - \exp(-2x^\lambda) \right]^{b-1} \exp(-x^\lambda)}{\left( 1 + \left[ 1 - \exp(-2x^\lambda) \right]^{b}\right)^{a+1}} \times \left( 1 - \left[ 1 - \exp(-2x^\lambda) \right]^{b} \right)^{a-1} \frac{1 - \theta \left[ \frac{1-\exp(-2x^\lambda)}{1+\exp(-2x^\lambda)} \right]^b}{-\log\left( 1 - \theta \left[ \frac{1-\exp(-2x^\lambda)}{1+\exp(-2x^\lambda)} \right]^b \right)^a}, \]
for $a, b, \theta, \lambda$, and $x > 0$. Figure 7 shows the pdfs of the TIEHL-TL-WL distribution. The pdf can take various shapes that include almost symmetric, reverse-J, left, or right-skewed. Furthermore, the hrfs for the TIEHL-TL-WL distribution exhibit increasing, reverse-J, bathtub, upside bathtub and upside bathtub followed by bathtub shapes.

We present in Figures 8 and 9, 3D plots of skewness and kurtosis of the TIEHL-TL-WL distribution. We observe that

- When we fix the parameters $\theta$ and $\lambda$, the skewness and kurtosis of the TIEHL-TL-WL distribution increases as $a$ and $b$ increase.

- When we fix the parameters $\theta$ and $b$, the skewness and kurtosis of the TIEHL-TL-WL distribution increases as $a$ and $\lambda$ increase.

3.2.2. Type II Exponentiated Half-Logistic Topp-Leone-Weibull Poisson (TIEHL-TL-WP) Distribution

The cdf and pdf of TIEHL-TL-WP distribution are given by

$$F_{X(1)}(x) = 1 - \frac{\exp\left(\frac{\theta \left[1 - \exp(-2x^\lambda)\right]^{b+1}}{1 + \left[1 - \exp(-2x^\lambda)\right]^b}\right)}{\exp(\theta - 1)},$$

and

$$f_{X(1)}(x) = \frac{4ab\theta\lambda x^{a-1} \exp(-x^\lambda) \left[1 - \exp(-2x^\lambda)\right]^{b-1} \exp(-x^\lambda)}{(1 + \left[1 - \exp(-2x^\lambda)\right]^b)^{a+1} \exp(\theta - 1)} \times \left(1 - \left[1 - \exp(-2x^\lambda)\right]^{b-1} \exp(\theta - 1)\right).$$
Figure 8: 3 D Plots of skewness and kurtosis for TIEHL-TL-WL distribution

respectively. The hrf is given by

\[
h_f(x) = \frac{4ab\theta \lambda x^{\lambda-1} \exp(-x^\lambda) [1 - \exp(-2x^\lambda)]^{b-1} \exp(-x^\lambda)}{(1 + [1 - \exp(-2x^\lambda)]^b)^{a+1}} \\
\times \left(1 - [1 - \exp(-2x^\lambda)]^b\right)^{a-1} \frac{\exp\left(\theta \left[\frac{1 - [1 - \exp(-2x^\lambda)]^b}{1 + [1 - \exp(-2x^\lambda)]^b}\right]^a\right)}{\exp\left(\theta \left[\frac{1 - [1 - \exp(-2x^\lambda)]^b}{1 + [1 - \exp(-2x^\lambda)]^b}\right]^a\right) - 1},
\]

for \(a, b, \theta, \lambda\) and \(x > 0\). Figure 10 shows the pdfs of the TIEHL-TL-WP distribution. The pdf can take various shapes that include almost symmetric, reverse-J, left, or right-skewed. Furthermore, the hrf’s for the TIEHL-TL-WP distribution exhibit increasing, reverse-J, bathtub, upside bathtub and upside bathtub followed by bathtub shapes.

We present in Figures 11 and 12, 3D plots of skewness and kurtosis of the TIEHL-TL-WP distribution. We observe that

- When we fix the parameters \(\theta\) and \(a\), the skewness and kurtosis of the TIEHL-TL-WP distribution decreases as \(b\) and \(\lambda\) increase.

- When we fix the parameters \(\theta\) and \(\lambda\), the skewness and kurtosis of the TIEHL-TL-WP distribution increases as \(a\) and \(b\) increase.
4. Simulation Study

In this section, the performance of the TIEHL-TL-WL distribution is examined by conducting various simulations for different sizes \( n = 25, 50, 100, 200, 400 \) and 800\). We simulate \( N = 1000 \) samples for the true parameters values given in Table 2. The table lists the mean MLEs of the model parameters along with the respective bias and root mean squared errors (RMSEs). The precision of the MLEs is discussed by means of the following measures: mean, mean square error (MSE) and average bias.
The bias and RMSE for the estimated parameter, say, $\hat{\theta}$, are given by:

$$Bias(\hat{\theta}) = \frac{1}{N} \sum_{i=1}^{N} \hat{\theta}_i - \theta,$$

and

$$RMSE(\hat{\theta}) = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2},$$

respectively. The results show that the estimation method used is appropriate for estimating the TIIEHL-TL-WL model parameters as the means of the parameters tend to be closer to the true parameter values when $n$ increases.

5. Application

In this section, we present two real data examples to demonstrate the importance and applicability of the TIIEHL-TL-WL distribution. The R software was used for data fitting and model diagnostics. The following goodness-of-fit statistics Cramer-von-Mises ($W^*$) and Andersen-Darling ($A^*$), -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Kolmogorov-Smirnov (K-S) statistic (and its p-value), and sum of squares (SS) are used to assess the performance of the model. The model with the smallest values of the goodness-of-fit statistics and a bigger p-value for the K-S statistic is regarded as the best model.

The TIIEHL-TL-WL distribution was compared to its nested models and to the following non-nested models: type II exponentiated half logistic Weibull (TIIEHLW) distribution by Al-Mofleh et al. (4) with the pdf

$$f_{\text{TIIEHLW}}(x; \alpha, \lambda, \delta, \gamma) = 2\alpha \lambda \delta \gamma x^{\gamma-1} \exp(-\delta x^\gamma) [1 - \exp(-\delta x^\gamma)]^{\lambda-1} \left[ 1 - [1 - \exp(-\delta x^\gamma)]^{\lambda} \right]^{a-1} \left[ 1 + [1 - \exp(-\delta x^\gamma)]^{\lambda} \right]^{a+1},$$

Figure 11: 3D Plots of skewness and kurtosis for TIIEHL-TL-WL distribution
**Type II Exponentiated Half-Logistic-Topp-Leone-G Power Series Class of Distributions with Applications**

Figure 12: 3 D Plots of skewness and kurtosis for TIIEHL-TL-WL distribution

### Table 2: Monte Carlo Simulation Results

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for $a, \lambda, \delta, \gamma > 0$ and $x > 0$, type II general inverse exponential Burr III (TIIGIE-BIII) distribution by Jamal et al. (13) with the pdf
\[
f_{\text{TIIGIE-BIII}}(x; \lambda, \theta, c, k) = \frac{\lambda \theta c x^{-c-1} (1 + x^{-c})^{-k-1} \left[1 - (1 + x^{-c})^{-k}\right]^{\theta-1}}{\left[1 - (1 + x^{-c})^{-k}\right]^2} \times \exp\left(-\frac{x}{\theta}\left[1 - (1 + x^{-c})^{-k}\right]^{\frac{\theta}{\theta}}\right),
\]
for $\lambda, \theta, c, k > 0$ and $x > 0$, and type II general inverse exponential Lomax (TIIGIE-Lx) distribution by Hamedani et al. (12) with the pdf
\[
f_{\text{TIIGIE-Lx}}(x; \lambda, \alpha, a, b) = \frac{\lambda \alpha a (1 + x)^{-(a+1)} (1 + x^{b})^{a(a+1)}}{\left[1 + x^{b}\right]^{a+1}} \times \exp\left(\frac{x}{\theta} \left(1 + x^{b}\right)^{a(a+1)}\right),
\]
for $\lambda, \alpha, a, b > 0$ and $x > 0$.

Application results are shown in Tables 3 and 4. Histogram of data, fitted densities and probability plots are shown in Figures 13 and 14.

5.1 Survival Times (in years)

The first real data set is a subset of data reported by Bekker et al. (7) which corresponds to the survival times (in years) of a group of patients given chemotherapy treatment alone. The data consisting of survival times (in years) for 46 patients are: 0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.694, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

From the results in Table 3, we conclude that the TIIEHL-TL-WL distribution is the “best” model compared to the selected models since it has the lowest values for the goodness-of-fit statistics $-2 \log L$, $AIC$, $AICC$, $BIC$, $A^*$, $W^*$ and K-S (and the largest p-value for the K-S statistic). Also, the plots in figure 13 show that the TIIEHL-TL-WL fitted the
from the results shown in Table 4, we conclude that the TIEHL-TL-WL distribution is indeed the “best” model.
compared to several selected models since it is associated with the lowest values for all the goodness-of-fit statistics (and the largest p-value for the K-S statistic). Also, the plots in figure 14 show that the TIIEHL-TL-WL fitted the data better than the other models of comparison and also has the smallest SS value.

5.3. Failure Times of 50 Components Data Set

The third data set taken from Murthy et al. (19) represents the failure times of 50 components (per 1000 hours). The data are

0.036, 0.148, 0.590, 3.076, 6.816, 0.058, 0.183, 0.618, 3.147, 7.896, 0.061, 0.192, 0.645, 3.625, 7.904, 0.074, 0.254, 0.961, 3.704, 8.022, 0.078, 0.262, 1.228, 3.931, 9.337, 0.086, 0.379, 1.600, 4.073, 10.940, 0.102, 0.381, 2.006, 4.393, 11.020, 0.103, 0.538, 2.054, 4.534, 13.880, 0.114, 0.570, 2.804, 4.893, 14.73, 0.116, 0.574, 3.058, 6.274, 15.08.

Figure 14: Fitted pdfs and probability plots for time to failure of kevlar 49/epoxy data set

<table>
<thead>
<tr>
<th>Model</th>
<th>μ</th>
<th>λ</th>
<th>θ</th>
<th>α</th>
<th>β</th>
<th>γ</th>
<th>δ</th>
<th>K-S p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>TIIEHL-TL-WL</td>
<td>0.0896</td>
<td>1.0133</td>
<td>0.9923</td>
<td>0.9839</td>
<td>0.1504</td>
<td>0.9845</td>
<td>0.3885</td>
<td>8.882×10⁻¹⁶</td>
</tr>
<tr>
<td>TIIEHL-TL-WL(1, b, λ)</td>
<td>0.1290</td>
<td>1.0621</td>
<td>1.3241</td>
<td>0.3977</td>
<td>3.1658</td>
<td>5.6788</td>
<td>3.6788</td>
<td>3.1658</td>
</tr>
<tr>
<td>TIIEHL-TL-WL(2, 0, λ)</td>
<td>9.4786</td>
<td>1.3272</td>
<td>1.3272</td>
<td>1.3272</td>
<td>1.3272</td>
<td>1.3272</td>
<td>1.3272</td>
<td>1.3272</td>
</tr>
<tr>
<td>TIIEHL-TL-WL(3, b, λ)</td>
<td>1.0397</td>
<td>1.0397</td>
<td>1.0397</td>
<td>1.0397</td>
<td>1.0397</td>
<td>1.0397</td>
<td>1.0397</td>
<td>1.0397</td>
</tr>
<tr>
<td>TIIEHL-TL-WL(3, 0, λ)</td>
<td>4.0696</td>
<td>1.0260</td>
<td>1.0451</td>
<td>1.0642</td>
<td>1.0833</td>
<td>1.1021</td>
<td>1.1212</td>
<td>1.1403</td>
</tr>
<tr>
<td>TIIEHL-BIII</td>
<td>1.3272</td>
<td>1.3272</td>
<td>1.3272</td>
<td>1.3272</td>
<td>1.3272</td>
<td>1.3272</td>
<td>1.3272</td>
<td>1.3272</td>
</tr>
<tr>
<td>TIIEHLW</td>
<td>1.0704</td>
<td>1.5713</td>
<td>1.5713</td>
<td>1.5713</td>
<td>1.5713</td>
<td>1.5713</td>
<td>1.5713</td>
<td>1.5713</td>
</tr>
<tr>
<td>TIEGE-BIII</td>
<td>1.4663</td>
<td>1.8263</td>
<td>1.8263</td>
<td>1.8263</td>
<td>1.8263</td>
<td>1.8263</td>
<td>1.8263</td>
<td>1.8263</td>
</tr>
<tr>
<td>TIEGE-Lx</td>
<td>1.4663</td>
<td>1.8263</td>
<td>1.8263</td>
<td>1.8263</td>
<td>1.8263</td>
<td>1.8263</td>
<td>1.8263</td>
<td>1.8263</td>
</tr>
</tbody>
</table>

Furthermore, from the results shown in Table 5, we conclude that the TIIEHL-TL-WL distribution is indeed the “best” model compared to several selected models since it is associated with the lowest values for all the goodness-of-fit statistics (and the largest p-value for the K-S statistic). Also, the plots in figure 15 show that the TIIEHL-TL-WL fitted the data better than the other models of comparison and also has the smallest SS value.
Figure 15: Fitted pdfs and probability plots for failure times of 50 components data set

Table 6: Likelihood ratio test results

<table>
<thead>
<tr>
<th>Model</th>
<th>df</th>
<th>$\chi^2$ ($p$-value)</th>
<th>$\chi^2$ ($p$-value)</th>
</tr>
</thead>
<tbody>
<tr>
<td>TIIIEHL-TL-WL(1, b, $\theta$, $\lambda$)</td>
<td>1</td>
<td>82.5525 (&lt;0.00001)</td>
<td>17.6272 (0.00003)</td>
</tr>
<tr>
<td>TIIIEHL-TL-WL(a, 1, $\theta$, $\lambda$)</td>
<td>1</td>
<td>153.6602 (&lt;0.00001)</td>
<td>494.3255 (&lt;0.00001)</td>
</tr>
<tr>
<td>TIIIEHL-TL-WL(a, b, $\theta$, 1)</td>
<td>1</td>
<td>60.4647 (&lt;0.00001)</td>
<td>135.1382 (&lt;0.00001)</td>
</tr>
<tr>
<td>TIIIEHL-TL-WL(1, 1, $\theta$, $\lambda$)</td>
<td>2</td>
<td>270.6975 (&lt;0.00001)</td>
<td>565.3379 (&lt;0.00001)</td>
</tr>
<tr>
<td>TIIIEHL-TL-WL(1, b, $\theta$, 1)</td>
<td>2</td>
<td>52.2514 (&lt;0.00001)</td>
<td>135.9695 (&lt;0.00001)</td>
</tr>
<tr>
<td>TIIIEHL-TL-WL(a, 1, $\theta$, 1)</td>
<td>2</td>
<td>97.1771 (&lt;0.00001)</td>
<td>7.7442 (0.0208)</td>
</tr>
</tbody>
</table>

Table 7: Likelihood ratio test results

<table>
<thead>
<tr>
<th>Model</th>
<th>df</th>
<th>$\chi^2$ ($p$-value)</th>
</tr>
</thead>
<tbody>
<tr>
<td>TIIIEHL-TL-WL(1, b, $\theta$, $\lambda$)</td>
<td>1</td>
<td>123.5628 (&lt;0.00001)</td>
</tr>
<tr>
<td>TIIIEHL-TL-WL(a, 1, $\theta$, $\lambda$)</td>
<td>1</td>
<td>140.5868 (&lt;0.00001)</td>
</tr>
<tr>
<td>TIIIEHL-TL-WL(a, b, $\theta$, 1)</td>
<td>1</td>
<td>346.4868 (&lt;0.00001)</td>
</tr>
<tr>
<td>TIIIEHL-TL-WL(1, 1, $\theta$, $\lambda$)</td>
<td>2</td>
<td>259.3636 (&lt;0.00001)</td>
</tr>
<tr>
<td>TIIIEHL-TL-WL(1, b, $\theta$, 1)</td>
<td>2</td>
<td>405.9026 (&lt;0.00001)</td>
</tr>
<tr>
<td>TIIIEHL-TL-WL(a, 1, $\theta$, 1)</td>
<td>2</td>
<td>126.1133 (&lt;0.00001)</td>
</tr>
</tbody>
</table>

5.4. Likelihood Ratio Test

The likelihood ratio test results in Table 6 and 7 indicates that the TIIIEHL-TL-WL performs better than its nested models at 5% level of significance, since all p-values are less than 0.05 among all the data sets considered.

6. Conclusion

A new class of distributions called the type II exponentiated half-logistic Topp-Leone-G power series (TIIIEHL-TL-GPS) class of distributions is introduced. Some mathematical properties including moments and moment generating function, order statistics, entropy and quantiles are provided. Model parameters are estimated using the maximum likelihood method and the performance of the estimates is assessed by means of a simulation study. The potentiality of the new model is demonstrated by means of three real data sets.
Appendix

The following URL contains the appendix material: https://drive.google.com/file/d/1TTrilDRb5cFul49QiVl7XwTM57c5Z3Gj/view?usp=sharing

7. Declarations
The authors have nothing to declare.

8. Funding
The research is not a funded research.

9. Conflict of Interest
Authors declare no conflict of interest.

10. Availability of data and material
Not applicable

11. Code availability
R codes provided in the appendix section.

References